

Topological Approaches to Volume

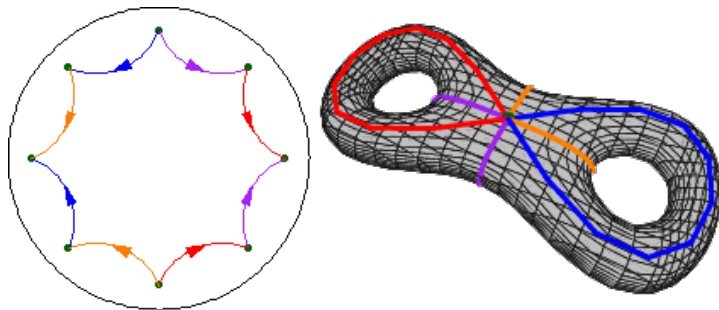
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If M is an aspherical manifold, then $M \simeq B\Gamma$ and M can be understood by studying the action of $\Gamma = \pi_1 M$ on the contractible manifold $\tilde{M}$.

- Toy Example:
 $M = T^n$ torus, $\Gamma = \mathbb{Z}^n$ acts on $\mathbb{R}^n$ by translations.
- Example:
 M surface of genus ≥ 2 : Γ acts on $\mathbb{H}^2$ by isometries.

Preparations II



If G is a semisimple Lie group and $K \subset G$ a maximal compact subgroup, then G/K is a **symmetric space of noncompact type**. The Killing form defines a G -invariant Riemannian metric on G/K . Its sectional curvature is nonpositive. G/K is contractible.

Examples:

- $\mathbb{H}^2 = PSL(2, \mathbb{R}) / PSO(2) = SL(2, \mathbb{R}) / SO(2)$
- $\mathbb{H}^3 = PSL(2, \mathbb{C}) / PSU(2) = SL(2, \mathbb{C}) / SU(2)$
- $\mathbb{H}^n = SO(n, 1) / SO(n) = Spin(n, 1) / Spin(n)$.

M is a **locally symmetric space of noncompact type** if $\tilde{M}$ is a symmetric space of noncompact type. Then

$$M = \Gamma \backslash G / K$$

with $\pi_1 M \cong \Gamma \subset G$ discrete.

For manifolds M with boundary ∂M we say that M is locally symmetric of noncompact type if $\text{int}(M) := M - \partial M$ it is.

- Γ is a lattice if $\text{vol}(M) < \infty$.
- Γ is a uniform lattice if M is compact.

Geometry of **simply connected manifolds $\tilde{M}$ of nonpositive sectional curvature**

- Ideal boundary at infinity: $\partial_\infty \tilde{M}$

Points of $\partial_\infty \tilde{M}$ are equivalence classes of geodesic rays. (Two rays are equivalent if they have bounded distance.) Can be topologized such that $S^{n-1} \cong \partial_\infty \tilde{M}$.

- Rank

$$\text{Rank}(\tilde{M}) := \max \left\{ r : \text{ex. isom. embedding } \mathbb{R}^r \rightarrow \tilde{M} \right\}.$$

- $\text{Rank}(\tilde{M}) = 1 \iff \text{sec} < 0$
- $\text{Rank}(SL(n, \mathbb{R})/SO(n)) = n - 1$

- Straight simplices

For each ordered set $(v_0, \dots, v_k)$ with $v_i \in \tilde{M}$ exists a straight simplex $\text{str}(v_0, \dots, v_k)$.

If $\text{sec} < 0$, then this works also for $v_i \in \partial_\infty \tilde{M}$, these simplices are called ideal simplices.

M is a hyperbolic manifold if $\tilde{M} = \mathbb{H}^n$. Thus

$$M = \Gamma \backslash \mathbb{H}^n$$

with $\Gamma \subset \text{Isom}(\mathbb{H}^n)$ discrete. If M is orientable, then $\Gamma \subset \text{Isom}^+(\mathbb{H}^n) = \text{SO}(n, 1)$.

