Generalizations of Agol's inequality and nonexistence of tight laminations

Thilo Kuessner

1 Results

Agol's inequality ([1], Theorem 2.1.) is the following:

Agol's inequality: *If M is a hyperbolic 3-manifold containing an incompressible, properly embedded surface F, then*

$$Vol\,(M) \geq -2V_3\chi\left(Guts\left(\overline{M-F}\right)\right),$$

where V_3 is the volume of a regular ideal tetrahedron in hyperbolic 3-space.

In [2], this inequality has been improved to

$$Vol\,(M) \geq Vol\left(Guts\left(\overline{M-F}\right)\right) \geq -V_{oct}\chi\left(Guts\left(\overline{M-F}\right)\right),$$

where V_{oct} is the volume of a regular ideal octahedron in hyperbolic 3-space.

In this paper we will, building on ideas from [1], prove a general inequality for the (transversal) Gromov norm $\parallel M \parallel_{\mathcal{F}}$ and the normal Gromov norm $\parallel M \parallel_{\mathcal{F}}^{norm}$ of laminations.

To state the result in its general form we first need two definitions.

Definition (Pared acylindrical): *Let Q be a manifold with a given decomposition*

$$\partial Q = \partial_0 Q \cup \partial_1 Q.$$

The pair $(Q, \partial_1 Q)$ is called a pared acylindrical manifold, if any continuous mapping of pairs $f : \left(\mathbf{S}^1 \times [0,1]\,, \mathbf{S}^1 \times \{0,1\}\right) \to (Q, \partial_1 Q)$, which is π_1-injective as a map of pairs, must be homotopic, as a map of pairs

$$\left(\mathbf{S}^1 \times [0,1]\,, \mathbf{S}^1 \times \{0,1\}\right) \to (Q, \partial_1 Q)\,,$$

into ∂Q.

Definition (Essential Decomposition): *Let $(N, \partial N)$ be a pair of topological spaces such that $N = Q \cup R$ for two subspaces Q, R. Let*

$$\partial_0 Q = Q \cap R, \partial_1 Q = Q \cap \partial N, \partial_1 R = R \cap \partial N, \partial Q = \partial_0 Q \cup \partial_1 Q, \partial R = \partial_0 Q \cup \partial_1 R.$$

We say that the decomposition $N = Q \cup R$ is an essential decomposition of $(N, \partial N)$ if the inclusions

$$\partial_1 Q \to Q \to N, \partial_1 R \to R \to N, \partial N \to N, \partial_0 Q \to Q, \partial_0 Q \to R$$

are each π_1-injective (for each path-component).

Theorem 1. *Let M be a compact, orientable, connected n-manifold and $\mathcal{F}$ a lamination (of codimension one) of M.*

Assume that $N := \overline{M - \mathcal{F}}$ has a decomposition $N = Q \cup R$ into orientable n-manifolds (with boundary) Q, R such that the following assumptions are satisfied for $\partial_0 Q = Q \cap R, \partial_1 Q = Q \cap \partial N, \partial_1 R = R \cap \partial N$:
i) each path-component of $\partial_0 Q$ has amenable fundamental group,
ii) $(Q, \partial_1 Q)$ is pared acylindrical, $\partial_1 Q$ is acylindrical
iii) $Q, \partial N, \partial_1 Q, \partial_1 R, \partial_0 Q$ are aspherical,
iv) the decomposition $N = Q \cup R$ is an essential decomposition of $(N, \partial N)$.

Then

$$\| M, \partial M \|_{\mathcal{F}}^{norm} \geq \frac{1}{n+1} \| \partial Q \| .$$

In the case of 3-manifolds M carrying an essential lamination $\mathcal{F}$, considering $Q = Guts\left(\overline{M - \mathcal{F}}\right)$ yields then as a special case:

Theorem 2. *Let M be a compact 3-manifold with (possibly empty) boundary consisting of incompressible tori, and let $\mathcal{F}$ be an essential lamination of M. Then*

$$\| M, \partial M \|_{\mathcal{F}}^{norm} \geq -\chi\left(Guts\left(\overline{M - \mathcal{F}}\right)\right).$$

More generally, if P is a polyhedron with f faces, then

$$\| M, \partial M \|_{\mathcal{F},P}^{norm} \geq -\frac{2}{f-2} \chi\left(Guts\left(\overline{M - \mathcal{F}}\right)\right).$$

The following corollary applies, for example, to all hyperbolic manifolds M obtained by Dehn-filling the complement of the figure-eight knot in $\mathbf{S}^3$. (It is known that each of these M contains tight laminations. By the following corollary, all these tight laminations have empty guts.)

Corollary 4: *If M is a finite-volume hyperbolic 3-manifold with $Vol(M) < 2V_3 = 2.02...$, then M carries no essential lamination $\mathcal{F}$ with $\parallel M \parallel_{\mathcal{F},P}^{norm} = \parallel M \parallel_P$ for all polyhedra P, and nonempty guts. In particular, there is no tight essential lamination with nonempty guts.*

It was observed by Calegari-Dunfield in [7] that a generalization of Agol's inequality to the case of tight laminations, together with the results in [7] about tight laminations with empty guts, would imply the following corollary.

Corollary 5 ([7], Conjecture 9.7.): *The Weeks manifold admits no tight lamination $\mathcal{F}$.*

Putting this together with the main result of a recent paper by Tao Li ([24]), one can even improve this result as follows.

Corollary 6: *The Weeks manifold admits no transversely orientable essential lamination.*

Finally, we also have an application of Theorem 1 to higher-dimensional manifolds.

Corollary 7: *Let M be a compact Riemannian n-manifold of negative sectional curvature and finite volume. Let $F \subset M$ be a geodesic $n-1$-dimensional hypersurface of finite volume. Then $\parallel F \parallel \leq \frac{n+1}{2} \parallel M \parallel$.*

The basic idea of Theorem 1, say for simplicity in the special situation of Corollary 7, is the following: a simplex which contributes to a normalized fundamental cycle of M should intersect $\partial Q = 2F$ in at most $n+1$ codimension one simplices. This is of course not true in general: simplices can wrap around M many times and intersect F arbitrarily often, and even a homotopy rel. vertices will not change this. As an obvious examle, look at the following situation: let γ be a closed geodesic transverse to F, and for some large N let σ be a straight simplex contained in a small neighborhood of γ^N. Then σ intersects F N times and, since σ is already straight, this number of intersections can of course not be reduced by straightening. This shows that some more involved straightening must take place, and that the acylindricity of F is an essential condition. The way to use acylindricity will be to find a normalization such that many subsets of simplices are mapped to cylinders, which degenerate and thus can be removed without changing the homology class.

We remark that many technical points, in particular the use of multicomplexes, can be omitted if (in the setting of Theorem 2) one does not consider incompressible surfaces or essential laminations, but just geodesic surfaces in hyperbolic manifolds. In this case, all essential parts of the proof of Theorem 1 enter without the notational complications caused by the use of multicomplexes. Therefore we have given a fairly detailed outline of the proof for this special case in Section 6.1. This should help to motivate the general proof in Section 6.2. (We mention that Theorem 1 is not true without assuming amenability of $\pi_1 \partial_0 Q$. This indicates that the proof of multicomplexes in the proof of Theorem 1 seems unavoidable.)

Acknowledgements: It is probably obvious that this paper is strongly influenced by Agol's preprint [1]. Moreover, the argument that a generalization of Agol's inequality would imply Corollary 5 is due to [7].

2 Preliminaries

2.1 Laminations

Let M be an n-manifold, possibly with boundary. In this paper all manifolds will be smooth and orientable. (Hence they are triangulable by Whitehead's theorem and possess a locally finite fundamental class.) A (codimension 1) lamination $\mathcal{F}$ of M is a foliation of a closed subset $\mathcal{F}$ of M, i.e., a decomposition of a closed subset $\mathcal{F} \subset M$ into immersed codimension 1 submanifolds (leaves) so that M is covered by charts $\phi_j : \mathbf{R}^{n-1} \times \mathbf{R} \rightarrow M$, the intersection of any leaf with the image of any chart ϕ_j being a union of plaques of the form $\phi_j \left(\mathbf{R}^{n-1} \times \{*\} \right)$. (We will denote by $\mathcal{F}$ both the lamination and the laminated subset of M, i.e. the union of leaves.) If M has boundary, we will always assume without further mentioning that $\mathcal{F}$ is *either transverse to* ∂M (that is, every leaf is transverse to $\mathcal{F}$) *or tangential to* ∂M (that is, ∂M is a leaf of $\mathcal{F}$). If neither of these two conditions were true, then the transverse and normal Gromov norm would be infinite, therefore all lower bounds will be trivially true.

To construct the leaf space T of $\mathcal{F}$, one considers the pull-back lamination $\widetilde{\mathcal{F}}$ on the universal covering $\widetilde{M}$. The space of leaves T is defined as the quotient of $\widetilde{M}$ under the following equivalence relation $\sim$. Two points $x, y \in \widetilde{M}$ are equivalent if either they belong to the same leaf of $\widetilde{\mathcal{F}}$, or they belong to the same connected component of the metric completion $\overline{\widetilde{M} - \widetilde{\mathcal{F}}}$ (for the path metric inherited by $\widetilde{M} - \widetilde{\mathcal{F}}$ from an arbitrary Riemannian metric on $\widetilde{M}$).

Laminations of 3-manifolds. A **lamination** $\mathcal{F}$ of a **3-manifold** M is called **essential** if no leaf is a sphere or a torus bounding a solid torus, $\overline{M-\mathcal{F}}$ is irreducible, and $\partial\left(\overline{M-\mathcal{F}}\right)$ is incompressible and end-incompressible in $\overline{M-\mathcal{F}}$, where again the metric completion $\overline{M-\mathcal{F}}$ of $M-\mathcal{F}$ is taken w.r.t. the path metric inherited from any Riemannian metric on M, see [13], ch.1. (Note that $\overline{M-\mathcal{F}}$ is immersed in M, the leaves of $\mathcal{F}$ in the image of the immersion are called boundary leaves.)

Examples of essential laminations are taut foliations or compact, incompressible, boundary-incompressible surfaces in compact 3-manifolds. (We always consider laminations without isolated leaves. If a lamination has isolated leaves, then it can be converted into a lamination without isolated leaves by replacing each two-sided isolated leaf S_i with the trivially foliated product $S_i \times [0,1]$, resp. each one-sided isolated leaf with the canonically foliated normal I-bundle, without changing the topological type of M.)

If $\mathcal{F}$ is an essential lamination, then the leaf space T is an order tree, with segments corresponding to directed, transverse, efficient arcs. (An order tree T is a set T with a collection of linearly ordered subsets, called segments, such that the axioms of [13], Def. 6.9., are satisfied.) Moreover, T is an $\mathbf{R}$-order tree, that is, it is a countable union of segments and each segment is order isomorphic to a closed interval in $\mathbf{R}$. T can be topologized by the order topology on segments (and declaring that a set is closed if the intersection with each segment is closed). For this topology, $\pi_0 T$ and $\pi_1 T$ are trivial (see, for example, [27], Chapter 5, and its references).

The order tree T comes with a fixed-point free action of $\pi_1 M$. Fenley ([9]) has exhibited hyperbolic 3-manifolds whose fundamental groups do not admit any fixed-point free action on $\mathbf{R}$-order trees. Thus there are hyperbolic 3-manifolds not carrying any essential lamination.

If M is hyperbolic and $\mathcal{F}$ an essential lamination, then $\overline{M-\mathcal{F}}$ has a characteristic submanifold which is the maximal submanifold that can be decomposed into I-bundles and solid tori, respecting boundary patterns (see [18], [19] for precise definitions). The complement of this characteristic submanifold is denoted by $Guts\,(\mathcal{F})$. It admits a hyperbolic metric with geodesic boundary and cusps. (Be aware that some authors, like [7], include the solid tori into the guts.) If $\mathcal{F}=F$ is a properly embedded, incompressible, boundary-incompressible surface, then **Agol's inequality** states that $Vol\,(M) \geq -2V_3\chi\,(Guts\,(F))$. This implies, for example, that a hyperbolic manifold of volume $< 2V_3$ can not contain any geodesic surface of finite area. Recently, this inequality has been improved to $Vol\,(M) \geq Vol\,(Guts\,(F)) \geq -V_{oct}\chi\,(Guts\,(F))$ in [2], using estimates coming from Perelman's work on the Ricci flow.

Assume that $\mathcal{F}$ is a codimension one lamination of an n-manifold M such that its leaf space T is an $\mathbf{R}$-order tree. (For example this is the case if $n=3$ and $\mathcal{F}$ is essential.) An essential lamination is called **tight** if T is Hausdorff. It is called unbranched if T is homeomorphic to $\mathbf{R}$. It is said to have two-sided branching ([5], Definition 2.5.2) if there are leaves $\lambda, \lambda_1, \lambda_2, \mu, \mu_1, \mu_2$ such that the corresponding points in the T satisfy $\lambda < \lambda_1, \lambda < \lambda_2, \mu > \mu_1, \mu > \mu_2$ but λ_1, λ_2 are incomparable and μ_1, μ_2 are incomparable. It is said to have one-sided branching if it is neither unbranched nor has two-sided branching.

If M is a hyperbolic 3-manifold and carries a tight lamination with empty guts, then Calegari and Dunfield have shown ([7], Theorem 3.2.) that $\pi_1 M$ acts effectively on the circle, i.e., there is an injective homomorphism $\pi_1 M \rightarrow Homeo\left(\mathbf{S}^1\right)$. This implies that the Weeks manifold (the closed hyperbolic manifold of smallest volume) can not carry a tight lamination with empty guts ([7], Corollary 9.4.). The aim of this paper is to find obstructions to the existence of laminations with nonempty guts.

2.2 Simplicial volume and refinements

Let M be a compact, orientable, connected n-manifold, possibly with boundary. Its top integer (singular) homology group $H_n\left(M, \partial M; \mathbf{Z}\right)$ is cyclic. The image of a generator under the change-of-coefficients homomorphism $H_n\left(M, \partial M; \mathbf{Z}\right) \rightarrow H_n\left(M, \partial M; \mathbf{R}\right)$ is called a fundamental class and is denoted $[M, \partial M]$. If M is not connected, we define $[M, \partial M]$ to be the formal sum of the fundamental classes of its connected components.

The simplicial volume $\parallel M, \partial M \parallel$ is defined as $\parallel M, \partial M \parallel = inf\left\{\sum_{i=1}^{r} \mid a_i \mid\right\}$ where the infimum is taken over all singular chains $\sum_{i=1}^{r} a_i \sigma_i$ (with real coefficients) representing the fundamental class in $H_n\left(M, \partial M; \mathbf{R}\right)$.

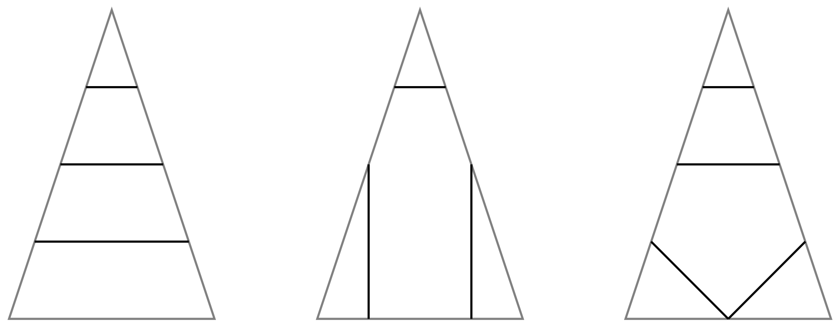
If $M - \partial M$ carries a complete hyperbolic metric of finite volume $Vol\left(M\right)$, then $\parallel M, \partial M \parallel = \frac{1}{V_n} Vol\left(M\right)$ with $V_n = sup\left\{Vol\left(\Delta\right) : \Delta \subset \mathbf{H}^n \text{ geodesic simplex}\right\}$ (see [14],[30],[3], [10]).

More generally, let P be any polyhedron. Then the invariant $\parallel M, \partial M \parallel_P$ is defined in [1] as follows: denoting by $C_*\left(M, \partial M; P; \mathbf{R}\right)$ the complex of P-chains with real coefficients, and by $H_*\left(M, \partial M; P; \mathbf{R}\right)$ its homology, there is a canonical chain homomorphism $\psi : C_*\left(M, \partial M; P; \mathbf{R}\right) \rightarrow C_*\left(M, \partial M; \mathbf{R}\right)$, given by some triangulations of P which is to be chosen such that all possible cancellations of boundary faces are preserved. $\parallel M, \partial M \parallel_P$ is defined as the infimum of $\sum_{i=1}^{r} \mid a_i \mid$ over all P-chains $\sum_{i=1}^{r} a_i P_i$ such that $\psi\left(\sum_{i=1}^{r} a_i P_i\right)$ represents the fundamental class $[M, \partial M]$. Let $V_P := sup\left\{Vol\left(\Delta\right)\right\}$, where the supremum is taken over all straight P-polyhedra $\Delta \subset \mathbf{H}^3$. Proposition 1 is Lemma 4.1. in [1]. (The proof in [1] is quite short, and it does not give details for the cusped case. However, the proof in the cusped case can be completed using the arguments in sections 5 and 6 of Francaviglia's paper [10].)

Proposition 1. *If $M - \partial M$ admits a hyperbolic metric of finite volume $Vol\,(M)$, then*

$$\parallel M, \partial M \parallel_P = \frac{1}{V_P} Vol\,(M)\,.$$

Let M be a manifold and $\mathcal{F}$ a **codimension one lamination** of M. Let Δ^n be the standard simplex in $\mathbf{R}^{n+1}$, and $\sigma : \Delta^n \rightarrow M$ some continuous singular simplex. The lamination $\mathcal{F}$ induces an equivalence relation on Δ^n by: $x \sim y \Longleftrightarrow \sigma\,(x)$ and $\sigma\,(y)$ belong to the same connected component of $L \cap \sigma\,(\Delta^n)$ for some leaf L of $\mathcal{F}$. We say that a singular simplex $\sigma : \Delta^n \rightarrow M$ is laminated if the equivalence relation $\sim$ is induced by a lamination $\mathcal{F} \mid_\sigma$ of Δ^n. We call a lamination $\mathcal{F}$ of Δ^n affine if there is an affine mapping $f : \Delta^n \rightarrow \mathbf{R}$ such that $x, y \in \Delta^n$ belong to the same leaf if and only if $f\,(x) = f\,(y)$. We say that a lamination $\mathcal{G}$ of Δ^n is conjugate to an affine lamination if there is a simplicial homeomorphism $H : \Delta^n \rightarrow \Delta^n$ such that $H^*\mathcal{G}$ is an affine lamination.
We say that a singular n-simplex $\sigma : \Delta^n \rightarrow M$, $n \geq 2$, is **transverse** to $\mathcal{F}$ if it is laminated and it is either contained in a leaf, or $\mathcal{F} \mid_\sigma$ is conjugate to an affine lamination $\mathcal{G}$ of Δ^n.
For $n = 1$, we say that a singular 1-simplex $\sigma : \Delta^1 \rightarrow M$ is transverse to $\mathcal{F}$ if it is either contained in a leaf, or for each lamination chart $\phi : U \rightarrow \mathbf{R}^{m-1} \times \mathbf{R}^1$ (with m-th coordinate map $\phi_m : U \rightarrow \mathbf{R}^1$) one has that $\phi_m \circ \sigma \mid_{\sigma^{-1}(U)} : \sigma^{-1}\,(U) \rightarrow \mathbf{R}^1$ is locally surjective at all points of $int\,(\Delta^1)$, i.e. for all $p \in int\,(\Delta^1) \cap \sigma^{-1}\,(U)$, the image of $\phi_m \circ \sigma \mid_{\sigma^{-1}(U)}$ contains a neighborhood of $\phi_m \circ \sigma\,(p)$.
We say that the simplex $\sigma : \Delta^n \rightarrow M$ is **normal** to $\mathcal{F}$ if, for each leaf F, $\sigma^{-1}\,(F)$ consists of normal disks, i.e. disks meeting each edge of Δ^n at most once. (If $F = \partial M$ is a leaf of $\mathcal{F}$ we also allow that $\sigma^{-1}\,(F)$ can be a face of Δ^n). In particular, any transverse simplex is normal.
In the special case of **foliations** $\mathcal{F}$ one has that the transversality of a singular simplex σ is implied by (hence equivalent to) the normality of σ, as can be shown along the lines of [23], section 1.3.
More generally, let P be any polyhedron. Then we say that a singular polyhedron $\sigma : P \rightarrow M$ is normal to $\mathcal{F}$ if, for each leaf F, $\sigma^{-1}\,(F)$ consists of normal disks, i.e. disks meeting each edge of P at most once (or being equal to a face of P, if F is a boundary leaf).



transverse | normal, not transverse | not normal

Definition 1. *Let M be a compact, oriented, connected n-manifold, possibly with boundary, and $\mathcal{F}$ a foliation or lamination on M. Let Δ^n be the standard simplex and P any polyhedron. Then*

$$\parallel M, \partial M \parallel_{\mathcal{F}} := inf \left\{ \sum_{i=1}^{r} \mid a_i \mid : \psi\left(\sum_{i=1}^{r} a_i \sigma_i \right) \ represents \ [M, \partial M], \sigma_i : \Delta^n \to M \ transverse \ to \ \mathcal{F} \right\}$$

and

$$\parallel M, \partial M \parallel_{\mathcal{F},P}^{norm} := inf \left\{ \sum_{i=1}^{r} \mid a_i \mid : \psi\left(\sum_{i=1}^{r} a_i \sigma_i \right) \ represents \ [M, \partial M], \sigma_i : P \to M \ normal \ to \ \mathcal{F} \right\}.$$

In particular, we define $\parallel M, \partial M \parallel_{\mathcal{F}}^{norm} = \parallel M, \partial M \parallel_{\mathcal{F},\Delta^n}^{norm}$.

All norms are finite, under the assumption that $\mathcal{F}$ is transverse or tangential to ∂M.
There is an obvious inequality

$$\parallel M, \partial M \parallel \leq \parallel M, \partial M \parallel_{\mathcal{F}}^{norm} \leq \parallel M, \partial M \parallel_{\mathcal{F}} .$$

In the case of foliations, equality $\parallel M, \partial M \parallel_{\mathcal{F}}^{norm} = \parallel M, \partial M \parallel_{\mathcal{F}}$ holds.
(We remark that all definitions extend in an obvious way to disconnected manifolds by summing over the connected components.)

Proposition 2 and Lemma 1 are a straightforward generalisation of [5], Theorem 2.5.9, and of arguments in [1].

Proposition 2. *Let M be a compact, oriented 3-manifold.*
a) If $\mathcal{F}$ is an essential lamination which is either unbranched or has one-sided branching such that the induced lamination of ∂M is unbranched, then

$$\parallel M, \partial M \parallel_{\mathcal{F},P}^{norm} = \parallel M, \partial M \parallel_P$$

for each polyhedron P.
b) If $\mathcal{F}$ is a tight essential lamination, then

$$\parallel M, \partial M \parallel_{\mathcal{F},P}^{norm} = \parallel M, \partial M \parallel_P$$

for each polyhedron P.

Proof. Since $\mathcal{F}$ is an essential lamination, we know from [13], Theorem 6.1., that the leaves are π_1-injective, the universal covering $\widetilde{M}$ is homeomorphic to $\mathbf{R}^3$ and that the leaves of the pull-back lamination are planes, in particular aspherical. Therefore Proposition 2 is a special case of Lemma 1. □

Lemma 1. *Let M be a compact, oriented, aspherical manifold, and $\mathcal{F}$ a lamination of codimension one.*
Assume that the leaves are π_1-injective and aspherical, and that the leaf space T is an $\mathbf{R}$-order tree.
a) If the leaf space T is either $\mathbf{R}$ or branches in only one direction, such that the induced lamination of ∂M has leaf space $\mathbf{R}$, then

$$\parallel M, \partial M \parallel_{\mathcal{F},P}^{norm} = \parallel M, \partial M \parallel_P$$

for each polyhedron P.
b) If the leaf space is a Hausdorff tree, then

$$\parallel M, \partial M \parallel_{\mathcal{F},P}^{norm} = \parallel M, \partial M \parallel_P$$

for each polyhedron P.

Proof. To prove the wanted equalities, it suffices in each case to show that any (relative) cycle can be homotoped to a cycle consisting of normal polyhedra.
We denote by $\widetilde{\mathcal{F}}$ the pull-back lamination of $\widetilde{M}$ and $p : \widetilde{M} \to T = \widetilde{M}/\widetilde{\mathcal{F}}$ the projection to the leaf space.
a) First we consider the case that P=simplex ([5], Section 4.1) and $\mathcal{F}$ unbranched. For this case, we can repeat the argument in [5], Lemma 2.2.8. Namely, let us be given a (relative) cycle $\sum_{i=1}^{r} a_i \sigma_i$, lift it to a $\pi_1 M$-equivariant (relative) cycle on $\widetilde{M}$ and then perform an (equivariant) straightening, by induction on the dimension of subsimplices of the lifts $\widetilde{\sigma_i}$ as follows: for each edge $\tilde{e}$ of any lift $\widetilde{\sigma_i}$, its projection $p(\tilde{e})$ to the leaf space T is homotopic to a unique straight arc

$str\,(p\,(\tilde{e}))$ in $T \simeq \mathbf{R}$. It is easy to see (covering the arc by foliation charts and then extending the lifted arc stepwise) that $str\,(p\,(\tilde{e}))$ can be lifted to an arc $str\,(\tilde{e})$ with the same endpoints as $\tilde{e}$, and that the homotopy between $str\,(p\,(\tilde{e}))$ and $p\,(\tilde{e})$ can be lifted to a homotopy between $str\,(\tilde{e})$ and $\tilde{e}$. $str\,(\tilde{e})$ is transverse to $\mathcal{F}$, because its projection is a straight arc in T. These homotopies of edges can be extended to a homotopy of the whole (relative) cycle. Thus we have straightened the 1-skeleton of the given (relative) cycle.

Now let us be given a 2-simplex $\tilde{f} : \Delta^2 \to \widetilde{M}$ with transverse edges. There is an obvious straightening $str\left(p\left(\tilde{f}\right)\right)$ of $p\left(\tilde{f}\right) : \Delta^2 \to T$ as follows: if, for $t \in T$, $\left(p\tilde{f}\right)^{-1}(t)$ has two preimages x_1, x_2 on edges of Δ^2 (which are necessarily unique), then $str\left(p\left(\tilde{f}\right)\right)$ maps the line which connects x_1 and x_2 in Δ^2 constantly to t. It is clear that this defines a continuous map $str\left(p\left(\tilde{f}\right)\right) : \Delta^2 \to T$.

Since leaves $\widetilde{F}$ of $\widetilde{\mathcal{F}}$ are connected ($\pi_0 \widetilde{F} = 0$), $str\left(p\left(\tilde{f}\right)\right)$ can be lifted to a map $str\left(\tilde{f}\right) : \Delta^2 \to \widetilde{M}$ with $p\left(str\left(\tilde{f}\right)\right) = str\left(p\left(\tilde{f}\right)\right)$. $str\left(\tilde{f}\right)$ is transverse to $\mathcal{F}$, because its projection is a straight simplex in T.

