Foliated norms on fundamental group and homology

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Abstract

This paper compares two invariants of foliated manifolds which seem to measure the non-Hausdorffness of the leaf space: the transversal length on the fundamental group and the foliated Gromov norm on the homology. We consider foliations with the property that the set of singular simplices strongly transverse to the foliation satisfies a weakened version of the Kan extension property. (We prove that this assumption is fairly general: it holds for all fibration-covered foliations, in particular for all foliations of 3-manifolds without Reeb components.) For such foliations we show that vanishing of the transversal length implies triviality of the foliated Gromov norm, and, more generally, that uniform bounds on the transversal length imply explicit bounds for the foliated Gromov norm. This is somewhat surprising in view of the fact that transversal length is defined in terms of 1- and 2-dimensional objects.

Key words: foliation, Gromov norm, Kan property
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Foliations are one of the tools to study the topology of 3-manifolds. Let M be a closed 3-manifold and $\mathcal{F}$ a C^∞-foliation without Reeb components. It is known that the pullback foliation $\widetilde{\mathcal{F}}$ of the universal covering $\widetilde{M}$ is a foliation of $\mathbb{R}^3$ by leaves homeomorphic to $\mathbb{R}^2$. By Palmeira's theorem, such foliations $\widetilde{\mathcal{F}}$ are completely classified by their leaf space, a simply connected non-Hausdorff 1-manifold. Therefore, to study codimension one foliations without Reeb components on 3-manifolds, it is useful to study the actions of 3-manifold fundamental groups on simply connected 1-manifolds. In particular, a foliation can be complicated in two ways: the leaf space of the pull-back foliation on $\widetilde{M}$ can

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be very branched (i.e., be non-Hausdorff with a complicated pattern of branching points), or the group acting on this 1-manifold can be complicated. The second point is of course inherent in M itself: the acting group is $\pi_1 M$, which does not depend on the specific foliation $\mathcal{F}$. Thus, an invariant measuring the complexity of $\mathcal{F}$ should be composed of $\pi_1 M$ and an invariant describing the branching of the leaf space of $\widetilde{\mathcal{F}}$.

Calegari defined in [1] an invariant $\parallel M \parallel_{\mathcal{F}}$, the foliated Gromov norm, and proved several theorems showing that the size of this invariant is related to the branching of the leaf space. His invariant is a refinement of the Gromov norm $\parallel M \parallel$, which measures the complexity of a manifold M and has subtle relations with the fundamental group $\pi_1 M$. The foliated Gromov norm $\parallel M \parallel_{\mathcal{F}}$ is at least as large as $\parallel M \parallel$ and it is the difference $\parallel M \parallel_{\mathcal{F}} - \parallel M \parallel$ which seems to be related to the branching of the leaf space of $\widetilde{\mathcal{F}}$. In particular, Calegari proved that $\parallel M \parallel_{\mathcal{F}} = \parallel M \parallel$ if the leaf space does not branch or branches in only one direction, and he exhibited large classes of branching foliations where the foliated Gromov norm is strictly larger than the simplicial volume.

We consider a second invariant $l_{\mathcal{F}}$, which is a pseudonorm on the fundamental group of a foliated manifold. (The definition is reminiscent of a similar definition in [1].) We show that, under a technical assumption, vanishing of $l_{\mathcal{F}}$ implies triviality of the foliated Gromov norm and, more generally, the foliated Gromov norm can be bounded in terms of $l_{\mathcal{F}}$ and $\parallel M \parallel$.

To describe the technical assumption, we need to sketch two definitions (which will be made precise in the first chapter). A singular simplex is said to be strongly transverse to $\mathcal{F}$ if the induced foliation is affine and there is no 'backtracking' (see section 1.1). We say that $\mathcal{F}$ satisfies the weak Kan property in degrees ≥ 2 if, for $2 \leq n \leq \dim(M)$, any n+1-tuple of strongly transversal n-simplices with compatible boundaries admits an n+1-simplex, with the given simplices as boundary faces, whose n+2-th boundary face is strongly transverse as well (see section 1.2).
For foliations satisfying this condition we have:

Corollary 11: *Let $(M, \mathcal{F})$ be a foliated manifold such that the set of strongly transversal simplices satisfies the weak Kan property in degrees ≥ 2. Then*

$$(\exists x_0 \in M \forall \gamma \in \pi_1(M, x_0) : l_{\mathcal{F}}(\gamma) = 0) \Longrightarrow \parallel M \parallel_{\mathcal{F}} = \parallel M \parallel .$$

More generally, we show that, under the same assumptions, the nontriviality of the foliated Gromov norm can be estimated in terms of the norm $l_{\mathcal{F}}$, if the latter happens to be uniformly bounded.

Theorem 10: *Let $(M, \mathcal{F})$ be a foliated manifold such that the set of strongly transversal simplices satisfies the weak Kan property in degrees ≥ 2. Then*

$$\parallel M \parallel_{\mathcal{F}} \leq (1 + (\dim(M) + 1)\, sup\{l_{\mathcal{F}}(\gamma) : \gamma \in \pi_1(M, x_0)\}) \parallel M \parallel .$$

The question of interest is then, of course, for which foliations the weak Kan property in degrees ≥ 2 holds. Let $K_{st}^{\mathcal{F}}$ denote the set of strongly transversal simplices and $K^{\mathcal{F}}$ the set of transversal simplices. We show that a fairly general class of foliations satisfies the weak Kan property in degrees ≥ 2, in particular

Lemma 13: *Let M^3 be a compact 3-manifold, $\mathcal{F}$ a codimension one foliation on M. Then*
a) the simplicial sets $K_{st}^{\mathcal{F}}$ and $K^{\mathcal{F}}$ satisfy the weak Kan property in degree 3,
b) the simplicial set $K^{\mathcal{F}}$ does not satisfy the weak Kan property in degree 2,
c) the simplicial set $K_{st}^{\mathcal{F}}$ satisfies the weak Kan property in degree 2 if and only if $\mathcal{F}$ has no Reeb component .

More generally, we have

Theorem 14: *If M is an m-dimensional manifold and $\mathcal{F}$ a fibration-covered codimension one foliation, then*

a) $K_{st}^{\mathcal{F}}$ satisfies the weak Kan property in degrees $2, \ldots, m$, and

b) $K^{\mathcal{F}}$ satisfies the weak Kan property in degrees $3, \ldots, m$.

Corollary 15: *If M is an m-dimensional manifold and $\mathcal{F}$ a codimension one foliation satisfying the following conditions:*
(i) for every leaf F is $\pi_1 F \rightarrow \pi_1 M$ injective,
(ii) for every leaf F, the universal covering $\widetilde{F}$ is homeomorphic to $\mathbb{R}^{m-1}$, then

a) $K_{st}^{\mathcal{F}}$ satisfies the weak Kan property in degrees $2, \ldots, m$, and

b) $K^{\mathcal{F}}$ satisfies the weak Kan property in degrees $3, \ldots, m$.

It seems worth mentioning that the assumption in theorem 10 is not just technical but can actually not be avoided. For example, it follows from [1], Thm.2.6.2., that there are foliations $\mathcal{F}$ of the 3-sphere $\mathbb{S}^3$ with arbitrarily large Gromov norm $\parallel \mathbb{S}^3 \parallel_{\mathcal{F}}$, but $\parallel \mathbb{S}^3 \parallel = 0$ and $l_{\mathcal{F}} = 0$ since $\pi_1 \mathbb{S}^3 = 0$. Thus an inequality as in theorem 10 can not hold true for foliations with Reeb components.

Our results and proofs suggest that the branching of the leaf space is directly related to the failure of the Kan extension property. (For example, the nontriviality of $l_{\mathcal{F}}$ expresses the failure of the weak Kan property in degree 1.) Thus it would be nice to have quantitative invariants of sets of simplices which measure the failure of the Kan extension property, as this would give interesting invariants of foliations.

Conventions: We assume all manifolds and foliations to be C^∞, tangentially and transversally. (This will be needed for applications of Palmeira's theorems.) We assume all manifolds to be orientable. All theorems generalize in an obvious way to non-orientable manifolds.
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1 Preparations

1.1 Basic definitions

Let M be a manifold and $\mathcal{F}$ a **codimension one foliation** of M. Let Δ^n be the standard simplex in $\mathbb{R}^{n+1}$, and $\sigma : \Delta^n \rightarrow M$ some singular simplex.

