Remark: Lemma 3 and hence Corollary 1/Theorem 1 are wrong. (I intend to give another construction of cycles that gives conditions for nontriviality of the Euler class.)

Lefschetz Fibrations with unbounded Euler Class - Work in progress

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Abstract. We investigate the bounded cohomology of Lefschetz fibrations: we show that the Euler class, of a Lefschetz fibration of fiber genus ≥ 2 having distinct vanishing cycles with nontrivial (homological) intersection, is not bounded. As a consequence, we exclude the existence of negatively curved metrics on such Lefschetz fibrations.

Contents

The bounded cohomology $H_b^*(X;\mathbb{Z})$ is an invariant of topological spaces, which was introduced by Gromov in his work about the simplicial volume and has since then shown to be useful also in group theory and dynamics of group actions.
A cohomology class $\beta \in H^*(X;\mathbb{Z})$ is said to be bounded if it is in the image of the natural map $H_b^*(X;\mathbb{Z}) \rightarrow H^*(X;\mathbb{Z})$. Among other results, Gromov proved that (real) characteristic classes in $H^*\left(BG^\delta;\mathbb{R}\right)$ are bounded, if G^δ is an algebraic subgroup of $GL(n,\mathbb{R})$ equipped with the discrete topology. This generalized the classical Milnor-Sullivan theorem which states that Euler classes of flat affine bundles are bounded.
In this article, we consider the Euler class of Lefschetz fibrations. A well-known theorem of Morita says that the Euler class of a surface bundle, with fiber of genus ≥ 2, is bounded. We prove a converse to Morita's theorem.

Theorem 1. *If a Lefschetz fibration, with regular fiber of genus ≥ 2, has bounded Euler class, then all pairs of vanishing cycles have vanishing homological intersection numbers.*

As an application, we can exclude the existence of negatively curved metrics on a large number of Lefschetz fibrations.

Corollary 2: *If a Lefschetz fibration, with regular fiber of genus ≥ 2, admits a Riemannian metric with negative sectional curvature everywhere, then all pairs of vanishing*

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cycles have vanishing homological intersection numbers.

We recall that any finitely presented group Γ can be realised as the fundamental group of a Lefschetz fibration ([3],[1]). If Γ happens to be word-hyperbolic, we will actually show that $\pi_2 M_\Gamma \neq 0$.

I thank the referee for pointing out a gap in a former version of the proof.

1. Preliminaries

Lefschetz fibrations. A smooth map $\pi : M \rightarrow B$ from a smooth (closed, oriented, connected) 4-manifold M to a smooth (closed, oriented, oriented) 2-manifold B is said to be a Lefschetz fibration, if it is surjective and $d\pi$ is surjective except at finitely many critical points $\{p_1, \ldots, p_k\} =: C \subset M$, having the property that there are complex coordinate charts (agreeing with the orientations of M and B), U_i around p_i and V_i around $\pi(p_i)$, such that in these charts f is of the form $f(z_1, z_2) = z_1^2 + z_2^2$, see [7]. After a small homotopy the **critical points are in distinct fibers**, we assume this to hold for the rest of the paper.

The preimages of points in $B - \pi(C)$ are called regular fibers. It follows from the definition that all regular fibers are diffeomorphic and that the restriction $\pi' := \pi \mid_{M'}$: $M' \rightarrow B'$ to $M' := \pi^{-1}\pi(M - C)$ **is a smooth fiber bundle over** $B' := B - \pi(C)$.

Let Σ_g be the regular fiber, a closed surface of genus g, and let, for an arbitrary point $* \in \Sigma_g$, be $Map_{g,*}$ the group of diffeomorphisms $f : \Sigma_g \rightarrow \Sigma_g$ with $f(*) = *$, modulo homotopies fixing $*$. It is well-known, cf. [13], that for any surface bundle one gets a monodromy $\rho : \pi_1 M' \rightarrow Map_{g,*}$, which fits into the commutative diagram

$$1 \longrightarrow \quad \pi_1\Sigma_g \quad \longrightarrow \quad \pi_1 M' \quad \longrightarrow \quad \pi_1 B' \longrightarrow 1 \quad {}^{id\rho} \quad 1 \longrightarrow \pi_1\Sigma_g \longrightarrow Map_{g,*} \longrightarrow Map_g \longrightarrow 1$$

It follows from the local structure of Lefschetz fibrations that, for a simple loop c_i surrounding $\pi(p_i)$ in B, its image under the monodromy, $\rho(c_i)$, is the Dehn twist at some closed curve $v_i \subset \Sigma_g$. v_i is called the 'vanishing cycle'.

We point out the following fact: for $\gamma \in \pi_1\Sigma_g \subset \pi_1 M'$, the pointed mapping class $\rho(\gamma)$ is a mapping which twists some loop representing $\gamma \in \pi_1(\Sigma_g, *)$ once along itself back and forth, such that it is homotopic (but not base-point preserving homotopic) to the identity. If Σ_g carries a hyperbolic metric, then there is a representative of $\rho(\gamma)$ which can be lifted to a hyperbolic isometry with axis $\tilde{\gamma} \subset \mathbb{H}^2$, mapping $\tilde{*}$ to $\gamma(\tilde{*})$, for any lift $\tilde{*}$ of $*$.