Theorem (Hyperbolization Theorem)

Compact, aspherical, atoroidal 3-manifolds are hyperbolic.

If M is a hyperbolic 3-manifold of finite volume, then ∂M is a (possibly empty) union of incompressible tori. (Ends corresponding to boundary components of genus ≥ 2 must have infinite volume.) If the end corresponding to $\partial_i M$ has finite volume, then the subgroup $\Gamma_i \subset \Gamma$ corresponding to $\pi_1 \partial_i M \subset \pi_1 M$ consists of parabolic isometries fixing some $c_i \in \partial_\infty \tilde{M}$. The ideal boundary point c_i is called a cusp of M .

Part I:

The simplicial volume and the rigidity of efficient fundamental cycles

- G/K (irreducible) symmetric space of noncompact type
e.g. $H^3 = SL(2, \mathbf{C})/SU(2)$.
- $\Gamma \subset G$ lattice

$$M^n = \Gamma \backslash G/K$$

Volume is a topological invariant of locally symmetric spaces:

- **Mostow Rigidity:** $n = \dim(M) \geq 3$:
homeomorphism $\Rightarrow$ isometry
- **Chern-Gauß-Bonnet** $\implies n = \dim(M)$ even: $Vol(M) \sim \chi(M)$

Topological definitions of volume?

Change of volume under topological operations, Cut and Paste?

Simplicial Volume

M^n compact, orientable manifold,
 $[M, \partial M] \in H_n(M, \partial M; \mathbf{R})$ its fundamental class

Definition (Gromov)

$$\| M \| = \inf \left\{ \sum_{i=1}^r |a_i| : \sum_{i=1}^r a_i \sigma_i \text{ represents } [M, \partial M] \right\}.$$

Example (Gromov)

$$\| S^n \| = 0, \| T^n \| = 0.$$

Example (Gromov)

For all G/K exists $c_{G/K}$ such that $M = \Gamma \backslash G/K$ implies

$$\text{Vol}(M) = c_{G/K} \| M \|.$$

$c_{H^n} = V_n := \text{volume of regular ideal simplex in } H^n.$

Gromov's Proof of $\text{Vol}(M) = V_n \|M\|$

M closed hyperbolic manifold.

- Proof of $\|M\| \geq \frac{\text{Vol}(M)}{V_n}$:
each (fundamental) cycle can be straightened, straight simplices have volume $\leq V_n$.
- Proof of $\|M\| \leq \frac{\text{Vol}(M)}{V_n}$:
there exist fundamental cycles approximating the Haar measure on $\{\text{regular ideal simplices}\} \simeq \text{Isom}(H^n)$. (In particular all involved simplices have volume nearly V_n .)

Similar argument for M cusped.

Remark (Rigidity of efficient fundamental cycles: Jungreis, K.)

If $n \geq 3$ and M has not the same cusps as the figure 8 knot complement, then this is the only construction of efficient fundamental cycles.

Gromov used the simplicial volume to give a topological proof of Mostow rigidity (homotopy equivalence implies isometry) in the constant curvature case:

- homotopy equivalences f preserve $\| M \|$
- hence simplices of maximal volume must be sent to simplices of maximal volume
- reflection group of regular ideal simplex (= simplex of maximal volume) has dense orbits on $\partial_\infty H^n$
- hence $\partial_\infty \tilde{f}$ is conformal, f is an isometry

Rigidity of efficient fundamental cycles implies a priori: $\tilde{f}$ even preserves the equidistribution on regular ideal simplices.

For a codimension 1 submanifold $A \subset M$ we denote M_A the manifold obtained by cutting M along A .

Theorem (Gromov 1982)

If $\pi_1 A$ is amenable and injects into $\pi_1 M$, then

$$\| M_A, \partial M_A \| = \| M, \partial M \| .$$

For example abelian groups are amenable.