There is an obvious homotopy between $p\left(\tilde{f}\right)$ and $str\left(p\left(\tilde{f}\right)\right)$. For each $t \in T$, the restriction of the homotopy to $\left(p\tilde{f}\right)^{-1}(t)$ can be lifted to a homotopy in $\widetilde{M}$, because $\pi_1 \widetilde{M} = 0$. Since $\pi_2 \widetilde{M} = 0$, these homotopies for various $t \in T$ fit together continuously to give a homotopy between $\tilde{f}$ and $str\left(\tilde{f}\right)$.

These homotopies of 2-simplices leave the (already transverse) boundaries pointwise fixed, thus they can be extended to a homotopy of the whole (relative) cycle. Hence we have straightened the 2-skeleton of the given (relative) cycle.

Assume that we have already straightened the k-skeleton, for some $k \in \mathbf{N}$. The analogous procedure, using $\pi_{k-1} \widetilde{F} = 0$ for all leaves, and $\pi_k \widetilde{M} = 0, \pi_{k+1} \widetilde{M} = 0$, allows to straighten the $(k+1)$-skeleton of the (relative) cycle. This finishes the proof in the case that $\mathcal{F}$ is unbranched.

The generalization to the case that $\mathcal{F}$ has one-sided branching such that the induced lamination of ∂M is unbranched works as in [5], Theorem 2.6.6.

We remark that in the case $P = simplex$ we get not only a normal cycle, but even a transverse cycle.

Now we consider the case of arbitrary polyhedra P. Let $\sum_{i=1}^r a_i \sigma_i$ be a P-cycle. It can be subtriangulated to a simplicial cycle $\sum_{i=1}^r a_i \sum_{j=1}^s \tau_{i,j}$. Again the argument in [5], Lemma 2.2.8 (resp. its version for manifolds with boundary), shows that this simplicial cycle can be homotoped such that each $\tau_{i,j}$ is transverse (and such that boundary cancellations are preserved). But transversality of each $\tau_{i,j}$ implies by definition that $\sigma_i = \sum_{j=1}^s \tau_{i,j}$ is normal (though in general not

transverse) to $\mathcal{F}$.
b) By assumption $\widetilde{M}/\widetilde{\mathcal{F}}$ is a Hausdorff tree. We observe that its branching points are the projections of complementary regions. Indeed, let F be a leaf of $\mathcal{F}$, then $\widetilde{F}$ is a submanifold of the contractible manifold $\widetilde{M}$. By asphericity and π_1-injectivity of F, $\widetilde{F}$ must be contractible. By Alexander duality it follows that $\widetilde{M}-\widetilde{F}$ has two connected components. Therefore the complement of the point $p\left(\widetilde{F}\right)$ in the leaf space has (at most) two connected components, thus $p\left(\widetilde{F}\right)$ can not be a branch point.
Again, to define a straightening of P-chains it suffices to define a canonical straightening of singular polyhedra P such that straightenings of common boundary faces will agree. Let $\tilde{v}_0,\ldots,\tilde{v}_n$ be the vertices of the image of P. For each pair $\{\tilde{v}_i,\tilde{v}_j\}$ there exists at most one edge $\tilde{e}_{ij}$ with vertices $\tilde{v}_i,\tilde{v}_j$ in the image of P. Since the leaf space is a tree, we have a unique straight arc $str\left(p\left(\tilde{e}_{ij}\right)\right)$ connecting the points $p\left(\tilde{v}_i\right)$ and $p\left(\tilde{v}_j\right)$ in the leaf space. As in a), one can lift this straight arc $str\left(p\left(\tilde{e}_{ij}\right)\right)$ to an arc $str\left(\tilde{e}_{ij}\right)$ in $\widetilde{M}$, connecting $\tilde{v}_i$ and $\tilde{v}_j$, which is transverse to $\mathcal{F}$. We define this arc $str\left(\tilde{e}_{ij}\right)$ to be the straightening of $\tilde{e}_{ij}$. As in a), we have homotopies of 1-simplices, which extend to a homotopy of the whole (relative) cycle. Thus we have straightened the 1-skeleton.

Now let us be given the 3 vertices $\tilde{v}_0,\tilde{v}_1,\tilde{v}_2$ of a 2-simplex $\tilde{f}$ with straight edges. If the projections $p\left(\tilde{v}_0\right),p\left(\tilde{v}_1\right),p\left(\tilde{v}_2\right)$ belong to a subtree isomorphic to a connected subset of $\mathbf{R}$, then we can straighten $\tilde{f}$ as in a). If not, we have that the projection of the 1-skeleton of this simplex has exactly one branch point, which corresponds to a complementary region. (The projection may of course meet many branch points of the tree, but the image of the projection, considered as a subtree, can have at most one branch point. In general, a subtree with n vertices can have at most $n-2$ branch points.) The preimage of the complement of this complementary region consists of three connected subsets of the 2-simplex ("corners around the vertices"). We can straighten each of these subsets and do not need to care about the complementary region corresponding to the branch point. Thus we have straightened the 2-skeleton.
Assume that we have already straightened the k-skeleton, for some $k\in\mathbf{N}$. Let us be given the $k+2$ vertices $\tilde{v}_0,\tilde{v}_1,\ldots,\tilde{v}_{k+1}$ of a $(k+1)$-simplex with straight faces. Then we have (at most k) branch points in the projection of the simplex, which correspond to complementary regions. Again we can straighten the parts of the simplex which do not belong to these complementary regions as in a), since they are projected to linearly ordered subsets of the tree. Thus we have straightened the $(k+1)$-skeleton.

Since, by the recursive construction, we have defined straightenings of simplices with common faces by first defining (the same) straightenings of their common faces, the straightening of a (relative) cycle will be again a (relative) cycle, in the same (relative) homology class. $\square$

Remark: For $\| M \|_{\mathcal{F}}$ instead of $\| M \|_{\mathcal{F}}^{norm}$, equality b) is in general wrong, and equality a) is unknown (but presumably wrong).
If $\mathcal{F}$ is essential but not tight, one may still try to homotope cycles to be transverse, by possibly changing the lamination. In the special case that the cycle is coming from a triangulation, this has been done in [4] and [11] by Brittenham resp. Gabai. It is not obvious how to generalize their arguments to cycles with overlapping simplices.

3 Retracting chains to codimension zero submanifolds

3.1 Definitions

The results of this section are essentially all due to Gromov, but we follow mainly our exposition in [22]. We start with some recollections about multicomplexes (cf. [14], Section 3, or [22], Section 1).

A multicomplex K is a topological space $\mid K \mid$ with a decomposition into simplices, where each n-simplex is attached to the $n-1$-skeleton K_{n-1} by a simplicial homeomorphism $f : \partial\Delta^n \rightarrow K_{n-1}$. (In particular, each n-simplex has $n+1$ distinct vertices.)

As opposed to simplicial complexes, in a multicomplex there may be n-simplices with the same $n-1$-skeleton.

We call a multicomplex minimally complete if the following holds: whenever $\sigma : \Delta^n \rightarrow \mid K \mid$ is a singular n-simplex, such that $\partial_0\sigma, \ldots, \partial_n\sigma$ are distinct simplices of K, then σ is homotopic relative $\partial\Delta^n$ to a *unique* simplex in K.

We call a minimally complete multicomplex K aspherical if all simplices $\sigma \neq \tau$ in K satisfy $\sigma_1 \neq \tau_1$. That means, simplices are uniquely determined by their 1-skeleton.
Orientations of multicomplexes are defined as usual in the simplicial theory. For a simplex σ, $\overline{\sigma}$ will denote the simplex with the opposite orientation.
A submulticomplex L of a multicomplex K consists of a subset of the set of simplices closed under face maps. (K, L) is a pair of multicomplexes if K is a multicomplex and L is a submulticomplex of K.
A group G acts simplicially on a pair of multicomplexes (K, L) if it acts on the set of simplices of K, mapping simplices in L to simplices in L, such that the action commutes with all face maps. For $g \in G$ and σ a simplex in K, we denote by $g\sigma$ the simplex obtained by this action.

3.2 Construction of $K(X)$

We recall the construction from [22], section 1.3 (originally due to [14], page 45-46).

For a topological space X, we denote by $S_*(X)$ the simplicial set of all singular simplices in X and $\mid S_*(X) \mid$ its geometric realization.

For a topological space X, a multicomplex $\widehat{K}(X) \subset \mid S_*(X) \mid$ is constructed as follows. The 0-skeleton $\widehat{K_0}(X)$ equals $S_0(X)$. The 1-skeleton $\widehat{K_1}(X)$ contains one element in each homotopy class (rel. $\{0,1\}$) of singular 1-simplices $f:[0,1] \to X$ with $f(0) \neq f(1)$. For $n \geq 2$, assuming by recursion that the n-1-skeleton is defined, the n-skeleton $\widehat{K_n}(X)$ contains *one* singular n-simplex in each homotopy class (rel. boundary) of singular n-simplices $f:\Delta^n \to X$ with $\partial f \in \widehat{K_{n-1}}(X)$. We can choose simplices in $\widehat{K}(X)$ such that $\sigma \in \widehat{K}(X) \Leftrightarrow \overline{\sigma} \in \widehat{K}(X)$, where $\overline{\sigma}$ denotes the simplex with the opposite orientation. We will henceforth assume that $\widehat{K}(X)$ is constructed according to this condition.

According to [14], $\mid \widehat{K}(X) \mid$ is weakly homotopy equivalent to X.

The multicomplex $K(X)$ is defined as the quotient

$$K(X) := \widehat{K}(X) / \sim$$

where simplices in $\widehat{K}(X)$ are identified if and only if they have the same 1-skeleton. Let p be the canonical projection $p:\widehat{K}(X) \to K(X)$.

$K(X)$ is minimally complete and aspherical.

If $X' \subset X$ is a subspace, then we have (not necessarily injective) simplicial mappings $\hat{j}:\widehat{K}(X') \to \widehat{K}(X)$ and $j:K(X') \to K(X)$.

If $\pi_1 X' \to \pi_1 X$ is injective (for each path-connected component of X'), then j is injective ([22], Section 1.3) and we can (and will) consider $K(X')$ as a submulticomplex of $K(X)$. (Since simplices in $\widehat{K}(X')$ have image in X', this means that we assume to have constructed $\widehat{K}(X)$ such that simplices in $\widehat{K}(X)$ have image in X' whenever this is possible.) If moreover $\pi_n X' \to \pi_n X$ is injective for all $n \geq 2$ (e.g. if X' is aspherical), then also $\hat{j}$ is injective and $\widehat{K}(X')$ can be considered as a submulticomplex of $\widehat{K}(X)$.

In particular, if X **and** X' **are aspherical and** $\pi_1 X' \to \pi_1 X$ **is injective**, then there is an inclusion

$$i_*: C_*^{simp}(K(X), K(X')) = C_*^{simp}\left(\widehat{K}(X), \widehat{K}(X')\right) \to C_*^{sing}(X, X')$$

into the relative singular chain complex of (X, X').

Infinite and locally finite chains. In this paper we will also work with infinite chains, and in particular with locally finite chains on noncompact manifolds, as introduced in [14], section 0.2.

For a topological space X, a formal sum $\sum_{i\in I} a_i\sigma_i$ of singular k-simplices with real coefficients (with a possibly infinite index set I, and the convention $a_i \neq 0$ for $i \in I$) is an infinite singular k-chain. It is said to be a locally finite chain if each point of X is contained in the image of at most finitely many σ_i. Infinite resp. locally finite k-chains form real vector spaces $C_k^{inf}(X)$ resp. $C_k^{lf}(X)$. The boundary operator maps locally finite k-chains to locally finite k-1-chains, hence, for a pair of spaces (X, X') the homology $H_*^{lf}(X, X')$ of the complex of locally finite chains can be defined.
For a noncompact, orientable n-manifold X with (possibly noncompact) boundary ∂X, one has a fundamental class $[X, \partial X] \in H_n^{lf}(X, \partial X)$. We will say that an infinite chain $\sum_{i\in I} a_i\sigma_i$ represents $[X, \partial X]$ if it is homologous to a locally finite chain representing $[X, \partial X] \in H_n^{lf}(X, \partial X)$.
For a simplicial complex K, we denote by $C_k^{simp,inf}(K)$ the $\mathbf{R}$-vector space of (possibly infinite) formal sums $\sum_{i\in I} a_i\sigma_i$ with $a_i \in \mathbf{R}$ and σ_i k-simplices in K. If $\pi_n X' \to \pi_n X$ is injective for $n \geq 1$, we have again the obvious inclusion $i_* : C_*^{simp,inf}\left(\widehat{K}(X), \widehat{K}(X')\right) \to C_*^{inf}(X, X')$.
The following observation is of course a well-known application of the **homotopy extension property**, but we will use it so often that we state it here for reference.

Observation 1. *: Let X be a topological space and $\sigma_0 : \Delta^n \to X$ a singular simplex. Let $H : \partial\Delta^n \times I \to X$ be a homotopy with $H(x, 0) = \sigma_0(x)$ for all $x \in \partial\Delta^n$. Then there exists a homotopy $\overline{H} : \Delta^n \times I \to X$ with $\overline{H}\mid_{\partial\Delta^n\times I} = H$ and $\overline{H}\mid_{\Delta^n\times\{0\}} = \sigma_0$.*

If $X' \subset X$ is a subspace and the images of σ_0 and H belong to X', then we can choose $\overline{H}$ such that its image belongs to X'.

Lemma 2. *Let (X, X') be a pair of topological spaces. Assume $\pi_n X' \to \pi_n X$ is injective for each path-component of X' and each $n \geq 1$.*
a) Let $\sum_{i\in I} a_i\tau_i \in C_n^{inf}(X, X')$ be a (possibly infinite) singular n-chain. Assume that I is countable, and that each path-component of X and each non-empty path-component of X' contain uncountably many points. Then $\sum_{i\in I} a_i\tau_i \in C_n^{inf}(X, X')$ is homotopic to a (possibly infinite) simplicial chain $\sum_i a_i\tau_i' \in C_n^{simp,inf}\left(\widehat{K}(X), \widehat{K}(X')\right)$. In particular,

$$\sum_i a_i\tau_i' \in C_n^{simp,inf}\left(\widehat{K}(X), \widehat{K}(X')\right) \subset C_*^{inf}(X, X')$$

is homologous to $\sum_{i\in I} a_i\tau_i$.
b) Let $\sigma_0 \in \widehat{K}(X)$ and $H : \Delta^n \times I \to X$ a homotopy with $H(., 0) = \sigma_0$. Consider a minimal triangulation $\Delta^n \times I = \Delta_0 \cup \ldots \Delta_n$ of $\Delta^n \times I$ into n+1 n+1-simplices. Assume that $H(\partial\Delta^n \times I)$ consists of simplices in $\widehat{K}(X)$. Then H is homotopic (rel. $\Delta^n \times \{0\} \cup \partial\Delta^n \times I$) to a map $\overline{H} : \Delta^n \times I \to X$ such that $\overline{H}\mid_{\Delta_i} \in \widehat{K}(X)$, in

particular $\sigma_1 := \overline{H}(.,1) \in \widehat{K}(X)$.

Proof. a) From the assumptions it follows that there exists a homotopy of the 0-skeleton such that each vertex is moved into a distinct point of X, and such that vertices in X' remain in X' during the homotopy. By Observation 1, this homotopy can by induction be extended to a homotopy of the whole chain.

Now we prove the claim by induction on k $(0 \leq k < n)$. We assume that the k-skeleton of $\sum_{i\in I} a_i\tau_i$ consists of simplices in $\widehat{K}(X)$ and we want to homotope $\sum_{i\in I} a_i\tau_i$ such that the homotoped k+1-skeleton consists of simplices in $\widehat{K}(X)$.

By construction, each singular k+1-simplex σ in X with boundary a simplex in $\widehat{K}(X)$ is homotopic (rel. boundary) to a unique k+1-simplex in $\widehat{K}(X)$. Since the homotopy keeps the boundary fixed, the homotopies of different k+1-simplices are compatible. By Observation 1, the homotopy of the k+1-skeleton can by induction be extended to a homotopy of the whole chain.

If the image of the k+1-simplex σ is contained in X', then it is homotopic rel. boundary to a simplex in $\widehat{K}(X')$, for a homotopy with image in X'. Thus we can realise the homotopy such that all simplices with image in X' are homotoped inside X'.
b) follows by the same argument as a), succesively applied to $\Delta_0, \ldots, \Delta_n$.

□

We remark that there exists a canonical simplicial map

$$p : C_*^{simp,inf}\left(\widehat{K}(X), \widehat{K}(X')\right) \to C_*^{simp,inf}\left(K(X), K(X')\right).$$

p is defined by induction. It is defined to be the identity on the 1-skeleton. If it is defined on the n-1-skeleton, for $n \geq 2$, then, for an n-simplex τ, $p(\tau) \in K(X)$ is the unique simplex with $\partial_i p(\tau) = p(\partial_i \tau)$ for $i = 0, \ldots, n$.

3.3 Action of $G = \Pi(A)$

We repeat the definitions from [22], section 1.5. (originally due to [14]), as they will be frequently used in the remainder of the paper.

Let (P, A) be a pair of minimally complete multicomplexes.
We define its nontrivial-loops space $\Omega^* A$ as the set of homotopy classes (rel. $\{0,1\}$) of continuous maps $\gamma : [0,1] \to \mid A \mid$ with $\gamma(0) = \gamma(1)$ and not homotopic (rel. $\{0,1\}$) to a constant map.

We define

$$\Pi(A) := \left\{ \begin{array}{c} \{\gamma_1, \ldots, \gamma_n\} : n \in N, \gamma_1, \ldots, \gamma_n \in A_1 \cup \Omega^* A \\ \gamma_i(0) \neq \gamma_j(0), \gamma_i(1) \neq \gamma_j(1) \text{ for } i \neq j, \\ \{\gamma_1(0), \ldots, \gamma_n(0)\} = \{\gamma_1(1), \ldots, \gamma_n(1)\}. \end{array} \right\}.$$

If γ, γ' are elements of A_1 with $\gamma' \neq \overline{\gamma}$ and $\gamma(0) = \gamma'(1)$, we denote[1] $\gamma * \gamma' \in A_1$ to be the unique edge of A in the homotopy class of the concatenation. If $\gamma \in A_1$ and $\gamma' \in \Omega^* A$ (or vice versa), with $\gamma(1) \neq \gamma(0) = \gamma'(1) = \gamma'(0)$, we also denote $\gamma * \gamma' \in A_1$ the unique edge in the homotopy class of the concatenation. If $\gamma, \gamma' \in \Omega^* A$ with $\gamma(1) = \gamma(0) = \gamma'(1) = \gamma'(0)$, we denote $\gamma * \gamma' \in \Omega^* A$ the concatenation of homotopy classes of loops.

This can be used to define a multiplication on $\Pi(A)$ as follows:

given $\{\gamma_1, \ldots, \gamma_m\}$ and $\{\gamma_1', \ldots, \gamma_n'\}$, we chose a reindexing of the unordered sets $\{\gamma_1, \ldots, \gamma_m\}$ and $\{\gamma_1', \ldots, \gamma_n'\}$ such that we have: $\gamma_j(1) = \gamma_j'(0)$ for $1 \leq j \leq i$ and $\gamma_j(1) \neq \gamma_k'(0)$ for $j \geq i+1, k \geq i+1$. (Since we are assuming that all $\gamma_j(1)$ are pairwise distinct, and also all $\gamma_j'(0)$ are pairwise distinct, such a reindexing exists for some $i \geq 0$, and it is unique up to permuting the indices $\leq i$ and permuting separately the indices of γ_j's and γ_k''s with $j \geq i+1, k \geq i+1$.)
Moreover we permute the indices $\{1, \ldots, i\}$ such that there exists some h with $0 \leq h \leq i$ satisfying the following conditions:
- for $1 \leq j \leq h$ we have either $\gamma_j' \neq \overline{\gamma_j} \in A_1$ or $\gamma_j' \neq \gamma_j^{-1} \in \Omega^* A$,
- for $h < j \leq i$ we have either $\gamma_j' = \overline{\gamma_j} \in A_1$ or $\gamma_j' = \gamma_j^{-1} \in \Omega^* A$.

With this fixed reindexing we define

$$\{\gamma_1, \ldots, \gamma_m\}\{\gamma_1', \ldots, \gamma_n'\} := \{\gamma_1' * \gamma_1, \ldots, \gamma_h' * \gamma_h, \gamma_{i+1}, \ldots, \gamma_m, \gamma_{i+1}', \ldots, \gamma_n'\}.$$

(Note that we have omitted all $\gamma_j' * \gamma_j$ with $j > h$. The choice of $\gamma_j' * \gamma_j$ rather than $\gamma_j * \gamma_j'$ is just because we want to define a left action on (P, A).)
We have shown in [22] (footnote to Section 1.5.1) that the product belongs to $\Pi(A)$. Moreover, the so-defined multiplication is independent of the chosen reindexing. It is clearly associative. A neutral element is given by the empty set. The inverse to $\{\gamma_1, \ldots, \gamma_n\}$ is given by $\{\gamma_1', \ldots, \gamma_n'\}$ with $\gamma_i' = \overline{\gamma_i}$ if $\gamma_i \in A_1$ resp. $\gamma_i' = \gamma_i^{-1}$ if $\gamma_i \in \Omega^* A$. (Indeed, in this case $h = 0$, thus $\{\gamma_1, \ldots, \gamma_n\}\{\gamma_1', \ldots, \gamma_n'\}$ is the empty set.) Thus we have defined a group law on $\Pi(A)$.

We remark that there is an inclusion $\Pi(A) \subset map_0\left(A_0, [[0,1], \mid A \mid]_{\mid P \mid}\right)$, where $[[0,1], \mid A \mid]_{\mid P \mid}$ is the set of homotopy classes (in $\mid P \mid$) rel. $\{0,1\}$ of maps from $[0,1]$ to $\mid A \mid$, and $map_0\left(A_0, [[0,1], \mid A \mid]_{\mid P \mid}\right)$ is the set of maps $f: A_0 \to [[0,1], \mid A \mid]_{\mid P \mid}$ with

[1] We follow the usual convention to define the concatenation of paths by $\gamma * \gamma'(t) = \gamma(2t)$ if $t \leq \frac{1}{2}$ and $\gamma * \gamma'(t) = \gamma'(2t-1)$ if $t \geq \frac{1}{2}$. Unfortunately this implies that, in order to let $\Pi(A)$ act on P, we will have the multiplication in $\Pi(A)$ such that, for example, $\{\gamma\}\{\gamma'\} = \{\gamma' * \gamma\}$. We hope that this does not lead to confusion.

- $f(y)(0) = y$ for all $y \in A_0$ and
- $f(.)(1) : A_0 \to A_0$ is a bijection.

This inclusion is given by sending $\{\gamma_1, \ldots, \gamma_n\}$ to the map f defined by $f(\gamma_i(0)) = [\gamma_i]$ for $i = 1, \ldots, n$, and $f(y) = [c_y]$ (the constant path) for $y \notin \{\gamma_1(0), \ldots, \gamma_n(0)\}$. The inclusion is a homomorphism with respect to the group law defined on $map_0\left(A_0, [[0,1], \mid A \mid]_{\mid P \mid}\right)$ by $[gf(y)] := [f(y)] * [g(f(y)(1))]$.

Action of $\Pi(A)$ on P:

From now on we assume: *P is aspherical.* We define an action of $map_0\left(A_0, [[0,1], \mid A \mid]_{\mid P \mid}\right)$ on P. This gives, in particular, an action of $\Pi(A)$ on P.

Let $g \in map_0\left(A_0, [(0,1), \mid A \mid]_{\mid P \mid}\right)$. Define $gy = g(y)(1)$ for $y \in A_0$ and $gx = x$ for $x \in P_0 - A_0$. This defines the action on the 0-skeleton of P.

We extend this to an action on the 1-skeleton of P: Recall that, by minimal completeness of P, 1-simplices σ are in 1-1-correspondence with homotopy classes (rel. $\{0,1\}$) of (nonclosed) singular 1-simplices in $\mid P \mid$ with vertices in P_0. Using this correspondence, define $g\sigma := \left[\overline{g(\sigma(0))}\right] * [\sigma] * [g(\sigma(1))]$, where $*$ denotes concatenation of (homotopy classes of) paths.

In [22], Section 1.5.1, we proved that this defines an action on P_1 and that there is an extension of ths action to an action on P. (The extension is unique because P is aspherical.)

We remark, because this will be one of the assumptions to apply Lemma 7, that the action of any element $g \in \Pi(A)$ is homotopic to the identity. The homotopy between the action of the identity and the action of $\{\gamma_1, \ldots, \gamma_r\}$ given by the action of $\{\gamma_1^t, \ldots, \gamma_r^t\}, 0 \le t \le 1$, with $\gamma_i^t(s) = \gamma_i(st)$.

The next Lemma follows directly from the construction, but we will use it so often that we want to explicitly state it.

Lemma 3. *Let (P, A) be a pair of aspherical, minimally complete multicomplexes, with the action of $G = \Pi(A)$. If $\sigma \in P$ is a simplex, all of whose vertices are not in A, then $g\sigma = \sigma$ for all $g \in G$.*

For a topological space and a subset $P \subset S_*(X)$, closed under face maps, the (antisymmetric) bounded cohomology $H_b^*(P)$ and its pseudonorm are defined literally like for multicomplexes in [14], Section 3.2. The following well-known fact will be needed for applications of Lemma 7 (to the setting of Theorem 1) with $P = K^{str}(\partial Q), G = \Pi(K(\partial_0 Q))$.

Lemma 4. *a) Let (P, A) be a pair of minimally complete multicomplexes. If each connected component of $\mid A \mid$ has amenable fundamental group, then $\Pi(A)$ is amenable.*
b) Let X be a topological space, $P \subset S_*(X)$ a subset closed under face maps, and

G an amenable group acting on P. Then the canonical homomorphism

$$id\otimes 1: C_*^{simp}(P)\rightarrow C_*^{simp}(P)\otimes_{\mathbf{Z}G}\mathbf{Z}$$

induces an isometric monomorphism in bounded cohomology.

Proof. a) The proof is an obvious adaptation of the proof of [22], Lemma 4.
b) This is proved by averaging bounded cochains, see [14]. $\square$

3.4 Retraction to central simplices

Lemma 5. *Let $(N,\partial N)$ be a pair of topological spaces with $N=Q\cup R$ for two subspaces Q,R. Let*

$$\partial_0 Q=Q\cap R, \partial_1 Q=Q\cap\partial N, \partial_1 R=R\cap\partial N, \partial Q=\partial_0 Q\cup\partial_1 Q, \partial R=\partial_0 Q\cup\partial_1 R.$$

Assume that $\partial_1 Q\rightarrow Q\rightarrow N, \partial_1 R\rightarrow R\rightarrow N, \partial N\rightarrow N, \partial_0 Q\rightarrow Q, \partial_0 Q\rightarrow R$ are π_1-injective, and that $\partial N, \partial_1 Q, \partial_1 R, \partial_0 Q$ are aspherical (and thus the corresponding $K(.)$ can be considered as submulticomplexes of $K(N)$.)

Consider the simplicial action of $G=\Pi(K(\partial_0 Q))$ on $K(N)$.