The foliation $\mathcal{F}$ induces an equivalence relation on Δ^n by: $x \sim y \Longleftrightarrow \sigma(x)$ and $\sigma(y)$ belong to the same connected component of $L \cap \sigma(\Delta^n)$ for some leaf L of $\mathcal{F}$.
This equivalence relation may or may not be induced by some foliation of the standard simplex Δ^n.

We say that a singular simplex $\sigma : \Delta^n \rightarrow M$ is **foliated** if the equivalence relation $\sim$ is induced by a foliation of Δ^n. We will denote this foliation of Δ^n by $\mathcal{F}\mid_\sigma$.

We call a foliation of Δ^n affine if there is an affine mapping $f : \Delta^n \rightarrow \mathbb{R}$ such that $x, y \in \Delta^n$ belong to the same leaf if and only if $f(x) = f(y)$.

We say that a singular n-simplex $\sigma : \Delta^n \rightarrow M$, $n \geq 2$, is **transverse** to $\mathcal{F}$ if it is foliated and it is
- either contained in a leaf,
- or the induced foliation $\mathcal{F}\mid_\sigma$ is topologically conjugate to an affine foliation $\mathcal{G}$ of Δ^n, i.e., there is a homeomorphism $h : \Delta^n \rightarrow \Delta^n$ mapping leaves of $\mathcal{F}$ to leaves of $\mathcal{G}$.
For $n = 1$, we say that a singular 1-simplex is transverse to $\mathcal{F}$ if its image is transverse to the leaves of $\mathcal{F}$ in the usual sense (i.e., the image of the tangent

vector is not contained in the tangent space of the leaves).

We say that a singular simplex $\sigma : \Delta^n \to M$ is **strongly transverse** if it is transverse and the mapping $\overline{\sigma} := \pi\sigma : \Delta^n \to M/\mathcal{F}$ is a submersion. (Here, $M/\mathcal{F}$ is the leaf space obtained by the quotient map which identifies points in the same leaf, and $\pi : M \to M/\mathcal{F}$ is the canonical projection. In general, $M/\mathcal{F}$ need not be a smooth manifold. We say that $\overline{\sigma}$ is a submersion if for each foliation chart U with transversal $t_U \simeq \mathbb{R}^1$, the restriction $\pi\sigma \mid_{\sigma^{-1}(U)}: \sigma^{-1}(U) \to t_U$ is a submersion.)

In general, if we have a preferred set of 'transversal' simplices T, we define the transversal Gromov norm of a compact, orientable manifold M with fundamental class $[M, \partial M] \in H_n(M, \partial M; \mathbb{R})$ as

$$\| M, \partial M \|_T := inf \left\{ \sum \mid a_i \mid : \sum a_i \sigma_i \text{ represents } [M, \partial M], \sigma_i \in T \right\}.$$

In the case of a codimension one foliation $\mathcal{F}$ on M, letting T be the set of singular simplices transverse to $\mathcal{F}$ (as defined above) and defining

$$\| M, \partial M \|_{\mathcal{F}} := \| M, \partial M \|_T,$$

one gets Calegari's definition of foliated Gromov norm ([1]).
If $\partial M = \emptyset$, we will omit ∂M from the notation.
We will also use the following notion: let T_{st} be the set of simplices strongly transverse to $\mathcal{F}$ and define $\| M, \partial M \|_{\mathcal{F}}^{st} := \| M, \partial M \|_{T_{st}}$. There is an obvious inequality $\| M, \partial M \|_{\mathcal{F}} \leq \| M, \partial M \|_{\mathcal{F}}^{st}$.

1.2 Weak Kan property and Gromov norm

Let X be a topological space and $S_k(X)$ its set of singular k-simplices with the face maps $\partial_i : S_k(X) \to S_{k-1}(X)$ and the degeneration maps $s_i : S_k(X) \to S_{k+1}(X)$ for $0 \leq i \leq k$. A set $T \subset S_*(X)$ is called a Kan complex if it is a simplicial set, i.e. stable with respect to face and degeneration maps, and if the following holds:
for any collection of $n+1$ n-simplices $\{\tau_0, \ldots, \tau_{k-1}, \tau_{k+1}, \ldots, \tau_{n+1}\} \subset T_n$ with $\partial_i \tau_j = \partial_{j-1} \tau_i$ for all $i < j$, there exists an $n+1$-simplex $\sigma \in T_{n+1}$ with $\partial_i \sigma = \tau_i$ for all $i \neq k$.

The theory of Kan complexes is well developed. However, we will need to work with sets of simplices which only satisfy the following condition.

Weak Kan property. We say that a set $T \subset S_*(X) = \cup_{k \in \mathbb{N}} S_k(X)$, satisfies

the weak Kan property in degree n if the following conditions hold:
- for any collection of $n+1$ n-simplices $\{\tau_0,\ldots,\tau_{k-1},\tau_{k+1},\ldots,\tau_{n+1}\}\subset T_n$ with $\partial_i\tau_j=\partial_{j-1}\tau_i$ for all $i<j$, there exists an $n+1$-simplex $\sigma\in S_{n+1}(X)$ with $\partial_i\sigma=\tau_i$ f or all $i\neq k$, such that $\partial_k\sigma\in T_n$ (not necessarily $\sigma\in T_{n+1}$),
- for any n-simplex $\sigma\in T_n$, $\partial_0\sigma,\ldots,\partial_n\sigma$ belong to T_{n-1},
- for any $n-1$-simplex $\tau\in T_{n-1}$, $s_0\tau,\ldots,s_{n-1}\tau$ belong to T_n.

Moreover, we will need the following notion: let Δ^n,Δ^{n-1} be standard simplices, $r:\Delta^n\rightarrow\Delta^{n-1}$ be any affine mapping with $r(v_i)=v_i$ for $0\leq i\leq n-1$ and $r(v_n)\in\Delta^{n-1}$ arbitrary, then we say that a singular n-simplex $\sigma:\Delta^n\rightarrow X$ is a **general degeneration** of a singular simplex $\tau:\Delta^{n-1}\rightarrow X$ if $\sigma=r\tau$.
We say that $K\subset S_*(X)$ is closed under general degenerations if $\tau\in K_{n-1}$ implies $r\tau\in K_n$ for any such r.

Simplicial approximation property. For a singular simplex $\sigma:\Delta^n\rightarrow X$ and $k\in\mathbb{N}$ let $sd^k(\sigma):sd^k(\Delta^n)\rightarrow X$ denote the k-th barycentric subdivision of σ, i.e., $sd^k(\sigma)=\sigma\Phi^k$ where $\Phi^k:sd^k(\Delta^n)\rightarrow\Delta^n$ is the canonical continuous projection. We say that a subset $K\subset S_*(X)$ satisfies the simplicial approximation property if the following holds true:
for each $j\in\mathbb{N}$ and each singular simplex $\sigma\in S_*(X)$ with j-skeleton in K, there is $k\in\mathbb{N}$ such that $sd^k(\sigma)$ is homotopic to a simplicial mapping $f:sd^k(\Delta^n)\rightarrow K$, by a homotopy which *leaves the j-skeleton of σ pointwise fixed.*

Geometric Realisation. For a subset $K\subset S_*(X)$, which is closed under degeneration maps, define its geometric realisation RK exactly as in [2], p.118, except for the following:
if, for some simplex $x\in K$, some boundary face $\partial_i x$ does not belong to K, then we erase this (open) boundary face from the image of x in RK (but we do not erase iterated boundaries of $\partial_i x$ in case they belong to K). That is, the image of x in RK will not necessarily be closed. (In other words, we consider RK as a subset $RS_*(X)$, with $RS_*(X)$ defined in [2], such that a point in $RS_*(X)$ belongs to RK if it is in the image of some simplex $x\in K$.)