Not applicable to hyperbolic 3-manifolds: no incompressible tori.

Observation

If $\Sigma \subset M$ is totally geodesic and M^τ obtained from M by cutting and pasting via an isometry $\tau : \Sigma \rightarrow \Sigma$, then $\text{vol}(M^\tau) = \text{vol}(M)$.

Obvious geometric fact, since $\text{Vol}(M) = \text{Vol}(M - \Sigma)$.

Topological proofs? No proof via simplicial volume, but simple proof via group homology (in the setting of generalized mutations).

Theorem (Jungreis 1997 for M closed, K. 2003 for M cusped)

If $\Sigma \subset M$ is totally geodesic in a finite-volume hyperbolic n -manifold, $n \geq 3$, then

$$\| M_\Sigma, \partial M_\Sigma \| > \| M, \partial M \| .$$

Theorem (Frigerio-Pagliantini 2010)

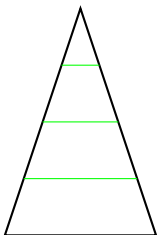
Inequality is sharp: $\frac{\| M_\Sigma, \partial M_\Sigma \|}{\| M, \partial M \|} - 1$ is bounded in terms of $\frac{\text{Vol}(\Sigma)}{\text{Vol}(M)}$.

Transverse simplices

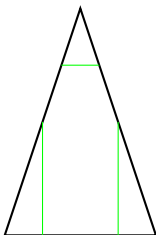
$\mathcal{F}$ codimension one foliation/lamination on M .

$\sigma : \Delta^n \rightarrow M$ is

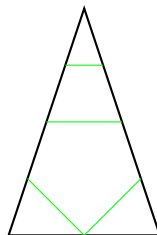
- $\dashv$ to $\mathcal{F}$ if $\mathcal{F}|_{\Delta^n}$ is affine foliation/lamination,
- **normal to $\mathcal{F}$** if each leaf is affine.



transverse



normal, not transverse



not normal

Another application of efficient fundamental cycles

Definition (Calegari)

$$\| M \|_{\mathcal{F}} = \inf \{ \sum_{i=1}^r |a_i| : \sum_{i=1}^r a_i \sigma_i \text{ represents } [M, \partial M], \sigma_i \cap \mathcal{F} \}$$

If M is a closed 3-manifold and $\mathcal{F}$ has no Reeb components, then the leaf space of $\tilde{\mathcal{F}}$ is a (non-Hausdorff) simply connected 1-manifold T .

$\| M \|_{\mathcal{F}}$ seems to relate to the branching of T :

Calegari (2000) proves

- $\| M \|_{\mathcal{F}} = \| M \|$ if T branches in at most one direction.

and (using rigidity of efficient fundamental cycles)

- $\| M \|_{\mathcal{F}} > \| M \|$ if $\tilde{\mathcal{F}}$ is asymptotically separated (which implies: T branches in both directions).

Result extends to cusped manifolds.

Part II:

The laminated Gromov norm and its application to nonexistence results for tight laminations on 3-manifolds of small volume

Let M be a compact hyperbolic 3-manifold and $\mathcal{F}$ an essential lamination, for example a π_1 -injective compact surface.

$N = \overline{M - \mathcal{F}}$ is a compact, atoroidal 3-manifold with boundary. Cutting along annuli yields the JSJ decomposition

$$N = Char(N) \cup Guts(N).$$

- $Char(N)$: I -bundles and solid tori
- $Guts(N)$: acylindrical, hence admits a hyperbolic metric with geodesic boundary (and cusps)

Agol's inequality for compact surfaces in 3-manifolds

Theorem (Agol-Storm-Thurston 2005, strengthening Agol 1999)

If M is a hyperbolic 3-manifold containing a π_1 -injective compact surface F , then

$$\text{Vol}(M) \geq \frac{V_{\text{oct}}}{4} \parallel \partial \text{Guts}(\overline{M - F}) \parallel$$

with $V_{\text{oct}} = 3.66 \dots$ the volume of a regular ideal octahedron in H^3 .