Euler class of Lefschetz fibrations. For a topological space X, and a rank-2-vector bundle ξ over X, one has an associated Euler class $e(\xi) \in H^2(X; \mathbb{Z})$.

If $\pi : M \rightarrow B$ is a Lefschetz fibration, we may consider the tangent bundle of the fibers, TF, except at points of C, where this is not well defined. We get a rank-2-vector bundle L' over $M - C$ with euler class $e' := e(TF) \in H^2(M - C; \mathbb{Z})$.
By a standard application of the Mayer-Vietoris sequence, there is an isomorphism $i^* : H^2(M; \mathbb{Z}) \rightarrow H^2(M - C; \mathbb{Z})$ induced by the inclusion. Hence, $e := (i^*)^{-1} e' \in H^2(M; \mathbb{Z})$ is well-defined. In what follows we will denote e as the Euler class of the Lefschetz fibration $\pi : M \rightarrow B$. It is actually true (but we will not need it) that there exists a rank-2-vector bundle ξ over M such that $\xi \mid_{M-C} \simeq TF$. It is the pull-back of the universal complex line bundle, pulled back via the map $f : M \rightarrow CP^\infty$ corresponding to $e \in H^2(M; \mathbb{Z})$ under

the bijection $H^2(M;\mathbb{Z}) \simeq [M, CP^\infty]$.

$\mathbb{S}^1$-bundles associated to surface bundles. For any surface bundle $\pi' : M' \to B'$ we may, after fixing a Riemannian metric, consider UTF, the unit tangent bundle of the fibers. We consider the case that the fiber has genus $g \geq 2$. Then this $\mathbb{S}^1$-bundle is, according to [13], equivalent to the flat $Homeo^+(\mathbb{S}^1)$-bundle with monodromy $\partial_\infty \rho$, where $\partial_\infty : Map_{g,*} \to Homeo^+(\mathbb{S}^1)$ is constructed as follows.

Recall that $\pi_1\Sigma_g$ is word-hyperbolic, since $g \geq 2$. For $f \in Map_{g,*}$ let $f_* : \pi_1(\Sigma_g, *) \to \pi_1(\Sigma_g, *)$ be the induced map of fundamental groups, and $\partial_\infty f_*$ the extension of f_* to the Gromov boundary $\partial_\infty \pi_1(\Sigma_g, *)$. It is well-known that $\partial_\infty f_*$ is a homeomorphism and that there is a canonical homeomorphism $\partial_\infty \pi_1(\Sigma_g, *) \simeq \mathbb{S}^1$.
If $\gamma \in \pi_1(\Sigma_g, *) \simeq ker(Map_{g,*} \to Map_g)$, then $\partial_\infty \rho(\gamma) \in PSL_2\mathbb{R} \subset Homeo^+(\mathbb{S}^1)$ is a hyperbolic map whose both fixed points are the ideal vertices of the lift $\tilde{\gamma} \subset \mathbb{H}^2$ of a representative of γ passing through $*$. An explicit diffeomorphism $F : (\Sigma_g, *) \to (\Sigma_g, *)$ representing γ in $Map_{g,*}$ can be constructed as follows. Fix some immersed loop c representing γ, extend to some immersion $C : \mathbb{R}/\mathbb{Z} \times [-1,1]$ and define $F(C([s], t)) = C([s+1- \mid t \mid], t)$ for $([s], t) \in \mathbb{R}/\mathbb{Z} \times [-1,1]$.
One should be aware that the extension of UTF to $M - C$ is not flat: a loop surrounding a singular fiber is trivial in $\pi_1(M - C)$ but its monodromy is a Dehn twist, giving a nontrivial homeomorphism of $\mathbb{S}^1$.

Bounded Cohomology. It will be important for us to distinguish between bounded cohomology with integer coefficients, $H_b^2(X;\mathbb{Z})$, and bounded cohomology with real coefficients, $H_b^2(X;\mathbb{R})$. We refer to [10] for definitions. To avoid too complicated notation, we use the following convention: for $\beta \in H^*(X;\mathbb{Z})$, we denote $\beta_\mathbb{R} \in H^*(X;\mathbb{R})$ its image under the canonical homomorphism $H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{R})$. Also, we will not distinguish between $H_b^*(X;\mathbb{R})$ and $H_b^*(\pi_1 X;\mathbb{R})$.

A cohomology class $\beta \in H^*(X;\mathbb{Z})$ is called bounded if it belongs to the image of the canonical homomorphism $H_b^*(X;\mathbb{Z}) \to H^*(X;\mathbb{Z})$.
We will use the following two facts. (A) is proved in Bouarich's thesis, see [2]. (B) is proved in [6].