Simplicial and singular Gromov norm. For any subset $K\subset S_*(X)$, there is its geometric realisation RK and the canonical inclusion

$$i_n:C_n(K;\mathbb{R})\rightarrow C_n^{sing}(RK;\mathbb{R})$$

of the simplicial chain complex of K into the singular chain complex of RK. Note that i_* factors over Φ_{K*}, where $\Phi_K:K\rightarrow S_*(RK)$ is the canonical simplicial mapping.

For a homology class $h \in H_n(K;\mathbb{R})$ define

$$\| h \|_{simp} := inf\left\{\sum \mid a_i \mid : \sum a_i\sigma_i \text{ represents } h\right\}$$

and

$$\| h \|_{sing} := inf\left\{\sum \mid a_i \mid : \sum a_i\sigma_i \text{ represents } \Phi_{K*}h \in H_*^{sing}(RK;\mathbb{R}), \sigma_i \text{ singular simplices}\right\}.$$

Of course, $\| h \|_{sing} \leq \| h \|_{simp}$.

We hope that it does not lead to confusion that the well-known Gromov norm on a topological space X is the **simplicial** Gromov norm of the simplicial set $K = S_*(X)$, meanwhile the singular Gromov norm on $S_*(X)$ might be smaller.

To motivate the results, we first discuss the case of Kan complexes.

For a simplicial set K, let $H_{b,simp}^*(K)$ be the simplicial bounded cohomology of K and $H_{b,sing}^*(RK)$ be the singular bounded cohomology of the geometric realization (see [6] for the definition of bounded cohomology). In general, of course, $H_{b,simp}^*$ and $H_{b,sing}^*$ are unrelated. (For example, if K is a finite simplicial complex, then $H_{b,simp}^*(K) = H^*(K)$ but, in general, $H_{b,sing}^*(RK) \neq H^*(RK)$.) However, for Kan complexes, there is the following lemma (which is similar in spirit, but not directly related, to the isometry lemma, [6], p.43).

Lemma 1 *If K is a Kan complex, then*
a)$H_{b,simp}^*(K)$ is isometrically isomorphic to $H_{b,sing}^*(RK)$,
b) $\| h \|_{sing} = \| h \|_{simp}$ for all $h \in H_*(K)$.

PROOF. Let $\Psi_{RK} : RS_*(RK) \rightarrow RK$ be the canonical continuous mapping which projects each singular simplex to its image. By the simplicial extension theorem (which is proved in [2], where it is attributed to unpublished work of Barratt and Kan), there exists a simplicial mapping $g : S_*(RK) \rightarrow K$ such that $R(g)$ is homotopic to Ψ_{RK}.

Let $\Phi_K : K \rightarrow S_*(RK)$ be the canonical simplicial mapping. It is shown in [2] that $\Phi_K g$ and $g\Phi_K$ are chain homotopic to the identities. By dualizing we get isometric isomorphisms of bounded cohomology. (g_* and Φ_{K*} do not increase norms, thus they must be isometries since their composition is the identity.) This proves claim a. By the well-known duality between the norm in bounded cohomology and the Gromov norm on homology (see [6]), claim b follows. □

Proposition 2 *Let X be a topological space, $n \in \mathbb{N}$, and $K \subset S_*(X)$ a set of singular simplices which satisfies the simplicial approximation prop-*

erty, satisfies the weak Kan property in degrees $\geq n$ *and is closed with respect to general degenerations. Let* $h \in H_*(K;\mathbb{R})$ *be a homology class and let* $\sum_{j=0}^{s} a_j \sigma_j \in C_*(X;\mathbb{R})$ *be a cycle such that*
i) the n-1-skeleta of $\sigma_1, \ldots, \sigma_s$ *belong to* K
ii) $\sum_{j=0}^{s} a_j \sigma_j$ *represents* $i_* h \in H_*(X;\mathbb{R})$.
Then $\| h \|_{simp} \leq \sum_{j=0}^{s} | a_j |$.

A special case (with $X = RK$) is the following corollary.

Corollary 3 *Let* Y *be a topological space,* $n \in \mathbb{N}$*, and* $K \subset S_*(Y)$ *a set of singular simplices which satisfies the simplicial approximation property, satisfies the weak Kan property in degrees* $\geq n$*, is closed with respect to general degenerations, and which is such that each cycle* $\sum_{j=0}^{s} a_j \sigma_j \in C_*^{sing}(RK;\mathbb{R})$ *is homologous to a cycle* $\sum_{j=0}^{s} a_j \sigma_j' \in C_*^{sing}(RK;\mathbb{R})$ *(with equal coefficients) with the n-1-skeleta of* $\sigma_1', \ldots, \sigma_s'$ *belonging to* K*. Then* $\| h \|_{sing} = \| h \|_{simp}$ *for all* $h \in H_*(K)$.

Proof of Proposition 2:
Let L be the simplicial set built of $\sigma_1, \ldots, \sigma_r$ together with all of their (iterated) boundaries and degenerations. Let $p : RL \rightarrow RS_*(X)$ be the canonical continuous mapping which projects each σ to its image. By assumption (i), p maps the n-1-skeleton of L to $RK \subset RS_*(X)$.

By the simplicial approximation property, there exists some $k \in \mathbb{N}$ and some simplicial mapping $f' : sd^k(L) \rightarrow K$ such that $R(f') \sim p\Phi^k$ and $R(f') = p\Phi^k$ on $\left(\Phi^k\right)^{-1}(L_{n-1})$. (We may choose k uniformly because L contains only finitely many nondegenerate simplices.)

As in [2], chapter 12, we are looking for a simplicial mapping $g : L \rightarrow K$ and a simplicial mapping $F : sd^k(L) \times I \rightarrow K$ such that $R(F)$ provides a homotopy between $R(f')$ and $R(g)\Phi^k$. Since $S_*(X)$ is a Kan complex, one can apply the construction in [2] to construct such mappings with image in $S_*(X)$, not necessarily in K. Our task is to prove that (under the assumptions of proposition 1), we can construct F and g with image in K. It suffices to consider the case $k = 1$, as the general case follows.

To this aim, we examine the construction in [2]. There one defined an ordering on the simplices of $sd^k(x) \times I$, for each simplex $x \in L$, which allows us to inductively define F (and finally $g(x)$), once F was defined on iterated boundaries of x.

For each simplex $x \in L$, let W_x be the simplices in the canonical triangulation of $sd(x) \times I$, as used in [2], p.204/205. There is a canonical continuous projection $\pi : sd(x) \times I \rightarrow sd(x)$ which is not simplicial, but which actually fails to be simplicial only for one simplex, namely the 'last' simplex $\overline{w}$ for the ordering of W_x given in [2], and for its last boundary face $\partial_{deg(x)+1}\overline{w}$. (Recall that $g(x)$ was defined to be $g(x) := \partial_{deg(x)+1}\overline{w}$.)

We wish to show that the construction in [2] can be carried out such that, for all $x \in L$ and $w \in W_x$, we have that all boundary faces are mapped to K, i.e. that $F(\partial_i w) \in K$. This implies especially $g(x) = F\left(\partial_{deg(x)+1}\overline{w}\right) \in K$.

We proceed by induction on the dimension of x. Assume we have proved $F(\partial_i w_y) \in K$ for $\dim(y) \leq m-1$. For the proof of the inductive step we will need to distinguish the cases $m \leq n-1, m = n$ and $m \geq n+1$.

Assume $\dim(x) = m \leq n-1$. We do already know by assumption (i) that x is mapped to K. For each $w \in W_x$, $F(w)$ is a general degeneration of $p(x)$. Hence $p(x) \in K$ implies $F(w) \in K$ for all $w \in W_x$. (In particular, $g(x) \in K$ which of course might have been achieved by setting $g(x) = p(x)$. We will however need $F(w) \in K$ for the inductive argument.)

Assume $\dim(x) = m = n$. Let $x \in W_x$. If either $w \neq \overline{w}$, or $w = \overline{w}$ but $i \neq n+1$, then $F(\partial_i w)$ is the general degeneration of an n-1-simplex in K, thus $F(\partial_i w) \in K$. In particular, $F(\partial_i \overline{w}) \in K$ for $i \neq n+1$. Since K satisfies the weak Kan property in degree n, we may choose $F(\overline{w})$ such that $\partial_i F(\overline{w}) = F(\partial_i \overline{w})$ for $i \leq n$ and $g(x) := \partial_{n+1} F(\overline{w}) \in K$.