The proof uses analytical methods (Perelman's entropy estimate for the Ricci flow, work of Bray and Miao on the Penrose conjecture) and does not seem to generalize to laminations. Using the original topological approach from Agol we give a generalization to laminations and to higher-dimensional hyperbolic manifolds.

General Theorem

Definition

$$\| M \|_{\mathcal{F}}^{normal} = \inf \{ \sum_{i=1}^r a_i \mid \sum_{i=1}^r a_i \sigma_i \text{ represents } [M, \partial M], \sigma_i \text{ normal to } \mathcal{F} \}$$

Theorem (K. 2007)

Let M^n compact, $\mathcal{F}$ codimension one lamination, $N := \overline{M - \mathcal{F}}$.

For a decomposition $N = Q \cup R$ into n -manifolds, denote

$\partial_0 Q = Q \cap R$, $\partial_1 Q = Q \cap \partial N$, $\partial_1 R = R \cap \partial N$, assume that:

- i) each component of $\partial_0 Q$ has amenable fundamental group,
- ii) $(Q, \partial_1 Q)$ is pared acylindrical, $\partial_1 Q$ is acylindrical
- iii) $Q, \partial N, \partial_1 Q, \partial_1 R, \partial_0 Q$ are aspherical,
- iv) all inclusions are π_1 -injective.

Then

$$\| M \|_{\mathcal{F}}^{normal} \geq \frac{1}{n+1} \| \partial Q \| .$$

M hyperbolic 3-manifold: choose $R = \text{Char}(N)$, $Q = \text{Guts}(N)$.

Corollary (Agol 1999 for surfaces, K. 2007 for laminations)

If M is a hyperbolic 3-manifold and $\mathcal{F}$ a tight lamination (the leaf space is Hausdorff), then

$$\text{Vol}(M) \geq -2V_3 \chi(\text{Guts}(\overline{M - \mathcal{F}})),$$

where $V_3 = 1,014\dots$ is the volume of a regular ideal simplex in H^3 .

As an application, the Weeks manifold W admits no tight laminations: Calegari-Dunfield (2003) have proved that W admits no tight lamination with empty guts, on the other hand non-empty guts would imply $\text{Vol}(W) \geq 2V_3$ contradicting

$$\text{Vol}(W) = 0.94\dots$$

(Together with results of Tao Li this implies that the Weeks manifold has no essential laminations at all.)

For hyperbolic 3-manifolds, Hass (1995) proved an upper bound on the genus of totally geodesic surfaces in terms of $\text{Vol}(M)$. We have a generalization to arbitrary dimensions:

Corollary (K. 2007)

If M is a negatively curved n -manifold and F a totally geodesic hypersurface, then

$$\| F \| \leq \frac{n+1}{2} \| M \| .$$

In this case the decomposition $N = R \cup Q$ of $N = \overline{M - F}$ is just $R = \emptyset, Q = N$.

Proof of Theorem in case of geodesic hypersurfaces:

Let F be a geodesic hypersurface in M .

Let $\sum_{i=1}^r a_i \sigma_i$ represent $[M]$ and all σ_i normal to F . We want to prove $\sum_{i=1}^r |a_i| \geq \frac{2}{n+1} \|F\|$.

We have $[M] \cap F = [F]$. We would like to prove: each σ_i intersects F at most $n+1$ times.

Our assumption (F geodesic) implies: $Q = \overline{M - F}$ is acylindrical. Whenever a normal simplex meets F more than $n+1$ times, then two intersecting triangles must have a parallel edge, i.e. cut out a square. If this square were mapped to a cylinder, then acylindricity implies that the cylinder degenerates after homotopy, hence can be removed without changing the homology class, thus the number of intersections can be reduced.