(A): If $1 \to N \to \Gamma \to G \to 1$ is an exact sequence of groups, then there is an exact sequence

$$0 \to H_b^2(G;\mathbb{R}) \to H_b^2(\Gamma;\mathbb{R}) \to H_b^2(N;\mathbb{R})^G \to H_b^3(G;\mathbb{R}).$$

(B): For any group Γ, there is an exact sequence, natural with respect to group homomorphisms,

$$H^1(\Gamma;\mathbb{R}/\mathbb{Z}) \to H_b^2(\Gamma;\mathbb{Z}) \to H_b^2(\Gamma;\mathbb{R}).$$

A commutative diagram. Let $\pi : M \to B$ be a Lefschetz fibration with fiber F, let $\pi\mid_{M'}: M' \to B'$ be the smooth fiber bundle obtained by removing the singular fibers and introduce the following notations: $N = \ker(\pi_1 M' \to \pi_1 M)$, $V = \ker(\pi_1 B' \to \pi_1 B)$, $\Gamma = \ker(N \to V)$. Then we have a commutative diagram

$$1 1 1 1 \Gamma N V 1 \pi_1 F \pi_1 M' \pi_1 B' 1$$

with all rows and columns being exact sequences.

A few remarks are in order about well-definedness of the involved homomorphisms. The second line is the long exact homotopy sequences of the surface bundle $M' \to B'$.

Inclusion maps $ker\left(N \rightarrow V\right)$ to $ker\left(\pi_1 M' \rightarrow \pi_1 B'\right)$, hence $\Gamma = ker\left(N \rightarrow V\right)$ is a subgroup of $\pi_1 F$. Clearly, the projection maps N to $ker\left(\pi_1 B' \rightarrow \pi_1 B\right) = V$. Surjectivity of this homomorphism does not follow from the commutative diagram, but is easy to see geometrically. Indeed, each simple loop c_i surrounding a puncture can be lifted to an element $\hat{c}_i \in N$, just working in coordinate charts. For $g \in \pi_1 B$, we fix some lift $\hat{g} \in \pi_1 M$. Then $\hat{g}\hat{c}_i\hat{g}^{-1}$ is an element of N, projecting to gc_ig^{-1}. Since V is generated by elements of the form gc_ig^{-1}, we have surjectivity.

2. Actions on the circle with nontrivial Euler class

We start with recalling the setting of [4].
Let

$$Homeo^+\mathbb{S}^1 := \{f : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z} \text{ orientation-preserving homeomorphism}\}$$

and

$$\overline{Homeo}^+\mathbb{S}^1 := \left\{\tilde{f} : \mathbb{R} \rightarrow \mathbb{R} \text{ orientation-preserving homeomorphism}, \tilde{f}\left(x+1\right) = \tilde{f}\left(x\right) + 1\ \forall x \in \mathbb{R}\right\}.$$

There is an exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \overline{Homeo}^+\mathbb{S}^1 \rightarrow Homeo^+\mathbb{S}^1 \rightarrow 0,$$

where $n \in \mathbb{Z}$ is mapped to translation by n.
Let $\chi \in H_b^2\left(Homeo^+\mathbb{S}^1\right)$ be the Euler class of this extension. An explicit representative c of χ is given as follows: we fix a set-theoretic section $s : Homeo^+\mathbb{S}^1 \rightarrow \overline{Homeo}^+\mathbb{S}^1$ by letting $s\left(f\right)$ be the unique lift of f with $0 \leq s\left(f\right)\left(0\right) < 1$. Then let

$$c\left(f,g\right) := s\left(fg\right)^{-1} s\left(f\right) s\left(g\right) \in ker\left(\overline{Homeo}^+\mathbb{S}^1 \rightarrow Homeo^+\mathbb{S}^1\right) = \mathbb{Z}$$

for all $f, g \in Homeo^+\mathbb{S}^1$. The bounded cocycle $c \in C_b^2\left(Homeo^+\mathbb{S}^1, \mathbb{Z}\right)$ represents the Euler class χ. One should observe that $c\left(f,g\right) = 0$ if $0, g\left(0\right), fg\left(0\right)$ are in clockwise order and $c\left(f,g\right) = 1$ if $0, g\left(0\right), fg\left(0\right)$ are in anti-clockwise order.

For a discrete group Γ, $H_b^2\left(\Gamma;\mathbb{Z}\right)$ "classifies" actions of Γ on $\mathbb{S}^1$ (see [5],thm.6.6). In particular ([4],p.35), for $\rho : \Gamma \rightarrow Homeo^+\left(\mathbb{S}^1\right)$, $\rho^*\chi = 0$ holds if and only if all $\rho\left(\gamma\right)$ with $\gamma \in \Gamma$ have a common fixed point on $\mathbb{S}^1$. (Note that the original statement in [4] is mistaken and would erroneously imply the existence of two common fixed points.)