Assume $\dim(x) = m \geq n+1$. Then apply the weak Kan property in degree m, as in [2], to get $F(w) \in K$ for all $w \in W_x$. This finishes the inductive argument.

Thus we arrive at a simplicial mapping $g : L \to K$ such that $R(F)$ provides a homotopy between $R(f')$ and $R(g)\Phi$. In particular, $g_*(\sum a_i\sigma_i) \in K \subset C_*(X)$ is homologous to $p_*(\sum a_i\sigma_i)$ in $C_*(X)$. This shows $\parallel h \parallel_{simp} \leq \sum \mid a_i \mid$. □

Relative version. If $L \subset K \subset S_*(X)$, then the geometric realisations satisfy $RL \subset RK$ and we have a canonical homomorphism $i_* : C_*(RK, RL) \to C_*(K, L)$ of the simplicial into the singular chain complex. We define, for $h \in H_*(K, L)$, the simplicial Gromov norm
$\parallel h \parallel_{simp} = inf\{\sum \mid a_i \mid : \sum a_i\sigma_i$ represents $h\}$ and the singular Gromov norm
$\parallel h \parallel_{sing} = inf\{\sum \mid a_i \mid : \sum a_i\sigma_i$ represents $i_* h\}$. A straightforward generalization of the proof of proposition 2 shows:

Lemma 4 *Let $X' \subset X$ be topological spaces, $n \in \mathbb{N}$, and $K \subset S_*(X)$ a set of singular simplices which satisfies the simplicial approximation property, satisfies the weak Kan property in degrees $\geq n$ and is closed with respect to general degenerations. Let $K' := K \cap S_*(X')$. Let $h \in H_*(K, K'; \mathbb{R})$ be a homology class and let $\sum_{j=0}^s a_j\sigma_j \in C_*(X, X'; \mathbb{R})$ be a relative cycle such that*
i) the n-1-skeleta of $\sigma_1, \ldots, \sigma_s$ belong to K
ii) $\sum_{j=0}^s a_j\sigma_j$ represents $i_* h \in H_*^{sing}(X, X'; \mathbb{R})$.

Then $\| h \|_{simp} \leq \sum_{j=0}^{s} | a_j |$.

Relation with [6]. Even though this is not related to the rest of our paper, we want to mention that the framework of Kan complexes can be used to give an alternative proof of results in [6], such as the theorem that the fundamental group determines the bounded cohomology. (This was brought to my attention by Elmar Vogt.) We outline the argument. Let $T = S_*(X)$. By the construction in [14], p.36, there is a minimal Kan subcomplex $M \subset T$. Let Γ be the set of simplicial automorphisms of M, $\Gamma_n \subset \Gamma$ the subgroup which fixes the n-skeleton pointwise, and $M_n = M/\Gamma_n$.
$M = M_\infty \to \ldots \to M_n \to M_{n-1} \to \ldots M_2 \to M_1$ is the **Postnikov system** considered e.g. in [14], par.8. Note that $M_1 = K(\pi_1 M, 1) = K(\pi_1 T, 1)$ follows from [14], thm.8.4.
Claim: $p_n : M_n \to M_{n-1}$ induces an isometric isomorphism in bounded cohomology.
Outline of proof: If σ is an n-simplex and $\gamma \in \Gamma_{n-1}$, then σ and $\gamma\sigma$ have the same boundary, hence define an n-simplex $\sigma * \gamma\sigma$ with boundary in some vertex v of σ. If $\gamma \in \Gamma_n$, then $\sigma * \gamma\sigma$ is homotopic to $s_0^n(v)$. This defines a map $I : \Gamma_{n-1}/\Gamma_n \to \bigoplus_{\sigma \in M_n} \pi_n(M_n, v_\sigma)$ which, analogously to [6], is an injective group homomorphism. With [14], prop.4.4 this implies that Γ_{n-1}/Γ_n is abelian, thus amenable. As in [6], one defines $A_n : C_n^b(M_n) \to C_n^b(M_{n-1})$ by averaging bounded cochains over the orbits of the amenable group Γ_{n-1}/Γ_n. Clearly, $A_n^* p_n^* = id$ and $\| A_n \| \leq 1, \| p_n \| \leq 1$, thus $\| A_n \| = \| p_n \| = 1$. It remains[1] to prove that $p_n^* A_n^* = id$.
Since $p_n : M_n \to M_{n-1}$ induces an isomorphism of π_1, we can lift p_n to a $\pi_1 M$-equivariant mapping $\tilde{p}_n : \tilde{M}_n \to \tilde{M}_{n-1}$. Let $C_b^n\left(\tilde{M}_n\right)^{\pi_1 M} \simeq C_b^n(M_n)$ be the $\pi_1 M$-invariant bounded cochains on $\tilde{M}_n$ and let $\tilde{A}_n : C_b^n\left(\tilde{M}_n\right) \to C_b^n\left(\tilde{M}_{n-1}\right)$ be defined by averaging over Γ_{n-1}/Γ_n. The restriction of $\tilde{A}_n$ to the $\pi_1 M$-invariant bounded cochains gives A_n. Now, $C_b^i\left(\tilde{M}_n\right)$ is a relatively injective $\pi_1 M$-module, and the resolution $\mathbb{R} \to C_b^0\left(\tilde{M}_n\right) \to C_b^1\left(\tilde{M}_n\right) \to \ldots$ admits a contracting homotopy, hence, by a standard argument in homological algebra ([10], p.1099), any two chain maps extending $id_{\mathbb{R}}$ are chain homotopic, in particular $\tilde{p}_n \tilde{A}_n \sim id$. Restricting to $\pi_1 M$-invariant bounded cochains we get $p_n A_n \sim id$. This shows that $M_n \to M_{n-1}$ induces an isometric isomorphism in bounded cohomology and we conclude:

Corollary 5 *If S and T are Kan complexes and $i : S \to T$ a simplicial mapping such that $i_* : \pi_1 S \to \pi_1 T$ is an isomorphism, then $i^* : H_b^*(T) \to H_b^*(S)$ is an isometric isomorphism.*

---

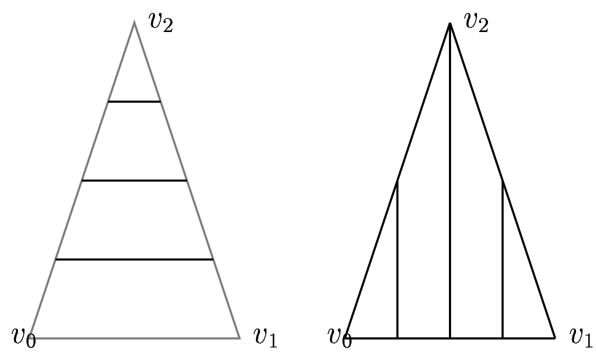
[1] Note that $\gamma \in \Gamma$ is not homotopic to the identity, so the argument in [6] does not work.

PROOF. We may without loss of generality assume that S and T are minimal. Consider the Postnikov systems $S = S_\infty \to \ldots \to S_n \to \ldots \to S_1$ and $T = T_\infty \to \ldots \to T_n \ldots \to T_1$ constructed above. Clearly, all S_i, T_i are minimal. In particular, S_1 and T_1 are weakly homotopy-equivalent minimal Kan complexes, thus are isomorphic (see [14]). This implies that $H_b^*(S_i)$ and $H_b^*(T_i)$ are isometrically isomorphic for any i.

Finally, for any $n \in \mathbb{N}$, $S \to S_n$ induces an isomorphism of $\pi_0, \ldots, \pi_n$ ([14]). By a simplicial version of the Hurewicz theorem (which *only in the case of Kan complexes* also holds true for bounded cohomology, here it would not suffice to assume the weak Kan property) we get an isometric isomorphism $H_b^n(S_n) \to H_b^n(S)$, the same way with T_n and T. This proves $H_b^n(S) \simeq H_b^n(T)$ in (arbitrary) degree n. □

1.3 Moduli space of affine foliations

In this section we study the question when two affine foliations of the standard simplex are topologically conjugate. For example, in dimension 2, there will be precisely the two non-conjugate affine foliations in the pictures below.