Then there is a chain homomorphism

$$r: C_*^{simp,inf}(K(N))\otimes_{\mathbf{Z}G}\mathbf{Z}\rightarrow C_*^{simp,inf}(K(Q))\otimes_{\mathbf{Z}G}\mathbf{Z}$$

in degrees $*\geq 2$, mapping $C_*^{simp,inf}(GK(\partial N))\otimes_{\mathbf{Z}G}\mathbf{Z}$ to $C_*^{simp,inf}(GK(\partial_1 Q))\otimes_{\mathbf{Z}G}\mathbf{Z}$ such that
- if σ is a simplex in $K(N)$, then $r(\sigma\otimes 1)=\kappa\otimes 1$, where either κ is a simplex in $K(Q)$ or $\kappa=0$,
- if σ is a simplex in $K(Q)$, then $r(\sigma\otimes 1)=\sigma\otimes 1$,
- if σ is a simplex in $K(R)$, then $r(\sigma\otimes 1)=0$.

Proof. This is [22], Proposition 6. (We have replaced the assumption $ker(\pi_1\partial_0 Q\rightarrow\pi_1 Q)= ker(\pi_1\partial_0 Q\rightarrow\pi_1 R)$ from [22] by the stronger assumption of π_1-injectivity, since this will be true in all our applications and we have no need for the more general assumption.) The Conclusion is stated in [22] for locally finite chains, but of course r extends linearly to infinite chains. $\square$

Remark: *If some edge of σ is contained in $K(\partial_0 Q)=K(Q)\cap K(R)$, then*

$$\sigma\otimes 1=0\in C_*^{simp,inf}(K(N))\otimes_{\mathbf{Z}G}\mathbf{Z},$$

see [22], Section 1.5.2. (The proof is essentially the same as that of Lemma 15 below.) In particular, if σ is contained in both $K(Q)$ and $K(R)$, then $r(\sigma\otimes 1)=$

$r(0) = 0$.

Fundamental cycles in $K(N)$ and $K(Q)$. Let N be a (possibly noncompact) connected, orientable n-manifold with (possibly noncompact) boundary ∂N. Then $H_n^{lf}(N, \partial N) \simeq \mathbf{Z}$ by Whitehead's theorem and a generator is called $[N, \partial N]$. (It is only defined up to sign, but this will not concern our arguments.) Recall that an infinite chain is said to represent $[N, \partial N]$ if it is homologous to a locally finite chain representing $[N, \partial N]$.

If $\partial N \to N$ is π_1-injective and ∂N is aspherical, then $C_*^{simp,inf}\left(\widehat{K}(N), \widehat{K}(\partial N)\right) \subset C_*^{sing,inf}(N, \partial N)$, see Section 3.2. Thus it makes sense to say that some chain $z \in C_*^{simp,inf}\left(\widehat{K}(N), \widehat{K}(\partial N)\right)$ represents the fundamental class $[N, \partial N]$.

If $\partial_1 Q \to Q$ is π_1-injective and Q and $\partial_1 Q$ are aspherical, and if $G := \Pi(K(\partial_0 Q))$, then $C_*^{simp,inf}(GK(\partial_1 Q)) = C_*^{simp,inf}\left(G\widehat{K}(\partial_1 Q)\right) \subset C_*^{sing,inf}(\partial Q)$, because G maps simplices in $im(K(\partial Q) \to K(Q))$ to simplices in $im(K(\partial Q) \to K(Q))$. Thus it makes sense to say that some chain $z \in C_*^{simp,inf}(K(Q), GK(\partial_1 Q))$ represents the fundamental class $[Q, \partial Q]$.

The projection $p : \widehat{K}(N) \to K(N)$ is defined at the end of 3.2.

Lemma 6. *Let $N^{n \geq 2}$ be an orientable n-manifold with boundary, and $Q, R \subset N$ orientable n-manifolds with boundary, such that $N = Q \cup R$ satisfies the assumptions of Lemma 5 and $\partial_0 Q, \partial_1 Q, \partial_1 R$ are n-1-dimensional submanifolds (with boundary) of ∂Q resp. ∂R. Assume, in addition, that Q is aspherical.*

If $\sum_i a_i \sigma_i \in C_n^{simp,inf}\left(\widehat{K}(N), \widehat{K}(\partial N)\right)$ represents $[N, \partial N]$, then

$$\sum_i a_i r(p(\sigma_i)) \otimes 1 \in C_n^{simp,inf}(K(Q), GK(\partial_1 Q)) \otimes_{\mathbf{Z}G} \mathbf{Z}$$

represents [2] $[Q, \partial Q] \otimes 1$

and

$$\partial \sum_i a_i r(p(\sigma_i)) \otimes 1 \in C_n^{simp,inf}(GK(\partial Q)) \otimes_{\mathbf{Z}G} \mathbf{Z}$$

represents [3] $[\partial Q] \otimes 1$.

[2] This means that it represents the image of $h \otimes 1$ under the canonical homomorphism $H_n^{sing,inf}(Q, \partial Q) \otimes_{\mathbf{Z}G} \mathbf{Z} \to H_n\left(C_*^{sing,inf}(Q, \partial Q) \otimes_{\mathbf{Z}G} \mathbf{Z}\right)$, where $h \in H_n^{simp,inf}(K(Q), GK(\partial_1 Q))$ represents $[Q, \partial Q] \in H_n^{sing}(Q, \partial Q)$

[3] This means that it represents the image of $h \otimes 1$ under the canonical homomorphism $H_n^{simp,inf}(GK(\partial_1 Q)) \otimes_{\mathbf{Z}G} \mathbf{Z} \to H_n\left(C_*^{simp,inf}(GK(\partial_1 Q)) \otimes_{\mathbf{Z}G} \mathbf{Z}\right)$, where $h \in H_n^{simp,inf}(GK(\partial_1 Q))$ represents $[\partial Q] \in H_n^{sing}(\partial Q)$.

Proof. Since p and r are chain maps, it suffices to check the claim for some chosen representative of $[N, \partial N]$. So let $z \in C_*^{simp,inf}\left(\widehat{K}(N), \widehat{K}(\partial N)\right)$ be a representative of $[N, \partial N]$ chosen such that

$$p(z) = z_Q + z_R$$

where z_Q represents $[Q, \partial Q]$ and z_R represents $[R, \partial R]$ and such that

$$\partial z_Q = w_1 + w_2, \partial z_R = -w_2 + w_3$$

with $w_1 \in C_{n-1}^{simp,inf}(K(\partial_1 Q)), w_2 \in C_{n-1}^{simp,inf}(K(\partial_0 Q)), w_3 \in C_{n-1}^{simp,inf}(K(\partial_1 R))$ representing $[\partial_1 Q], [\partial_0 Q], [\partial_1 R]$, respectively.

From Lemma 5:

$$r(p(z) \otimes 1) = z_Q \otimes 1,$$

which implies the first claim, and

$$\partial r(p(z) \otimes 1) = \partial z_Q \otimes 1 = w_1 \otimes 1 + w_2 \otimes 1.$$

Since $w_1 + w_2$ represents $[\partial Q]$, this implies the second claim.

(Remark: From the Remark after Lemma 5 we have $w_2 \otimes 1 = 0$. This implies $\partial r(p(z) \otimes 1) = \partial z_Q \otimes 1 = w_1 \otimes 1$, that is, $\partial r(p(z) \otimes 1)$ represents at the same time $[\partial Q] \otimes 1$ and $[\partial_1 Q] \otimes 1$.)

□

3.5 Using amenability

Lemma 7 is well-known in slightly different formulations and we reprove it here only for completeness. We will apply[4] Lemma 7 in the proof of Theorem 1 with $X = \partial Q, G = q_*(\Pi(K(\partial_0 Q)))$ and $K = GK^{str}(\partial_1 Q)$. (The following lemma has of course also a relative version, but we will not need that for our argument.)

Lemma 7. *: Let X be a closed, orientable manifold and $K \subset S_*(X)$ closed under face maps. Assume that*
- there is an amenable group G acting on K, such that the action of each $g \in G$ on $\mid K \mid$ is homotopic to the identity

[4] If a group G acts simplicially on a multicomplex M, then $C_*(M) \otimes_{\mathbf{Z}G} \mathbf{Z}$ are abelian groups with well-defined boundary operator $\partial_* \otimes 1$, even though M/G may not be a multicomplex, like for the action of $G = \Pi_X(X)$ on $K(X)$, for a topological space X.

We remark that $C_*(M) \otimes_{\mathbf{Z}G} \mathbf{Z} \simeq C_*(M) \otimes_{\mathbf{R}G} \mathbf{R}$ is just the quotient chain complex for the G-action. In particular, even though $C_*(M)$ is an $\mathbf{R}G$-module, it does not make any difference whether we tensor over $\mathbf{Z}G$ or $\mathbf{R}G$.

- *there is a fundamental cycle* $z \in C_*^{simp}(K)$ *such that* $z \otimes 1$ *is homologous to a cycle* $h = \sum_{j=1}^{s} b_j \tau_j \otimes 1 \in C_*^{simp}(K) \otimes_{\mathbf{Z}G} \mathbf{Z}$.
Then

$$\parallel X \parallel \leq \sum_{j=1}^{s} \mid b_j \mid .$$

Proof. If $\parallel X \parallel = 0$, there is nothing to prove.
Thus we may assume $\parallel X \parallel \neq 0$, which implies ([14], p.17) that there is $\beta \in H_b^n(X)$, a bounded cohomology class dual to $[X] \in H_n(X)$, with $\parallel \beta \parallel = \frac{1}{\parallel X \parallel}$.

Let $p : C_*^{simp}(K) \rightarrow C_*^{simp}(K) \otimes_{\mathbf{Z}G} \mathbf{Z}$ be the homomorphism defined by $p(\sigma) = \sigma \otimes 1$. Since G is amenable we have, by the proof of Lemma 4b) in [14], an 'averaging homomorphism' $Av : H_b^*(K) \rightarrow H_b^*(C_*(K) \otimes_{\mathbf{Z}G} \mathbf{Z})$ such that Av is left-inverse to p^* and Av is an isometry. Hence we have

$$\parallel Av(\beta) \parallel = \parallel \beta \parallel = \frac{1}{\parallel X \parallel}.$$

Moreover, denoting by $\left[\sum_{j=1}^{s} b_j \tau_j \otimes 1\right]$ the homology class of $\sum_{j=1}^{s} b_j \tau_j \otimes 1$, we have obviously

$$\mid Av(\beta) \left[\sum_{j=1}^{s} b_j \tau_j \otimes 1\right] \mid \leq \parallel Av(\beta) \parallel \sum_{j=1}^{s} \mid b_j \mid$$

and therefore

$$\parallel X \parallel = \frac{1}{\parallel Av(\beta) \parallel} \leq \frac{\sum_{j=1}^{s} \mid b_j \mid}{\mid Av(\beta) \left[\sum_{j=1}^{s} b_j \tau_j \otimes 1\right] \mid}.$$

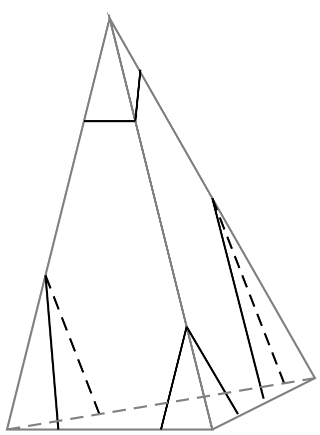
It remains to prove $Av(\beta) \left[\sum_{j=1}^{s} b_j \tau_j \otimes 1\right] = 1$.
For this we have to look at the definition of Av, which is as follows: Let $\gamma \in C_b^*(K)$ be a bounded cochain. By amenability there exists a bi-invariant mean $av : B(G) \rightarrow \mathbf{R}$ on the bounded functions on G with $inf_{g \in G} \delta(g) \leq av(\delta) \leq sup_{g \in G} \delta(g)$ for all $\delta \in B(G)$. Then, given any $p(\sigma) \in C_*(K) \otimes_{\mathbf{Z}G} \mathbf{Z}$ one can fix an identification between G and $G\sigma$, the set of all σ' with $p(\sigma') = p(\sigma)$, and thus consider the restriction of γ to $G\sigma$ as a bounded cochain on G. Define $Av(\gamma)(p(\sigma))$ to be the average av of this bounded cochain on $G \simeq G\sigma$. (This definition is independent of all choices, see [17].)
Now, if $z = \sum_{j=1}^{s} b_j \tau_j$ is a fundamental cycle, then we have $\beta(z) = 1$.
If $g \in G$ is arbitrary, then left multiplication with g is a chain map on $C_*^{simp}(K)$, as well as on $C_*^{sing}(X)$. Since the action of g on $\mid K \mid$ is homotopic to the identity, it induces the identity on the image of $C_*^{simp}(K) \rightarrow C_*^{sing}(X)$. Thus, for each

cycle $z \in C_*^{simp}(K)$ representing $[X] \in H_*^{sing}(X)$, the cycle $gz \in C_*^{simp}(K)$ must also represent $[X]$.
If gz represents $[X]$, then $\beta(gz) = \beta([X]) = 1$. In conclusion, we have $\beta(p(z')) = 1$ for each z' with $p(z') = p(z)$. By definition of Av, this implies $Av(\beta)(p(z)) = 1$ for each fundamental cycle z.
In particular, $Av(\beta)\left[\sum_{j=1}^{s} b_j \tau_j \otimes 1\right] = 1$, finishing the proof of the lemma. $\square$

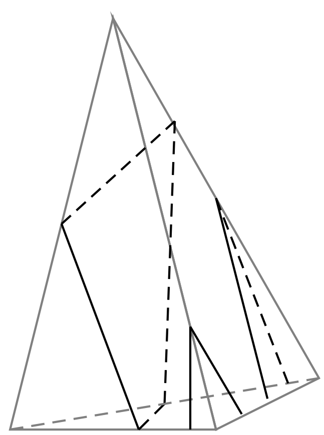
Remark: In the proof of Theorem 1, we will work with $C_*^{simp}(K) \otimes_{\mathbf{Z}G} \mathbf{Z}$ rather than $C_*^{simp}(K)$. This is analogous to Agol's construction of "crushing the cusps to points" in [1]. However $C_*^{simp}(K(Q)) \otimes_{\mathbf{Z}\Pi(\partial_0 Q)} \mathbf{Z} \neq C_*^{simp}(K(Q/\partial_0 Q))$, thus one can not simplify our arguments by working directly with $Q/\partial_0 Q$.

4 Disjoint planes in a simplex

In this section, we will discuss the possibilities how a simplex can be cut by planes without producing parallel arcs in the boundary. (More precisely, we pose the additional condition that the components of the complement can be coloured by black and white such that all vertices belong to black components, and we actually want to avoid only parallel arcs in the boundary of white components.) For example, for the 3-simplex, it will follow that there is essentially only the possibility in Case 1, pictured below, meanwhile in Case 2 each triangle has a parallel arc with another triangle, regardless how the quadrangle is triangulated.



Case 1



Case 2

Let $\Delta^n \subset \mathbf{R}^{n+1}$ be the standard simplex[5] with vertices $v_0, \ldots, v_n$. It is con-

[5]As usual, v_i is the vertex with all coordinates, except the i-th, equal to zero, and $\partial_i \Delta^n$

tained in the plane $E=\{(x_1,\ldots,x_{n+1})\in\mathbf{R}^{n+1}: x_1+\ldots+x_{n+1}=1\}$.

In this section we will be interested in n-1-dimensional affine planes $P\subset E$ whose intersection with Δ^n either contains no vertex, consists of exactly one vertex, or consists of a face of Δ^n. For such planes we define their type as follows.

Definition 2. *Let $P\subset E$ be an n-1-dimensional affine plane such that $P\cap\Delta^n$ either contains no vertex, consists of exactly one vertex, or consists of a face of Δ^n.*
If $P\cap\Delta^n=\partial_0\Delta^n$, then we say that P is of type $\{0\}$.
If $P\cap\Delta^n=\partial_j\Delta^n$ with $j\geq 1$, then we say that P is of type $\left\{01\ldots\hat{j}\ldots n\right\}$.
If $P\cap\{v_0,\ldots,v_n\}=\{v_0\}$, then we say that P is of type $\{0\}$.
If $P\cap\{v_0,\ldots,v_n\}=\emptyset$ or $P\cap\{v_0,\ldots,v_n\}=\{v_j\}$ with $j\geq 1$, then we say that P is of type $\{0a_1\ldots a_k\}$ with $a_1,\ldots,a_k\in\{1,\ldots,n\}$ if:
v_i belongs to the same connected component of $\Delta^n-(P\cap\Delta^n)$ as v_0 if and only if $i\in\{a_1,\ldots,a_k\}$.

Observation 2. *Let P_1,P_2 be two planes of type $\{0a_1\ldots a_k\}$ resp. $\{0b_1\ldots b_l\}$ and let $Q_1=P_1\cap\Delta^n\neq\emptyset, Q_2=P_2\cap\Delta^n\neq\emptyset$. Then $Q_1\cap Q_2=\emptyset$ implies that either $\{a_1,\ldots,a_k\}=\{b_1,\ldots,b_l\}$ or exactly one of the following conditions holds:*
- $\{a_1,\ldots,a_k\}\subset\{b_1,\ldots,b_l\}$,
- $\{b_1,\ldots,b_l\}\subset\{a_1,\ldots,a_k\}$,
- $\{a_1,\ldots,a_k\}\cup\{b_1,\ldots,b_l\}=\{1,\ldots,n\}$.

Proof. Δ^n-Q_1 consists of two connected components, C_1 and C_2. W.l.o.g. assume that $v_0\in C_1$. Δ^n-Q_2 consists of two connected components, D_1 and D_2. W.l.o.g. assume that $v_0\in D_1$. In particular, $C_1\cap D_1\neq\emptyset$.

Since $Q_1\cap Q_2=\emptyset$, it follows that Q_2 is contained in one of C_1 or C_2, and Q_1 is contained in one of D_1 or D_2.
Case 1: $Q_1\subset D_1$. Then either we have $C_1\subset D_1$, which implies $\{a_1,\ldots,a_k\}\subset\{b_1,\ldots,b_l\}$, or we have $C_2\subset D_1$, which implies $\{1,\ldots,n\}-\{a_1,\ldots,a_k\}\subset\{b_1,\ldots,b_l\}$, hence $\{a_1,\ldots,a_k\}\cup\{b_1,\ldots,b_l\}=\{1,\ldots,n\}$.
Case 2: $Q_1\subset D_2$. This implies $Q_2\subset C_1$ and after interchanging Q_1 and Q_2 we are in Case 1. $\square$

Notational remark: *'arc'* will mean the intersection of an n-1-dimensional affine plane $P\subset E$ (such that $P\cap\Delta^n\neq\emptyset$ either contains no vertex, consists of exactly one vertex or consists of a face) with a 2-dimensional subsimplex $\tau^2\subset\Delta^n$. If an arc consists of only one vertex, we call it a *degenerate arc*.

---

denotes the subsimplex spanned by all vertices except v_i. We will occasionally identify singular 1-simplices $\sigma:\Delta^1\to M$ with paths $e:[0,1]\to M$ by the rule $e(t)=\sigma(t,1-t)$. In particular, $e(0)=\sigma(v_0)=\partial_1\sigma$ and $e(1)=\sigma(v_1)=\partial_0\sigma$.

Definition 3. (Parallel arcs) *Let $P_1, P_2 \subset E$ be n-1-dimensional affine planes. Let τ^2 be a 2-dimensional subsimplex of Δ^n with vertices v_r, v_s, v_t. We say that disjoint arcs e_1, e_2 obtained as intersections of P_1 resp. P_2 with (the same) τ^2 are parallel arcs if one of the following holds:*
- both are nondegenerate and any two of $\{v_r, v_s, v_t\}$ belong to the same connected component of $\tau^2 - e_1$ if and only if they belong to the same connected component of $\tau^2 - e_2$,
- one, say e_1 is nondegenerate, the other, say with vertices v_s, v_t is contained in a face, and v_r belongs to another connected component of $\tau^2 - e_1$ as both v_s and v_t,
- one, say e_1, is nondegenerate, the other is degenerate, say equal to v_r, and both v_s, v_t belong to another connected component of $\tau^2 - e_1$ as v_r,
- both are degenerate and equal,
- both are contained in a face and equal,
- one is degenerate, the other is contained in a face.

Lemma 8. *Let $\Delta^n \subset \mathbf{R}^{n+1}$ be the standard simplex. Let $P_1, P_2 \subset E$ be n-1-dimensional affine planes with $Q_i = P_i \cap \Delta^n \neq \emptyset$ for $i = 1, 2$.*
Let P_1 be of type $\{0a_1 \dots a_k\}$ with $1 \leq k \leq n-2$ and P_2 of type $\{0b_1 \dots b_l\}$ with l arbitrary.
Then either $Q_1 \cap Q_2 \neq \emptyset$, or Q_1 and Q_2 have a parallel arc.

Proof. Assume that $Q_1 \cap Q_2 = \emptyset$.
By Observation 2, there are 4 possible cases if $Q_1 \cap Q_2 = \emptyset$.
Case 1: $\{0a_1 \dots a_k\} = \{0b_1 \dots b_l\}$. Then we clearly have parallel arcs.
Case 2: $\{0a_1 \dots a_k\}$ is a proper subset of $\{0b_1 \dots b_l\}$, i.e. $1 \leq k < l \leq n-1$ and $a_1 = b_1, \dots, a_k = b_k$. There is at least one index, say i, not contained in $\{0b_1 \dots b_l\}$. Consider the 2-dimensional subsimplex $\tau^2 \subset \Delta^n$ with vertices v_0, v_{a_1}, v_i. It intersects P_1 and P_2 in parallel arcs, because P_1 and P_2 both separate v_0 and v_{a_k} from v_i.
Case 3: $\{0b_1 \dots b_l\}$ is a proper subset of $\{0a_1 \dots a_k\}$, i.e. $0 \leq l < k \leq n-2$ and $a_1 = b_1, \dots, a_l = b_l$. There are two indices i, j not contained in $\{0a_1 \dots a_k\}$. Consider the 2-dimensional subsimplex $\tau^2 \subset \Delta^n$ with vertices v_0, v_i, v_j. It intersects P_1 and P_2 in parallel arcs, because P_1 and P_2 both separate v_0 from v_i and v_j.
Case 4: $\{a_1, \dots, a_k\} \cup \{b_1, \dots, b_l\} = \{1, \dots, n\}$. By $k \leq n-2$, there are two indices i, j with $i, j \notin \{0a_1 \dots a_k\}$. Hence $i, j \in \{b_1, \dots, b_l\}$. Moreover, there exists an index h such that $h \in \{a_1, \dots, a_k\}$ but $h \notin \{b_1, \dots, b_l\}$. (If not, we would have $\{a_1, \dots, a_k\} \subset \{b_1, \dots, b_l\}$, hence $\{1, \dots, n\} = \{a_1, \dots, a_k\} \cup \{b_1, \dots, b_l\} \subset \{b_1, \dots, b_l\}$, contradicting $Q_2 \neq \emptyset$.) Consider the 2-dimensional subsimplex $\tau^2 \subset \Delta^n$ with vertices v_i, v_j, v_h. It intersects P_1 and P_2 in parallel arcs, because both P_1 and P_2 separate v_i and v_j from v_h..

□

Definition 4. (Canonical colouring of complementary regions) *Let $P_1, P_2, \ldots \subset E$ be a (possibly infinite) set of n-1-dimensional affine planes with $Q_i := P_i \cap \Delta^n \neq \emptyset$ and $Q_i \cap Q_j = \emptyset$ for all $i \neq j$. Assume that each Q_i either contains no vertices or consists of exactly one vertex.*

A colouring of
- the connected components of $\Delta^n - \cup_i Q_i$ by colours black and white, and
- of all Q_i by black,
is called a canonical colouring (associated to $P_1, P_2, \ldots$) if:
- all vertices of Δ^n are coloured black,
- each Q_i is incident to at least one white component.

Definition 5. (White-parallel arcs) *Let $\{P_i : i \in I\}$ be a set of of $n-1$-dimensional affine planes $P_i \subset E$, with $Q_i := P_i \cap \Delta^n \neq \emptyset$ for $i \in I$. Assume that $Q_i \cap Q_j = \emptyset$ for all $i \neq j \in I$, and that we have a canonical colouring associated to $\{P_i : i \in I\}$. We say that arcs e_i, e_j obtained as intersections of P_i, P_j $(i, j \in I)$ with some 2-dimensional subsimplex τ^2 of Δ^n are white-parallel arcs if they are parallel arcs and, moreover, belong to the boundary of the closure of the same white component.*

We mention two consequences of Lemma 8. These will not be needed for the proof of Lemma 10, but they will be necessary for the proof of Theorem 1.

Corollary 1. *Let $\Delta^n \subset \mathbf{R}^{n+1}$ be the standard simplex. Let $P_1, \ldots, P_m \subset E$ be a finite set of n-1-dimensional affine planes and let $Q_i = P_i \cap \Delta^n$ for $i = 1, \ldots, m$.*

Assume that $Q_i \cap Q_j = \emptyset$ for all $i \neq j$, and that we have an associated canonical colouring, such that Q_i and Q_j do not have a white-parallel arc for $i \neq j$.

Then either $m = 0$, or
$m = n+1$ and P_1 is of type $\{0\}$, P_{n+1} is of type $\{0\ 1 \ldots n-1\}$, and P_i is of type $\left\{01 \ldots \widehat{i-1} \ldots n\right\}$ for $i = 2, \ldots, n$.

Proof. If the conclusion were not true, there would exist a plane P_1 of type $\{0a_1 \ldots a_k\}$ with $1 \leq k \leq n-2$. Let W be the white component of the canonical colouring, which is incident to P_1. Because, for a canonical colouring, no vertex belongs to a white component, there must be at least one more plane P_2 incident to W. Since $Q_1 \cap Q_2 = \emptyset$, from Lemma 8 we get that Q_1 and Q_2 have a parallel arc. Because Q_1 and Q_2 are incident to W, the arc is white-parallel. $\square$

Corollary 2. *Let $\Delta^n \subset \mathbf{R}^{n+1}$ be the standard simplex. Let $P_1, P_2, \ldots \subset E$ be a (possibly infinite) set of n-1-dimensional affine planes and let $Q_i = P_i \cap \Delta^n$ for $i = 1, 2, \ldots$. Assume that we have an associated canonical colouring.*
Let P_i be of type $\left\{0a_1^i \ldots a_{c(i)}^i\right\}$, for $i = 1, 2, \ldots$. Then
- either $c(1) \in \{0, n-1\}$,

- or whenever, for some $i \in \{2, 3, \ldots\}$, P_1 and P_i bound a white component of $\Delta^n - \cup_j Q_j$, then they must have a white-parallel arc.

Proof. Assume that $c(1) \notin \{0, n-1\}$. Let W be the white component bounded by P_1. W is bounded by a finite number of planes, thus we can apply Corollary 1, and conclude that P_1 has a white-parallel arc with each other plane adjacent to W. $\square$

Definition 6. *Let $P \subset E$ be an n-1-dimensional affine plane, and T a triangulation of the polytope $Q := P \cap \Delta^n$. We say that T is minimal, if all vertices of T are vertices of Q. We say that an edge of some simplex in T is an exterior edge if it is an edge of Q.*

Observation 3. *Let $P \subset E$ be an n-1-dimensional affine plane, and T a triangulation of the polytope $Q := P \cap \Delta^n$. If T is minimal, then each edge of Q is an (exterior) edge of (exactly one) simplex in T.*

Proof. By minimality, the triangulation does not introduce new vertices. Thus every edge of Q is an edge of some simplex. $\square$

Observation 4. *Let $P \subset E$ be an n-1-dimensional affine plane with $Q := P \cap \Delta^n \neq \emptyset$. Assume that P is of type $\{0a_1 \ldots a_k\}$.*
a) Each vertex of Q arises as the intersection of P with an edge e of Δ^n. The vertices of e are v_i and v_j with $i \in \{0, a_1, \ldots, a_k\}$ and $j \notin \{0, a_1, \ldots, a_k\}$. (We will denote such a vertex by $(v_i v_j)$.)
b) Two vertices $(v_{i_1} v_{j_1})$ and $(v_{i_2} v_{j_2})$ of Q are connected by an edge of Q (i.e. an exterior edge of any triangulation) if either $i_1 = i_2$ or $j_1 = j_2$.