Lemma 1. *Let Γ be a group, $\mathcal{A}$ a (possibly infinite) set of generators of Γ and $\partial_\infty\rho : \Gamma \rightarrow Homeo^+\mathbb{S}^1$ be a representation such that*
a) for all $a \in \mathcal{A}$ the rotation number of $\partial_\infty\rho\left(a\right)$ is zero, and
b) there is no common fixed point on $\mathbb{S}^1$, that is,
there is no $x \in \mathbb{S}^1$ with $\partial_\infty\rho\left(a\right)\left(x\right) = x$ for all $a \in \mathcal{A}$.
Then the Euler class of ρ does not belong to the kernel of the canonical homomorphism $H_b^2\left(\Gamma;\mathbb{Z}\right) \rightarrow H_b^2\left(\Gamma;\mathbb{R}\right)$.

Proof. For $\gamma \in \mathcal{A}$ let $j_\gamma : \mathbb{Z} \rightarrow \Gamma$ be the unique homomorphism with $j_\gamma\left(1\right) = \gamma$. By functoriality of the exact sequence (B) (section 1), we have a commutative diagram

$$\Pi_{\gamma\in\mathcal{A}}\mathrm{H}^1\left(\mathbb{Z};\mathbb{R}/\mathbb{Z}\right)^{\simeq}\Pi_{\gamma\in\mathcal{A}}\mathrm{H}_b^2\left(\mathbb{Z};\mathbb{Z}\right)\Pi_{\gamma\in\mathcal{A}}\mathrm{H}_b^2\left(\mathbb{Z};\mathbb{R}\right)^{\Pi\ j_\gamma^*\Pi\ j_\gamma^*\Pi\ j_\gamma^*}\mathrm{H}^1\left(\Gamma;\mathbb{R}/\mathbb{Z}\right)\mathrm{H}_b^2\left(\Gamma;\mathbb{Z}\right)\mathrm{H}_b^2\left(\Gamma;\mathbb{R}\right),$$

where the isomorphism

$$H_b^2\left(\mathbb{Z};\mathbb{Z}\right) \simeq \mathbb{R}/\mathbb{Z} \simeq H^1\left(\mathbb{Z};\mathbb{R}/\mathbb{Z}\right)$$

follows from prop. 3.1. in [4].

Let $e \in H_b^2(\Gamma;\mathbb{Z})$ be the Euler class of ρ. Its image $j_\gamma^* e \in H_b^2(\mathbb{Z};\mathbb{Z})$ is the Euler class of the representation of $\mathbb{Z}$ mapping 1 to $\partial_\infty \rho(\gamma_i)$. By theorem A3 in [4], $j_\gamma^* e$ is mapped to the rotation number of $\partial_\infty \rho(\gamma)$ under the isomorphism $H_b^2(\mathbb{Z};\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}$. The rotation number of $\partial_\infty \rho(\gamma_i)$ is zero, hence $j_\gamma^* e = 0$ for all $\gamma \in \mathcal{A}$.

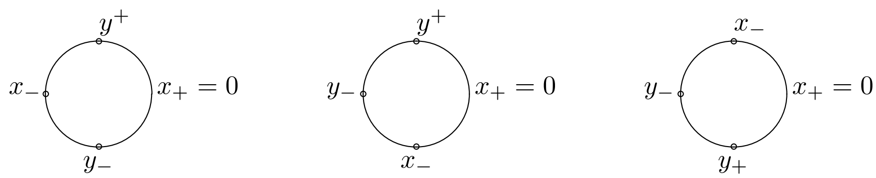
Now assume that e, the Euler class of ρ, belonged to the kernel of the canonical homomorphism $H_b^2(\Gamma;\mathbb{Z}) \to H_b^2(\Gamma;\mathbb{R})$. It follows that $e \in H_b^2(\Gamma;\mathbb{Z})$ has a preimage

$$E \in H^1(\Gamma;\mathbb{R}/\mathbb{Z}).$$

Since $\mathcal{A}$ generates Γ, the homomorphism $\Pi_{\gamma\in\mathcal{A}} j_\gamma^* : H^1(\Gamma;\mathbb{R}/\mathbb{Z}) \to \Pi_{\gamma\in\mathcal{A}} \mathrm{H}^1(\mathbb{Z};\mathbb{R}/\mathbb{Z})$ is injective. Hence, $\Pi_{\gamma\in\mathcal{A}} j_\gamma^* e = 0$ implies $E = 0$. Therefore, also $e = 0$.

