# Volumes of hyperbolic manifolds

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The aim of this talk was to show that odd-dimensional hyperbolic manifolds $M$ of finite volume give rise to nontrivial classes in the algebraic K-theory of number fields, by the following theorem of Goncharov, which is the main result of [1].

**Theorem 1** *: To each compact hyperbolic manifold of odd dimension $n$, there exists an element*

$$\gamma(M) \in K_n\left(\overline{\mathbb{Q}}\right) \otimes \mathbb{Q}$$

*such that the Borel regulator $r_n : K_n\left(\overline{\mathbb{Q}}\right) \otimes \mathbb{Q} \to \mathbb{R}$ maps*

$$r_n(\gamma(M)) = vol(M).$$

There are two constructions of $\gamma(M)$, a homological and a Hodge-theoretic one.

We discussed in our talk the homological construction ([1], section 2), which uses the description of $K_n\left(\overline{\mathbb{Q}}\right) \otimes \mathbb{Q}$ as the subspace of indecomposable elements in $H_n\left(GL\left(\overline{\mathbb{Q}}\right);\mathbb{Q}\right)$ (see [3], p.29), and of the Borel regulator $r_n$ as being defined by pairing with the Borel class $b_n \in H_c^n(GL(\mathbb{C});\mathbb{Q})$. ($H_c^*$ denotes the continuous cohomology. It is known that $H_c^*(GL(N;\mathbb{C});\mathbb{Q}) \cong H^*(U(N);\mathbb{Q})$ and $b_{2m-1}$ is defined to be the pull-back of the fundamental class $\left[S^{2m-1}\right] \in H^{2m-1}\left(S^{2m-1}\right)$ under the canonical map $U(m) \to S^{2m-1}$.)

Each closed hyperbolic manifold $M = \Gamma\backslash\mathbb{H}^n$ gives rise, using an embedding $j : \Gamma \to SO\left(n,1;\overline{\mathbb{Q}}\right)$, to a class $j_*[M] \in H_n\left(SO\left(n,1;\overline{\mathbb{Q}}\right)\right)$. In the case $n = 3$ one can use the isomorphism $SO(n,1;\mathbb{R}) \cong PSL(2,\mathbb{C})$ to make $j_*[M]$ a class in the homology of $GL$ and hence in K-theory. In general, however, the construction is more complicated: one uses the isomorphism $H_n\left(SO\left(n,1;\overline{\mathbb{Q}}\right)\right) \cong H_n\left(Spin\left(n,1;\overline{\mathbb{Q}}\right)\right)$ and constructs the homology class as $s_{+*}j_*[M]$, where $s_+ : Spin(n,1) \to GL(N,\mathbb{C})$ is the spin representation (for $n$ odd and $N = 2^{\frac{n+1}{2}+1}$).

We now explain the proof of Theorem 1 in some detail.
Recall that a manifold $M^n$ is called hyperbolic if it is diffeomorphic to $\Gamma\backslash\mathbb{H}^n$,

for $\Gamma < SO(n,1)$ discrete and torsionfree. If $\Gamma \backslash \mathbb{H}^n$ has finite volume, then $\Gamma$ can be conjugated to be a subgroup of $SO\left(n,1;\overline{\mathbb{Q}}\right)$ (Weil). There is a canonical identification $H_*(M) \cong H_*(\Gamma)$. In particular, if $M^n$ is compact and orientable, then the fundamental class $[M] \in H_n(M)$ corresponds to a class in $H_n(\Gamma)$.

One can define a continuous $n$-cocycle on $Isom^+(\mathbb{H}^n) = SO(n,1)$ by sending each $(g_0, \ldots, g_n)$ to the volume of the geodesic simplex with vertices $g_0x, \ldots, g_nx$, for a fixed $x \in \mathbb{H}^n$. (The cocycle property follows from the additivity of volume.) Its cohomology class is called the Lobachevsky class $v_n \in H_c^n(SO(n,1))$. The proof of Theorem 1 will follow from Lemma 1 and Lemma 2.

**Lemma 1** *: If $M$ is a compact, orientable, hyperbolic n-manifold, then $< v_n, j_*[M] >= vol(M)$ for the inclusion $j : \Gamma \to SO(n,1)$.*

*Proof:* There exists a triangulation of $M$ by geodesic simplices $\sigma_1, \ldots, \sigma_r$, such that all vertices of all $\sigma_i$ are in $x_0 \in M$. We have that $[M]$ is represented by $\sigma_1 + \ldots + \sigma_r$. Moreover, each $\sigma_i$ has n+1 edges which are closed loops and thus can be considered as an n+1-tuple of elements of $\Gamma = \pi_1(M, x_0)$. Hence, we get an element $j_*\sigma_i$ in the bar resolutions of $\Gamma$ resp. $SO(n,1)$. (According to Eilenberg-McLane, this map defines the isomorphism $H_n(M) \to H_n(\Gamma)$.) From the definition of the Lobachevsky class we have $< v_n, j_*(\sigma_i) >= vol(\sigma_i)$, since the geodesic simplex with vertices $g_0\tilde{x}_0, g_1\tilde{x}_0, \ldots, g_n\tilde{x}_0$, where $g_0, \ldots, g_n \in \pi_1(M, x_0)$ are represented by the edges of $\sigma_i$, is a lift of $\sigma_i$ to $\widetilde{M} = \mathbb{H}^n$. Hence $< v_n, j_*([M]) >= vol(\sigma_1) + \ldots + vol(\sigma_r) = vol(M)$. □