If a foliation is as in the right picture, we will say that the affine map has two isolated extrema (here v_0 and v_1). If the foliation is as in the left picture, we will say that it has non-isolated extrema.
The result of this section which we are actually going to need is corollary 7, which will be important for the proof of theorem 14.

Lemma 6 *: Let $f, g : \Delta^n \to \mathbb{R}$ be affine mappings such that*

$$f(v_0) \geq f(v_1) \geq \ldots \geq f(v_n), g(v_0) \geq g(v_1) \geq \ldots \geq g(v_n)$$

and such that for all $0 \leq i \leq n-1$ we have $f(v_i) = f(v_{i+1})$ if and only if $g(v_i) = g(v_{i+1})$. Then there is a piecewise linear homeomorphism $h : \Delta^n \to \Delta^n$ such that for all $x, y \in \Delta^n$ we have:

$$f(x) = f(y) \Longleftrightarrow g(hx) = g(hy).$$

PROOF. Define $h(v_i) = v_i$ for $i = 0, \ldots, n$.
We extend h to the 1-skeleton. We will use that an affine mapping from a 1-simplex to $\mathbb{R}$ is either constant or injective. Consider $[v_i, v_j]$ some 1-simplex with $i < j$.
If $f(v_i) = f(v_j)$, then f must be constant along $[v_i, v_j]$. By assumption $g(v_i) = g(v_j)$, thus also g is constant along $[v_i, v_j]$. Then define h to be the identity on $[v_i, v_j]$. Thus $f(x) = f(y) \Longleftrightarrow g(hx) = g(hy)$ holds for $x, y \in [v_i, v_j]$.
If $f(v_i) > f(v_j)$, then f is strictly increasing along $[v_i, v_j]$, hence there exist unique $w_{i+1}, \ldots, w_{j-1} \in [v_i, v_j]$ with $f(w_k) = f(v_k)$ for $i+1 \leq k \leq j-1$. The same way, since g is strictly increasing (by assumption), we find unique $u_{i+1}, \ldots, u_{j-1}$ with $g(u_k) = g(v_k)$ for $i+1 \leq k \leq j-1$. Define h on $[v_i, v_j]$ by $h(w_k) = u_k$ for $i+1 \leq k \leq j-1$ and by piecewise linear extension, i.e.,

$$h(tw_k + (1-t)w_{k+1}) = tu_k + (1-t)u_{k+1}$$

for all $t \in [0,1]$ and $i+1 \leq k \leq j-1$.

We are given h on the 1-skeleton such that it is linear on $\left[w_k^{(i,j)}, w_{k+1}^{(i,j)}\right] \subset [v_i, v_j]$. We want to extend h to Δ^n. Consider

$$W_i := \{x \in \Delta^n : f(v_i) \geq x \geq f(v_{i+1})\}, U_i := \{x \in \Delta^n : g(v_i) \geq x \geq g(v_{i+1})\}.$$

Since f is affine, W_i is a polytope with vertices $v_i, v_{i+1}, w_i^{(j,k)}$ for all $j < i < k$ and $w_{i+1}^{(j,k)}$ for all $j < i+1 < k$. The same way, U_i is a polytope with vertices $v_i, v_{i+1}, u_i^{(j,k)}$ for all $j < i < k$ and $u_{i+1}^{(j,k)}$ for all $j < i+1 < k$. We are given a bijection $h : w_i^{(j,k)} \to u_i^{(j,k)}$. Fix some triangulation of W_i. Since W_i and U_i are combinatorially equivalent polytopes, we may choose an equivalent triangulation for U_i. Moreover we may choose the triangulations of the W_i's such that the triangulations of W_i and W_{i+1} coincide on $W_i \cap W_{i+1}$ for all i. An affine mapping between simplices is uniquely determined by their vertices. Therefore we can define $h : W_i \to U_i$ as the unique piecewise linear mapping (with respect to the fixed triangulations) with

$$h(v_i) = v_i, h(v_{i+1}) = v_{i+1},$$

$$h\left(w_i^{(j,k)}\right) = u_i^{(j,k)} \; for \; j < i < k, h\left(w_{i+1}^{(j,k)}\right) = u_{i+1}^{(j,k)} \; for \; j < i+1 < k.$$

We have

$$h_i \mid_{V_i \cap V_{i+1}} = h_{i+1} \mid_{V_i \cap V_{i+1}}$$

because the piecewise linear mapping is uniquely determined by its vertices. Thus the h_i fit together to a piecewise linear homeomorphism $h : \Delta^n \to \Delta^n$. We have:

$$f(x) = f(y) \Rightarrow x, y \in V_i \text{ for some } i \Rightarrow g(hx) = g(hy).$$

and vice versa. h is a continuous bijection between compact Hausdorff spaces, thus a homeomorphism. □

Referring to the title of this section, lemma 6 shows that the moduli space of affine foliations consists of a finite number of points. More important for us will be the following corollary.

Corollary 7 *Let $\Delta^n \subset \mathbb{R}^{n+1}$ be the standard simplex, $v_0, \ldots, v_n$ its vertices, and v some point in the interior of Δ^n. Let τ_i be the straight simplex in $\mathbb{R}^{n+1}$ spanned by $v, v_0, \ldots, \hat{v}_i, \ldots, v_n$. Assume that affine mappings $f_i : \tau_i \rightarrow \mathbb{R}$ are defined such that the following conditions a),b),c),d) hold:*
a) if $f_i(v_k) > f_i(v_l)$ for some i, k, l then $f_j(v_k) > f_j(v_l)$ for all j,
b) if $f_i(v_k) = f_i(v_l)$ for some i, k, l, then $f_j(v_k) = f_j(v_l)$ for all j,
c) there do not exist i, j, k, p, r, s such that
$$f_i(v_p) < f_i(v_r), f_j(v_r) < f_j(v_s), f_k(v_s) < f_k(v_p)$$
d) if we reindex $v_0, \ldots, v_n$ such that $f_i(v_k) \geq f_i(v_{k+1})$ for all i (this is possible by a,b), then there is some k with $f_i(v_k) \geq f_i(v) \geq f_i(v_{k+1})$ for all i.

Then there is an affine mapping $f : \Delta^n \rightarrow \mathbb{R}$ and a homeomorphism $h : \Delta^n \rightarrow \Delta^n$ such that $f|_{\tau_i} = f_i h$.

PROOF. We may define an ordering of $\{v_0, \ldots, v_n\}$ by $v_r \geq v_s \Leftrightarrow f_i(v_r) \geq f_i(v_s)$ and $v_r > v_s \Leftrightarrow f_i(v_r) > f_i(v_s)$. The assumption says that this is well-defined and transitive. Therefore there exists some affine mapping $f : \Delta^n \rightarrow \mathbb{R}$ with $f(v_r) > f(v_s) \Leftrightarrow v_r > v_s$ for all $1 \leq r, s \leq n$. By d), we may choose some $v' \in int(\Delta^n)$ such that $f(v_r) \geq f(v') \Leftrightarrow f_i(v_r) \geq f_i(v')$ and $f(v_r) > f(v) \Leftrightarrow f_i(v_r) > f_i(v)$ for all $0 \leq r \leq n$ and $i \neq r$.