Proof. a) holds because e has to connect vertices in distinct components of $\Delta^n - Q$. b) holds because the edge of Q has to belong to some 2-dimensional subsimplex of Δ^n, with vertices either $v_{i_1}, v_{j_1}, v_{j_2}$ or $v_{i_1}, v_{i_2}, v_{j_1}$. $\square$

Remark: if, for an affine hyperplane $P \subset E$, $Q = P \cap \Delta^n$ consists of exactly one vertex, then we will consider the minimal triangulation of Q to consist of one (degenerate) n-1-simplex. This convention helps to avoid needless case distinctions.

Lemma 9. *Let $\{P_i \subset E : i \in I\}$ be a set of n-1-dimensional affine planes and let $Q_i := P_i \cap \Delta^n$ for $i \in I$. Assume that $Q_i \cap Q_j = \emptyset$ for all $i \neq j$ and that we have an associated canonical colouring. Assume that we have fixed, for each $i \in I$, a minimal triangulation $Q_i = \cup_a \tau_{ia}$ of Q_i.*

If P_1 is of type $\left\{0a_1^1 \ldots a_{c(1)}^1\right\}$ with $1 \leq c(1) \leq n-2$, then for each simplex $\tau_{1a} \subset Q_1$ there exists some $j \in I$ and some simplex $\tau_{jb} \subset Q_j$ (of the fixed triangulation of Q_j) such that τ_{ia} and τ_{jb} have a white-parallel arc.

Proof. Let $w_1, \ldots, w_n$ be the n vertices of the n-1-simplex τ_{1k}. By Observation 4a), each w_l arises as intersection of Q_1 with some edge $(v_{r_l} v_{s_l})$ of Δ^n, and the vertices v_{r_l}, v_{s_l} satisfy $r_l \in \left\{0, a_1^1, \ldots, a_{c(1)}^1\right\}$ and $s_l \notin \left\{0, a_1^1, \ldots, a_{c(1)}^1\right\}$.

For the canonical colouring, there must be a white component W bounded by P_1. We distinguish the cases whether W and v_0 belong to the same connected component of $\Delta^n - Q_1$ or not.

Case 1: W and v_0 belong to the same connected component of $\Delta^n - Q_1$.

Since $c(1) \leq n-2$, there exist at most $n-1$ possible values for r_l. Hence there exists $l \neq m \in \{1, \ldots, n\}$ such that $v_{r_l} = v_{r_m}$.

Let e be the edge of $\tau_{1k} \subset Q_1$ connecting w_l and w_m. By Observation 4b), e is an exterior edge. Consider the 2-dimensional subsimplex $\tau^2 \subset \Delta^n$ with vertices $v_{r_l}, v_{s_l}, v_{s_m}$. We have that P_1 intersects τ^2 in e, i.e. in an arc separating v_{r_l} from the other two vertices of τ^2.

Note that $r_l \in \left\{0, a_1^1, \ldots, a_{c(1)}^1\right\}$, hence v_{r_l} belongs to the same component of $\Delta^n - Q_1$ as v_0. In particular, v_{r_l} belongs to the same component of $\Delta^n - Q_1$ as W. On the other hand, since the colouring is canonical, all vertices are coloured black and v_{r_l} can not belong to the white component W. Thus there must be some plane P_j such that Q_j bounds W and separates v_{r_l} from Q_1. (The possiblity $P_j \cap \Delta^n = \{v_{r_l}\}$ is allowed.) In particular, some (possibly degenerate) exterior edge f of Q_j separates v_{r_l} from v_{s_l}, v_{s_m}. Thus e and f are white-parallel arcs. By Observation 3, f is an edge of some τ_{jl}.

Case 2: W and v_0 don't belong to the same connected component of $\Delta^n - Q_1$.

Since $n - c(1) \leq n-1$, there exist some $l \neq m \in \{1, \ldots, n\}$ such that $v_{s_l} = v_{s_m}$.

Let e be the edge of $\tau_{1k} \subset Q_1$ connecting w_l and w_m. e is an exterior edge by Observation 4b). Consider the 2-dimensional subsimplex $\tau^2 \subset \Delta^n$ with vertices $v_{r_l}, v_{r_m}, v_{s_l}$. P_1 intersects τ^2 in e, i.e. in an arc separating v_{s_l} from the other two vertices of τ^2.

We have that $s_l \notin \left\{0, a_1^1, \ldots, a_{c(1)}^1\right\}$, hence v_{s_l} does not belong to the same component of $\Delta^n - Q_1$ as v_0. This implies that v_{s_l} belongs to the same component of $\Delta^n - Q_1$ as W. On the other hand, since the colouring is canonical, v_{s_l} can not belong to the white component W and there must be some plane P_j such that Q_j bounds W and separates v_{s_l} from Q_1. In particular, some exterior edge f of Q_j separates v_{s_l} from v_{r_l}, v_{r_m}. Thus e and f are white-parallel arcs. By Observation 3, f is an edge of some τ_{jl}. $\square$

Lemma 10. *Let $\{P_i : i \in I\}$ be a set of n-1-dimensional affine planes with $Q_i := P_i \cap \Delta^n \neq \emptyset$ for $i \in I$. Let P_i be of type $\left\{0 a_1^{(i)} \ldots a_{k_i}^{(i)}\right\}$ for $i \in I$. Assume that $Q_i \cap Q_j = \emptyset$ for $i \neq j \in I$, and that we have an associated canonical colouring. Assume that for each Q_i one has fixed a minimal triangulation $Q_i = \cup_{k=1}^{t(i)} \tau_{ik}$.*

For each $i \in I$, let

$$D_i = \sharp \left\{ \tau_{ik} \subset Q_i : \text{ there is no } \tau_{jl} \subset Q_j \text{ such that } \tau_{ik}, \tau_{jl} \text{ have a white-parallel arc} \right\}.$$

Then

$$\sum_{i \in I} D_i = 0 \quad or \quad \sum_{i \in I} D_i = n + 1.$$

Proof. First we remark that the number of planes may be infinite, but we may of course remove pairs of planes P_i, P_j whenever they are of the same type and bound the same white component. This removal (of P_i, P_j and the common white component) does not affect $\sum_{i \in I} D_i$. Since there are only finitely many different types of planes, we may w.l.o.g. assume that we start with a finite number $P_1, \ldots, P_m$ of planes. (It may happen that after this removal no planes and no white components remain. In this case $\sum_{i \in I} D_{i \in I} = 0$.) So we assume now that we have a finite number of planes $P_1, \ldots, P_m$, and no two planes of the same type bound a white region.

The first case to consider is that all planes are of type $\{0 a_1 \ldots a_k\}$ with $k = 0$ or $k = n - 1$. Since all vertices are coloured black, this means that $m = n + 1$ and (upon renumbering) P_1 is of type $\{0\}$, P_{n+1} is of type $\{0\ 1 \ldots n-1\}$, and P_i is of type $\left\{ 0 1 \ldots \widehat{i-1} \ldots n \right\}$ for $i = 2, \ldots, n$. Hence $D_1 = \ldots = D_{n+1} = 1$ and $\sum_{i=1}^{n+1} D_i = n + 1$.

Now we assume that there exists P_i, w.l.o.g. P_1, of type $\left\{ 0 a_1^{(1)} \ldots a_{k_1}^{(1)} \right\}$ with $1 \leq c(1) \leq n - 2$. Let W be the white component bounded by P_1 and let w.l.o.g. $P_2, \ldots, P_l$ be the other planes bounding W. Then Lemma 9 says that each simplex in the chosen triangulation of Q_1 has a parallel arc with some simplex in the chosen triangulation of each of $Q_2, \ldots, Q_l$. In particular, $D_1 = 0$. For $j \in \{2, \ldots, l\}$, if $1 \leq c(j) \leq n - 2$, the same argument shows that $D_j = 0$. If $j \in \{2, \ldots, l\}$ and $c(j) = 0$ or $c(j) = n - 1$, then Q_j consists of only one simplex. By Corollary 2, this simplex has a parallel arc with (some exterior edge of) Q_1 and thus (by Observation 3) with (some) simplex of the chosen triangulation of Q_1. This shows $D_j = 0$ also in this case. Altogether we conclude $\sum_{j=1}^{l} D_j = 0$ and thus $\sum_{i=1}^{m} D_i = \sum_{i=l+1}^{m} D_i$. Hence we can remove[6] the white component W and its bounding planes $P_1, \ldots, P_l$ to obtain a smaller number of planes and a new canonical colouring without changing $\sum_{i=1}^{m} D_i$. Since we start with finitely many planes, we can repeat this reduction finitely many times and will end up either with an empty set of planes or with a set of planes of type $\{0 a_1 \ldots a_k\}$ with $k = 0$ or $k = n - 1$. Thus either $\sum_{i=1}^{m} D_i = 0$ or $\sum_{i=1}^{m} D_i = n + 1$. $\square$

[6] To remove a white component means that this component together with the neighbouring black components will form **one** new black component.

We have thus proved that, in presence of a canonical colouring, the number of n-1-simplices without white-parallel arcs in a minimal triangulation of the Q_i's is 0 or $n+1$. We remark that in the proof of Theorem 1 we will actually count only those triangles which neither have a white-parallel arc nor a degenerate arc. Thus, in general, we may remain with even less than $n+1$ n-1-simplices.

5 A straightening procedure

In this section we will always work with the following set of assumptions.

Assumption I: *Q is an aspherical n-dimensional manifold with aspherical boundary ∂Q. We have n-1-dimensional submanifolds $\partial_0 Q, \partial_1 Q \subset \partial Q$ such that $\partial Q = \partial_0 Q \cup \partial_1 Q, \partial\partial_0 Q = \partial\partial_1 Q$ and $\partial_1 Q \neq \emptyset$ is aspherical.*

The example that one should have in mind is a nonpositively curved manifold Q with totally geodesic boundary $\partial_1 Q$ and cusps corresponding to $\partial_0 Q$.

In the case of nonpositively curved manifolds with totally geodesic boundary, there is a well-known straightening procedure (explained for closed hyperbolic manifolds in [3], Lemma C.4.3.), which homotopes each relative cycle into a straight relative cycle.

However, we will need a more subtle straightening procedure, which considers relative cycles with a certain 0-1-labeling of their edges and straightens the 1-labeled edges into certain distinguished 1-simplices. This straightening procedure will be explained in Section 5.3. Before, we explain a construction which will morally (although not literally) "reduce" the proof of Theorem 1 to the case that $\partial_0 Q \cap C$ is path-connected, for each path-component C of ∂Q.

5.1 Making $\partial_0 Q \cap C$ connected

Construction 1. *Let Assumption I be satisfied. Then there exists a continuous map of triples $q : (Q, \partial Q, \partial_1 Q) \to (Q, \partial Q, \partial_1 Q)$ which is (as a map of triples) homotopic to the identity and such that, for each path-component C of ∂Q, the image $A := q(\partial_0 Q \cap C)$ is path-connected.*

Moreover, for each path-component F of $\partial_1 Q$, the path-components of $\partial F \subset \partial_0 Q \cap \partial_1 Q$ can be numbered by $E_0^F, \ldots, E_s^F$ and one can choose points $x_{E_i^F} \in E_i^F$ such that $q\left(x_{E_i^F}\right) \equiv x_{E_0^F}$ for $i = 0, \ldots, s$.

Proof. For each path-component F of $\partial_1 Q$, number the path-components of $\partial F \subset \partial_0 Q \cap \partial_1 Q$ by $E_0^F, \ldots, E_s^F$, where s depends on F. Choose one point $x_E^F \in E$ for each path-component $E \subset F$ of $\partial_0 Q \cap \partial_1 Q$. Whenever E_0, E_i is a pair of path-components of $\partial_0 Q \cap \partial_1 Q$ adjacent to the same path-component F of $\partial_1 Q$, choose a 1-dimensional submanifold $l_{E_0^F E_i^F} \subset \partial_1 Q$ with $\partial l_{E_0^F E_i^F} = \left\{x_{E_0^F}\right\} \cup \left\{x_{E_i^F}\right\}$. The $l_{E_0^F E_i^F}$ may be chosen succesively such that they are disjoint from each other (apart from the common vertex $x_{E_0^F}$) and disjoint from $\partial_0 Q$ (apart from the vertices $x_{E_0^F}$ and $x_{E_i^F}$).

For each pair $\left\{E_0^F, E_i^F\right\}$ let $h : l_{E_0^F E_i^F} \to \left\{x_{E_0^F}\right\}$ be the constant map from $l_{E_0^F E_i^F}$ to $x_{E_0^F}$. For each path-component F of $\partial_1 Q$, the union

$$\bigcup_{i=1}^{s} l_{E_0^F E_i^F}$$

is an embedded wedge of arcs in $\partial_1 Q$, hence it is contractible. In particular, h is homotopic to the identity. By the homotopy extension property exists $g : F \to F$ with $g \mid_{l_{E_0^F E_i^F}} = h \equiv x_{E_0}$ for all $l_{E_0^F E_i^F}$, and $g \sim id$ by a homotopy extending the homotopies between h and id.

Thus we defined g on each path-component F of $\partial_1 Q$ with $F \cap \partial_0 Q \neq \emptyset$. On path-components F of $\partial_1 Q$ with $F \cap \partial_0 Q = \emptyset$ we define $g = id$. Hence we have defined g on all of $\partial_1 Q$.

On path-components C of $\partial_0 Q$ with $C \cap \partial_1 Q = \emptyset$, we define $f = id$. Again by the homotopy extension property exists $f : \partial Q \to \partial Q$ with $f \mid_{\partial_1 Q} = g$, $f \mid_C = id$ for path-components C of $\partial_0 Q$ with $C \cap \partial_1 Q = \emptyset$, and $f \sim id$ by a homotopy extending the homotopy of g. (Of course, f does not preserve those path-components of $\partial_0 Q$ which intersect $\partial_1 Q$.)

Once again by the homotopy extension property exists $q : Q \to Q$ with $q \sim id$ such that q extends f and the homotopy between q and id extends the homotopy between f and id.

Due to the stepwise construction, q is a map of triples, homotopic to the identity by a homotopy of triples. Moreover, $A := q\left(\partial_0 Q \cap C\right)$ is path-connected for each component C of ∂Q. Indeed, any two points in $\partial_0 Q \cap C$ can be connected by a sequence of paths which either have image in $\partial_0 Q$ or belong to $\cup_{i=1}^{s} l_{E_0^F E_i^F}$ for some path-component F of $\partial_1 Q \cap C$. The image of these paths under q, in both cases, is in A. $\square$

Remark: *q induces a simplicial map $q : K(Q) \to K(Q)$ and a homomorphism $q_* : \Pi\left(K\left(\partial_0 Q\right)\right) \to \Pi\left(K(A)\right)$ defined by $q_*\left(\left\{\gamma_1, \ldots, \gamma_n\right\}\right) := \left\{q\left(\gamma_1\right), \ldots, q\left(\gamma_n\right)\right\}$ such that*

$$q_*(g)\, q(\sigma) = q(g\sigma)$$

holds for each $\sigma \in K(Q), g \in \Pi\left(K\left(\partial_0 Q\right)\right)$.

Proof. Continuous maps $q : Q \to Q$ induce simplicial maps $q : K(Q) \to K(Q)$. (The simplicial map agrees with q on the 0-skeleton, and it maps each 1-simplex

$e \in K_1(Q)$ to the unique 1-simplex of $K_1(Q)$ that is in the homotopy class rel. $\{0,1\}$ of $q(e)$.)

Let $e \in K_1(Q)$. By construction $\{\gamma_1, \ldots, \gamma_n\} e = \left[\alpha * e * \overline{\beta}\right]$ for some $\alpha, \beta \in \{\gamma_1, \ldots, \gamma_n\} \cup \left\{c_{e(0)}, c_{e(1)}\right\}$. Thus

$$\{q(\gamma_1), \ldots, q(\gamma_n)\} q(e) = \left[q(\alpha) * q(e) * q\left(\overline{\beta}\right)\right\} = q(\{\gamma_1, \ldots, \gamma_n\} e).$$

This implies the claim for the 1-skeleton and thus, by asphericity of $K(Q)$, for all $\sigma \in K(Q)$. □

5.2 Definition of $K^{str}(Q)$

Let $Q, \partial Q, \partial_1 Q, \partial_0 Q$ satisfy Assumption I.

Recall that we have defined in Section 3.2 an aspherical multicomplex $K(Q) \subset S_*(Q)$ with the property that (for aspherical Q) each singular simplex in Q, with boundary in $K(Q)$ and pairwise distinct vertices, is homotopic rel. boundary to a unique simplex in $K(Q)$.

The aim of this subsection is to describe a selection procedure yielding a subset $K_*^{str}(Q) \subset S_*(Q)$. The final purpose of the straightening procedure will be to produce a large number of (weakly) degenerate simplices, in the sense of the following definition.

Definition 7. *Let Q be an compact manifold with boundary ∂Q. We say that a simplex in $S_*(Q)$ is degenerate if one of its edges is a constant loop. We say that it is weakly degenerate if it is degenerate or its image is contained in ∂Q.*

Notational remark: for subsets $K_*^{str}(Q) \subset S_*(Q)$ we will denote $K_*^{str}(\partial_0 Q) := K_*^{str}(Q) \cap S_*(\partial_0 Q)$, $K_*^{str}(\partial_1 Q) := K_*^{str}(Q) \cap S_*(\partial_1 Q)$, $K_*^{str}(\partial_0 Q Q) := K_*^{str}(Q) \cap S_*(\partial_0 Q)$.

Lemma 11. *Let $Q, \partial Q, \partial_1 Q, \partial_0 Q$ satisfy Assumption I. Let $K(Q) \subset S_*(Q)$ be as defined in Section 3.2. Let $q: Q \to Q$ and $\left\{x_{E_i^F} \in \partial_0 Q \cap \partial_1 Q : 0 \leq i \leq s\right\}$ be given by Construction 1.*

Then there exists a subset $K_*^{str}(Q) \subset S_*(Q)$, closed under face maps, such that:
i) If C is a path-component of $\partial_0 Q$ with $C \cap \partial_1 Q = \emptyset$, then $K_0^{str}(Q)$ contains each point in C,
ii) for a path-component F of $\partial_1 Q$ with $F \cap \partial_0 Q = \emptyset$, there is exactly one point $x_F \in K_0^{str}(Q) \cap F$,
for a path-component F of $\partial_1 Q$ with $F \cap \partial_0 Q \neq \emptyset$, we have $K_0^{str}(Q) \cap F = \left\{x_{E_0^F}, \ldots, x_{E_s^F}\right\}$,

iii) $K_0^{str}(Q) = K_0^{str}(\partial Q)$,
iv) $K_1^{str}(Q)$ *consists of*
- all 1-simplices $e \in K(Q)$ *with* $\partial e \in K_0^{str}(Q)$, *and*
- exactly one 1-simplex for each nontrivial homotopy class (rel. boundary) of loops e *with* $\partial_0 e = \partial_1 e \in K_0^{str}(Q)$,
- the constant loop for the homotopy class of the constant loop at x, *if* $x \in K_0^{str}(Q)$,
v) for $n \geq 2$, *if* $\sigma \in S_n(Q)$ *is an n-simplex with* $\partial\sigma \in K_{n-1}^{str}(Q)$, *then* σ *is homotopic rel. boundary to a unique* $\tau \in K_n^{str}(Q)$,
vi) if $\sigma \in K_n^{str}(Q)$ *is homotopic rel. boundary to some* $\tau \in K_n(Q)$, *then* $\sigma = \tau$,
vii) if $\sigma \in K_n^{str}(Q)$ *is homotopic rel. boundary to a simplex* $\tau \in S_n(\partial_1 Q)$, *then* $\sigma \in K_n^{str}(\partial_1 Q)$; *if* $\sigma \in K_1^{str}(Q)$ *is homotopic rel. boundary to a simplex* $\tau \in S_1(\partial_0 Q)$, *then* $\sigma \in K_1^{str}(\partial_0 Q)$,
viii) $K_*^{str}(Q)$ *is aspherical, i.e. if* $\sigma, \tau \in K_*^{str}(Q)$ *have the same 1-skeleton, then* $\sigma = \tau$.

Proof. $K_*^{str}(Q)$ is defined by induction on the dimension of simplices as follows.

Definition of $K_0^{str}(Q)$:
Choose $K_0^{str}(Q)$ such that conditions i),ii),iii) are satisfied. Note that we have chosen a nonempty set of 0-simplices since we are assuming $\partial_1 Q \neq \emptyset$.

Definition of $K_1^{str}(Q)$:
For an ordered pair

$$(x, y) \in K_0^{str}(Q) \times K_0^{str}(Q)$$

with $x \neq y$, there exists in each homotopy class (rel. boundary) of arcs e with

$$e(0) = x, e(1) = y$$

a unique simplex in $K_1(Q)$. Choose these 1-simplices to belong to $K_1^{str}(Q)$. (Uniqueness implies that vi)) is true for $n = 1$.) Moreover, for pairs

$$(x, x) \in K_0^{str}(Q) \times K_0^{str}(Q)$$

choose one simplex in each homotopy class (rel. boundary) of loops e with

$$e(0) = e(1) = x.$$

For the homotopy class of the constant loop choose the constant loop.
Choose the 1-simplices in $\partial_0 Q$ and/or $\partial_1 Q$ whenever this is possible. (If a 1-simplex is homotopic into both $\partial_0 Q$ and $\partial_1 Q$, then it is necessarily homotopic into $\partial_0 Q \cap \partial_1 Q$. Indeed, a disk realizing a homotopy between 1-simplices in $\partial_0 Q$ and $\partial_1 Q$ can be made transversal to $\partial_0 Q \cap \partial_1 Q$ and then intersects $\partial_0 Q \cap \partial_1 Q$ in

an arc resp. loop.) Hence vii) is satisfied for $n=1$.

Definition of $K_n^{str}(Q)$ for $n\geq 2$, assuming that $K_{n-1}^{str}(Q)$ is defined:
For an $n+1$-tuple $\kappa_0,\ldots,\kappa_n$ of $n-1$-simplices in $K_{n-1}^{str}(Q)$, satisfying

$$\partial_i\kappa_j=\partial_{j-1}\kappa_i$$

for all i,j, there are two possibilities:
- if no edge of any κ_i is a loop, then, by asphericity of Q, there is a unique n-simplex

$$\sigma\in K_n(Q)$$

with

$$\partial_i\sigma=\kappa_i$$

for $i=0,\ldots,n$. In this case set $\kappa:=\sigma$. Uniqueness implies that vi) is satisfied for n. (By the construction in Section 3.2 $\kappa\in K_n(\partial_1 Q)$ if κ is homotopic rel. boundary into $\partial_1 Q$.)

- otherwise, choose an n-simplex

$$\kappa\in S_n(Q)$$

with

$$\partial_i\kappa=\kappa_i$$

for $i=0,\ldots,n$. By asphericity of Q, κ exists and is unique up to homotopy rel. boundary. Choose the simplices in $\partial_1 Q$ whenever this is possible.

By construction, $K_*^{str}(Q)$ is closed under face maps and satisfies the conditions i)-vii). Condition viii) follows by induction on the dimension of subsimplices of σ and τ from condition v). $\square$

The simplices in $K_*^{str}(Q)$ will be called the **straight simplices**.
We remark that $K_*^{str}(Q)$ is not a multicomplex because simplices in $K_*^{str}(Q)$ need not have pairwise distinct vertices. (Note also that simplices in $K(Q)$ belong to $K^{str}(Q)$ if and only if all their vertices belong to $K_0^{str}(Q)$, by construction.)

Observation 5. *Let $Q,\partial Q,\partial_1 Q,\partial_0 Q$ satisfy Assumption I. Let $K_*^{str}(Q)\subset S_*(Q)$ satisfy the conditions i)-viii) from Lemma 11. Then*
$q:Q\rightarrow Q$ induces a simplicial map $q:K^{str}(Q)\rightarrow K^{str}(Q)$, compatible with the simplicial map $q:K(Q)\rightarrow K(Q)$ from Section 5.1.

Proof. By construction, q maps $K_0^{str}(Q)$ to itself. Indeed:
- if C is a path-component of $\partial_0 Q$ with $C \cap \partial_1 Q = \emptyset$, then $q(v) = v$ for each $v \in C$,
- if F is a path-component F of $\partial_1 Q$ with $F \cap \partial_0 Q = \emptyset$, then $q(v) = v$ for *each* $v \in F$ (in particular for the unique $v \in F \cap K_0^{str}(Q)$),
- if F is a path-component of $\partial_1 Q$ with $F \cap \partial_0 Q \neq \emptyset$, then we have $K_0^{str}(Q) \cap F = \left\{x_{E_0^F}, \ldots, x_{E_s^F}\right\}$, and $q\left(x_{E_i^F}\right) = x_{E_0^F}$ for $i = 0, \ldots, s$ by Construction 1.

Hence q induces a simplicial map on $K^{str}(Q)$. (The simplicial map agrees with q on the 0-skeleton, and it maps each 1-simplex $e \in K_1^{str}(Q)$ to the unique 1-simplex of $K_1^{str}(Q)$ that is in the homotopy class rel. $\{0,1\}$ of $q(e)$. Since $K^{str}(Q)$ is aspherical, this determines the simplicial map q uniquely.) $\square$

5.3 Definition of the straightening

Definition 8. *Let $(Q, \partial_1 Q)$ be a pair of topological spaces and let $z = \sum_{i \in I} a_i \tau_i \in C_n^{inf}(Q)$ a (possibly infinite) singular chain.*
a) A set of cancellations of z is a symmetric set $\mathcal{C} \subset S_{n-1}(Q) \times S_{n-1}(Q)$ with $(\eta_1, \eta_2) \in \mathcal{C} \Rightarrow \eta_1 = \eta_2$ and $\eta_1 = \partial_k \tau_{i_1}, \eta_2 = \partial_l \tau_{i_2}$ for some $i_1, i_2 \in I, k, l \in \{0, \ldots, n\}$.
b) If $z = \sum_{i \in I} a_i \tau_i \in C_n^{inf}(Q)$ and $\mathcal{C}$ is a set of cancellations for z, then the associated simplicial set $\Upsilon_{z,\mathcal{C}}$ is the simplicial set generated[7] by $\{\Delta_i : i \in I\}$, subject to the identifications $\partial_k \Delta_{i_1} = \partial_l \Delta_{i_2}$ if and only if $(\partial_k \tau_{i_1}, \partial_l \tau_{i_2}) \in \mathcal{C}$.
c) Let $z = \sum_{i \in I} a_i \tau_i \in C_n^{inf}(Q)$. Choose a minimal presentation for ∂z (i.e. no further cancellation is possible). Let

$$J = J_{\partial z} := \left\{ \begin{array}{c} (i,a) \in I \times \{0, \ldots, n\} : \\ \partial_a \tau_i \textit{ occurs with non-zero coefficient in the chosen presentation of } \partial z \end{array} \right\}.$$

Let $\mathcal{C}$ be a set of cancellations for z. Then the simplicial set $\partial \Upsilon_{z,\mathcal{C}}$ is defined as the set consisting of $\mid J \mid$ n-1-simplices $\Delta_{i,a}, (i,a) \in J$, together with all their iterated faces and degenerations, subject to the identifications $\partial_a \partial_{a_1} \tau_{i_1} = \partial_a \partial_{a_2} \tau_{i_2}$ for all $a = 0, \ldots, n-1$, whenever $(\partial_{a_1} \tau_{i_1}, \partial_{a_2} \tau_{i_2}) \in \mathcal{C}$ and $(i_1, a_1) \in J$.
d) If $z = \sum_{i \in I} a_i \tau_i \in C_n^{inf}(Q)$ is a relative cycle, then a set of cancellations $\mathcal{C}$ is called sufficient if the formal sum $\sum_{i \in I} \sum_{k=0}^{n} (-1)^k a_i \partial_k \tau_i$ can be reduced to a chain in $C_{n-1}^{inf}(\partial Q)$ by substracting (possibly infinitely many) multiples of $(\partial_{a_1} \tau_{i_1} - \partial_{a_2} \tau_{i_2})$ with $(\partial_{a_1} \tau_{i_1}, \partial_{a_2} \tau_{i_2}) \in \mathcal{C}$.