Using the spin representation $s_+ : Spin(n,1) \to GL(N;\mathbb{C})$, one can relate the Lobachevsky class to the Borel class.

**Lemma 2** *: $s_+^* b_n = c_n v_n \in H_c^n(SO(n,1))$ with $c_n \neq 0$.*

Now one defines $\gamma(M) := \frac{1}{c_n} s_{+*} j_*[M]$ and gets

$$r_n(\gamma(M)) = \frac{1}{c_n} < b_n, s_{+*}j_*[M] >= vol(M).$$

This proves Theorem 1. (For noncompact hyperbolic manifolds of finite volume, the proof becomes more involved because one has to work with the relative fundamental class.) □

The second (Hodge-theoretic) construction, which we did not have the time to discuss, uses a motivic interpretation of the scissors congruence groups. It associates to each simplex a mixed Hodge-Tate structure such that the volume of the simplex is the period of a certain framing. The Dehn-invariant, considered as a coproduct on the scissors congruence group, corresponds to the well-known coproduct $\tilde{\nu}$ on the Hopf algebra of all framed mixed Hodge-Tate structures. A hyperbolic manifold gives an element in the scissors congruence group, which is in the kernel of the Dehn-invariant. Hence the corresponding Hodge-Tate structure is in the kernel of $\tilde{\nu}$. But, according to Beilinson, $ker\,(\tilde{\nu}) \cong K_*\left(\overline{\mathbb{Q}}\right) \otimes \mathbb{Q}$ and the period (i.e. the volume) corresponds to the Borel regulator. ([1], section 3)

Finally, we explain how Theorem 1 connects hyperbolic volume to polylogarithms.

Zagier's conjecture states that $K_{2n-1}(F) \otimes \mathbb{Q}$ is isomorphic to the Bloch group $\mathcal{B}_n(F)$ and that, under this isomorphism, the Borel regulator $r_{2n-1} : K_{2n-1}(F) \otimes \mathbb{Q} \to \mathbb{R}$ corresponds to the single-valued polylogaritm $P_n : \mathcal{B}_n(F) \to \mathbb{R}$.

In view of Theorem 1, Zagier's conjecture would imply that volumes of odd-dimensional hyperbolic manifolds are related to polylogaritms as follows.

**Conjecture 1** *: Let $M^{2n-1}$ be a 2n-1-dimensional hyperbolic manifold of finite volume. Then*

$$vol\,(M) = \sum_i P_n(z_i)$$

*for some $z_i \in \overline{\mathbb{Q}}$ with $\delta_n\left(\sum_i z_i\right) = 0 \in B_{n-1}\left(\overline{\mathbb{Q}}\right) \otimes \overline{\mathbb{Q}}^*$.*

Conjecture 1 is well-known in the case of 3-manifolds. Indeed, if $\Delta$ is an ideal geodesic simplex in $\mathbb{H}^3$ with vertices $v_0, v_1, v_2, v_3 \in \partial_\infty \mathbb{H}^3 = \mathbb{C}P^1$, then $vol\,(\Delta) = P_2(r(v_0, v_1, v_2, v_3))$, where $r$ denotes the cross ratio. Assuming an ideal geodesic triangulation on $M$, this gives the first equality of conjecture 1. The second equality is true because $\delta_2(z_i)$ factors over the Dehn invariant $D_3$ of the geodesic simplex with vertices $\infty, 0, 1, z_i$. (Precisely, if we define a map $\lambda : \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Z} \to \Lambda^2\mathbb{C}$ by $\lambda(r \otimes \theta) = e^r \wedge e^{i\theta}$, then $\delta_2 \sim \lambda D_3$.) If we have a geodesic triangulation of the closed manifold $M$, then the angles around any edge add up to $2\pi$, hence the Dehn invariants add up to 0. This implies $\sum_i \delta_2(z_i) = 0$.

Conjecture 1 is also known in the case of 5-manifolds, which follows from the results in [2].

# References


[1] A. GONCHAROV, 'Volumes of hyperbolic manifolds and mixed Tate motives', *J. AMS* 12 (1999).

[2] A. GONCHAROV, 'Geometry of configurations, polylogarithms, and motivic cohomology', *Adv. Math.* 114 (1995).

[3] V. SRINIVAS, 'Algebraic K-theory', Birkhäuser (1996).