According to [5], this contradicts assumption b). □

Definition 1. *We say that $f \in Homeo^+\mathbb{S}^1$ is a hyperbolic map if it has exactly two fixed points x_- and x_+ and if $\lim_{n\to-\infty} f^n(x) = x_-$, $\lim_{n\to\infty} f^n(x) = x_+$ holds for each $x \notin \{x_-, x_+\}$.*



Observations. We consider two hyperbolic maps g and h which do not have common fixed points. Let x_- and x_+ be the repelling resp. attracting fixed point of g. Let y_- and y_+ be the repelling resp. attracting fixed point of h. We choose an identification of $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z} = [0,1]/\sim$ such that $x_+ = 0 = 1$. Up to possibly replacing h by h^{-1} we have after this identification three possibilities: either $0 = x_+ < x_- < y_- < y_+$ or $0 = x_+ < y_+ < x_- < y_-$ or $0 = x_+ < y_+ < y_- < x_-$. (The other three possibilities are obtained after replacing h by h^{-1}, which will not affect our arguments.)

It follows directly from definition 1 that in each of the three cases the following is true: for each $0 < \epsilon < y_-$ there exist some m, n such that $1 - \epsilon < g^m h^{-n} g^{-m}(\epsilon) < 1$. This leads to the following observations:

(A) *If $f \in Homeo^+(\mathbb{S}^1)$ satisfies $y_- > f(x_+) > x_+$, then $c(f, g^m h^{-n} g^{-m} f) = 1$.*
Indeed, $f(0) > 0$ implies $f(1-\epsilon) > \epsilon > 0$ for some small ϵ. Depending on ϵ we choose m and n so large that $g^m h^{-n} g^{-m}(\epsilon) > 1 - \epsilon$. Then we get the claim.
(B) *If $f \in Homeo^+(\mathbb{S}^1)$ satisfies $y_- > f^2(x_+) > x_+$, then $c(g^m h^{-n} g^{-m} f, f) = 0$.*
This is obvious because 1 is the attracting fixed point of g, hence $g^m h^{-n} g^{-m} f^2$ can not exceed $x_+ = 1$.

In the statements of the following lemmas we will identify elements $x \in \pi_1(\Sigma_g, *)$ with their images in $Map_{g,*}$. Moreover, $t_x \in Map_{g,*}$ will denote the (isotopy class of the) Dehn twist at x. $i(x,y)$ denotes the homological intersection number of $x, y \in \pi_1(\Sigma_g, *)$.

Lemma 2. *Let N be a group, and $\rho : N \to Map_{g,*}$ be a representation such that there exist elements $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in N$ and $x, y \in \pi_1(\Sigma_g, *)$ with $\rho(\gamma_1) = x, \rho(\gamma_2) = y, \rho(\gamma_3) = t_x, \rho(\gamma_4) = t_y$.*
If $i(x,y) \neq 0$, then the bounded real Euler class of $\partial_\infty \rho$ does not belong to the kernel of the canonical homomorphism $H_b^2(N;\mathbb{R}) \to H^2(N;\mathbb{R})$.

Proof. To show that $e \neq 0 \in H^2(N;\mathbb{R})$, it suffices to exhibit some 2-cycle z on N with $c(\partial_\infty \rho(z)) \neq 0$.

$i(x,y) \neq 0$ implies that x_+ is not a fixed point of t_y. We consider first the case that $t_y(x_+) > x_+$.

Define (for m, n large enough to make the above observations (A) and (B) valid):

$$z = \left(t_y, x^{-m} y^n x^m t_y\right) - \left(x^{-m} y^n x^m t_y, t_y\right)$$

(This is again to be considered as a chain in the bar resolution.)

We will show in corollary 1 below that z is a cycle.

Since y_- is a fixed point of t_y, we have that $t_y(x_+) < y_-$ and $t_y^2(x_+) < y_-$. So we may apply the above observations (A) and (B) to get $c(t_y, x^{-m}y^n x^m t_y) = 1$ and $c(t_y x^{-m} y^n x^m, t_y) = 0$, implying

$$c(z) \neq 0.$$

Now consider the case that $t_y(x_+) < x_+$. Then $t_{y^{-1}}(x_+) > x_+$. We consider then $z = \left(t_{y^{-1}}, x^{-m}y^n x^m t_{y^{-1}}\right) - \left(x^{-m}y^n x^m t_{y^{-1}}, t_{y^{-1}}\right)$ and get, with the same reasoning as before, $c(z) \neq 0$.

□

To prove corollary 1 we need the following lemma which ought to be well-known.

Lemma 3. *Let $x, y \in \pi_1(\Sigma_g, *)$ and $i := i(x,y)$ their homological intersection number. Then $t_y x t_y^{-1} = y^i x$.*

Proof. Recall that the isomorphism $I : ker(Map_{g,*} \rightarrow Map_g) \rightarrow \pi_1(\Sigma_g, *)$ is constructed as follows: let $f \in ker(Map_{g,*} \rightarrow Map_g)$ and H_t a homotopy in $Diff(\Sigma_g)$ between f and id. Then $t \rightarrow H_t(*)$ is a loop in Σ_g, representing $I(f) \in \pi_1(\Sigma_g, *)$.
Now, if g is any diffeomorphism of Σ_g which fixes $*$, then gH_tg^{-1} is a homotopy between gfg^{-1} and id, and $gH_tg^{-1}(*) = gH_t(*)$ for all t. If g is the Dehn twist at y, then g spins $H_t(*)$ along y for every intersection of $H_t(*)$ with y, hence the claim of the lemma. □