The proof of $\sum_{i=1}^r |a_i| \geq \frac{2}{n+1} \|F\|$ proceeds by defining a straightening which produces the maximally possible number of cylinders.

Part III:

The image of the fundamental class in the homology of the linear group and its application to the invariance of volume under generalized mutations

K-theoretic invariants of manifolds - Example

If $M = \Gamma \backslash SL(2, \mathbf{C}) / SU(2)$ is a hyperbolic 3-manifold, then a K-theoretic invariant $\beta(M)$ can be defined as follows.

Definition (Bloch)

$\mathcal{P}(\mathbf{C})$ is the quotient of $Z[\mathbf{C} - \{0, 1\}]$ via all 5-term relations.

$\mathcal{B}(\mathbf{C})$ is the kernel of the Dehn invariant $D : \mathcal{P}(\mathbf{C}) \rightarrow \mathbf{C}^* \wedge \mathbf{C}^*$.

Suslin (1990): $K_3^{\text{ind}}(\mathbf{C}) \otimes \mathbf{Q} \cong \mathcal{B}(\mathbf{C}) \otimes \mathbf{Q}$.

Definition (Neumann-Yang)

An ideal simplex Δ has its 4 vertices in $\mathbf{C} \cup \{\infty\}$, their cross-ratio $z(\Delta)$ in $\mathcal{P}(\mathbf{C})$.

A hyperbolic 3-manifold has trivial Dehn invariant, thus an ideal triangulation $\sum_{i=1}^r \Delta_i$ yields $\beta(M) := \sum_{i=1}^r z(\Delta_i) \in \mathcal{B}(\mathbf{C})$.

SnapPea computes $\beta(M)$ and $\text{vol}(M) = \sum_{i=1}^r D_2(z(\Delta_i))$.

Neumann (2009): $\beta(M)$ preserved under mutation of hyperbolic knots.

Group Homology and volume cocycle

In general one has

$$K_*(\mathbf{C}) \otimes \mathbf{Q} \cong PH_*(BGL(\mathbf{C}); \mathbf{Q}) \subset H_*(BGL(\mathbf{C}); \mathbf{Q}).$$

How to define an invariant in $H_*(BG; \mathbf{Q})$?

- $\Gamma \backslash G/K$ is an Eilenberg-MacLane space $B\Gamma$.

$$j : M = \Gamma \backslash G/K \simeq B\Gamma \rightarrow BG$$

M closed: $j_*[M] \in H_n(BG; \mathbf{R})$ determines $vol(M)$.

- Let $v_n \in H_c^n(G)$ be the volume class, represented by

$$v_n(g_1, \dots, g_n) = vol(str(\tilde{x}, g_1\tilde{x}, \dots, g_n\tilde{x}))$$

for $\tilde{x} \in G/K$, $str(\dots)$ denoting the straight simplex.

Then for closed M :

$$\langle v_n, j_*[M] \rangle = vol(M).$$

Example: hyperbolic 3-manifolds and the Bloch group

If $M = \Gamma \backslash PSL(2, \mathbf{C}) / PSU(2)$ is a closed hyperbolic 3-manifold, then we have defined

$$j_*[M] \in H_3(PSL(2, \mathbf{C}))$$

and the Bloch invariant

$$\beta(M) \in \mathcal{B}(\mathbf{C}).$$

Bloch-Wigner isomorphism

$$H_3(PSL(2, \mathbf{C}); \mathbf{Z}) / \text{Torsion} \cong \mathcal{B}(\mathbf{C})$$

Neumann-Yang (1999): **The isomorphism maps $j_*[M]$ to $\beta(M)$.**
(In particular, $\beta(M)$ is triangulation-independent.)

$$\langle v_n, j_* [M] \rangle = \text{vol} (M)$$

implies that M_1, M_2 have the same volume if

$$j_* [M_1] = j_* [M_2] \in H_* (BG).$$

We are going to apply this to generalized mutations.