Observation 6. *Let $(Q, \partial_1 Q)$ be a pair of topological spaces.*
a) If $z = \sum_{i \in I} a_i \tau_i \in C_n^{inf}(Q)$ is a singular chain, $\mathcal{C}$ a set of cancellations, and

[7] That is, the subset of $S_*^{sing}(Q)$ which contains the $\mid I \mid$ n-simplices $\Delta_i, i \in I$, together with all simplices obtained by iterated applications of face and degeneracy operators, cf. [25], Example 1.5.

$\Upsilon := \Upsilon_{z,\mathcal{C}}$ the associated simplicial set, then the geometric realisation $\mid \Upsilon \mid$ is obtained from $\mid I \mid$ copies of the standard n-simplex $\Delta_i, i \in I$, with identifications $\partial_{a_1}\Delta_{i_1} = \partial_{a_2}\Delta_{i_2}$ if and only if $(\partial_{a_1}\tau_{i_1}, \partial_{a_2}\tau_{i_2}) \in \mathcal{C}$. Moreover, for a minimal presentation of ∂z and $\partial\Upsilon := \partial\Upsilon_{z,\mathcal{C}}$, $\mid \partial\Upsilon \mid$ is the subspace of $\mid \Upsilon \mid$ containing all simplices $\partial_{a_1}\Delta_{i_1}$ with $(i_1, a_1) \in J$.
b) There exists an associated continuous map $\tau :\mid \Upsilon \mid \to Q$ with $\tau \mid \Delta_i = \tau_i$ (upon the identification $\Delta_i = \Delta^n$). If z is a relative cycle, i.e. $\partial z \in C_{n-1}^{inf}(\partial_1 Q)$, then τ maps $\mid \partial\Upsilon \mid$ to $\partial_1 Q$.
c) Let $z_1 = \sum_{i \in I} a_i\tau_i, z_2 = \sum_{i\in I} a_i\sigma_i \in C_n^{inf}(Q, \partial_1 Q)$ be relative cycles and $\mathcal{C}_1, \mathcal{C}_2$ sufficient sets of cancellations of z_1 resp. z_2. Assume that $(\partial_{a_1}\tau_{i_1}, \partial_{a_2}\tau_{i_2}) \in \mathcal{C}_1$ if and only if $(\partial_{a_1}\sigma_{i_1}, \partial_{a_2}\sigma_{i_2}) \in \mathcal{C}_2$, and that there exist minimal presentations of $\partial z_1, \partial z_2$ such that $J_{z_1} = J_{z_2}$.
If the associated continuous maps
$\tau, \sigma :\mid \Upsilon \mid \to Q$ are homotopic,
for a homotopy mapping $\mid \partial\Upsilon \mid$ to ∂Q, then $\sum_{i\in I} a_i\tau_i$ and $\sum_{i\in I} a_i\sigma_i \in C_*^{inf}(Q, \partial Q)$ are relatively homologous.

We emphasize that we do not assume that $\mathcal{C}$ is a complete list of cancellations, the simplicial map $\tau_* : C_*^{simp}(\Upsilon) \to C_*^{sing}(Q)$ need not be injective.

After having set up the necessary notations, we now start with the actual definition of the straightening. We first mention that there is of course an analogue of the classical straightening ([3], Lemma C.4.3.) in our setting.

Observation 7. *Let $Q, \partial_1 Q, \partial_1 Q, \partial_0 Q$ satisfy Assumption I. Let $K_*^{str}(Q) \subset S_*(Q)$ satisfy the conditions i)-viii) from Lemma 11.*

Then there exists a 'canonical straightening' map

$$str_{can} : C_*^{simp,inf}(K(Q)) \to C_*^{simp,inf}\left(K^{str}(Q)\right),$$

mapping $C_*^{simp,inf}(K(\partial_1 Q))$ to $C_*^{simp,inf}(K^{str}(\partial_1 Q))$, with the following properties:
i) str_{can} is a chain map,
ii) if $z = \sum_{i\in I} a_i\tau_i \in C_*^{simp,inf}(K(Q))$ and $\sum_{i\in I} a_i\sigma_i := \sum_{i\in I} a_i str_{can}(\tau_i)$, then the maps

$$\tau, \sigma :\mid \Upsilon \mid \to Q$$

(defined by Observation 6b) after fixing a set of cancellations $\mathcal{C}$ and a minimal presentation of ∂z) are homotopic.

Moreover, if $z = \sum_{i\in I} a_i\tau_i$ is a relative cycle with $\partial z \in C_*^{simp,inf}(K(\partial_1 Q))$, then the same is true for $\sum_{i\in I} a_i\sigma_i$ and

$$\tau, \sigma : (\mid \Upsilon \mid, \mid \partial\Upsilon \mid) \to (Q, \partial_1 Q)$$

are homotopic as maps of pairs.

In particular, $\sum_{i\in I} a_i str_{can}(\tau_i)$ *is relatively homologous to* $\sum_{i\in I} a_i\tau_i$*,*

Proof. We define str_{can}, and the homotopy to the identity, by induction on the dimension of simplices. (During the construction we take care that str_{can} and the homotopy preserve $K(\partial_1 Q)$.)

0-simplices.
If C is a path-component of $\partial_0 Q$ with $C\cap\partial_1 Q=\emptyset$, then we define $str_{can}(v)=v$ for each 0-simplex v in C. The homotopy $H(v)$ is for each v given by the constant map.

If C is a path-component of $\partial_0 Q$ with $C\cap\partial_1 Q\neq\emptyset$, then there is at least one path-component F of $\partial_1 Q$ with $C\cap F\neq\emptyset$. By Construction 1 and condition ii) from Lemma 11, for each such F, there is a straight 0-simplex $x_{E_i^F}\in C\cap F$. Choose one such straight 0-simplex (among the $x_{E_i^F}$'s) for each path-component C of $\partial_0 Q$, denote it x_C, and for each $v\in C$ we define $str_{can}(v):=x_C\in K_0^{str}(Q)\cap C$ and we choose the homotopy $H(v)$ to belong to C.

If $v\in\partial_1 Q$, then there is (at least) one straight 0-simplex in the same path-component F of $\partial_1 Q$, we choose $str_{can}(v)\in F\cap K_0^{str}(Q)$ and there exists $H(v)\in K_1(\partial_1 Q)$ with $\partial H(v)=v-str_{can}(v)$.

If $v\notin\partial Q$, then we define $str_{can}(v)$ to be some straight 0-simplex in ∂Q and we fix arbitrarily some $H(v)\in K_1(Q)$ with $\partial H(v)=v-str_{can}(v)$.

1-simplices.
For $e\in K_1(Q)$ we define

$$str_{can}(e):=\left[\overline{H(\partial_1 e)}*e*H(\partial_0 e)\right],$$

where, as always, $[.]$ denotes the unique 1-simplex in $K_1^{str}(Q)$, which is homotopic rel. boundary to the path in the brackets.

e is homotopic to $str_{can}(e)$ by the canonical homotopy which is inverse to the homotopy moving $\overline{H(\partial_1 e)}$ resp. $H(\partial_0 e)$ into constant maps. In particular, the restriction of this homotopy to $\partial_1 e,\partial_0 e$ gives $H(\partial_1 e),\overline{H(\partial_0 e)}$. Thus, for different edges with common vertices, the homotopies are compatible. We thus have constructed a homotopy for the 1-skeleton Υ_1.

We note that, for $v\in\partial_1 Q$, the homotopy $H(v)$ is either constant or $H(v)\in K_1(\partial_1 Q)$, Thus if $\tau\in K_1(\partial_1 Q)$ then $str_{can}(\tau)\in K_1^{str}(\partial_1 Q)$ and the homotopy between τ and $str_{can}(\tau)$ takes place in $\partial_1 Q$.

n-simplices.

We assume inductively, that for some $n \geq 1$, we have defined str_{can} on $K_{*\leq n}(Q)$, mapping $K_{*\leq n}(\partial_1 Q)$ to $K^{str}_{*\leq n}(\partial_1 Q)$, and satisfying i),ii),iii).

Let $\tau \in K(Q)$ be an n+1-simplex. Then we have by ii) a homotopy between $\partial\tau$ and $str_{can}(\partial\tau)$. By Observation 1 this homotopy extends to τ. The resulting simplex τ' satisfies $\partial\tau' \in K^{str}_n(Q)$. Condition v) from Lemma 11 means that τ' is homotopic rel. boundary to a unique simplex $str_{can}(\tau) \in K^{str}_{n+1}(Q)$. This proves the inductive step.

If $\tau \in K(\partial_1 Q)$, then we can inductively assume that the homotopy of $\partial\tau$ has image in $\partial_1 Q$. Then condition vii) from Lemma 11 implies $str_{can}(\tau) \in K^{str}_{n+1}(\partial_1 Q)$. Moreover, since $\partial_1 Q$ is aspherical, the homotopy of τ can be chosen to have image in $\partial_1 Q$.

By construction, for any set of cancellations $\mathcal{C}$, the induced maps τ and σ are homotopic. In particular, if we chose a sufficient set of cancellations in the sense of Definition 8d), then Observation 6c) implies that $\sum_{i=1}^r a_i str_{can}(\tau_i)$ is (relatively) homologous to $\sum_{i=1}^r a_i\tau_i$.

□

However, we want to define a more refined straightening, which will be defined only on relative cycles with some kind of additional information.

Before stating the definition of "distinguished 1-simplices" we remark that there is a left and right action of the pseudogroup $\Gamma := \Omega(\partial Q)$ (as defined in Section 3.3) on $K^{str}_1(Q)$: if $e \in K^{str}_1(Q), \gamma_1 \in \pi_1(\partial Q, \partial_1 e), \gamma_2 \in \pi_1(\partial Q, \partial_0 e)$, then let $\gamma_1 e \gamma_2$ be the unique straight 1-simplex homotopic rel. $\{0,1\}$ to $\gamma_1 * e * \gamma_2$. (The left action agrees with the action defined in Section 3.3.) The cosets $\Gamma K^{str}_1(Q)\Gamma$ in Definition 9 are with respect to this action.

For $x, y \in K^{str}_0(Q)$ we will denote $K^{str}_{1,xy} := \{e \in K^{str}_1(Q) : \partial_1 e = x, \partial_0 e = y\}$.

Definition 9. *Let $Q, \partial Q, \partial_1 Q, \partial_0 Q$ satisfy Assumption I.*

Let $q : Q \to Q$ and $\left\{x_{E_i^F} \in \partial_0 Q \cap \partial_1 Q\right\}$ be given by Construction 1.

Let $K^{str}_*(Q) \subset S_*(Q)$ satisfy conditions i)-viii) from Lemma 11.

A set $D \subset K^{str}_1(Q)$ is called a set of distinguished 1-simplices if

ix) $\partial_0 e, \partial_1 e \in K^{str}_0(Q)$ for each $e \in D$,

x) for each

$$(x, y) \in K^{str}_0(Q) \times K^{str}_0(Q)$$

we have that

$$D_{xy} := \{e \in D : \partial_1 e = x, \partial_0 = y\}$$

contains exactly one element in each double coset (w.r.t. $\Gamma = \Omega(\partial Q)$)

$$\Gamma f \Gamma \in \Gamma K_{1,xy}^{str}(Q)\,\Gamma,$$

xi) for all $x \in K_0^{str}(Q)$, the constant loop c_x belongs to D,
xii) if $e \in D$, then $\overline{e} \in D$, where $\overline{e}$ denotes the 1-simplex with the opposite orientation,
xiii) if F, F' are path-components of $\partial_1 Q$ and $\left\{x_{E_i^F} \in \partial_0 Q \cap F\right\}, \left\{x_{E_j^{F'}} \in \partial_0 Q \cap F'\right\}$ are given[8] by Construction 1, then $q\left(D_{x_{E_i^F} x_{E_j^{F'}}}\right) = D_{x_{E_0^F} x_{E_0^{F'}}}$ for all $x_{E_i^F}, x_{E_j^{F'}}$,
xiv) if $x_1, x_2 \in C_1, y_1, y_2 \in C_2$ for some path-components C_1, C_2 of ∂Q, then for each $e_1 \in D_{x_1 y_1}$ exists some $e_2 \in D_{x_2 y_2}$ with $q(e_2) = g q(e_1)$ for some $g \in H := q_*(\Pi(K(\partial_0 Q)))$.

Observation 8. *Let the assumptions of Definition 9 be satisfied. Then a set D of distinguished 1-simplices exists.*

Proof. For each path-component C of ∂Q we fix some $x_C \in K_0^{str}(C)$.

For each pair $\{C_1, C_2\}$ of path-components we fix one simplex e with

$$\partial_1 e = x_{C_1}, \partial_0 e = x_{C_2}$$

in each coset of $\Gamma K_{1,x_{C_1} x_{C_2}}^{str}(Q)\,\Gamma$ to belong to $D_{x_{C_1} x_{c_2}}$.
(For all chosen 1-simplices $e \in D_{x_{C_1} x_{C_2}}$, we choose $\overline{e}$ to belong to $D_{x_{C_2} x_{C_1}}$. If $C_1 = C_2$, then in particular for the coset of the constant loop we choose the constant loop to belong to $D_{x_{C_1} x_{C_2}}$.)

For each path-component C of ∂Q and each path-component F of $C \cap \partial_1 Q$, we have that $q(x_C)$ and $q\left(x_{E_0^F}\right)$ belong to the path-connected set $q(\partial_0 Q \cap C)$. Therefore we have a sequence of 1-simplices $\alpha_1, \ldots, \alpha_m \in K_1(\partial_0 Q)$ with images in distinct path-components of $\partial_0 Q \cap C$, such that

$$\partial_1 q(\alpha_1) = q(x_C), \partial_0 q(\alpha_1) = \partial_1 q(\alpha_2), \ldots, \partial_0 q(\alpha_{m-1}) = \partial_1 q(\alpha_m), \partial_0 q(\alpha_m) = q\left(x_{E_0^F}\right).$$

In order to prepare the definition of the $D_{x,y}$'s, we first describe, for each $x \in C \cap K_0^{str}(Q)$ a sequence $\{\alpha_1, \ldots, \alpha_k\}$ of 1-simplices:
- if $C \cap \partial_1 Q = \emptyset$, then $k = 1$ and for each $x \in C$ we choose arbitrarily a 1-simplex α_1 in C with $\partial_1 \alpha_1 = x_C, \partial_0 \alpha_1 = x$,

[8] If $F \cap \partial_0 Q = \emptyset$ and/or $F' \cap \partial_0 Q = \emptyset$, then there is only one straight 0-simplex $x_{E_0^F}$ resp. $x_{E_0^{F'}}$ in F resp. F'. In particular, if $F \cap \partial_0 Q = \emptyset$ and $F' \cap \partial_0 Q = \emptyset$, then condition xiii) is empty.

- if $C \cap \partial_0 Q = \emptyset$, then $C \cap K_0^{str}(Q) = \{x_C\}$ by Lemma 11, condition ii), and we let $k = 0$,
- if $C \cap \partial_0 Q \cap \partial_1 Q \neq \emptyset$, then by condition ii) from Lemma 11 we have $x = x_{E_i^F}$ for some path-component F of $\partial_1 Q$ and some i, thus we have the above-constructed sequence $\alpha_1, \ldots, \alpha_m$ with $\partial_1 q(\alpha_1) = q(x_C), \partial_0 q(\alpha_1) = \partial_1 q(\alpha_2), \ldots, \partial_0 q(\alpha_{m-1}) = \partial_1 q(\alpha_m), \partial_0 q(\alpha_m) = q\left(x_{E_i^F}\right)$, where the last equality holds true because $q\left(x_{E_i^F}\right) = x_{E_0^F} = q\left(x_{E_0^F}\right)$.

Let $x, y \in K_0^{str}(Q)$. Let C_1, C_2 be the path-components of ∂Q with $x \in C_1, y \in C_2$.

We have constructed sequences of 1-simplices $\alpha_1, \ldots, \alpha_k \in K_1(\partial Q)$ resp. $\beta_1, \ldots, \beta_l \in K_1(\partial Q)$, such that $\partial_1 q(\alpha_1) = q(x_{C_1}), \partial_0 q(\alpha_1) = \partial_1 q(\alpha_2), \ldots, \partial_0 q(\alpha_{k-1}) = \partial_1 q(\alpha_k), \partial_0 q(\alpha_k) = q(x)$ resp. $\partial_1 q(\beta_1) = q(x_{C_2}), \partial_0 q(\beta_1) = \partial_1 q(\beta_2), \ldots, \partial_0 q(\beta_{k-1}) = \partial_1 q(\beta_k), \partial_0 q(\beta_k) = q(y)$. Note that all $q(\alpha_i)$ and $q(\beta_i)$ are either constant or contained in $q(K_1(\partial_0 Q))$.

Let $H := q_*(\Pi(K(\partial_0 Q)))$. Define

$$g := \{q(\alpha_1), q(\overline{\alpha}_1)\} \ldots \{q(\alpha_k), q(\overline{\alpha}_k)\} \left\{q(\beta_l), q\left(\overline{\beta}_l\right)\right\} \ldots \left\{q(\beta_1), q\left(\overline{\beta}_1\right)\right\} \in H.$$

(If $k = l = 0$, this means just $g = 1$.)

We have that $g = g^{-1}$ and that

$$ge \in K_{1,q(x)q(y)}^{str}(Q) \Longleftrightarrow e \in K_{1,q\left(x_{C_1}\right)q\left(x_{C_2}\right)}^{str}(Q).$$

By construction, the g **associated to $x_{E_i^F}, x_{E_j^{F'}}$ agrees with the g associated to $x_{E_0^F}, x_{E_0^{F'}}$.**

We are given $D_{x_{C_1} x_{C_2}}$ and we want to define D_{xy} such that condition xiii) is satisfied.

First, if $C_1 \cap \partial_1 Q = \emptyset$ or $C_2 \cap \partial_1 Q = \emptyset$, then we can fix an arbitrary choice of $D_{x,y}$ satisfying conditions x),xi,xii). (Condition xiii) is empty in this case.)

So let us assume $C_1 \cap \partial_1 Q \neq \emptyset$ and $C_2 \cap \partial_1 Q \neq \emptyset$. We note that

$$q : (Q, \partial Q, \partial_1 Q) \to (Q, \partial Q, \partial_1 Q)$$

is homotopic to the identity as a map of triples, by the construction in Section 5.1. This implies that cosets of $\Gamma K_{1,xy}^{str}(Q) \Gamma$ are in 1-1-correspondence (by applying q) to cosets of $\Gamma K_{1,q(x)q(y)}^{str} \Gamma$. It is thus sufficient to describe $q(D_{xy}) \subset K_{1,q(x)q(y)}^{str}$.

Let

$$\Gamma f \Gamma \in \Gamma K_{1,q(x)q(y)}^{str}(Q) \Gamma$$

be a double coset. Then the double coset

$$\Gamma(gf)\Gamma \in \Gamma K_{1,q\left(x_{C_1}\right)q\left(x_{C_2}\right)}^{str}(Q) \Gamma$$

is the image under q of some double coset

$$\Gamma e'\Gamma \in \Gamma K^{str}_{1,x_{C_1}x_{C_2}}(Q)\,\Gamma$$

Let e be the unique distinguished simplex in $\Gamma e'\Gamma$. Then we choose $gq(e)$ to be the distinguished simplex in $\Gamma f\Gamma$. This is possible because $gq(e)$ belongs to the double coset $\Gamma f\Gamma$. Indeed

$$q(e) \in \Gamma(gf)\,\Gamma$$

means that $q(e) = q_*(\gamma_1)\,gfq_*(\gamma_2)$ for some loops γ_1 and γ_2 based at x_{C_1} resp. x_{C_2}, and this implies $gq(e') = q_*(\gamma_1')\,fq_*(\gamma_2')$ with

$$\gamma_1' := \left[\overline{\alpha}_m * \ldots * \overline{\alpha}_1 * \gamma_1 * \alpha_1 * \ldots * \alpha_m\right], \gamma_2' := \left[\overline{\beta}_n * \ldots * \overline{\beta}_1 * \gamma_2 * \beta_1 * \ldots * \beta_n\right].$$

This defines D_{xy}. By construction, condition xiv) is satisfied if $e_1 \in D_{x_{C_1}x_{C_2}}$. In general, if $e_1 \in D_{x_1y_1}$, then we get $e \in D_{x_{C_1}x_{C_2}}$ and $g_1 \in H$ with $q(e_1) = g_1q(e)$ and $e_2 \in D_{x_2y_2}, g_2 \in H$ with $q(e_2) = g_2q(e)$, thus $q(e_2) = g_2g_1^{-1}q(e_1)$.

Condition xiii) is implied because $q\left(x_{E_i^F}\right) = x_{E_0^F}, q\left(x_{E_j^{F'}}\right) = x_{E_0^{F'}}$ and the g associated to $x_{E_i^F}, x_{E_j^{F'}}$ agrees with the g associated to $x_{E_0^F}, x_{E_0^{F'}}$.
One checks easily that xi) and xii) are true for D_{xy} since they are true for $D_{x_{C_1}x_{C_2}}$. □

Definition 10. *Let $Q, \partial Q, \partial_0 Q, \partial_1 Q$ satisfy Assumption I. Let $z = \sum_{i\in I} a_i\tau_i \in C_n^{inf}(Q)$ be a singular chain and Υ the associated simplicial set (for some set of cancellations $\mathcal{C}$).*

We say that a labeling of the elements of the 1-skeleton Υ_1 by 0's and 1's is admissible, if $\partial e_1 \cap \partial e_2 = \emptyset$ for all 1-labeled vertices e_1, e_2.

Lemma 12. *Let $Q, \partial Q, \partial_1 Q, \partial_0 Q$ satisfy Assumption I. Let $q : Q \to Q$ be given by Construction 1.*

Let $K_*^{str}(Q) \subset S_*(Q)$ satisfy conditions i)-viii) from Lemma 11, and let $D \subset K_1^{str}(Q)$ be a set of distinguished 1-simplices.

Let $z = \sum_{i\in I} a_i\tau_i \in C_*^{simp,inf}(K(Q))$ be a relative cycle with $\partial z \in C_*^{simp,inf}(K(\partial_1 Q))$.

Let a set of cancellations $\mathcal{C}$ for z and a minimal presentation of ∂z be given. Let $\Upsilon, \partial\Upsilon$ be the associated simplicial sets, $\tau : (\mid \Upsilon \mid, \mid \partial\Upsilon \mid) \to (Q, \partial_1 Q)$ the associated continuous mapping.

Assume that we have an admissible 0-1-labeling of Υ_1.
Then there exists a relative cycle

$$z' = \sum_{i\in I} a_i\tau_i' \in C_*^{simp,inf}\left(K^{str}(Q), K^{str}(\partial_1 Q)\right)$$

such that:
i) the associated continuous mappings

$$\tau, \tau' : (\mid \Upsilon \mid, \mid \partial\Upsilon \mid) \rightarrow (Q, \partial_1 Q)$$

are homotopic by a homotopy mapping $\mid \partial\Upsilon \mid$ *to* ∂Q,
ii) if an edge of some τ_i *is labeled by 1, then the corresponding edge of* τ_i' *belongs to* D,

(Remark: The homotopy in i) does not necessarily map $\mid \partial\Upsilon \mid$ to $\partial_1 Q$, but to ∂Q.)

Proof. First we apply the 'canonical straightening' str_{can} from Observation 7. The resulting chain $\sum_{i\in I} a_i str_{can}(\tau_i)$ satisfies i), but not necessarily ii).

$\sum_{i\in I} a_i str_{can}(\tau_i)$ inherits the admissible labeling from $\sum_{i\in I} a_i \tau_i$. Thus we can w.l.o.g. restrict to the case that all τ_i belong to $K^{str}(Q)$.

Let
$$e \in K_1^{str}(Q)$$
be a 1-labeled edge, let $x = \partial_1 e \in K_0^{str}(Q), y = \partial_0 e \in K_0^{str}(Q)$. By Definition 9, the coset $\Gamma e \Gamma$ contains a unique distinguished 1-simplex $str(e) \in D_{xy}$. (We use the notation from Definition 9, in particular $\Gamma := \Omega(\partial Q)$.)

$str(e) \in \Gamma e \Gamma$ means[9] that there are loops $\gamma_1, \gamma_2 \subset \partial Q$ based at x resp. y such that $str(e) \sim \gamma_1 * e * \gamma_2$ rel. $\{0,1\}$. There is an obvious homotopy between e and $\gamma_1 * e * \gamma_2$, which moves $\partial_1 e$ along $\overline{\gamma}_1$ and $\partial_0 e$ along γ_2. (Of course, we change the homotopy class relative boundary, so we can not keep the endpoints fixed during the homotopy.) If e and/or $\partial_0 e$ and/or $\partial_1 e$ have image in $\partial_1 Q$, then their images remain in ∂Q (and end up in $\partial_1 Q$) during the homotopy.

Using Observation 1, the so-constructed homotopy between e and $str(e)$ can be extended to a homotopy from

$$\tau : (\mid \Upsilon \mid, \mid \partial\Upsilon \mid) \rightarrow (Q, \partial_1 Q)$$

to some
$$\hat{\tau} : (\mid \Upsilon \mid, \mid \partial\Upsilon \mid) \rightarrow (Q, \partial_1 Q),$$

such that $\hat{\tau}$ is a simplicial map from Υ to $S_*(Q)$. (If a 0-labeled edge has one or both vertices adjacent to 1-labeled edges, then the 0-labeled edge just follows the homotopy of the vertices. 0-labeled edges that are not adjacent to 1-labeled edges can remain fixed during the homotopy.) The homotopy maps $\mid \partial\Upsilon \mid$ to ∂Q.

[9] If $\partial_0 e, \partial_1 e \notin \partial_1 Q$, then $str(e) \in \Gamma e \Gamma$ means, of course, $str(e) = e$. Similarly, if only one vertex of e belongs to $\partial_1 Q$, then only that vertex is moved during the homotopy.

Next we apply homotopies rel. boundary to the (already homotoped images of) all 0-labeled edges $f \in K_1^{str}(Q)$, to homotope them to edges in $K_1^{str}(Q)$. If f and/or $\partial_0 f$ and/or $\partial_1 f$ have image in $\partial_1 Q$, then their images remain in ∂Q (and end up in $\partial_1 Q$) during the homotopy.