Corollary 1. *Assume that the assumptions of lemma 2 are satisfied. Let $x, y \in \pi_1(\Sigma_g, *)$ and $z = (t_y, t_y x^{-m} y^n x^m) - (t_y x^{-m} y^n x^m, t_y)$. Then $\partial c = 0$.*

Proof.

$$\partial c = t_y + t_y x^{-m} y^n x^m - t_y^2 x^{-m} y^n x^m - \left(t_y x^{-m} y^n x^m + t_y - t_y x^{-m} y^n x^m t_y\right) =$$

$$= t_y^2 x^{-m} y^n x^m - t_y x^{-m} y^n x^m t_y = 0,$$

where the last identity $t_y^2 x^{-m} y^n x^m - t_y x^{-m} y^n x^m t_y = 0$ follows from lemma 3 because $i(y, x^{-m} y^n x^m) = i(y, x^{-m}) + i(y, x^m) = 0$. □

Finally we explain some elementary (and surely well-known) facts, concerning the action of Dehn twists on $\partial_\infty \pi_1(\Sigma_g, *)$, which will be of importance in the proof of the backward direction of lemma 5.
Let x_- and x_+ be the repelling resp. attracting fixed point for the action of $x \in \pi_1(\Sigma_g, *)$ on $\partial_\infty \pi_1(\Sigma_g, *)$. Then $x_- = x^{-\infty}$ and $x_+ = x^\infty$, meaning that x_- is the limit of the sequence x^{-n} and x_+ is the limit of x^n.
Let t_y be the Dehn twist at y. t_y acts on the fundamental group by $t_y(x) = y^{i(x,y)}x$. In particular, if $i(x,y) = 0$, then the induced action of t_y on $\partial_\infty \pi_1(\Sigma_g, *)$ fixes x_- and x_+. What is more, if in addition $i(y,w) = 0$ holds for some $w \in \pi_1(\Sigma_g, *)$, then t_y fixes

also $w(x_\pm)$. We will apply this in the proof of theorem 1 to the situation where the intersection form vanishes on all of Γ, to conclude that t_y fixes the whole Γ-orbit of x_+, for $y \in \Gamma$.

3. Proof of Theorem 1 and corollaries

Theorem 1 *If a Lefschetz fibration with regular fiber Σ_g of genus $g \geq 2$, has bounded Euler class, then all pairs of vanishing cycles have vanishing homological intersection numbers.*

Theorem 1 will follow from the following two lemmata. ($\chi_{\mathbb{R}}$ denotes the real Euler class of the representation $\partial_\infty \rho : \pi_1 M' \to Homeo^+\left(\mathbb{S}^1\right)$, and N denote the kernel of the homomorphism $\pi_1 M' \to \pi_1 M$ induced by inclusion.)

Lemma 4. *If a Lefschetz fibration with regular fiber Σ_g of genus $g \geq 2$, has bounded real Euler class, then $\chi_{\mathbb{R}} \in ker\left(H_b^2\left(\pi_1 M'; \mathbb{R}\right) \to H^2\left(\pi_1 M'; \mathbb{R}\right)\right) + ker\left(H_b^2\left(\pi_1 M'; \mathbb{R}\right) \to H_b^2\left(N; \mathbb{R}\right)\right)$. In particular, boundedness of the real Euler class implies $\chi_{\mathbb{R}} \mid_N \in ker\left(H_b^2\left(N; \mathbb{R}\right) \to H^2\left(N; \mathbb{R}\right)\right)$.*

Lemma 5. *For a Lefschetz fibration with regular fiber Σ_g of genus $g \geq 2$, we have $\chi_{\mathbb{R}} \mid_N \in ker\left(H_b^2 N \to H^2 N\right)$ if and only if all pairs of vanishing cycles have vanishing homological intersection.*

Proof of Lemma 4. Assume that $e_{\mathbb{R}}$ is bounded. Let $e_b \in H_b^2\left(M, \mathbb{R}\right)$ be a bounded cohomology class which maps to $e_{\mathbb{R}}$ under the homomorphism $H_b^2\left(M, \mathbb{R}\right) \to H^2\left(M, \mathbb{R}\right)$. It follows from commutativity of

$$H_b^2\left(M; \mathbb{R}\right)^{i^*} H_b^2\left(M'; \mathbb{R}\right) H^2\left(M; \mathbb{R}\right)^{i^*} H^2\left(M'; \mathbb{R}\right)$$