Cut and Paste III: Generalized Mutation

Recall: $\Sigma \subset M$ codim 1 submanifold, $\tau : \Sigma \rightarrow \Sigma$ diffeo, then M^τ is the manifold obtained by cutting M along Σ and regluing via τ .

Definition

$M_1 = \Gamma_1 \backslash G/K$, $M_2 = \Gamma_2 \backslash G/K$ locally symmetric spaces,
Assume $\Sigma \subset M_1$, and $M_2 = M_1^\tau$ for some finite order diffeomorphism $\tau : \Sigma \rightarrow \Sigma$.

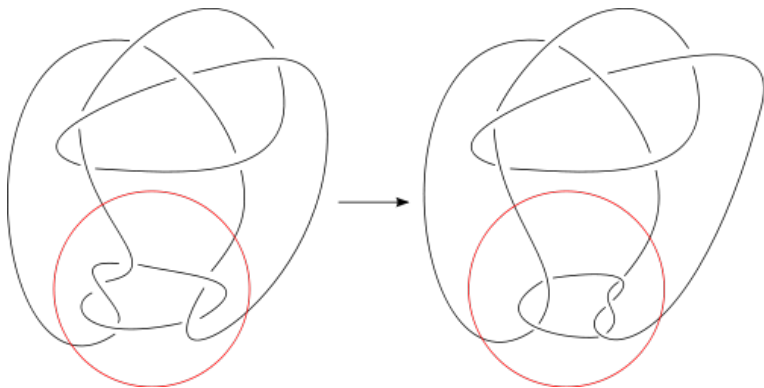
Then M_2 is obtained from M_1 by generalized mutation, if there is some $A \in G$ with

$$\tau_*(\sigma) = A\sigma A^{-1}$$

for all $\sigma \in \pi_1 \Sigma$.

- Toy Example: $H \subset G$, $\Sigma = \Gamma \backslash H / (K \cap H)$ totally geodesic, $\tau : \Sigma \rightarrow \Sigma$ isometry.
Then τ_* is realised by conjugation with some $A \in H$.
- M^3 hyperbolic, Σ^2 pretzel, $\tau : \Sigma \rightarrow \Sigma$ hyperelliptic involution.
Then exists $A \in SL(2, \mathbb{C})$ with $\tau_*(\sigma) = A\sigma A^{-1}$ (Ruberman).

Mutation of knots



- disc in projection plane, boundary intersects K four times
- part of the knot inside the disc is a tangle
- two reflections that switch pairs of endpoints of the tangle, composition is a rotation
- replace the original tangle by the rotated tangle

Example

Mutation of knots:

If $M = S^3 - K$ is a knot complement and K' the mutant of K , then $S^3 - K' = M^\tau$ for a 4-punctured sphere $\Sigma \subset M$ and a hyperelliptic involution $\tau : \Sigma \rightarrow \Sigma$.

Ruberman (1989) proved:

- If $M = S^3 - K$ is a hyperbolic knot complement, then τ is a generalized mutation, M^τ is hyperbolic and $\text{vol}(M^\tau) = \text{vol}(M)$.

Proof uses heavy analytical machinery (existence and uniqueness results for minimal surfaces).

Remark: Hyperbolicity of M^τ can be proved topologically using the Maskit combination theorem.

K.: alternative proof of $\text{vol}(M^\tau) = \text{vol}(M)$ using group homology; proof extends to locally symmetric spaces.