For $0 \leq i \leq n$ let $\tau_i' \subset \Delta^n$ be the straight simplex spanned by $v', v_0, \ldots, \hat{v}_i, \ldots, v_n$. Let $g_i : \tau_i \rightarrow \tau_i'$ be the unique affine mapping which sends v to v' and fixes v_j for $j \neq i$. We are going to construct $h : \Delta^n \rightarrow \Delta^n$.

fg_i is affine. By lemma 6, we know that there exist homeomorphisms $h_i : \tau_i \rightarrow \tau_i$ such that $fg_i = f_i h_i$. The homeomorphisms h_i are constructed in the proof of lemma 6 in such a way that they are uniquely defined as soon as triangulations of the different U_k's and W_k's are fixed. If we choose the triangulations in such a way that they coincide on the common boundary faces $\tau_i \cap \tau_j$, then the so constructed homeomorphisms h_i, h_j coincide on $\tau_i \cap \tau_j$: $h_i|_{\tau_i \cap \tau_j} = h_j|_{\tau_i \cap \tau_j}$. Therefore the h_i's fit together to give a piecewiese linear homeomorphism $h : \Delta^n \rightarrow \Delta^n$.
Moreover, by construction, we have $g_i = g_j$ on $\tau_i \cap \tau_j$, therefore the $fg_i : \tau_i \rightarrow \mathbb{R}$ fit together to a well-defined affine mapping. □

2 The norm on the fundamental group ...

2.1 ... and its relation with the foliated Gromov norm

Definition 8 *Let $(M,\mathcal{F})$ be a foliated manifold, $x_0 \in M$ and $\gamma \in \pi_1(M,x_0)$. Then define*

$$l_{\mathcal{F}}(\gamma) = inf\,\{k:\ \text{exists representative } c \text{ of } \gamma \text{ and } k \text{ points } \{p_1,\ldots,p_k\} \text{ on } c \text{ such that each segment of } c - \{p_1,\ldots,p_k\} \text{ is strongly transverse to } \mathcal{F}.$$

This invariant may depend on the choice of base point $x_0 \in M$ and may differ from the invariant $l(\alpha)$ which was defined in [1] as follows:

Definition 9 *Let $(M,\mathcal{F})$ be a foliated manifold and $\alpha : \mathbb{S}^1 \to M$ a closed loop. Then define*

$$l(\alpha) = inf\,\{k:\ \text{exists closed loop } \beta \text{ freely homotopic to } \alpha \text{ and } k \text{ points } \{p_1,\ldots,p_k\} \text{ on } \beta \text{ such that each segment of } \beta - \{p_1,\ldots,p_k\} \text{ is strongly transverse to } \mathcal{F}.$$

There is an obvious inequality $l_{\mathcal{F}}([\alpha]) \geq l(\alpha)$, where $[\alpha]$ denotes the class of α in $\pi_1(M,\alpha(0))$.
It may happen that $l_{\mathcal{F}}([\alpha])$ and $l(\alpha)$ are not bounded. This is for the example the case with the foliation discussed in [1], ex.3.11. On the other hand, classes of foliations with uniform bounds are exhibited in [1]. It is not clear to me what are conditions on a foliation to give uniform *nonzero* bounds on $l_{\mathcal{F}}$.

We say that a codimension one foliation of an m-manifold M satisfies the weak Kan property in degrees ≥ 2 if the union (over $n \in \mathbb{N}$) of

$$K^{\mathcal{F}}_{n,st} := \{\sigma : \Delta^n \to M \text{ singular simplex strongly transverse to } \mathcal{F}\}$$

satisfies the weak Kan property in degrees $2, 3, \ldots, m$.

We observe that the inclusion $i : K^{\mathcal{F}}_{*,st} \to S_*(M)$ induces an **isomorphism** $i_* : H^{simp}_*\left(K^{\mathcal{F}}_{*,st}\right) \to H_*(M)$. Indeed, let $\tau = \sum_{j=1}^r a_j\sigma_j$ be a singular chain with $\partial\tau = 0$. We may perform barycentrical subdivision (which preserves the homology class) sufficiently often such that all simplices of the subdivision are contained in a foliation chart and therefore can be homotoped to be strongly transverse. If we perform the homotopies succesively on simplices of increasing dimensions, these homotopies are compatible (boundary cancellations are preserved), because the homotopy may leave the $k-1$-skeleton pointwise fixed if it is already strongly transverse. Hence the homotoped chain is a cycle in the homology class of τ, showing that i_* is surjective. Similarly, if

$\tau = \partial\left(\sum_{j=1}^{s} b_j \kappa_j\right)$, we may perform barycentrical subdivision and homotopies to get some strongly transversal chain with boundary τ, showing that i_* is injective.
Since i_* is an isomorphism, we have that, for any homology class $h \in H_*(M;\mathbb{R})$, the equality

$$\| h \|_{\mathcal{F}}^{st} = \| i_*^{-1} h \|_{simp}$$

holds, where $\| \, . \, \|_{simp}$ denotes the simplicial Gromov norm on $K_{*,st}^{\mathcal{F}}$. This will enable us to apply the results of section 1.2., especially proposition 2. (By the way, a similar argument shows that $\| i_*^{-1} h \|_{sing} = \| h \|$, i.e. relations between $\| h \|_{\mathcal{F}}^{st}$ and $\| h \|$ are actually relations between $\| i_*^{-1} h \|_{simp}$ and $\| i_*^{-1} h \|_{sing}$.)

The following theorem connects $l_{\mathcal{F}}$ with the foliated Gromov norm. The technical assumption (of the weak Kan property being satisfied for degrees ≥ 2) is satisfied for fairly general foliations, as will be explained in chapter 3. In particular, it is satisfied for any foliation without Reeb components on a 3-manifold.

Theorem 10 *Let $(M, \mathcal{F})$ be a foliated manifold such that the set of strongly transversal simplices satisfies the weak Kan property in degrees ≥ 2. Then*

$$\| h \|_{\mathcal{F}} \leq \left(1 + (n+1)\, sup \left\{ l_{\mathcal{F}}(\gamma) : \gamma \in \pi_1(M, x_0) \right\}\right) \| h \|$$

holds for any $x_0 \in M$ and any $h \in H_n(M;\mathbb{R})$. In particular, if M is closed and oriented,

$$\| M \|_{\mathcal{F}} \leq \left(1 + (\dim(M) + 1)\, sup \left\{ l_{\mathcal{F}}(\gamma) : \gamma \in \pi_1(M, x_0) \right\}\right) \| M \| .$$

Corollary 11 *Let $(M, \mathcal{F})$ be a foliated, closed and oriented manifold such that the set of strongly transversal simplices satisfies the weak Kan property in degrees ≥ 2. If $l_{\mathcal{F}}(\gamma) = 0$ for some $x_0 \in M$ and all $\gamma \in \pi_1(M, x_0)$, then*

$$\| M \|_{\mathcal{F}} = \| M \| .$$

PROOF. Let $\sum_{i=1}^{r} a_i \sigma_i$ be a cycle representing the homology class h. We may homotope the σ_i such that all vertices of all σ_i are in the base point x_0. After this homotopy, all edges are closed loops γ, representing classes $[\gamma] \in \pi_1(M, x_0)$, and we may further homotope, keeping x_0 fixed, such that the homotoped edges can be subdivided into $l_{\mathcal{F}}([\gamma])$ transverse arcs (resp., if $l_{\mathcal{F}}([\gamma]) = 0$, such that the homotoped γ is transverse to $\mathcal{F}$). It is straightforward to see that these homotopies can actually be extended to homotopies of $\sigma_1, \ldots, \sigma_r$ such that the homotopies do not affect cancellation of boundary faces. We continue to denote the homotoped singular simplices by $\sigma_1, \ldots, \sigma_r$.

Let $L = sup \left\{ l_{\mathcal{F}}(\gamma) : \gamma \in \pi_1(M, x_0) \right\}$. We claim that we may subdivide each σ_i into $1 + (n+1) L$ simplices τ_{ij} such that σ_i is homologous to $\sum_j \tau_{ij}$ and such that each τ_{ij} has transversal 1-skeleton.

Given a simplex σ_i, we can subdivide each of its edges e by $l_{\mathcal{F}}([e])$ points into transversal arcs. Denote by P this set of points. There is a (not unique) subtriangulation T of σ_i such that the vertices of simplices in T are exactly the points in P together with the vertices of σ_i. (We remark that these triangulations can be performed such that cancelling boundary faces of σ_i and σ_j are triangulated compatibly. This can just be achieved after prescribing an ordering on the total set of vertices.) The number of simplices in this triangulation is at most $1+(n+1)L$. We claim that we can realise this subtriangulation of σ_i such that its 1-skeleton is transverse.