that $i^* e_b$ is mapped to $e'_{\mathbb{R}} = i^* e_{\mathbb{R}}$ under the homomorphism $H_b^2\left(M', \mathbb{R}\right) \to H^2\left(M, \mathbb{R}\right)$. On the other hand, consider the class $\chi \in H_b^2\left(\pi_1 M', \mathbb{Z}\right) = H_b^2\left(M', \mathbb{Z}\right)$ discussed in section 2, i.e. the Euler class associated to the representation $\partial_\infty \rho : \pi_1 M' \to Homeo^+ \mathbb{S}^1$. Its associated real class $\chi_{\mathbb{R}} \in H_b^2\left(\pi_1 M', \mathbb{R}\right) = H_b^2\left(M', \mathbb{R}\right)$ is mapped to $e'_{\mathbb{R}}$ by the homomorphism $H_b^2\left(M', \mathbb{R}\right) \to H^2\left(M', \mathbb{R}\right)$. Hence,

$$\chi_{\mathbb{R}} - i^* e_b \in \ker\left(H_b^2\left(M', \mathbb{R}\right) \to H^2\left(M', \mathbb{R}\right)\right).$$

We observe that the canonical homomorphism $H_b^2\left(\pi_1 M', \mathbb{R}\right) \to H^2\left(\pi_1 M', \mathbb{R}\right)$ factors over the canonical homomorphism $H_b^2\left(M', \mathbb{R}\right) \to H^2\left(M', \mathbb{R}\right)$. (This is because the classifying map $M' \to K\left(\pi_1 M', 1\right)$ induces an isomorphism on bounded cohomology.) Therefore we have

$$\chi_{\mathbb{R}} - i^* e_b \in \ker\left(H_b^2\left(\pi_1 M', \mathbb{R}\right) \to H^2\left(\pi_1 M', \mathbb{R}\right)\right).$$

Consider $N = \ker\left(i_* : \pi_1 M' \to \pi_1 M\right)$. Since e_b is a class in $H_b^2\left(\pi_1 M, \mathbb{R}\right)$, we have that the restriction of $i^* e_b$ to N is trivial in $H_b^2\left(N, \mathbb{R}\right)$. Hence

$$\chi_{\mathbb{R}} \mid_N \in \ker\left(H_b^2\left(N, \mathbb{R}\right) \to H^2\left(N, \mathbb{R}\right)\right).$$

We note that $\chi_{\mathbb{R}} \mid_N$ is the real Euler class of

$$\partial_\infty \rho \mid_N : N \to Homeo^+ \mathbb{S}^1,$$

constructed as in section 2.

Bouarichs's exact sequence gives $im\left(i^*\right) = ker\left(H_b^2\left(\pi_1 M'; \mathbb{R}\right) \to H_b^2\left(N; \mathbb{R}\right)\right)$, hence the claim of lemma 4. □

Proof of Lemma 5. As explained in section 1, if we are given a hyperbolic metric on Σ_g, then, for any $\gamma \in \pi_1 \Sigma_g$, $\rho(\gamma)$ can be represented by a mapping which lifts to a hyperbolic isometry of $\mathbb{H}^2$. (The axis of the hyperbolic isometry projects to a loop representing γ.) This implies that $\partial_\infty \rho(\gamma)$ is a hyperbolic map in the sense of definition 1.

Let $x, y \in \pi_1(\Sigma_g, *)$ represent vanishing cycles with $i(x,y) \neq 0$. By the discussion at the end of section 1 we have $x, y \in \Gamma \subset N$. According to lemma 2 this contradicts $\chi_{\mathbb{R}} \mid_N \in ker\left(H_b^2(N;\mathbb{R}) \to H^2(N;\mathbb{R})\right)$, giving one direction of lemma 5.

Conversely, assume that $i(x,y) = 0$ holds for all $x, y \in \Gamma$. (In particular, Γ must have infinite index in $\pi_1(\Sigma_g, *)$, thus $H^2(\Gamma;\mathbb{R}) = 0$.)

If x_-, x_+ are the fixed points of $\rho(x)$, $i(x,y) = 0$ implies $t_y(x_\pm) = x_\pm$ for all $y \in \Gamma$. It is immediate from the definition of the Euler cocycle c that this implies $c(f, gt_y) = c(f,g)$ for all $y \in \Gamma$ and arbitrary f, g. Moreover, t_y fixes not only x_+ but also each point in the Γ-orbit of x_+, since $i(x,y) = 0$ for all $x \in \Gamma$. This implies then that also $c(ft_y, g) = c(f,g)$ holds for all $f, g \in \Gamma$ (although not for arbitrary f, g). We remark that each element of $N \subset Map_{g,*}$ is a product of Dehn twists (at vanishing cycles) and elements in $\Gamma \subset \pi_1(\Sigma_g, *)$. From lemma 3 we get (argueing by induction) that each such product can be written in the form $\gamma t_{y_1} \ldots t_{y_r}$ with $\gamma \in \Gamma$ and $y_1, \ldots, y_r$ vanishing cycles. We define a map $p : N \to \Gamma$ by $p(\gamma t_{y_1} \ldots t_{y_r}) = \gamma$. Denoting $i : \Gamma \to N$ the inclusion, we have just shown that $(ip)^* c = c$. But, on the cohomology level, $(ip)^*$ factors over $H^2(\Gamma;\mathbb{R}) = 0$, hence $(ip)^* = 0$ in cohomology. Thus $(ip)^* \chi \mid_N = \chi \mid_N$ implies $\chi \mid_N = 0 \in H^2(N;\mathbb{R})$. $\square$

We close this section by proving some corollaries.