Now we have a simplicial map $\hat{\tau} : \Upsilon \to S_*(Q)$, such that all 1-simplices are mapped to $K_1^{str}(Q)$, and such that

$$\hat{\tau}(e) \in D \subset K_1^{str}(Q)$$

holds for all 1-labeled edges e. Then we can, as in the proof of Observation 7, by induction on n, apply homotopies rel. boundary to all n-simplices to homotope them into $K_n^{str}(Q)$. Simplices in $\partial_1 Q$ remain in ∂Q (and end up in $\partial_1 Q$) during the homotopy.

We obtain a homotopy (of pairs), which keeps the 1-skeleton fixed, to a simplicial map

$$\tau' : \Upsilon \to K^{str}(Q),$$

mapping $\partial\Upsilon$ to $K^{str}(\partial_1 Q)$ and satisfying i),ii). $\square$

A somewhat artificial formulation of the conclusion of Lemma 12 is that we have constructed a chain map

$$str : C_*^{simp,inf}(\Upsilon, \partial\Upsilon) \to C_*^{simp,inf}\left(K^{str}(Q), K^{str}(\partial_1 Q)\right).$$

Unfortunately, this somewhat artificial formulation can not be simplified because str depends on the chain $\sum_{i\in I} a_i \tau_i$. That is, we do not get a chain map $str : C_*^{simp,inf}(K(Q), K(\partial_1 Q)) \to C_*^{simp,inf}(K^{str}(Q), K^{str}(\partial_1 Q))$.

5.4 Straightening of crushed cycles

Recall from Section 3.5 that $. \otimes_{\mathbf{Z}G} \mathbf{Z}$ means the tensor product with the trivial $\mathbf{Z}G$-module $\mathbf{Z}$, that is, the quotient under the G-action. We first state obvious generalizations of Observation 6 to the case of tensor products with a factor with trivial G-action.

Observation 9. *Let $(Q, \partial_1 Q)$ be a pair of topological spaces. Let G be a group acting on a pair $(K, \partial K)$ with $K \subset S_*(Q)$ and $\partial K \subset S_*(\partial_1 Q)$ both closed under face maps.*
i) If

$$z = \sum_{i\in I} a_i \tau_i \otimes 1 \in C_*^{simp,inf}(K, \partial K) \otimes_{\mathbf{Z}G} \mathbf{Z}$$

is a relative cycle, then

$$\hat{z} = \sum_{i\in I} \sum_{g\in G} a_i (g\tau_i) \in C_*^{simp,inf}(K, \partial K)$$

is a relative cycle.
If $\mathcal{C}$ is a sufficient set of cancellations for z, then there exists a set of cancellations $\widehat{\mathcal{C}}$ for $\hat{z}$ such that $(\eta_1, \eta_2) \in \widehat{\mathcal{C}}$ implies $(\eta_1 \otimes 1, \eta_2 \otimes 1) \in \mathcal{C}$.
If $\partial z = \sum_{a,i} c_{ai} \partial_a \tau_i \otimes 1$ is a minimal presentation for ∂z, then $\partial \hat{z} = \sum_{g \in G} \sum_{a,i} c_{ai} \partial_a (g \tau_i)$ is a minimal presentation for $\hat{z}$.
ii) Let $\widehat{\Upsilon}, \partial \widehat{\Upsilon}$ be the simplicial sets associated to $\hat{z}$, the sufficient set of cancellations $\widehat{\mathcal{C}}$ and the minimal presentation of $\partial \hat{z}$. They come with an obvious G-action. Then we have an associated continuous mapping $\hat{\tau} : \left(\mid \widehat{\Upsilon} \mid, \mid \partial \Upsilon \mid \right) \rightarrow (Q, \partial_1 Q)$.

Corollary 3. *Let $Q, \partial Q, \partial_1 Q, \partial_0 Q$ satisfy Assumption I. Let $q : Q \rightarrow Q$ be given by Construction 1.*

Let $K_*^{str}(Q) \subset S_*(Q)$ satisfy conditions i)-viii) from Lemma 11, and let $D \subset K_1^{str}(Q)$ be a set of distinguished 1-simplices.

Let $G := \Pi(K(\partial_0 Q))$ with its action on $K^{str}(Q)$ defined in Observation 5, and let $H := q_*(G)$ as defined in Section 5.1. Let

$$\sum_{i \in I} a_i \tau_i \otimes 1 \in C_n^{simp,inf}(K(Q), GK(\partial_1 Q)) \otimes_{\mathbf{Z}G} \mathbf{Z}$$

be a relative cycle. Fix a sufficient set of cancellations $\mathcal{C}$ and a minimal presentation for ∂z. Let $\widehat{\Upsilon}, \partial \widehat{\Upsilon}$ be defined by Observation 9. Assume that we have a G-invariant admissible 0-1-labeling of the edges of $\widehat{\Upsilon}$.

Then there is a well-defined chain map

$$q \circ str : C_*^{simp,inf}\left(\widehat{\Upsilon}\right) \otimes_{\mathbf{Z}G} \mathbf{Z} \rightarrow C_*^{simp,inf}\left(HK^{str}(Q)\right) \otimes_{\mathbf{Z}H} \mathbf{Z},$$

mapping $C_*^{simp,inf}\left(\partial \widehat{\Upsilon}\right) \otimes_{\mathbf{Z}G} \mathbf{Z}$ to $C_*^{simp,inf}(GK^{str}(\partial_1 Q)) \otimes_{\mathbf{Z}H} \mathbf{Z}$, such that:
i) if $e \in \widehat{\Upsilon}_1$ is a 1-labeled edge, $str(e \otimes 1) = f \otimes 1$, then $f \in D$.
ii) if Q is an orientable manifold with boundary ∂Q, and if

$$\sum_{i \in I} a_i \tau_i \otimes 1 \in C_*^{simp,inf}(K(Q), GK(\partial_1 Q)) \otimes_{\mathbf{Z}G} \mathbf{Z}$$

represents[10] *the image of $[Q, \partial Q] \otimes 1$, then*

$$\sum_{i \in I} a_i q \circ str(\tau_i \otimes 1) \in C_*^{simp,inf}\left(HK^{str}(Q), HK^{str}(\partial_1 Q)\right) \otimes_{\mathbf{Z}H} \mathbf{Z}$$

represents the image of $[Q, \partial Q] \otimes 1$ and

$$\partial \sum_{i \in I} a_i q \circ str(\tau_i \otimes 1) \in C_*^{simp,inf}\left(HK^{str}(\partial_1 Q)\right) \otimes_{\mathbf{Z}H} \mathbf{Z}$$

represents the image of $[\partial Q] \otimes 1$.

[10] Cf. the footnotes in Section 3.4

Proof. We can apply Lemma 12 to the infinite chain $\sum_{i\in I,g\in H} a_i(g\tau_i)$. Thus Lemma 12 provides us with a chain map $str: C_*^{simp,inf}\left(\widehat{\Upsilon}\right) \to C_*^{simp,inf}\left(K^{str}(Q)\right)$, given by

$$str\left(g\tau_i\right) := \left(g\tau_i\right)'.$$

$q: \left(K^{str}(Q), K^{str}(\partial_1 Q)\right) \to \left(K^{str}(Q), K^{str}(\partial_1 Q)\right)$ is defined by Observation 5. (Remark: we actually have $q\circ str\left(g\tau_i\right)\in K^{str}(Q)$. We need $HK^{str}(Q)$ in the statement of Corollary 3 just to have the tensor product well-defined.)

We are going to define $q\circ str\left(\sigma\otimes z\right) := q\left(str\left(\sigma\right)\right)\otimes z$ for each $\sigma\in\widehat{\Upsilon}, z\in\mathbf{Z}$. For this to be well-defined, we have to check the following claim:

for each $\sigma\in K, g\in G$, there exists $h\in H$ with $q\left(str\left(g\sigma\right)\right) = hq\left(str\left(\sigma\right)\right)$.

By condition viii) from Lemma 11 (asphericity of $K^{str}(Q)$), it suffices to check this for the 1-skeleton.

It is straightforward to check the claim for the 0-skeleton.

If $\sigma = v\in S_0\left(\partial_0 Q\right)$ then v and gv belong to the same path-component C of $\partial_0 Q$, hence $str(v)$ and $str(gv)$ belong to the same path-component C. Let $\gamma: [0,1]\to\partial_0 Q$ be a path with $\gamma(0) = str(v), \gamma(1) = str(gv)$. Let γ' be the unique 1-simplex in $K\left(\partial_0 Q\right)$ which is homotopic rel. boundary to γ. Let $g' := \left\{\gamma', \overline{\gamma'}\right\}\in G = \Pi\left(K\left(\partial_0 Q\right)\right)$. Then $g' str(v) = str(gv)$, which implies $q\left(str(gv)\right) = hq\left(str(v)\right)$ with $h = q_*\left(g'\right)\in H$.

If $\sigma = v\notin\partial_0 Q$, then $gv = v$, hence $q\left(str(gv)\right) = q\left(str(v)\right)$.

The proof for 1-simplices consists of two steps. In the first step we prove that for $e\in K_1(Q), g\in G$ we have $str_{can}(ge) = g' str_{can}(e)$ with $g'\in G$. In the second step we show that, if $e\in K_1^{str}(Q)$ and $g\in G$, then there exists $h\in H$ with $q\left(str(ge)\right) = hq\left(str(e)\right)$. Hence altogether we will get $q\left(str(ge)\right) = q\left(str\left(str_{can}(ge)\right)\right) = q\left(str\left(g'\ str_{can}(e)\right)\right) = h\ q\left(str\left(str_{can}(e)\right)\right) = h\ q\left(str(e)\right)$.

First step: This is fairly obvious.
First case: If both vertices of e do not belong to $\partial_0 Q$, then also both vertices of $str_{can}(e)$ do not belong to $\partial_0 Q$, and we have $ge = e, gstr_{can}(e) = str(e)$, which implies the claim.
Second Case: If both vertices of e belong to $\partial_0 Q$, then $str_{can}(e)\sim\alpha_1 * e * \alpha_2, str_{can}(ge)\sim\beta_1 * ge * \beta_2$ for some paths $\alpha_1,\alpha_2,\beta_1,\beta_2$ in $\partial_0 Q$. Moreover, by the definition of the action (Section 3.3) we have $ge\sim\gamma_2 * e * \gamma_1$ for some $\gamma_1,\gamma_2\in K_1\left(\partial_0 Q\right)$. Thus $str_{can}(ge)\sim\beta_1 * \gamma_1 * \alpha_1^{-1} * str_{can}(e) * \alpha_2^{-1} * \gamma_2 * \beta_2$, in particular $str_{can}(ge) = g' str_{can}(e)$ for some $g'\in G$.
Third case: Finally we consider the case that one vertex, say $\partial_0 e$ belongs to

$\partial_0 Q$, but $\partial_1 e$ does not belong. Then we are in the situation of the second case with $\gamma_2 = 1, \alpha_2 = \beta_2$ (except that α_2 is not contained in $\partial_0 Q$). We get $str_{can}(ge) \sim \beta_1 * \gamma_1 * \alpha_1^{-1} * str_{can}(e)$. Since $\beta_1 * \gamma_1 * \alpha_1^{-1}$ is contained in $\partial_0 Q$, this implies that $str_{can}(ge) = g' str_{can}(e)$ for some $g' \in G$.

Second step: Let $e \in K_1^{str}(Q)$.

If e is a 1-labeled edge, with $x = \partial_1 e, y = \partial_0 e \in K_0^{str}(Q)$, then we have by condition xiv) from Definition 9 that

$$q\left(str\left(ge\right)\right) = hq\left(e_2\right)$$

for some $e_2 \in D_{xy}$ and some $h \in H$. But e_2 belongs to the same coset in $\Gamma K_1^{str}(Q)\Gamma$ as e, thus $e_2 = str(e)$ which proves the claim for e.

If f is adjacent to one 1-labeled edge e and $q(str(ge)) = hq(str(e))$, then $q(str(gf)) = hq(str(f))$ because the homotopy of f resp. gf just followed the homotopy of e resp. ge: e.g. if $\partial_1 f = \partial_1 e$ and $q(str(ge)) \sim q_*(\alpha) * q(str(e)) * q_*(\beta)$ with $\alpha, \beta \in K_1(\partial_0 Q)$, then $q(str(gf)) \sim q_*(\alpha) * q(str(f))$. Similarly if f is adjacent to two 1-labeled edges.

Finally, if a 0-labeled straight 1-simplex f is not adjacent to a 1-labeled edge, then $str(f) = f$ and $str(gf) = gf$, which implies $str(gf) = gstr(f)$ and $q(str(gf)) = q_*(g)\, str(f)$.

Thus we have proved $q(str(gf)) = hq(str(f))$ with some $h \in H$ for any 0-labeled edges f.

Thus $q \circ str$ is well-defined and satisfies i) by Lemma 12. To prove ii), we first observe that, if $\sum_{i \in I} a_i \tau_i$ represents $[Q, \partial Q]$, then, by Observation 6c) and condition i) from Lemma 12 (together with $q \sim id$), we have that

$$\sum_{i \in I} a_i q \circ str\left(\tau_i\right) = \sum_{i=1}^{r} a_i q\left(\tau_i'\right)$$

represents $[Q, \partial Q]$ and the claim follows. Thus it suffices to check: if $\sum_{i \in I} a_i \tau_i \otimes 1$ is (relatively) homologous to $\sum_{j \in J} b_j \kappa_j \otimes 1$, then $q \circ str\left(\sum_{i \in I} a_i \tau_i \otimes 1\right)$ is (relatively) homologous to $q \circ str\left(\sum_{j \in J} b_j \kappa_j \otimes 1\right)$.

So let

$$\sum_{i \in I} a_i \tau_i \otimes 1 - \sum_{j \in J} b_j \kappa_j \otimes 1 = \partial \sum_{k \in K} c_k \eta_k \otimes 1 \ mod\ C_*^{simp,inf}\left(GK\left(\partial_1 Q\right)\right) \otimes_{\mathbf{Z}G} \mathbf{Z}$$

for some chain $\sum_{k \in K} c_k \eta_k \otimes 1 \in C_*^{simp,inf}(K(Q)) \otimes_{\mathbf{Z}G} \mathbf{Z}$. In complete analogy with Lemma 12, we may extend str to the simplicial set built by the $g\eta_k$'s, their faces and degenerations, and obtain a singular chain $q\left(str\left(\sum_{k\in K}c_k\eta_k\right)\right)$ whose boundary is

$$\partial q\circ str\left(\sum_{k\in K}c_k\eta_k\right)=q\circ str\left(\sum_{i\in I}a_i\tau_i\otimes 1\right)-q\circ str\left(\sum_{j\in J}b_j\kappa_j\otimes 1\right)\ mod\, C_*^{simp,inf}\left(HK^{str}\left(\partial_1 Q\right)\right)\otimes_{\mathbf{Z}H}\mathbf{Z}.$$

This gives the first claim of ii). The second claim of ii) follows because ∂ maps $[Q,\partial Q]$ to $[\partial Q]$.

$\square$

5.5 Removal of 0-homologous chains

Definition 11. *Let Q be an n-dimensional compact manifold with boundary ∂Q. We define $rmv: S_*(Q)\to S_*(Q)$ by*

$$rmv(\sigma)=0$$

if σ is weakly degenerate (Definition 7) and

$$rmv(\sigma)=\sigma$$

else.

Lemma 13. *Assume that Q is a n-dimensional compact manifold with boundary ∂Q. Let $K_*^{str}(Q)\subset S_*(Q)$ satisfy the conditions i)-viii) from Lemma 11. Then*

$$rmv: C_*^{simp}\left(K^{str}(Q),K^{str}(\partial_0 Q)\cup K^{str}(\partial_1 Q)\right)\to C_*^{simp}\left(K^{str}(Q),K^{str}(\partial_0 Q)\cup K^{str}(\partial_1 Q)\right),$$

defined by

$$rmv([\sigma]):=[rmv(\sigma)],$$

is a well-defined chain map. Moreover, if

$$\sum_{j=1}^r a_j\tau_j\in C_*^{simp}\left(K^{str}(Q),K^{str}(\partial_0 Q)\cup K^{str}(\partial_1 Q)\right)\subset C_*^{sing}(Q,\partial Q)$$

represents $[Q,\partial Q]$, then $\sum_{j=1}^r a_j rmv(\tau_j)$ represents $[Q,\partial Q]$.

Proof. If $\sigma\in K^{str}(\partial_0 Q)\cup K^{str}(\partial_1 Q)$, then $rmv(\sigma)\in K^{str}(\partial_0 Q)\cup K^{str}(\partial_1 Q)$, thus rmv is well-defined.
In a first step, we prove that rmv is a chain map.

Assume that $rmv(\sigma)=0$.
If σ has image in ∂Q, then $rmv(\sigma)=0$ and $rmv(\partial\sigma)=0$, thus $\partial rmv(\sigma)=rmv(\partial\sigma)$.

If some edge e of σ, say connecting the i-th and j-th vertex, is a constant loop, then all faces of σ except possibly $\partial_i\sigma$ and $\partial_j\sigma$ have a constant edge. Thus $rmv(\partial_k\sigma) = 0$ if $k \notin \{i,j\}$. Moreover, since e is constant, corresponding edges of $\partial_i\sigma$ and $\partial_j\sigma$ are homotopic rel. boundary and thus agree (possibly up to orientation) by condition v) from Lemma 11. By induction on the dimension of subsimplices we get, again using condition v) from Lemma 11, that $\partial_i\sigma = (-1)^{i-j}\partial_j\sigma$. Altogether we get $rmv(\partial\sigma) = 0$, thus $\partial rmv(\sigma) = rmv(\partial\sigma)$.

Assume that $rmv(\sigma) = \sigma$. Since no edge of σ is a constant loop, of course also no edge of a face $\partial_i\sigma$ is a constant loop. If the image of $\partial_i\sigma$ is not contained in ∂Q, this implies $rmv(\partial_i\sigma) = \partial_i\sigma = \partial_i rmv(\sigma)$. If $\partial_i\sigma$ has image in ∂Q, then of course $[\partial_i\sigma] = [0] = [\partial_i rmv(\sigma)]$, which implies $rmv(\partial_i\sigma) = \partial_i rmv(\sigma)$.
Now we prove that rmv sends relative fundamental cycles to relative fundamental cycles.

Let $\sum_{j=1}^r a_j\tau_j$ be a straight relative cycle, representing the relative homology class $[Q,\partial Q]$.

We denote by $J_1 \subset \{1,\ldots,r\}$ the indices of those τ_j which have a constant edge. The sum $\sum_{j\in J_1} a_j\tau_j$ is a relatively 0-homologous relative cycle. Indeed, each face of $\partial_i\tau_k$ not contained in ∂Q has to cancel against some face of some τ_l, because $\sum_{j=1}^r a_j\tau_j$ is a relative cycle. If $\partial_i\tau_k$ is degenerate, then necessarily $l \in J_1$. Moreover, if τ_k is degenerate and $\partial_i\tau_k$ is nondegenerate, then we have proved in the first step that $\partial_i\tau_k$ cancels against some $\partial_j\tau_k$.

Thus $\sum_{j\in J_1} a_j\tau_j$ represents some relative homology class. The isomorphism $H_n\left(C_*^{sing}(Q,\partial Q)\right) \to \mathbf{R}$ is given by pairing with the volume form of an arbitrary Riemannian metric. After smoothing the relative cycle, we can apply Sard's lemma, and conclude that degenerate simplices have volume 0. Thus $\sum_{j\in J_1} a_j\tau_j$ is 0-homologous.

We denote by $J_2 \subset \{1,\ldots,r\}$ the indices of those τ_j which are contained in ∂Q. For $j \in J_2$ we have $[\tau_j] = [0] \in C_*^{sing}(Q,\partial Q)$.

Thus $\sum_{j\notin J_1\cup J_2} a_j\tau_j$ is another representative of the homology class $[Q,\partial Q]$. But, by Definition 11, it also represents $(rmv)_*([Q,\partial Q])$. $\square$

Consider a subgroup $H \subset \Pi(K(A))$ for some $A \subset \partial Q$. (E.g. $A = q(\partial_0 Q)$ in the setting of Construction 1, and $H = q_*(\Pi(K(\partial_0 Q))) \subset \Pi(K(A))$.

A 1-simplex e is a constant loop if and only if he is a constant loop for all $h \in H$. This implies that a simplex σ is degenerate if and only if $h\sigma$ is degenerate for all $h\sigma$. Moreover, H maps simplices in ∂Q to simplices in ∂Q. Thus $rmv(\sigma) = 0$ if and only if $rmv(h\sigma) = 0$ for all $h \in H$, that is, rmv is well defined on $C_*^{simp,inf}\left(HK^{str}(Q)\right) \otimes_{\mathbf{Z}H} \mathbf{Z}$ for each subgroup H.

Lemma 14. *Assume that Q is a n-dimensional compact manifold with boundary ∂Q. Let the assumptions of Corollary 3 be satisfied. Then we can extend rmv to* *a well-defined chain map*

$$rmv : C_*^{simp,inf}\left(HK^{str}(Q), HK^{str}(\partial_1 Q)\right)\otimes_{\mathbf{Z}H}\mathbf{Z} \to C_*^{simp,inf}\left(HK^{str}(Q), HK^{str}(\partial_1 Q)\right)\otimes_{\mathbf{Z}H}\mathbf{Z}$$

by defining

$$rmv(\sigma\otimes z) = \left\{ \begin{array}{cc} 0: & rmv(\sigma)=0 \\ \sigma\otimes z: & else \end{array} \right\}.$$

Moreover, if $\sum_{j\in J} a_j\tau_j \otimes 1 \in C_*^{simp,inf}(HK^{str}(Q), HK^{str}(\partial_1 Q)) \otimes_{\mathbf{Z}H} \mathbf{Z}$ *represents the image of* $[Q,\partial Q]\otimes 1$*, then* $\sum_{\in J} a_j rmv(\tau_j\otimes 1)$ *represents the image of* $[Q,\partial Q]\otimes 1$*.*

Proof. Well-definedness of rmv follows from the remark before Lemma 14. The same proof as for Lemma 13 shows that rmv is a chain map.

If $\sum_{j=1}^r a_j\tau_j$ represents $[Q,\partial Q]$, then the second claim follows from Lemma 13. If $\sum_{j\in J} a_j\tau_j\otimes 1$ is homologous to $\sum_{i=1}^s b_i\kappa_i\otimes 1$ and $\sum_{i=1}^s b_i\kappa_i$ represents $[Q,\partial Q]$, then (because rmv is a chain map) $rmv\left(\sum_{j\in J} a_j\tau_j\otimes 1\right)$ is homologous to $rmv\left(\sum_{i=1}^s b_i\kappa_i\otimes 1\right)$, which implies the second claim. □

The proof of Theorem 1 will pursue the idea of straightening a given cycle such that many simplices either become weakly degenerate or will have an edge in $\partial_0 Q$. In the first case, they will disappear after application of rmv. In the second case, they disappear in view of the following observation, which is a variant of an argument used in [14].

Lemma 15. *a) Let Assumption I be satisfied for a manifold Q and consider the action of $G=\Pi(K(\partial_0 Q))$ on $K(Q)$. Let $\sigma\in K(Q)$ be a simplex.*

If $str(\sigma)$ has an edge in $\partial_0 Q$, then

$$str(\sigma\otimes 1)=0\in C_*^{simp,inf}(K(Q))\otimes_{\mathbf{Z}G}\mathbf{Z}.$$

b) If $q:Q\to Q$ is given by Construction 1, $H=q_*(G)$, and $\sigma\in K(Q)$ a simplex such that $q(str(\sigma))$ has an edge in $q(\partial_0 Q)$, then

$$q(str(\sigma\otimes 1))=0\in C_*^{simp,inf}(K(Q))\otimes_{\mathbf{Z}H}\mathbf{Z}.$$

Proof. a) Let γ be the edge of $str(\sigma)$ with image in $\partial_0 Q$, then $g=\{\gamma,\overline{\gamma}\}$ is an element of $G=\Pi(K(\partial_0 Q))$ and $gstr(\sigma)=\overline{str(\sigma)}$. In the simplicial chain complex $C_*^{simp,inf}(K(Q))$, one has $\overline{str(\sigma)}=-str(\sigma)$. Thus $gstr(\sigma)=-str(\sigma)$, which implies $str(\sigma\otimes 1)=str(\sigma)\otimes 1=0$.
b) Let γ be the edge of $q(str(\sigma))$ with image in $q(\partial_0 Q)$. Let γ' be the corresponding edge of $str(\sigma)$. Let $g=\{\gamma',\overline{\gamma}'\}\in G$ and $h=q_*(g)=\{\gamma,\overline{\gamma}\}\in H$. The same argument as in a) shows $hq(str(\sigma))=-q(str(\sigma))$. □

6 Proof of Main Theorem

6.1 Motivating examples

Example 1: *Let M be a connected, orientable, hyperbolic n-manifold, F an orientable, geodesic n-1-submanifold, $Q = \overline{M - F}$. For simplicity we assume that M and F are closed, thus Q is a hyperbolic manifold with geodesic boundary $\partial_1 Q \neq \emptyset$, and $\partial_0 Q = \emptyset$.*

Outline of proof of $\parallel M \parallel_F^{norm} \geq \frac{1}{n+1} \parallel \partial Q \parallel$: Start with a fundamental cycle $\sum_{i=1}^r a_i \sigma_i$ of M, such that $\sigma_1, \ldots, \sigma_r$ are normal to F. Since we want to consider laminations without isolated leaves, we replace F by a trivially foliated product neighborhood $\mathcal{F}$. We can assume after a suitable homotopy that each component of $\sigma_i^{-1}(\partial Q)$ either contains no vertex of Δ^n or consists of exactly one vertex, and that each vertex of Δ^n belongs to $\sigma_i^{-1}(\mathcal{F})$, for $i = 1, \ldots, r$.

Each $\sigma_i^{-1}(Q)$ consists of polytopes, which can each be further triangulated (without introducing new vertices) in a coherent way (i.e., such that boundary cancellations between different σ_i's will remain) into $\tau_{i1}, \ldots, \tau_{is(i)}$.

$\sum_{i=1}^r a_i \left(\tau_{i1} + \ldots + \tau_{is(i)}\right)$ is a relative fundamental cycle for Q.

For each σ_i, preimages of the boundary leaves of $\mathcal{F}$ cut Δ^n into regions which we colour with black (components of $\sigma_i^{-1}(\mathcal{F})$) and white (components of $\sigma_i^{-1}(Q)$). Moreover, if $\sigma_i^{-1}(\partial Q)$ contains vertices, these vertices are coloured black. This is a canonical colouring (Definition 4).

The edges of the simplices $\tau_{i,j}$ fall into two classes: 'old edges', i.e. subarcs of edges of σ_i, and 'new edges', which are contained in the interior of some subsimplex of σ_i of dimension ≥ 2.

We define the labeling of the edges of τ_{ij} such that 'old edges' are labelled 1 and 'new edges' are labelled 0. This is an admissible labeling (Definition 10). With this labeling, we apply the straightening procedure[11] from Section 5 to get a straight cycle $\sum_{i=1}^r a_i \left(str\left(\tau_{i1}\right) + \ldots + str\left(\tau_{is(i)}\right)\right)$. (Thus 'old edges' are straightened to distinguished 1-simplices.)