To show this claim, we lift σ to the universal covering $\widetilde{M}$ and observe that the leaf space $\widetilde{M}/\widetilde{\mathcal{F}}$ for the pull-back foliation $\mathcal{F}$ is a simply connected 1-manifold (since every loop can be lifted). Therefore we can say that each projection of a simplex has (at least two) outermost vertices in $\widetilde{M}/\widetilde{\mathcal{F}}$. Thus it suffices to show the following: whenever an n-simplex τ is given with the property that for one of its outermost vertices $v \in \tau$ all edges emanating from v are transverse, we can homotope τ to a simplex with transversal 1-skeleton. (This implies the former claim, as we may apply the latter claim succesively to the simplices in the subtriangulation of σ_i, using that the 1-skeleton of σ_i is already transverse. At each step we have to take an outermost vertex of the remaining simplices in the subtriangulation σ_i, after the subsimplices whose 1-skeleton is already transverse have been removed.) So given an outermost vertex $v \in \tau$ and two other vertices p_i, p_j of τ, our task is to show that the arc connecting p_i and p_j can be homotoped to be transverse.

Consider the image of τ in $\widetilde{M}/\widetilde{\mathcal{F}}$. If v is the outermost point, then, w.l.o.g., the projection of the arc connecting v to p_j passes through p_i, hence the arc connecting v and p_j has to pass through the leaf F_{p_i} containing p_i. Hence there is an arc connecting p_i and p_j which is composed by an arc in F_{p_i} and by a transversal arc. A small perturbation makes this composed arc transverse to $\mathcal{F}$. (This argument shows also that, for any subtriangulation obtained after some removals, the vertex with the outermost projection to $\widetilde{M}/\widetilde{\mathcal{F}}$ is an exterior vertex. Namely, if it was lying on an edge between two points p_i and p_j, then the same argument would show that there is a transversal arc connecting p_i and p_j, i.e., we could reduce the number of points on this edge.) This finishes the proof of the claim.

Now we want to apply proposition 2. We observe that $K_{st}^{\mathcal{F}}$ obviously is closed with respect to general degenerations, and that it also satisfies the simplicial approximation property. To see the latter statement, observe that each simplex can be barycentically subdivided sufficiently often such that all simplices of the iterated subdivision are contained in foliation charts. Inside these foliation charts, each simplex can be homotoped to a strongly transverse one. If we perform the homotopies succesively on simplices of increasing dimensions, these homotopies are compatible (boundary cancellations are preserved). This works because the homotopy may leave the $k-1$-skeleton pointwise fixed if it is already strongly transverse.

Thus we have checked the assumptions of proposition 2 and may apply proposition 2 (with $n = 2$) to conclude $\| M \|_{\mathcal{F}}^{st} \leq (1 + (n+1) L) \| M \|$. This implies $\| M \|_{\mathcal{F}} \leq (1 + (n+1) L) \| M \|$. □

It should be mentioned that, for taut foliations of 3-manifolds, the implication

$$l(\alpha) = 0 \ \ \forall \alpha \Longrightarrow \| M \|_{\mathcal{F}} = \| M \|$$

already follows from results in [1].

Remark: The proof of theorem 10 would actually also work if we assumed that the set of transversal simplices satisfies the weak Kan property in degrees ≥ 2. However, as we will see in the beginning of section 3, there is *no* foliation $\mathcal{F}$ such that $K_{\mathcal{F}}$ satisfies the weak Kan property in degree 2. Thus, theorem 10 would be (correct but) meaningless if we replaced 'strongly transversal' by 'transversal' in the statement of the theorem.

2.2 (Sub)Additivity of foliated Gromov norms

Using lemma 4, one obtains the obvious generalization of theorem 10 to foliated manifolds with boundary. This is needed for the following corollary.

Corollary 12 *Let M_1, M_2 be compact, orientable n-manifolds with foliations $\mathcal{F}_i, i = 1, 2$ transverse to the connected boundaries. Assume that the boundaries are π_1-injective and have amenable fundamental groups.*
Let $f : \partial M_1 \to \partial M_2$ a homeomorphism which is compatible with $\mathcal{F}_1 \mid_{\partial M_1}$ and $\mathcal{F}_2 \mid_{\partial M_2}$, $M = M_1 \cup_f M_2$ and $\mathcal{F}$ the glued foliation on M. If all foliations satisfy the weak Kan property in degrees $n \geq 2$, then

$$\frac{1}{1 + (n+1) L_1} \| M_1 \|_{\mathcal{F}_1} + \frac{1}{1 + (n+1) L_2} \| M_2 \|_{\mathcal{F}_2} \leq$$

$$\| M \|_{\mathcal{F}} \leq (1 + (n+1) L) (\| M_1 \|_{\mathcal{F}_1} + \| M_2 \|_{\mathcal{F}_2}) .$$

with $L = sup \{ l_{\mathcal{F}}(\gamma) : \gamma \in \pi_1 (M, x_0) \}$ and $L_i = sup \{ l_{\mathcal{F}_i}(\gamma) : \gamma \in \pi_1 (M_i, x_0) \}$.

In particular, if all foliations satisfy the weak Kan property in degrees $n \geq 2$, and $l_{\mathcal{F}}(\gamma) = 0$ for all $\gamma \in \pi_1 (M, x_0)$ (for some $x_0 \in M$), then

$$\| M \|_{\mathcal{F}} = (\| M_1 \|_{\mathcal{F}_1} + \| M_2 \|_{\mathcal{F}_2}) .$$

PROOF. The right hand inequality follows from theorem 10 together with $\| M \|=\| M_1, \partial M_1 \| + \| M_2, \partial M_2 \|$ ([6],[13]) and the obvious inequality $\| M_i, \partial M_i \| \leq \| M_i, \partial M_i \|_{\mathcal{F}_i}$ for $i = 1, 2$. Similarly one gets the left hand inequality. □

An analogous statement can be proved if the glueing only takes place along some connected components of M_1 resp. M_2, i.e., if M happens to have nonempty boundary.
Moreover, if the foliated manifold M_1 has (at least) two (amenable, π_1-injective) boundary components $\partial_1 M_1, \partial_2 M_1$ and one glues with a homeomorphism $h : \partial_1 M_1 \rightarrow \partial_2 M_1$ to get a foliated manifold $(M = M_1/h, \mathcal{F})$, one obtains $(1 + (n+1) L_1)^{-1} \| M_1 \|_{\mathcal{F}_1} \leq \| M \|_{\mathcal{F}} \leq (1 + (n+1) L) \| M_1 \|_{\mathcal{F}_1}$
by the same arguments.

Corollary 12 applies in particular to the JSJ-decomposition of 3-manifolds carrying foliations without Reeb components.

3 Weak Kan property for taut foliations

Let M be a compact manifold and $\mathcal{F}$ a codimension one foliation of M.

Denote

$$K_n^{\mathcal{F}} := \{\sigma : \Delta^n \rightarrow M \text{ singular simplex transverse to } \mathcal{F}\}$$

and

$$K_{n,st}^{\mathcal{F}} := \{\sigma : \Delta^n \rightarrow M \text{ singular simplex strongly transverse to } \mathcal{F}\}.$$

It is easy to see that the union $K_{st}^{\mathcal{F}} := \cup_{n \geq 0} K_{n,st}^{\mathcal{F}}$ is stable with respect to boundary maps and degeneracy maps, meanwhile, in general, the faces of simplices in $K_2^{\mathcal{F}}$ do not necessarily belong to $K_1^{\mathcal{F}}$. (Note that each foliated *degenerate* 2-simplex belongs to $K_2^{\mathcal{F}}$, but not necessarily to $K_{2,st}^{\mathcal{F}}$.)
We mention that, for an arbitrary foliation $\mathcal{F}$, the simplicial set $K_{st}^{\mathcal{F}}$ does not satisfy the weak Kan property in degree 1, and $K^{\mathcal{F}}$ does not satisfy the weak Kan property neither in degree 1 nor in degree 2. To prove the first statement, just consider two transversal 1-simplices whose composition is not transverse. To prove the latter statement, consider (for the foliation of $\mathbb{R}^n$ by horizontal hyperplanes with the induced total ordering of the leaf space) 4 points v_0, v_1, v_2, v_3 such that, with respect to the total ordering, $v_0 > v_1 > v_2 = v_3$ (for the leaves containing the respective vertices) and consider the straight simplices σ_2 resp. σ_3 with vertices v_0, v_1, v_3 resp. v_0, v_1, v_2. Let σ_0 be the degenerate 2-simplex with edges (v_1, v_2) and (v_1, v_3). Although σ_0 is

transverse (but not strongly transverse) to $\mathcal{F}$, it is clear that there is no transverse 2-simplex σ_1 with $\partial_0\sigma_1 = \partial_0\sigma_0, \partial_1\sigma_1 = \partial_0\sigma_2, \partial_2\sigma_1 = \partial_0\sigma_3$. This would contradict the weak Kan property. Clearly, the same argument works for any foliated manifold $(M, \mathcal{F})$ because it can be realised inside a foliation chart.