Corollary 2. *: Let Γ be a word-hyperbolic group and M_Γ a Lefschetz fibration (with regular fiber of genus $g \geq 2$) with $\pi_1 M_\Gamma = \Gamma$, which has two vanishing cycles with nontrivial homological intersection. Then $\pi_2 M_\Gamma \neq 0$.*

Remark: To any finitely presented group Γ, there exists some Lefschetz fibration M_Γ with $\pi_1 M_\Gamma = \Gamma$ ([3],[1]).

Proof. If $\pi_1 M$ is word-hyperbolic, then $H_b^2(\pi_1 M, \mathbb{Z}) \to H^2(\pi_1 M, \mathbb{Z})$ is surjective, by the Gromov-Mineyev theorem ([12]).

Assume $\pi_2 M = 0$. Then, by the Hopf-identity,

$$H^2(\pi_1 M;\mathbb{Z}) \simeq H^2(M;\mathbb{Z}) / \pi_2 M = H^2(M, \mathbb{Z}).$$

Thus, surjectivity of $H_b^2(\pi_1 M, \mathbb{Z}) \to H^2(\pi_1 M, \mathbb{Z})$ would imply surjectivity of $H_b^2(M, \mathbb{Z}) \to H^2(M, \mathbb{Z})$. In particular, the Euler class would be bounded. This contradicts theorem 1. $\square$

Corollary 3. *If a Lefschetz fibration, with regular fiber of genus ≥ 2, admits a Riemannian metric with negative sectional curvature everywhere, then all vanishing cycles have vanishing homological intersection.*

Proof. If M admitted a metric of negative sectional curvature, then $H_b^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z})$ would be surjective ([8]). $\square$

4. Relation with simplicial volume

For a closed, orientable manifold M, we consider the simplicial volume $\parallel M \parallel$, defined in [8]. It is well-known ([8]) that $\parallel M \parallel > 0$ if and only if the fundamental class $\omega_M \in H^{dim(M)}(M;\mathbb{R})$ is bounded.

Lemma 6. *Let $\pi : M \to B$ be a Lefschetz fibration with regular fiber F of genus $g(F) \neq 1$. Let $e \in H^2(M; \mathbb{Z})$ be the Euler class. Then*
$\| M \| > 0$ *if and only if the cup-product $e_{\mathbb{R}} \cup \pi^* \omega_B \in H^*(M; \mathbb{R})$ is bounded.*

Proof. It is well-known that $\| M \| > 0$ if and only if the fundamental class is bounded. Thus, to show equivalence of $\| M \| > 0$ with boundedness of $e_{\mathbb{R}} \cup \pi^* \omega_B$, it suffices to show that $e \cup \pi^* \omega_B$ is a *non-zero* multiple of the fundamental class, i.e., that

$$< e \cup \pi^* \omega_B, [M] > \neq 0.$$

The proof of this inequality is a minor generalisation of the argument in [9].

We work with de Rham-cohomology. Define $\pi_* : H^2(M) \to H^0(B)$ by $\pi_* = D_B^{-1} \pi_* D_M$, where D_B resp. D_M are the Poincare duality maps. One has
$< \pi^* \alpha \cup \beta, c > = < \alpha \cup \pi_* \beta, \pi_* c >$ for any $\alpha, \beta \in H^*(M), c \in H_*(M)$.

We continue to denote e resp. ω_B some differential forms representing the cohomology classes $e_{\mathbb{R}}$ resp. ω_B. At each point, $e \cup \pi^* \omega_B$ is a multiple of the volume form. Therefore its value on $[M]$ does not depend on the zero-volume set $\pi^{-1}\pi(C)$. Hence,

$$e \cup \pi^* \omega_B([M]) = \int_{M - \pi^{-1}\pi(C)} e \cup \pi^* \omega_B = \int_{B - \pi(C)} \pi_* e \cup \omega_B.$$

Using, for $b \in B$, $< \pi_* e, [b] > = < \pi_* e, \pi_* [F] > = < e, [F] > = \chi(F)$, we get

$$e \cup \pi^* \omega_B([M]) = \chi(F) \int_B \omega_B \neq 0$$

because $\chi(F) \neq 0$. □

If genus$(B) \geq 2$, then $\pi^* \omega_B$ is bounded and, thus, a sufficient condition for $\| M \| > 0$ is boundedness of the Euler class e. One should not expect this condition to be necessary. To understand under which conditions boundedness of $e \cup \pi^* \omega_B$ (and hence nontriviality of $\| M \|$) holds, it would be necessary to understand the fourth bounded cohomology of Lefschetz fibrations better.

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