Theorem (K. 2011)

If M_1 and M_2 are locally symmetric spaces,

- either closed,

- or cusped and $\text{rank}(G/K) = 1$,

*such that $M_2 = M_1^\tau$ is a **generalized mutation***

(for some $\Sigma \subset M$ and some finite order diffeomorphism

$\tau : \Sigma \rightarrow \Sigma$),

then

$$\text{vol}(M^\tau) = \text{vol}(M).$$

Cut and Paste VI: Proof of Volume Equality - closed case

- Volume of M is determined by $j_* [M] \in H_n(BG; \mathbf{Q})$, therefore it suffices to prove

$$j_* [M] = j_* [M^\tau] \in H_n(BG; \mathbf{Q}).$$

- Consider the space X obtained by identifying M and M^τ along $M - nbh(\Sigma)$, it contains an embedded copy of the mapping torus T^τ . Clearly

$$[M] - [M^\tau] = [T^\tau] \in H_n(X).$$

- Definition of generalized mutation implies:
 $j_* : \pi_1 M \rightarrow G$ extends to a well-defined homomorphism
 $j_* : \pi_1 X \rightarrow G$.

$$j_* [M] - j_* [M^\tau] = j_* [T^\tau] \in H_n(BG; \mathbf{Q})$$

- Using $\tau^m = id$ prove $j_* [T^\tau] = 0 \in H_n(BG; \mathbf{Q})$.
(Not true with $\mathbf{Z}$ -coefficients, except for classical mutation by Neumann.)

Manifolds with boundary

M^n manifold with boundary, $\text{int}(M) \cong \Gamma \backslash G/K$.

- G/K symmetric space, define

$$BG^{comp} = BG \bigcup_{c \in \partial_\infty G/K} \text{Cone}(BG),$$

the volume cocycle $v_n \in H^n(BG)$ extends to

$\bar{v}_n \in H^n(BG^{comp})$ by

$$\bar{v}_n(g_0, \dots, g_{n-1}, c) = \text{vol}(\text{str}(g_0\tilde{x}, \dots, g_{n-1}\tilde{x}, c)).$$

If $\text{rank}(G/K) = 1$ and $\text{int}(M) \cong \Gamma \backslash G/K$ for a lattice $\Gamma \subset G$, then

$$j_*[M, \partial M] \in H_n(BG^{comp})$$

can be defined such that

$$\langle \bar{v}_n, j_*[M, \partial M] \rangle = \text{vol}(M).$$

In particular, $\text{vol}(M)$ depends only on $j_*[M, \partial M] \in H_n(BG^{comp})$.

Cut and Paste VII: Proof of volume equality - cusped case

Recall: X obtained by identifying M and M^τ along $M - nbh(\Sigma)$, contains embedded copy of the mapping torus T^τ . For generalized mutations $\rho : \pi_1 M \rightarrow G$ extends to $\rho^X : \pi_1 X \rightarrow G$.

Let $t \in \pi_1 \partial X$ be the extending element in $\pi_1 \partial T^\tau \subset \pi_1 \partial X$.

$\rho^X(t) \in G$ is not parabolic (therefore $B\rho^X$ does not extend to $B(\pi_1 X)^{comp}$), but elliptic: t lifts to 1 in some finite covering $\hat{X}$.

Let $\hat{M}, \hat{M}^\tau$ be the preimages of M, M^τ in $\hat{X}$. Prove

$$j_* [\hat{M}, \partial \hat{M}] = j_* [\hat{M}^\tau, \partial \hat{M}^\tau] \in H_n(BG^{comp}; \mathbf{Q})$$

$$\implies \text{vol}(\hat{M}) = \text{vol}(\hat{M}^\tau) \implies \text{vol}(M) = \text{vol}(M^\tau).$$

$M = \Gamma \backslash G / K$ closed

$$\rho : G \rightarrow GL(N, \mathbf{C})$$

Weil rigidity: w.l.o.g. $\rho(\Gamma) \subset GL(N, \overline{\mathbf{Q}})$, hence

$$(B\rho)_* j_* [M] \in H_n(BGL(N, \overline{\mathbf{Q}})) .$$

$(B\rho)_* j_* [M]$ contains more information than the volume, for example the volume of Galois conjugates (if they are discrete).

