Theorem (**Agol-Storm-Thurston**, [2], Cor.2.2.): *If M is a hyperbolic 3-manifold containing an incompressible surface F, then*

$$Vol(M) \geq Vol\left(Guts\left(\overline{M-F}\right)\right) \geq -V_{oct}\chi\left(Guts\left(\overline{M-F}\right)\right) = V_{oct}\frac{1}{4} \parallel \partial Guts\left(\overline{M-F}\right) \parallel$$

with $V_{oct} = 3.66..$ the volume of a regular ideal octahedron in $\mathbb{H}^3$.

Notation:

- $N = \overline{M-F}$ compact, atoroidal 3-manifold with boundary,
  JSJ-decomposition (cut along annuli): $N = Char(N) \cup Guts(N)$,
  $Char(N)$ consists of I-bundles and solid tori, $Guts(N)$ is acylindrical, hence admits a hyperbolic metric with geodesic boundary and cusps
- simplicial volume:
  X compact, orientable manifold, $[X, \partial X] \in H_n(X, \partial X; \mathbb{R})$ fundamental class
  $\parallel X \parallel := inf\left\{\sum \mid a_i \mid : \sum a_i\sigma_i \text{ represents } [X, \partial X]\right\}$
  Theorem (Gromov): X^n hyperbolic of finite volume: $\parallel X \parallel = \frac{1}{V_n}Vol(X)$,
  $V_3 = 1.01..$ volume of regular ideal tetrahedron in $\mathbb{H}^3$

Its proof in [2] uses analytical methods (Perelman's entropy estimate for the Ricci flow, work of Bray and Miao on the Penrose conjecture) and does not seem to generalize to laminations. Using the original topological approach of [1], we want to give a generalization to laminations and to higher-dimensional non-hyperbolic manifolds.

Let $\mathcal{F}$ be a lamination on a manifold M. A singular simplex $\sigma : \Delta^n \to M$ is **transverse** to $\mathcal{F}$ if the pull-back of $\mathcal{F}$ to Δ^n is an affine lamination, it is **normal** to $\mathcal{F}$ if the pull-back of each single leaf to Δ^n is an affine lamination.

It was first observed in [3] that a refinement of the simplicial volume gives a meaningful invariant for foliations and laminations.

Definition 1 : *Let M be a compact, oriented, connected manifold, possibly with boundary, and $\mathcal{F}$ a foliation or lamination on M. Then define*

$$\parallel M \parallel_{\mathcal{F}} := inf\left\{\sum_{i=1}^r \mid a_i \mid : \psi\left(\sum_{i=1}^r a_i\sigma_i\right) \text{ represents } [M, \partial M], \sigma_i : \Delta^n \to M \text{ transverse to } \mathcal{F}\right\}$$

$$\parallel M \parallel_{\mathcal{F}}^{normal} := inf\left\{\sum_{i=1}^r \mid a_i \mid : \psi\left(\sum_{i=1}^r a_i\sigma_i\right) \text{ represents } [M, \partial M], \sigma_i : \Delta^n \to M \text{ normal to } \mathcal{F}\right\}.$$

There is an obvious inequality

$$\parallel M \parallel \leq \parallel M \parallel_{\mathcal{F}}^{normal} \leq \parallel M \parallel_{\mathcal{F}} .$$

In the case of foliations, equality $\parallel M \parallel_{\mathcal{F}}^{normal} = \parallel M \parallel_{\mathcal{F}}$ holds.

Theorem 1 *(K., [5], Thm.1): Let M be a compact, orientable n-manifold and $\mathcal{F}$ a lamination (of codimension one) of M. Assume that there exists a compact, aspherical, n-dimensional submanifold $Q \subset \overline{M-\mathcal{F}}$ such that, if we let* $N = \overline{M-\mathcal{F}}, \partial_0 N = \partial N \cap \partial M, \partial_1 N = \overline{\partial N - \partial_0 N}, \partial_{00} Q = \overline{\partial Q - (\partial N \cap \partial Q)}, \partial_{01} Q = \partial Q \cap \partial_0 N, \partial_0 Q = \partial_{00} Q \cup \partial_{01} Q, \partial_1 Q = \partial_1 N \cap \partial Q$*, then*
i) each component of $\partial_{00} Q := \overline{\partial Q - (\partial M \cap \partial Q)}$ has amenable fundamental group,
ii) $(Q, \partial_1 Q)$ is pared acylindrical,
iii) the decomposition $N = Q \cup \overline{(N-Q)}$ is essential (i.e. the inclusions $Q \rightarrow N, \overline{N-Q} \rightarrow N, \partial_0 Q \rightarrow Q, \partial_0 Q \rightarrow \overline{N-Q}$ are π_1-injective) . Then

$$\| M \|_{\mathcal{F}}^{normal} \geq \frac{1}{n+1} \| \partial Q \|, \| M \|_{\mathcal{F}} \geq \frac{1}{\left[\frac{n}{2}\right]+1} \| \partial Q \| .$$

Proof: To give a flavor of the argument, we describe it in the simplest case: M is a hyperbolic 3-manifold, $\mathcal{F} = F$ a geodesic surface (i.e. $\partial Guts\left(\overline{M-\mathcal{F}}\right) = 2F$).

Let $\sum a_i \sigma_i$ be a normal cycle representing $[M]$. Then $\sum a_i \left(\sigma_i \cap F\right)$ represents $[F]$ and to get the wanted inequality $\sum \mid a_i \mid \geq \frac{1}{4} \| 2F \|$ it would suffice to have that each σ_i intersects F in at most 4 simplices.

Of course, there is a priori no reason to have an upper bound on the number of intersections that a normal simplex may have with the geodesic surface.