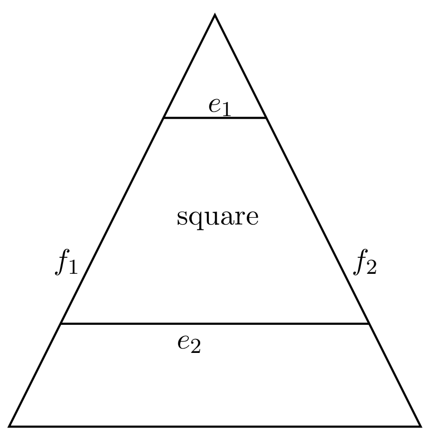
After straightening we remove all weakly degenerate simplices (simplices contained in ∂Q or having a constant edge), i.e. we apply the map rmv from Section 5.4. By Lemma 13, this does not change the homology class. In particular, the boundary of the relative cycle, $\partial \sum_{i,j} a_i rmv\left(str\left(\tau_{ij}\right)\right)$ still represents the fundamental class $[\partial Q]$ of ∂Q.

Claim: for each σ_i, after straightening there remain at most $n+1$ faces of nondegenerate simplices $str\left(\tau_{ij}\right)$ contributing to $\partial \sum_{i,j} a_i rmv\left(str\left(\tau_{ij}\right)\right)$.

[11] Under the assumptions of Example 1, straight simplices can be chosen to be the totally geodesic simplices with vertices in $S_0^{str}(Q)$. Distinguished simplices are chosen according to Observation 8.

In view of Lemma 10, it suffices to show the following subclaim: if, for a fixed i, $T_1 = \partial_{k_1} \tau_{ij_1}, T_2 = \partial_{k_2} \tau_{ij_2}$ are faces of some τ_{ij_1} resp. τ_{ij_2} such that T_1, T_2 have a white-parallel arc (Definition 6), then $rmv\left(str\left(\tau_{ij_1}\right)\right) = 0, rmv\left(str\left(\tau_{ij_2}\right)\right) = 0$. and in particular the corresponding straightened faces $str\left(T_1\right), str\left(T_2\right)$ do not occur (with nonzero coefficient) in $\partial \sum_{i,j} rmv\left(str\left(\tau_{ij}\right)\right)$. (Notational remark: for a subsimplex T of an affine subset $S \subset \Delta^n$ we get a singular simplex $\sigma_i \mid_T$ by restricting σ_i to T. We denote by $str\left(T\right)$ the straightening of $\sigma_i \mid_T$.)

To prove the subclaim, let W be the white region of Δ^n containing T_1 and T_2 in its boundary. By assumption of the subclaim, there is a white square bounded by two arcs $e_1 \subset T_1, e_2 \subset T_2$ and two arcs f_1, f_2 which are subarcs of edges of Δ^n. (The square is a formal sum of two triangles, $U_1 + U_2$, which are 2-dimensional faces of some τ_{ij}'s.)



We want to show that all edges of $str\left(\tau_{ij_1}\right)$ belong to $S_1^{str}\left(\partial Q\right)$. Note that $T_1, T_2 \subset \partial W$ are mapped to ∂Q. Let $x_1 \in S_0^{str}\left(Q\right)$ resp. $x_2 \in S_0^{str}\left(Q\right)$ be the unique elements of $S_0^{str}\left(Q\right)$ in the same connected component C_1 resp. C_2 of ∂Q as $\sigma_i\left(T_1\right)$ resp. $\sigma_i\left(T_2\right)$. In particular $\partial_0 str\left(e_1\right) = x_1 = \partial_1 str\left(e_1\right)$ and $\partial_0 str\left(e_2\right) = x_2 = \partial_1 str\left(e_2\right)$. Thus e_1 and e_2 are straightened to loops $str\left(e_1\right)$ resp. $str\left(e_2\right)$ based at x_1 resp. x_2. The straightenings of the other two arcs, $str\left(f_1\right), str\left(f_2\right)$ connect x_1 to x_2, and they are distinguished 1-simplices because they arise as straightenings of 'old edges'. Thus $str\left(f_1\right) = str\left(f_2\right)$, by uniqueness of distinguished 1-simplices in each coset $\Gamma K_1^{str}\left(Q\right)\Gamma$ of $\Gamma = \Omega\left(\partial Q\right)$. This is why we have performed the straightening construction in Section 5 such that there should be only one distinguished 1-simplex, in each coset, for any given pair of connected components.

This means that the square is straightened to a cylinder.

But $(Q, \partial Q)$ is acylindrical, thus either both $str\,(e_1)$ and $str\,(e_2)$ are constant (in which case $rmv\,(str\,(\tau_{ij_1})) = rmv\,(str\,(\tau_{ij_2})) = 0$), or the cylinder must be homotopic into ∂Q. In the latter case, $str\,(f_1)$ must (be homotopic into and therefore) be contained in ∂Q. In particular, $\partial_0 str\,(f_1)$ and $\partial_1 str\,(f_1)$ belong to the same component of ∂Q. This implies $\partial_0 str\,(f_1) = \partial_1 str\,(f_1)$. Since $str\,(f_1)$ is a distinguished 1-simplex, this implies that $str\,(f_1)$ is constant.

Let P_1, P_2 be the affine planes whose intersections with Δ^n contain T_1 resp. T_2. We have now that there is an arc f_1 connecting $P_1 \cap \Delta^n$ to $P_2 \cap \Delta^n$ such that $str\,(f_1)$ is contained in ∂Q. This implies that for each other arc f connecting $P_1 \cap \Delta^n$ to $P_2 \cap \Delta^n$ its straightening $str\,(f)$ must (be homotopic into and therefore) be contained in ∂Q.

If P_1 and P_2 are of the same type, then all edges of $str\,(\tau_{ij_1})$ connect $P_1 \cap \Delta^n$ to $P_2 \cap \Delta^n$, hence all edges of $str\,(\tau_{ij_1})$ belong to $S_1^{str}\,(\partial Q)$. If P_1 and P_2 are not of the same type, then existence of a parallel arc implies that at least one of them, say P_1, must be of type $\{0a_1 \ldots a_k\}$ with $k \notin \{0, n-1\}$. Then, if P_3 is any other plane bounding W, it follows from Corollary 2 that P_3 has a white-parallel arc with P_1. Thus, repeating the argument in the last paragraph with P_1 and P_3 in place of P_1 and P_2, we obtain that for each arc f connecting $P_1 \cap \Delta^n$ to $P_3 \cap \Delta^n$ its straightening $str\,(f)$ must (be homotopic into and therefore) be contained in ∂Q. Hence, for each τ_{ij_1} in the chosen triangulation of W, its 1-skeleton is straightened into ∂Q.

Since straight simplices σ (of dimension ≥ 2) with $\partial \sigma$ in the geodesic boundary ∂Q, must be in ∂Q, this implies by induction that the k-skeleton of $str\,(\tau_{ij_1})$ is in ∂Q, for each k. In particular, $str\,(\tau_{ij_1}) \in S_n^{str}\,(\partial Q)$. Hence $rmv\,(str\,(\tau_{ij_1})) = 0$. We have proved the subclaim.

By Lemma 10, the *subclaim* implies the *claim*. Since $\sum_{i=1}^r a_i \partial \sum_j rmv\,(str\,(\tau_{ij}))$ represents the fundamental class $[\partial Q]$, we conclude $\parallel \partial Q \parallel \leq (n+1) \sum_{i=1}^r \mid a_i \mid$.

The simplifications of Example 1 in comparison to the proof in Section 6.2 are essentially all due to the fact that $\partial_0 Q = \emptyset$. We remark that in Example 2, if F is not geodesic, then $Q \neq N$ and thus $\partial_0 Q \neq \emptyset$ (even though $\partial M = \emptyset, \partial F = \emptyset$). Thus the generalization to $\partial_0 Q \neq \emptyset$ would be necessary even if one only wanted to consider closed manfifolds M and F.

Example 2: *Let M be a connected, closed, hyperbolic 3-manifold, $F \subset M$ a closed, incompressible surface, $N = \overline{M - F}, Q = Guts\,(N)$.*

Outline of proof of $\| M \|_F^{norm} \geq \frac{1}{4} \| \partial Q \|$: Start with a fundamental cycle $\sum_{i=1}^r a_i \sigma_i$ of M, such that $\sigma_1, \ldots, \sigma_r$ are normal to F. As in Example 1 we get a relative fundamental cycle $\sum_{i=1}^r a_i \left(\tau_{i1} + \ldots + \tau_{is(i)} \right)$ of N. We can not apply the argument from Example 1 to N because N is not acylindrical. Therefore we would like to work with a relative fundamental cycle for the acylindrical manifold Q.

N is aspherical. Using Lemma 2, we can assume that all τ_{ij} belong to $K(N)$. Then we can apply the retraction r from Lemma 5. Since r is only defined after tensoring $. \otimes_{\mathbf{Z}G} \mathbf{Z}$, we get $r(\tau_{ij} \otimes 1) = \kappa_{ij} \otimes 1$ with $\kappa_{ij} \in K(Q)$ only determined up to choosing one κ_{ij} in its G-orbit.

Since Q is aspherical, we have $K(Q) = \widehat{K}(Q)$, that is, the κ_{ij} can be considered as simplices in Q and we can apply Lemma 6b) to obtain a fundamental cycle for ∂Q.

The rest of the proof then basically boils down to copying the proof of Example 1 (with τ_{ij} replaced by κ_{ij}), but taking care of the ambiguity in the choice of κ_{ij}. The details can be found in the next section.

6.2 Proof

We refer to the introduction for the statement of Theorem 1 and the relevant definitions. In this section we are going to prove Theorem 1.

Proof:

If $n = 1$, then Theorem 1 is trivially true. Hence we can restrict to the case $n \geq 2$.

If $\partial_1 Q$ were empty, then $\partial Q = \partial_0 Q$ and amenability of $\pi_1 \partial_0 Q$ would imply $\| \partial Q \| = 0$, in particular Theorem 1 would be trivially true. Hence we can restrict to the case $\partial_1 Q \neq \emptyset$. In particular, Q satisfies Assumption I from Section 5.

Consider a relative cycle $\sum_{i=1}^r a_i \sigma_i$, representing $[M, \partial M]$, such that $\sigma_1, \ldots, \sigma_r$ are normal to $\mathcal{F}$. Our aim is to show: $\sum_{i=1}^r | a_i | \geq \frac{1}{n+1} \| \partial Q \|$.

Denote

$$N = \overline{M - \mathcal{F}}.$$

Since each σ_i is normal to $\mathcal{F}$, we have for each $i = 1, \ldots, r$ that, after application of a simplicial homeomorphism $h_i : \Delta^n \to \Delta^n$, the image of $\sigma_i^{-1}(N)$ consists of polytopes, which can each be further triangulated in a coherent way (i.e., such that boundary cancellations between different σ_i's will remain) into simplices $\theta_{ij}, j \in \hat{J}_i$. (It is possible that $| \hat{J}_i | = \infty$, because N may be noncompact.) We choose these triangulations of the $\sigma_i^{-1}(N)$ to be minimal (Definition 6), that is, we do not introduce new vertices. (Indeed, compatible minimal triangulations of the $\sigma_i^{-1}(N)$ do exist: one starts with common minimal triangulations of the common faces and extends them to minimal triangulations of each polytope.)

Because boundary cancellations are preserved, we have that

$$\sum_{i=1}^{r} a_i \sum_{j\in\hat{J}_i} \theta_{ij}$$

is a countable (possibly infinite) relative cycle representing the fundamental class $[N, \partial N]$ in the sense of section 3.2.
We fix a sufficient set of cancellations $\mathcal{C}^M$ for the relative cycle $\sum_{i=1}^{r} a_i\sigma_i$, in the sense of Definition 8. This induces a sufficient set of cancellations $\mathcal{C}^N$ for the relative cycle $\sum_{i=1}^{r}\sum_{j\in\hat{J}_i} a_i\theta_{ij}$.

If ∂M is a leaf of $\mathcal{F}$, then all faces of z contributing to ∂z are contained in ∂N. We call these faces exterior faces. We can assume that, for each i,
- each component of $\sigma_i^{-1}(\partial N)$ either contains no vertex of Δ^n, or consists of exactly one vertex, or consists of an exterior face,
- and each vertex of Δ^n belongs to $\sigma_i^{-1}(\mathcal{F})$.
Indeed, by a small homotopy of the relative fundamental cycle $\sum_{i=1}^{r} a_i\sigma_i$, preserving normality, we can obtain that no component of $\sigma_i^{-1}(\partial N)$ contains a vertex of Δ^n, except for exterior faces. Afterwards, if some vertices of $\sum_{i=1}^{r} a_i\sigma_i$ do not belong to $\mathcal{F}$, we may homotope a small neighborhood of the vertex, until the vertex (and no other point of the neighborhood) meets ∂N. This, of course, preserves normality to $\mathcal{F}$.

Since each σ_i is normal to $\mathcal{F}$, in particular each σ_i is normal to the union of boundary leaves

$$\partial_1 N := \overline{\partial N - (\partial M \cap \partial N)}.$$

Thus for each σ_i, after application of a simplicial homeomorphism $h_i : \Delta^n \to \Delta^n$, the image of $\sigma_i^{-1}(\partial_1 N)$ consists of a (possibly infinite) set

$$Q_1, Q_2, \ldots \subset \Delta^n,$$

such that

$$Q_i = P_i \cap \Delta^n$$

for some affine hyperplanes $P_1, P_2, \ldots$. We define a colouring by declaring that (images under h_i of) components of

$$\sigma_i^{-1}(int(N)) := \sigma_i^{-1}(N - \partial_1 N)$$

are coloured white and (images under h_i of) components of $\sigma_i^{-1}(\mathcal{F})$ are coloured black. (In particular, all Q_i are coloured black.) Since we assume that all vertices of Δ^n belong to $\sigma_i^{-1}(\mathcal{F})$, and since each boundary leaf is adjacent to at least one component of $\sigma_i^{-1}(int(N))$, this is a canonical colouring (Definition 4).

By Lemma 2a), we can homotope the relative cycle $\sum_{i=1}^r \sum_{j\in\hat{J}_i} \theta_{ij} \in C_n^{inf}(N, \partial N)$ to a relative cycle

$$\sum_{i=1}^r a_i \sum_{j\in\hat{J}_i} \hat{\theta}_{ij}$$

such that each $\hat{\theta}_{ij}$ is a simplex of $\widehat{K}(N)$, as defined in Section 3.2, and such that the boundary $\partial \sum_{i=1}^r \sum_{j\in\hat{J}_i} \theta_{ij}$ is homotoped into $\widehat{K}(\partial N)$. Then consider

$$\sum_{i=1}^r \sum_{j\in\hat{J}_i} a_i \tau_{ij} := \sum_{i=1}^r \sum_{j\in\hat{J}_i} a_i p\left(\hat{\theta}_{ij}\right) \in C_n^{simp,inf}(K(N)),$$

where $p : \widehat{K}(N) \to K(N)$ is the projection defined at the end of Section 3.2, and $\tau_{ij} := p\left(\hat{\theta}_{ij}\right)$ for all i, j.

Consider $Q \subset N$ as in the assumptions of Theorem 1. We denote

$$G := \Pi(K(\partial_0 Q)).$$

We have by assumption that $N = Q \cup R$ is an essential decomposition (as defined in the introduction), which means exactly that the assumptions of Lemma 5 are satisfied. Thus, according to Lemma 5, there exists a retraction

$$r : C_n^{simp,inf}(K(N)) \otimes_{\mathbf{Z}G} \mathbf{Z} \to C_n^{simp,inf}(K(Q)) \otimes_{\mathbf{Z}G} \mathbf{Z}$$

for $n \geq 2$, mapping $C_n^{simp,inf}(GK(\partial N)) \otimes_{\mathbf{Z}G} \mathbf{Z}$ to $C_n^{simp,inf}(GK(\partial_1 Q)) \otimes_{\mathbf{Z}G} \mathbf{Z}$, such that, for each simplex $\tau_{ij} \in K(N)$, we either have $r(\tau_{ij} \otimes 1) = 0$ or

$$r(\tau_{ij} \otimes 1) = \kappa_{ij} \otimes 1$$

for some simplex $\kappa_{ij} \in K(Q)$. (Recall that we assume from the beginning $n \geq 2$.)

Thus

$$r\left(\sum_{i=1}^r a_i \sum_{j\in\hat{J}_i} \tau_{ij} \otimes 1\right) = \sum_{i=1}^r a_i \sum_{j\in J_i} \kappa_{ij} \otimes 1$$

with $J_i \subset \hat{J}_i$ for all i. (It may still be possible that $|J_i| = \infty$.)
We remark that κ_{ij} is only determined up to choosing one κ_{ij} in its G-orbit.

Since r is a chain map, we get a sufficient set of cancellations for $\sum_{i=1}^r a_i \sum_{j\in J_i} \kappa_{ij} \otimes 1$ by

$$\mathcal{C}^Q := \left\{ (\partial_k \kappa_{i_1 j_1} \otimes 1, \partial_l \kappa_{i_2 j_2} \otimes 1) : (\partial_k \tau_{i_1 j_1}, \partial_l \tau_{i_2 j_2}) \in \mathcal{C}^N \right\}.$$

By assumption, Q is aspherical. We can therefore apply Lemma 6 and have that

$$\partial\left(\sum_{i=1}^{r} a_i \sum_{j\in J_i} \kappa_{ij}\otimes 1\right)\in C_*^{simp,inf}\left(GK\left(\partial_1 Q\right)\right)\otimes_{\mathbf{Z}G}\mathbf{Z}$$

represents (the image of) $[\partial Q]\otimes 1$.

Lemma 4a) gives that G is amenable. Together with Lemma 7 this implies

$$\parallel \partial Q \parallel \leq \sum_{i=1}^{r} \mid a_i \mid (n+1) \mid J_i \mid .$$

In the remainder of the proof, we will use Lemma 14 to improve this inequality and, in particular, get rid of the unspecified (possibly infinite) numbers $\mid J_i \mid$.

$Q, \partial Q, \partial_0 Q, \partial_1 Q$ satisfy Assumption I from Section 5. Thus there exists a simplicial set

$$K_*^{str}\left(Q\right)\subset S_*\left(Q\right)$$

satisfying conditions i)-viii) from Lemma 11, and a set

$$D\subset K_1^{str}\left(Q\right)$$

of distinguished 1-simplices (Definition 9).

Recall that, for each i,

$$\sum_{j\in\hat{J}_i}\theta_{i,j}$$

was defined by choosing a triangulation of $\sigma_i^{-1}\left(N\right)$. The simplices $\theta_{i,j}$ thus have 'old edges', i.e. subarcs of edges of σ_i, and 'new edges', whose interior is contained in the interior of some subsimplex of σ_i of dimension ≥ 2.

Associated to $z=\sum_{i=1}^{r} a_i \sum_{j\in\hat{J}_i}\theta_{ij}$ and $\mathcal{C}^N$ (and an arbitrary minimal presentation of ∂z) are, by Definition 8, simplicial sets $\Upsilon^N, \partial\Upsilon^N$.

The only possibility that two 'old edges' have a vertex in Υ^N in common is that this vertex is a vertex of σ_i.

So the labeling of edges of

$$\sum_{i=1}^{r} a_i \sum_{j\in\hat{J}_i}\theta_{ij}$$

by labeling 'old edges' not containig a vertex of any σ_i with label 1 and all other edges with label 0 is an admissible labeling (Definition 10).

Associated to

$$w = \sum_{i=1}^{r} a_i \sum_{j \in J_i} \kappa_{ij} \otimes 1$$

and $\mathcal{C}^Q$ (and an arbitrary minimal presentation of ∂w) there are simplicial sets $\Upsilon, \partial\Upsilon$. By our definition of $\mathcal{C}^Q$, Υ is isomorphic to a simplicial subset of Υ^N, namely to the subset generated by the set

$$\left\{\tau \in \Upsilon^N : r\left(\tau \otimes 1\right) \neq 0\right\}$$

together with all iterated faces and degenerations. In particular, the admissible 0-1-labeling of Υ^N induces an admissible 0-1-labeling of Υ.

By Construction 1, there is a map of triples $q : (Q, \partial Q, \partial_1 Q) \to (Q, \partial Q, \partial_1 Q)$ which is (as a map of triples) homotopic to the identity, and such that $q\left(\partial_0 Q \cap C\right)$ is path-connected for each path-component C of ∂Q.

We denote

$$A := q\left(\partial_0 Q\right), H := q_*\left(G\right) = q_*\left(\Pi\left(K\left(\partial_0 Q\right)\right)\right) \subset \Pi\left(K\left(A\right)\right).$$

We observe that H is a quotient of G, hence amenable, even though $\Pi\left(K\left(A\right)\right)$ need not be amenable.

Let $\widehat{\Upsilon}, \partial\widehat{\Upsilon}$ be defined by Observation 9. By Corollary 3, there is a chain map

$$q \circ str : C_*^{simp,inf}\left(\widehat{\Upsilon}\right) \otimes_{\mathbf{Z}G} \mathbf{Z} \to C_*^{simp,inf}\left(HK^{str}\left(Q\right)\right) \otimes_{\mathbf{Z}H} \mathbf{Z},$$

mapping $C_*^{simp,inf}\left(\partial\widehat{\Upsilon}\right) \otimes_{\mathbf{Z}G} \mathbf{Z}$ to $C_*^{simp,inf}\left(HK^{str}\left(\partial_1 Q\right)\right) \otimes_{\mathbf{Z}H} \mathbf{Z}$, such that

$$\partial \sum_{i=1}^{r} a_i \sum_{j \in J_i} q\left(str\left(\kappa_{ij}\right)\right) \otimes 1$$

represents (the image of) $[\partial Q] \otimes 1$ and such that 1-labeled edges are mapped to distinguished 1-simplices. (We keep in mind that κ_{ij} is only determined up to G-action, thus $q\left(str\left(\kappa_{ij}\right)\right)$ is determined only up to choosing one simplex in its H-orbit.)

We then apply Lemma 14 to get the cycle

$$\partial \sum_{i=1}^{r} a_i \sum_{j \in J_i} rmv\left(q\left(str\left(\kappa_{ij}\right)\right) \otimes 1\right) \in C_*^{simp,inf}\left(HK^{str}\left(\partial_1 Q\right)\right) \otimes_{\mathbf{Z}H} \mathbf{Z}$$

representing (the image of) $[\partial Q] \otimes 1$. We want to show that it is actually a finite chain of l^1-norm at most $(n+1)\sum_{i=1}^{r} \mid a_i \mid$.

Claim: *For each i,*

$$\partial \sum_{j\in J_i} rmv\left(q\left(str\left(\kappa_{ij}\right)\right)\otimes 1\right)$$

is the formal sum of at most $n+1$ n-1-simplices $L\otimes 1$ with coefficient 1.

The claim will be a consequence of the following subclaim and Lemma 10.

Subclaim: *Assume that for some fixed $i\in I$, for the chosen triangulation*

$$\sigma_i^{-1}\left(N\right)=\bigcup_{j\in\hat{J}_i}\theta_{ij},$$

and the associated canonical colouring, there exist $j_1,j_2\in\hat{J}_i, k_1,k_2\in\{0,\ldots,n\}$ such that the faces

$$T_1=\partial_{k_1}\theta_{ij_1}\in S_{n-1}\left(\partial N\right), T_2=\partial_{k_2}\theta_{ij_2}\in S_{n-1}\left(\partial N\right)$$

have a white-parallel arc (Definition 6). Then

$$rmv\left(q\left(str\left(\kappa_{ij_1}\right)\right)\otimes 1\right)=0, rmv\left(q\left(str\left(\kappa_{ij_2}\right)\right)\otimes 1\right)=0.$$

We are going to prove the subclaim.

$$\partial_{k_l}\theta_{ij_l}\in S_{n-1}\left(\partial N\right)$$

implies (by Lemma 5 and Construction 1)

$$\partial_{k_l}q\left(str\left(\kappa_{ij_l}\right)\right)\in HK_*^{str}\left(\partial_1 Q\right)$$

for $l=1,2$. Argueing by contradiction, we assume that

$$rmv\left(q\left(str\left(\kappa_{ij_1}\right)\right)\otimes 1\right)\neq 0.$$

By assumption of the subclaim, there are white-parallel arcs e_1,e_2 of T_1 resp. T_2. This means that there are arcs e_1,e_2 in a 2-dimensional subsimplex $\tau^2\subset\Delta^n$ of the standard simplex, and that there are arcs f_1,f_2, which are subarcs of some edge of τ^2, such that

$$\partial_0 e_1=\partial_1 f_2, \partial_0 f_2=\partial_0 e_2, \partial_1 e_2=\partial_0 f_1, \partial_1 f_1=\partial_1 e_1$$

and such that

$$e_1, f_2, e_2, f_1$$

bound a square in the boundary of a white component. (Cf. the picture in section 6.1. We will use the same letter for an affine subset of Δ^n and for the singular simplex obtained by restricting σ_i to this subset.) The square is of the form U_1+U_2, where U_1, U_2 are n-2-fold iterated faces of some θ_{ij}'s. Hence

$$\partial U_1 = e_1 + f_2 + \partial_2 U_1$$

and

$$\partial U_2 = -e_2 - f_1 - \partial_2 U_1,$$

i.e.

$$\partial\left(U_1+U_2\right) = e_1 + f_2 - e_2 - f_1$$

and

$$\partial_2 U_1 = -\partial_2 U_2.$$

We emphasize that we assume e_1 resp. e_2 to be edges of θ_{ij_1} resp. θ_{ij_2} but that f_1, f_2 need not be edges of θ_{ij_1} or θ_{ij_2}.

Notational convention: *for each iterated face $f=\partial_{k_1}\ldots\partial_{k_l}\theta_{ij}$ with $i\in I, j\in J_i$, we will denote f' the $n-l$-simplex with*

$$f'\otimes 1 = \partial_{k_1}\ldots\partial_{k_l}\kappa_{ij}\otimes 1 = r\left(\partial_{k_1}\ldots\partial_{k_l}\tau_{ij}\otimes 1\right) = r\left(\partial_{k_1}\ldots\partial_{k_l}p\left(\hat{\theta}_{ij}\right)\otimes 1\right).$$

(The last two equations are true because r, p and the homotopy from $\sum_{i,j} a_i\theta_{ij}$ to $\sum_{i,j}\hat{\theta}_{ij}$ are chain maps.) In other words, if f is an iterated face of some τ_{ij}, then f' is, up to the ambiguity by the H-action, the corresponding iterated face of κ_{ij}.

By Lemma 5 we have $e_1', e_2' \in GK\left(\partial_1 Q\right)$. Thus we can (and will) choose $\kappa_{ij_1}, \kappa_{ij_2}$ in their G-orbits such that we have $e_1', e_2' \in K\left(\partial_1 Q\right)$, hence $str\left(e_1'\right), str\left(e_2'\right) \in K^{str}\left(\partial_1 Q\right)$.

Since r, p and the homotopy are chain maps, we have

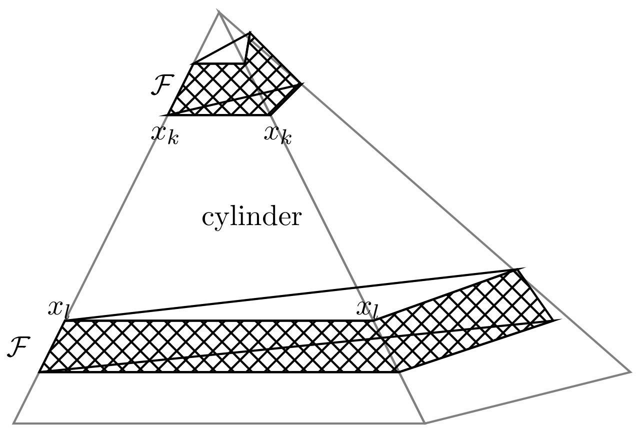
$$\partial_2 U_1' \otimes 1 = -\partial_2 U_2' \otimes 1.$$

That is,

$$\partial_2 U_1' = g\overline{\partial_2 U_2'}$$

for some $g\in G$.