$$\rho : G \rightarrow GL(N, \mathbf{C})$$

$$(B\rho)_* j_* [M] \in H_n(BGL(N, \overline{\mathbf{Q}})) .$$

$H_c^*(GL(N, \mathbf{C})) = \Lambda(b_1, b_3, \dots, b_{2N-1})$ generated by Borel classes

$$\rho^* b_n = c_\rho v_n, c_\rho \in \mathbf{R}$$

Hence, if M^n closed, then

$$\langle b_n, (B\rho)_* j_* [M] \rangle = c_\rho \text{vol}(M) .$$

K-theoretic invariant:

There exists a projection $H_n(BGL(N, \overline{\mathbf{Q}}); \mathbf{Q}) \rightarrow K_n(\overline{\mathbf{Q}}) \otimes \mathbf{Q}$ preserving values of Borel classes. The image of $(B\rho)_* j_* [M]$ under this projection is the Goncharov invariant $\gamma(M)$.

Nontriviality

The following is a complete list of irreducible symmetric spaces G/K of noncompact type and of fundamental representations $\rho : G \rightarrow GL(N, \mathbb{C})$ with $\rho^* b_n \neq 0$ for $n = \dim(G/K)$.

Symmetric Space	Representation
$SL_l(\mathbb{R})/SO_l, l \equiv 0, 3, 4, 7 \pmod{8}$	any fund. repres.
$SL_l(\mathbb{C})/SU_l, l \equiv 0 \pmod{2}$	any fund. repres.
$SL_{2l}(\mathbb{H})/Sp_l, l \equiv 0 \pmod{2}$	any fund. repres.
$Spin_{p,q}/(Spin_p \times Spin_q),$ $p, q \equiv 1 \pmod{2}, p \not\equiv q \pmod{4}$	any fund. repres.
$Spin_{p,q}/(Spin_p \times Spin_q),$ $p, q \equiv 1 \pmod{2}, p \equiv q \pmod{4}$	+/- half-spinor repres.
$SO_l(\mathbb{C})/SO_l, l \equiv 3 \pmod{4}$	any fund. repres.
$Sp_l(\mathbb{C})/Sp_l, l \equiv 1 \pmod{4}$	any fund. repres.
$E_7(\mathbb{C})/E_7$	any fund. repres.

In these cases, $\gamma(M) \neq 0$ whenever $M = \Gamma \backslash G/K$ is closed.

In particular, for hyperbolic space

$$H^n = Spin(n, 1) / Spin(n)$$

one obtains:

- If $n \equiv 3 \pmod{4}$, then $\gamma(M) \neq 0$ for every hyperbolic n -manifold $M = \Gamma \backslash H^n$ and every nontrivial representation $\rho : Spin(n, 1) \rightarrow GL(N, \mathbf{C})$.
- If $n \equiv 1 \pmod{4}$, then $\gamma(M) \neq 0$ for every hyperbolic n -manifold $M = \Gamma \backslash H^n$ and for the half-spinor representations.

The latter was proved by Goncharov (1999).

K-theoretic invariant - the cusped case

Assume again $\text{rank}(G/K) = 1$ and $\text{int}(M) \cong \Gamma \backslash G/K$ for a lattice $\Gamma \subset G$, thus $j_*[M, \partial M] \in H_n(BG^{\text{comp}})$ is defined.

Lemma (K. 2009)

If $\rho : G \rightarrow SL(N, \mathbf{C})$, then

$$(B\rho)_* j_*[M, \partial M] \in H_*(BSL(N, \mathbf{C})^{\text{comp}})$$

has a preimage

$$\bar{\gamma}(M) \in H_*(SL(N, \mathbf{C}))$$

such that $\langle b_n, \bar{\gamma}(M) \rangle = c_\rho \text{vol}(M)$.

In particular, the Goncharov invariant $\gamma(M) \in K_*(\overline{\mathbf{Q}}) \otimes \mathbf{Q}$ can be defined.

We have $\gamma(M^\tau) = \gamma(M)$ if M^τ is a generalized mutation of M .