However, one can easily see that, whenever a 3-simplex intersects the surface more than 4 times, than two of the intersection triangles must have a parallel edge, i.e. cut out a square on one boundary face of the standard 3-simplex.

If σ_i mapped this square to a cylinder (i.e. mapped the opposite edges to the same edge), then one could use the acylindricity of $\overline{M-F}$ to argue that the cylinder degenerates after homotopy, hence can be removed without changing the homology class, and thus the number of intersections can be reduced.

Then the proof consists of defining a straightening which produces the maximally possible number of cylinders. (Some care is needed because the subdivided 1-skeleton can, of course, not be straightened arbitrarily. Even though each 1-simplex can be moved freely, the 2-skeleton imposes homotopy relations between concatenations of 1-simplices, which have to be respected by the straightening.)

Roughly the same argument works whenever $Q = N$. To handle the general case $Q \neq N$, one would like to define a retraction $r : C_*(N) \rightarrow C_*(Q)$. It seems not possible to define r directly, but, using Gromov's work on multicomplexes, one can at least define it up to some 'amenable ambiguity' and use this to prove the general case. $\square$

Corollary 1 *: Let M be a compact Riemannian n-manifold of negative sectional curvature and finite volume. Let $F \subset M$ be a geodesic $n-1$-dimensional hypersurface of finite volume. Then $\| F \| \leq \frac{n+1}{2} \| M \|$.*

Proof: : Consider the lamination given by F. Its complement $N = \overline{M-F}$ is aspherical and pared acylindrical. If N is compact we can choose $Q = N$, in which case the other assumptions of Theorem 1 are trivially satisfied. If N is not compact, we cut off the cuspical ends along submanifolds with amenable fundamental groups to get Q. From Theorem 1 we conclude $\| M \|_F^{normal} \geq \frac{1}{n+1} \| \partial N \|$. The boundary of N consists of two copies of F, hence $\| \partial N \| = 2 \| F \|$. Moreover $\| M \|_F^{normal} = \| M \|$, because any cycle can be homotoped to be normal to F. The claim follows. $\square$

Corollary 2 *: Let M be a compact hyperbolic 3-manifold and $\mathcal{F}$ an essential lamination of M. Then*

$$\parallel M \parallel_{\mathcal{F}}^{normal} \geq \frac{1}{2} \parallel \partial Guts\left(\overline{M-\mathcal{F}}\right) \parallel .$$

Proof: The inequality $\parallel M \parallel_{\mathcal{F}}^{normal} \geq \frac{1}{4} \parallel \partial Guts\left(\overline{M-\mathcal{F}}\right) \parallel$ follows from Theorem 1. The improvement by a factor 2 can be obtained as in [1] by considering more general polyhedral (rather than simplicial) norms. □

If $\mathcal{F}$ consists of one compact, incompressible surface, then $\parallel M \parallel_{\mathcal{F}}^{normal} = \parallel M \parallel$, and the above inequality is exactly the weaker form of Agol's inequality, from [1].
We discuss the application to non-existence results for laminations on 3-manifolds.

For a lamination $\mathcal{F}$ on a 3-manifold M, its leaf space is defined as $T = \widetilde{M}/\sim$, where two points of the universal covering $\widetilde{M}$ are identified if either they belong to the same leaf of the pull-back lamination $\widetilde{\mathcal{F}}$, or they belong to the same connected component of the complement $\overline{\widetilde{M}-\widetilde{\mathcal{F}}}$. (One assumes that $\mathcal{F}$ has no isolated leaf, which can be assured by replacing each isolated leaf with a one-parameter family of leaves.)

Corollary 3 *: If M is a hyperbolic 3-manifold and $\mathcal{F}$ a tight lamination (i.e. the leaf space T is Hausdorff), then $Vol(M) \geq V_3 \frac{1}{2} \parallel \partial Guts\left(\overline{M-\mathcal{F}}\right) \parallel$.*

Proof: If $\mathcal{F}$ is tight, then each cycle can be homotoped to be normal to $\mathcal{F}$, thus $\parallel M \parallel_{\mathcal{F}}^{normal} = \parallel M \parallel$. The claim follows then from corollary 2. □

Tao Li has proved that the existence of a transv. orientable essential lamination on a given hyperbolic 3-manifold M implies that the same M must also carry a tight lamination.

If M is hyperbolic and carries a tight lamination with empty guts, then Calegari and Dunfield have shown ([4], Theorem 3.2.) that there is an injective homomorphism $\pi_1 M \rightarrow Homeo\left(S^1\right)$. Calegari and Dunfield also observed that a generalization of Agol's inequality to tight laminations would give obstructions to existence of laminations with nonempty guts, and, e.g., exclude existence of tight laminations on the Weeks manifold.

The following corollary applies, for example, to all hyperbolic manifolds M obtained by Dehn-filling the complement of the figure-eight knot in S^3. (It follows from work of Hatcher that each of these M contains tight laminations. By the following corollary, all these tight laminations have empty guts.)

Corollary 4 *: If M is a closed hyperbolic 3-manifold with $Vol(M) < 2V_3 = 2.02...$, then each tight lamination has empty guts.*

Proof: If $Guts\left(\overline{M-\mathcal{F}}\right)$ is not empty, then $\parallel \partial Guts\left(\overline{M-\mathcal{F}}\right) \parallel \geq 4$. □

Corollary 5 *: The Weeks manifold W admits no transv. orientable essential lamination.*

Proof: Calegari-Dunfield have shown that $\pi_1 W$ is not a subgroup of $Homeo^+ \mathbb{S}^1$. Moreover, $Vol(W) = 0.94.. < 2.02...$ Apply corollary 4. □

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