Since U_1' and U_2' belong to different κ_{ij}'s, say κ_{ij_1} and κ_{ij_2}, we can, upon replacing κ_{ij_2} by $g\kappa_{ij_2}$, assume that $\partial_2 U_1' = \overline{\partial_2 U_2'}$, that is, $U_1'+U_2'$ is a square. (Since g maps $\partial e_2'$ to $\partial e_1'$, this second choice of κ_{ij_2} in its G-orbit preserves the condition that $e_2' \in K^{str}\left(\partial_1 Q\right)$.)



Let F resp. F' be the path-components of $\partial_1 Q$ with $e_1' \subset F$ resp. $e_2' \subset F'$. Then we have $\partial_1 str\left(f_1'\right), \partial_0 str\left(f_2'\right) \in F, \partial_0 str\left(f_1'\right), \partial_1 str\left(f_2'\right) \in F'$.

We note that f_1' and f_2' are edges with label 1. By condition (i) of Corollary 3, this implies that $str\left(f_1'\right)$ and $str\left(f_2'\right)$ are distinguished 1-simplices.

By Condition ix) and Condition xiii) of Definition 9 we have that

$$\partial_1 q\left(str\left(f_1'\right)\right) = x_{E_0^F} = \partial_0 q\left(str\left(f_2'\right)\right), \partial_0 q\left(str\left(f_1'\right)\right) = x_{E_0^{F'}} = \partial_1 q\left(str\left(f_2'\right)\right).$$

That is, $q\left(str\left(e_1'\right)\right)$ and $q\left(str\left(e_2'\right)\right)$ are loops in $\partial_1 Q$, based at $x_{E_0^F}$ resp. $x_{E_0^{F'}}$.

Since the square $q\left(str\left(U_1' + U_2'\right)\right)$ realizes a homotopy between $q\left(str\left(f_1'\right)\right)$ and $q\left(str\left(f_2'\right)\right)$, we have that

$$q\left(str\left(f_1'\right)\right) = \gamma_1 q\left(str\left(f_2'\right)\right)\gamma_2$$

with

$$\gamma_1 = q\left(str\left(e_1'\right)\right), \gamma_2 = q\left(str\left(e_2'\right)\right) \in \Omega\left(\partial_1 Q\right) \subset \Gamma = \Omega\left(\partial Q\right).$$

By condition x) from Definition 9 this implies

$$q\left(str\left(f_1'\right)\right) = q\left(str\left(f_2'\right)\right).$$

This means that $q\left(str\left(U_1'\right)\right) + q\left(str\left(U_2'\right)\right)$ is a cylinder with the boundary circles $q\left(str\left(e_1'\right)\right)$ and $q\left(str\left(e_2'\right)\right)$ in $\partial_1 Q$.

(This is why we have performed the straightening construction in Section 5 such that there should be only one distinguished 1-simplex in each coset.)

The assumption $rmv\left(q \circ str\left(\kappa_{ij_1}\right) \otimes 1\right) \neq 0$ implies that the loops $q\left(str\left(e_1'\right)\right)$ and $q\left(str\left(e_2'\right)\right)$ are not 0-homotopic.
Indeed, if one of them, say $q\left(str\left(e_1'\right)\right)$, were a 0-homotopic (thus constant) loop, then also $q\left(str\left(e_2'\right)\right)$ would be a 0-homotopic (thus constant) loop, because they are homotopic through the cylinder. But $q\left(str\left(e_1'\right)\right), q\left(str\left(e_2'\right)\right)$ are edges of $q\left(str\left(\kappa_{ij_1}\right)\right)$ resp. $q\left(str\left(\kappa_{ij_2}\right)\right)$. In particular, $q\left(str\left(\kappa_{ij_1}\right)\right)$ and $q\left(str\left(\kappa_{ij_2}\right)\right)$ would have a constant loop as an edge. By Lemma 14 and Definition 7, this would prove the wanted equalities $rmv\left(q \circ str\left(\kappa_{ij_1}\right) \otimes 1\right) = 0, rmv\left(q \circ str\left(\kappa_{ij_2}\right) \otimes 1\right) = 0$.

Thus we can assume that $q\left(str\left(e_1'\right)\right)$ and $q\left(str\left(e_2'\right)\right)$ are not 0-homotopic, that is, the cylinder

$$q\left(str\left(U_1'\right)\right) + q\left(str\left(U_2'\right)\right)$$

is π_1-injective as a map of pairs. Since $\left(Q, \partial_1 Q\right)$ is a pared acylindrical manifold, the cylinder must then be homotopic into ∂Q, as a map of pairs

$$\left(\mathbf{S}^1 \times [0,1], \mathbf{S}^1 \times \{0,1\}\right) \to \left(Q, \partial_1 Q\right).$$

Since $\partial_1 Q$ is acylindrical, the cylinder must then either degenerate ($\mathbf{S}^1 \times [0,1] \to \partial Q$ homotopes to a map that factors over the projection $\mathbf{S}^1 \times [0,1] \to \mathbf{S}^1$, in particular $q\left(str\left(e_1'\right)\right) = q\left(str\left(e_2'\right)\right)$) or be homotopic into $\partial_0 Q$ (and hence into $q\left(\partial_0 Q\right)$, since $q \sim id$). In the second case the vertices $x_{E_0^F}, x_{E_0^{F'}}$ must belong to $\partial_0 Q$ and we get by condition vii) from Lemma 11 that $q\left(str\left(e_1'\right)\right), q\left(str\left(e_2'\right)\right) \in K_1^{str}\left(\partial_0 Q\right)$. By Lemma 15 this implies that $q\left(str\left(\kappa_{ij_1}\right)\right) \otimes 1 = 0, q\left(str\left(\kappa_{ij_2}\right)\right) \otimes 1 = 0$.
Thus we can assume that the cylinder degenerates. In particular $q\left(str\left(f_1'\right)\right), q\left(str\left(f_2'\right)\right) \in K_1^{str}\left(\partial_1 Q\right)$.

Let P_1, P_2 be the affine planes whose intersections with Δ^n contain T_1 resp. T_2. Let W be the white component whose boundary contains the white-parallel arcs of T_1, T_2. We have seen that there is are arcs f_1, f_2 connecting $P_1 \cap \Delta^n$ to $P_2 \cap \Delta^n$ such that

$$q\left(str\left(f_1'\right)\right), q\left(str\left(f_2'\right)\right) \in K_1^{str}\left(\partial_1 Q\right).$$

This implies that for each other arc f connecting $P_1 \cap \Delta^n$ to $P_2 \cap \Delta^n$ the straightening $q\left(str\left(f'\right)\right)$ must be (homotopic into and therefore by condition vii) from Lemma 11) contained in $\partial_1 Q$.

If P_1 and P_2 are of the same type (Definition 2), then this shows that for all arcs $f \subset W$:

$$q\left(str\left(f'\right)\right) \in K_1^{str}\left(\partial_1 Q\right)$$

If P_1 and P_2 are not of the same type, then the existence of a parallel arc implies that at least one of them, say P_1, must be of type $\{0 a_1 \ldots a_k\}$ with $k \notin \{0, n-1\}$.

Then, for each plane $P_3 \neq P_1$ with $P_3 \cap \Delta^n \subset \partial W$, it follows from Corollary 2 that $P_3 \cap \Delta^n$ has a white-parallel arc with $P_1 \cap \Delta^n$. Thus, repeating the argument with P_1 and P_3 in place of P_1 and P_2, we prove that there are arcs in $\partial_1 Q$ connecting $P_1 \cap \Delta^n$ to $P_3 \cap \Delta^n$, and consequently for each arc $f \subset W$ connecting $P_1 \cap \Delta^n$ to $P_3 \cap \Delta^n$, the straightening $str\,(f')$ must be (homotopic into and therefore) contained in $\partial_1 Q$.

Consequently, also for arcs connecting $P_2 \cap \Delta^n$ to $P_3 \cap \Delta^n$, we have that $q\,(str\,(f'))$ must be (homotopic into and therefore) contained in $\partial_1 Q$. This finally shows that the 1-skeleta of $q\,(str\,(\kappa_{ij_1}))$ and $q\,(str\,(\kappa_{ij_2}))$ belong to $K_1^{str}\,(\partial_1 Q)$. By π_1-injectivity of $\partial_1 Q \to Q$, asphericity of $K\,(\partial_1 Q)$, and condition vii) from Lemma 11, this implies that the 2-skeleta of $q\,(str\,(\kappa_{ij_1}))$ and $q\,(str\,(\kappa_{ij_2}))$ belong to $K_1^{str}\,(\partial_1 Q)$. Inductively, if the k-skeleta of $q\,(str\,(\kappa_{ij_1}))$ and $q\,(str\,(\kappa_{ij_2}))$ belong to $K_k^{str}\,(\partial_1 Q)$, then by asphericity of $K\,(Q)$, asphericity of $K\,(\partial_1 Q)$, and condition vii) from Lemma 11 we obtain that the $k+1$-skeleta of $q\,(str\,(\kappa_{ij_1}))$ and $q\,(str\,(\kappa_{ij_2}))$ belong to $K_{k+1}^{str}\,(\partial_1 Q)$. This provides the inductive step and thus our inductive proof shows that $q\,(str\,(\kappa_{ij_1}))$ and $q\,(str\,(\kappa_{ij_2}))$ belong to $K^{str}\,(\partial_1 Q)$.

By Definition 7, Definition 11 and Lemma 14 this implies

$$rmv\,(q\,(str\,(\kappa_{ij_1}))\otimes 1)=0, rmv\,(q\,(str\,(\kappa_{ij_2}))\otimes 1)=0.$$

So we have shown the subclaim: if $T_1=\partial_{k_1}\theta_{ij_1}, T_2=\partial_{k_2}\theta_{ij_2}$ have a white-parallel arc, then $rmv\,(q\,(str\,(\kappa_{ij_1}))\otimes 1)=0, rmv\,(q\,(str\,(\kappa_{ij_2}))\otimes 1)=0$. In particular,

$$q\,(str\,(T_1')),\, q\,(str\,(T_2'))$$

do not occur (with non-zero coefficient) in

$$\partial \sum_{j\in J_i} rmv\,(q\,(str\,(\kappa_{ij}))\otimes 1)\,.$$

By Lemma 10, for a canonical colouring associated to a set of affine planes $P_1, P_2, \ldots$, and a fixed triangulation of each $Q_i = P_i \cap \Delta^n$, we have at most $n+1$ n-1-simplices whose 1-skeleton does not contain a white-parallel arc. Therefore the *subclaim* implies the *claim*.

Thus we have presented $[\partial Q]\otimes 1$ as a finite chain of l^1-norm at most $(n+1)\sum_{i=1}^{r} \mid a_i \mid$. By Lemma 4a) we know that $G=\Pi\,(K\,(\partial_0 Q))$ is amenable. Hence $H=q_*\,(G)$ is amenable. Thus Lemma 7, applied to $X=\partial Q$ and $K=HK^{str}\,(\partial_1 Q)$ with its H-action, implies

$$\parallel \partial Q \parallel \leq (n+1)\sum_{i=1}^{r} \mid a_i \mid .$$

QED

We remark that Theorem 1 is not true without assuming amenability of $\pi_1\partial_0 Q$. Counterexamples can be found, for example, using [20] or [21], Theorem 6.3.

In [1], Theorem 1 has been proven for incompressible surfaces in hyperbolic 3-manifolds. We compare the steps of the proof in [1] with the arguments in our paper:
Step 1 in [1] is the normalization procedure, which we have restated in Lemma 1.
Step 2 in [1] consists in choosing compatible triangulations of the polytopes $\sigma_i^{-1}(N)$.
Step 3 in [1] boils down to the statement that, for each component Q_i of Q, there exists a retraction $r : \hat{N} \rightarrow p^{-1}(Q_i)$, for the covering $p : \hat{N} \rightarrow N$ corresponding to $\pi_1 Q_i$. Such a statement can not be correct because it would (together with step 7 from [1]) imply $\parallel N \parallel \geq \parallel Q \parallel$ whenever Q is a π_1-injective submanifold of N. This inequality is true for submanifolds with amenable boundary, but not in general. In fact, one only has the more complicated retraction $r : C_*(K(N), K(N')) \otimes_{\mathbf{Z}G} \mathbf{Z} \rightarrow C_*(K(Q), K(\partial Q)) \otimes_{\mathbf{Z}G} \mathbf{Z}$, with $G = \Pi(K(\partial_0 Q))$. This more complicated retraction is the reason that much of the latter arguments become notationally awkward, although conceptually not much is changing. Moreover, the action of the group G is basically the reason that Theorem 1 is true only for amenable G.
Basically, the reason why the retraction $r : \hat{N} \rightarrow Q$ does not exist, is as follows. Let R_j be the connected components of $\hat{N} - p^{-1}(Q_i)$. Then R_j is homotopy equivalent to each connected component of ∂R_j. If ∂R_j were connected for each j, this homotopy equivalence could be extended to a homotopy equivalence $r : \hat{N} \rightarrow p^{-1}(Q_i)$. However, in most cases ∂R_j will be disconnected, and then such an r can not exist.
We note that also the weaker construction of cutting off simplices does not work. A simplex may intersect Q_i in many components and it is not clear which component to choose.
Step 5 in [1] is the straightening procedure, it corresponds to sections 5.2-5.4 in this paper. We remark that the straightening procedure must be slightly more complicated than in [1] because it is not possible, as suggested in [1], to homotope all edges between boundary components of ∂Q into shortest geodesics. This is the reason why we can only straighten chains with an admissible 0-1-labeling of their edges (and why our straightening homomorphism in Section 5.3 is only defined on $C_*^{simp}(\mid \Upsilon \mid)$ and not on all of $C_*^{sing}(Q)$).
Step 6 in [1] consists in removing degenerate simplices. This corresponds to Sec-

tion 5.5 in this paper.
Step 7 in [1] proves that each triangle in $\sigma_i^{-1}(\partial N)$ contributes only once to the constructed fundamental cycle of ∂Q. Since, in our argument, we do not work with the covering $p : \hat{N} \rightarrow N$, we have no need for this justification.
Step 8 in [1] counts the remaining triangles per simplex (after removing degenerate simplices). It seems to have used the combinatorial arguments which we work out for arbitrary dimensions in Section 4.
We mention that the arguments of Section 4 are the only part of the proof which gets easier if one restricts to 3-manifolds rather than arbitrary dimensions. Moreover, the proof for laminations is the same as for hypersurfaces except for Lemma 1. Thus, upon these two points it seems that even in the case of incompressible surfaces in 3-manifolds the proof of Theorem 1 can not be further simplified.

7 Specialization to 3-manifolds

Guts of essential laminations. We start with recalling the guts-terminology. Let M be a compact 3-manifold with (possibly empty) boundary consisting of incompressible tori, and $\mathcal{F}$ an essential lamination transverse or tangential to the boundary. $N = \overline{M - \mathcal{F}}$ is a, possibly noncompact, irreducible 3-manifold with incompressible, aspherical boundary ∂N. We denote $\partial_0 N = \partial N \cap \partial M$ and $\partial_1 N = \overline{\partial N - \partial_0 N}$. (Thus $\partial_1 N$ is the union of boundary leaves of the lamination.) By the proof of [12], Lemma 1.3., the noncompact ends of N are essential I-bundles over noncompact subsurfaces of $\partial_1 N$. After cutting off each of these ends along an essential, properly fibered annulus, one obtains a compact 3-manifold to which one can apply the JSJ-decomposition of [18], [19]. Hence we have a decomposition of N into the characteristic submanifold $Char\,(N)$ (which consists of I-bundles and Seifert fibered solid tori, where the fibrations have to respect boundary patterns as defined in [19], p.83) and the guts of N, $Guts\,(N)$. The I-fibered ends of N will be added to the characteristic submanifold, which thus may become noncompact, while $Guts\,(N)$ is compact. (We mention that there are different notions of guts in the literature. Our notion is compatible with [1], [2], but differs from the definition in [12] or [7] by taking the Seifert fibered solid tori into the characteristic submanifold and not into the guts. Thus, solid torus guts in the paper of Calegari-Dunfield is the same as empty guts in our setting.) If $\partial_0 N \cap \partial Q \neq \emptyset$ consists of annuli $A_1, \ldots, A_k$, then, to be consistent with the setting of Theorem 1, we add components $A_i \times [0,1]$ to $Char\,(N)$ (without changing the homeomorphism type of N), which implies $\partial_0 N \cap \partial Q = \emptyset$.

For $Q = \underline{Guts\,(N)}$ we denote $\partial_1 Q = \partial_1 N \cap \partial Q = \partial N \cap \partial Q = Q \cap \partial N$ and $\partial_0 Q = \overline{\partial Q - \partial_1 Q}$. For $R = Char\,(N)$ we denote $\partial_1 R = \partial N \cap \partial R$ and

$\partial_0 R = \overline{\partial R - \partial_1 R}$. $\partial_0 N \cap \partial Q = \emptyset$ implies then $\partial_0 Q = Q \cap R$.

$\partial_0 Q$ consists of essential tori and annuli, in particular $\pi_1 \partial_0 Q$ is amenable. The guts of N has the following properties: the pair $(Q, \partial_1 Q)$ is a pared acylindrical manifold as defined in Definition 3, $Q, \partial_1 Q, \partial_1 R$ are aspherical, and the inclusions $\partial_0 Q \to Q, \partial_1 Q \to Q, Q \to N$, $\partial_0 R \to R, \partial_1 R \to R, R \to N$ are π_1-injective (see [18],[19]). It follows from Thurston's hyperbolization theorem for Haken manifolds that Q admits a hyperbolic metric with geodesic boundary $\partial_1 Q$ and cusps corresponding to $\partial_0 Q$. (In particular, $\chi(\partial Q) \leq 0$, thus ∂Q is aspherical, and $\partial_1 Q$ is a hyperbolic surface, thus acylindrical.)

Theorem 2 : *Let M be a compact 3-manifold with (possibly empty) boundary consisting of incompressible tori, and let $\mathcal{F}$ be an essential lamination of M. Then*

$$\| M, \partial M \|_{\mathcal{F}}^{norm} \geq -\chi\left(Guts\left(\mathcal{F}\right)\right).$$

More generally, if P is a polyhedron with f faces, then

$$\| M, \partial M \|_{\mathcal{F},P}^{norm} \geq -\frac{2}{f-2}\chi\left(Guts\left(\mathcal{F}\right)\right).$$

Proof. Let $N = \overline{M - \mathcal{F}}$. Since $\mathcal{F}$ is essential, N is irreducible (hence aspherical, since $\partial N \neq \emptyset$) and has incompressible, aspherical boundary. Let $R = Char(N)$ be the characteristic submanifold and $Q = Guts(N)$ be the complement of the characteristic submanifold of N. The discussion before Theorem 2 shows that the decomposition $N = Q \cup R$ satisfies the assumptions of Theorem 1.

From the computation of the simplicial volume for surfaces ([14], section 0.2.) and $\chi(Q) = \frac{1}{2}\chi(\partial Q)$ (which is a consequence of Poincare duality for the closed 3-manifold $Q \cup_{\partial Q} Q$), it follows that

$$-\chi\left(Guts\left(\mathcal{F}\right)\right) = -\frac{1}{2}\chi\left(\partial Guts\left(\mathcal{F}\right)\right) = \frac{1}{4} \| \partial Guts\left(\mathcal{F}\right) \| .$$

Thus, the first claim is obtained as application of Theorem 1 to $Q = Guts(\mathcal{F})$.

The second claim, that is the generalisation to arbitrary polyhedra, is obtained as in [1]. Namely, one uses the same straightening as above, and asks again how many nondegenerate 2-simplices may, after straightening, occur in the intersection of ∂Q with some polyhedron P_i. In [1], p. 11, it is shown that this number is at most $2f-4$, where f is the number of faces of P_i. The same argument as above shows then $\sum_{i=1}^{r} | a_i | \geq \frac{1}{2f-4} \| \partial Guts(\mathcal{F}) \|$, giving the wanted inequality. $\square$

The following corollary applies, for example, to all hyperbolic manifolds obtained by Dehn-filling the complement of the figure-eight knot in $\mathbf{S}^3$. (Note that Hatcher has proved in [16] that each hyperbolic manifold obtained by Dehn-filling the complement of the figure-eight knot in $\mathbf{S}^3$ carries essential laminations.)

Corollary 4. *If M is a finite-volume hyperbolic manifold with $Vol\,(M) < 2V_3 = 2.02...$, then M carries no essential lamination $\mathcal{F}$ with $\parallel M, \partial M \parallel_{\mathcal{F},P}^{norm} = \parallel M, \partial M \parallel_P$ for all polyhedra P, and nonempty guts. In particular, there is no tight essential lamination with nonempty guts.*

Proof. The derivation of Corollary 4 from Theorem 2 is exactly the same as in [1] for the usual (non-laminated) Gromovnorm. Namely, by [29] (or [1], end of Section 6) there exists a sequence P_n of straight polyhedra in $\mathbf{H}^3$ with $\lim_{n\to\infty} \frac{Vol(P_n)}{f_n - 2} = V_3$, with f_n denoting the number of faces of P_n. Assuming that M carries a lamination $\mathcal{F}$ with $\parallel M, \partial M \parallel_{\mathcal{F},P_n}^{norm} = \parallel M, \partial M \parallel_{P_n}$ for all n, one gets

$$-\chi\,(Guts\,(\mathcal{F})) \leq \frac{f_n - 2}{2} \parallel M, \partial M \parallel_{\mathcal{F},P_n} = \frac{f_n - 2}{2} \parallel M, \partial M \parallel_{P_n}$$

$$\leq \frac{f_n - 2}{2} \frac{Vol\,(M)}{Vol\,(P_n)} \to \frac{Vol\,(M)}{2V_3} < 1.$$

On the other hand, if $Guts\,(\mathcal{F})$ is not empty, then it is a hyperbolic manifold with nonempty geodesic boundary, hence

$$\chi\,(Guts\,(\mathcal{F})) \leq -1,$$

giving a contradiction. $\square$

Definition 12. *The Weeks manifold is the closed 3-manifold obtained by $\left(-\frac{5}{1}, -\frac{5}{2}\right)$-surgery at the Whitehead link ([28], p.68).*

It is known that the Weeks manifold is hyperbolic and that its hyperbolic volume is approximately 0.94... (It is actually the hyperbolic 3-manifold of smallest volume.)

Corollary 5. *([7], Conjecture 9.7.): The Weeks manifold admits no tight lamination $\mathcal{F}$.*

Proof. According to [7], the Weeks manifold can not carry a tight lamination with empty guts. Since tight laminations satisfy $\parallel M \parallel_{\mathcal{F},P}^{norm} = \parallel M \parallel$ for each polyhedron (see Lemma 1), and since the Weeks manifold has volume smaller than $2V_3$, it follows from Corollary 4 that it can not carry a tight lamination with nonempty guts neither. $\square$

The same argument shows that a hyperbolic 3-manifold M with
- $Vol(M) < 2V_3$, and
- no injective homomorphism $\pi_1 M \rightarrow Homeo^+(\mathbf{S}^1)$

can not carry a tight lamination, because it was shown by Calegari-Dunfield in [7] that the existence of a tight lamination with empty guts implies the existence of an injective homomorphism $\pi_1 M \rightarrow Homeo^+(\mathbf{S}^1)$. Some methods for excluding the existence of injective homomorphisms $\pi_1 M \rightarrow Homeo^+(\mathbf{S}^1)$ have been developed in [7] (which yielded in particular the nonexistence of such homomorphisms for the Weeks manifold, used in the corollary above), but in general it is still hard to apply this criterion to other hyperbolic 3-manifolds of volume $< 2V_3$.

As indicated in [6], an approach to a generalization of some of the above arguments to essential, non-tight laminations, yielding possibly a proof for nonexistence of essential laminations on the Weeks manifold, could consist in trying to define a straightening of cycles (as in the proof of Lemma 1) upon possibly changing the essential lamination.

As a consequence of a recent paper of Tao Li, one can at least exclude the existence of transversely orientable essential laminations on the Weeks manifold.

Corollary 6. *The Weeks manifold admits no transversely orientable essential lamination $\mathcal{F}$.*

Proof. According to [24], Theorem 1.1, the following statement is true: if a closed, orientable, atoroidal 3-manifold M contains a transversely orientable essential lamination, then it contains a transversely orientable tight essential lamination. Hence Corollary 6 is a direct consequence of Corollary 5. □

8 Higher dimensions

We want to finish this paper with showing that Theorem 1 is interesting also in higher dimensions. While in dimension 3 the assumptions of Theorem 1 hold for each essential lamination, it is likely that this will not be the case for many laminations in higher dimensions. However, the most straightforward, but already interesting application of the inequality is Corollary 7 which means that, for a given negatively curved manifold M, we can give an explicit bound on the topological complexity of geodesic hypersurfaces. Such a bound seems to be new except, of course, in the 3-dimensional case where it is due to Agol ([1]) and (with nonexplicit constants) to Hass ([15]).

Corollary 7. *Let M be a compact Riemannian n-manifold of negative sectional curvature and finite volume. Let $F \subset M$ be a geodesic $n-1$-dimensional hypersurface of finite volume. Then $\parallel F \parallel \leq \frac{n+1}{2} \parallel M \parallel$.*

Proof. Consider $N = \overline{M - F}$. $(N, \partial N)$ is acylindrical. This is well-known and can be seen as follows: assume that N contained an essential cylinder, then the double $DN = N \cup_{\partial_1 N} N$ would contain an essential 2-torus. But, since N is a negatively curved manifold with geodesic boundary, we can glue the Riemannian metrics to get a complete negatively curved Riemannian metric on DN. In particular, DN contains no essential 2-torus, giving a contradiction.

Moreover, the geodesic boundary ∂N is π_1-injective and negatively curved, thus aspherical. Therefore we can choose $Q = N$, in which case the other assumptions of Theorem 1 are trivially satisfied. From Theorem 1 we conclude $\parallel M \parallel_F^{norm} \geq \frac{1}{n+1} \parallel \partial N \parallel$. The boundary of N consists of two copies of F, hence $\parallel \partial N \parallel = 2 \parallel F \parallel$. The leaf space of $\widetilde{F} \subset \widetilde{M}$ is a Hausdorff tree, thus Lemma 1b) implies $\parallel M \parallel_F^{norm} = \parallel M \parallel$. The claim follows. $\square$

This statement should be read as follows: for a given manifold M (with given volume) one has an upper bound on the topological complexity of compact geodesic hypersurfaces.

For hyperbolic manifolds one can use the proportionality principle and the Chern-Gauß-Bonnet Theorem to reformulate Corollary 7 as follows: If M is a closed hyperbolic n-manifold and F a closed n-1-dimensional geodesic hypersurface, then $Vol\,(M) \geq C_n \chi\,(F)$ for a constant C_n depending only on n.

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Thilo Kuessner
Mathematisches Institut, Universität Münster
Einsteinstraße 62
D-48149 Münster
Germany
e-mail: kuessner@math.uni-muenster.de