Lemma 13 *Let M^3 be a compact 3-manifold, $\mathcal{F}$ a codimension one foliation on M. Then*
a) the simplicial sets $K_{st}^{\mathcal{F}}$ and $K^{\mathcal{F}}$ satisfy the weak Kan property in degree 3,
b) the simplicial set $K^{\mathcal{F}}$ does not satisfy the weak Kan property in degree 2,
c) the simplicial set $K_{st}^{\mathcal{F}}$ satisfies the weak Kan property in degree 2 if and only if $\mathcal{F}$ has no Reeb component.

We will deduce lemma 13 from the following, more general, theorem 14.
We say that a foliation $\mathcal{F}$ of a manifold is **fibration-covered** if there is some covering $\widehat{M} \to M$ with pull-back foliation $\hat{\mathcal{F}}$ on $\widehat{M}$, such that the projection $\hat{\pi} : \widehat{M} \to \widehat{M}/\hat{\mathcal{F}}$ from $\widehat{M}$ to its leaf space is a locally trivial fibration. (An equivalent condition is that just $\tilde{\pi} : \widetilde{M} \to \widetilde{M}/\tilde{\mathcal{F}}$ is a locally trivial fibration, where $\widetilde{M}$ denotes the universal covering and $\tilde{\mathcal{F}}$ the pull-back foliation.) This property is by far more common than one might expect in view of the following observation:

Observation. If M is an m-dimensional manifold and $\mathcal{F}$ a codimension one foliation satisfying the following conditions:
(i) for every leaf F is $\pi_1 F \to \pi_1 M$ injective,
(ii) for every leaf F, the universal covering $\tilde{F}$ is homeomorphic to $\mathbb{R}^{m-1}$,
then $\mathcal{F}$ is fibration-covered.

PROOF. Let $\widetilde{M}$ be the universal covering with the pull-back foliation $\tilde{\mathcal{F}}$. The assumptions imply that all leaves of $\tilde{\mathcal{F}}$ are homeomorphic to $\mathbb{R}^{m-1}$. These foliations have been investigated in [15] and it has been shown, in particular, that the projection $\tilde{\pi} : \widetilde{M} \to \widetilde{M}/\tilde{\mathcal{F}}$ is a locally trivial fibration (this is the corollary to the trivialization lemma on [15], p.117). □

Theorem 14 *If M is an m-dimensional manifold and $\mathcal{F}$ a fibration-covered codimension one foliation, then*

a) $K_{st}^{\mathcal{F}}$ satisfies the weak Kan property in degrees $2, \ldots, m$, and

b) $K^{\mathcal{F}}$ satisfies the weak Kan property in degrees $3, \ldots, m$.

Corollary 15 *If M is an m-dimensional manifold and $\mathcal{F}$ a codimension one*

foliation satisfying the following conditions:
(i) for every leaf F, $\pi_1 F \to \pi_1 M$ is injective,
(ii) for every leaf F, the universal covering $\tilde{F}$ is homeomorphic to $\mathbb{R}^{m-1}$,
then

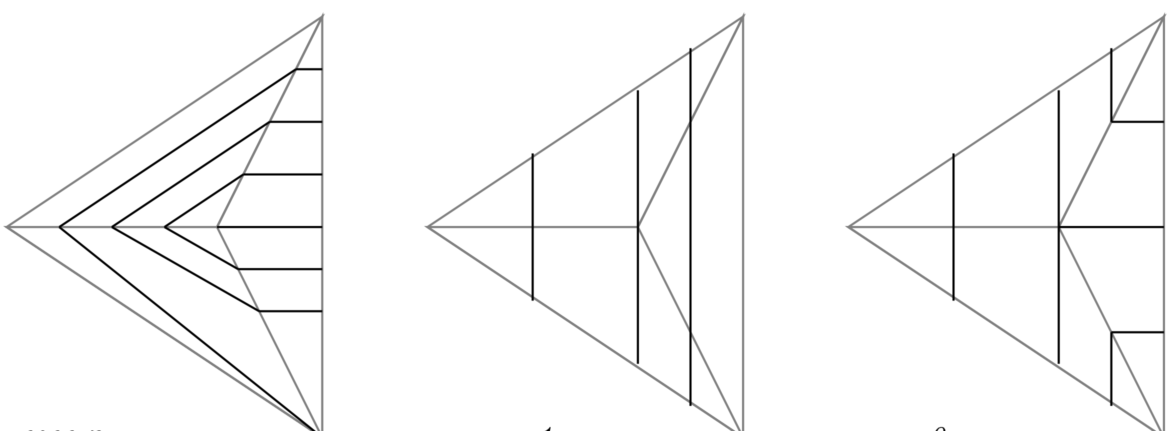
a) $K_{st}^{\mathcal{F}}$ satisfies the weak Kan property in degrees $2, \ldots, m$, and

b) $K^{\mathcal{F}}$ satisfies the weak Kan property in degrees $3, \ldots, m$.

PROOF.

Of course, whenever n-simplices $\tau_0, \ldots, \tau_n$ satisfy $\partial_i \tau_j = \partial_{j-1} \tau_i$ for all $i < j$, we have the canonical degenerate $n+1$-simplex σ with $\partial_i \sigma = \tau_i$ for all $i \leq n$ which is defined by precomposing $\tau_0 \cup \ldots \cup \tau_n$ with the canonical retraction from the standard $n+1$-simplex Δ^{n+1} to the union of $\partial_0 \Delta^{n+1}, \ldots, \partial_n \Delta^{n+1}$. We will denote this simplex by $\sigma_{deg}(\tau_0, \ldots, \tau_n)$. The cases of interest will be those where $\partial_{n+1} \sigma_{deg}(\tau_0, \ldots, \tau_n)$ is not transverse to $\mathcal{F}$ even though $\tau_0, \ldots, \tau_n$ are.

Given $n+1$ n-simplices $\tau_0, \ldots, \tau_n$ with compatible boundaries, their union $K := \tau_0 \cup \ldots \cup \tau_n$ is a singular n-simplex with the common vertex $v := \partial_0^n \tau_i$ in its interior. K is exactly the image of $\partial_{n+1} \sigma_{deg}(\tau_0, \ldots, \tau_n)$. We assume to have foliations by level sets of affine maps $f_i : \tau_i \to \mathbb{R}, 0 \leq i \leq n$.



case n *case n-1* *case 0*

Let $v := \partial_0^n \tau_i$ be the common vertex of $\tau_0, \ldots, \tau_n$, and denote by v_i the unique vertex $\neq v$ which is not a vertex of τ_i. We will distinguish two possibilities. Possiblity A is that there exists some simplex τ_i such that v is not an extremum of f_i. (Some examples are pictured above, where case k means that v is an extremum of k simplices.) In this case we will show, without needing any assumption on the foliation $\mathcal{F}$, that $\partial_{n+1} \sigma_{deg}(\tau_0, \ldots, \tau_n)$ is strongly transverse to $\mathcal{F}$ if $\tau_0, \ldots, \tau_n$ are. (If $n \geq 3$, we will also get transversality of $\partial_{n+1}\sigma_{deg}(\tau_0, \ldots, \tau_n)$ under the weaker assumption that $\tau_0, \ldots, \tau_n$ are transverse, not necessarily strongly transverse.) Possibility B is that v is an extremum of all f_i. In this case we will use the assumptions on $\mathcal{F}$ to construct a simplex τ_{n+1} (strongly) transverse to $\mathcal{F}$.

A: v is not an extremum of f_0.

We claim that in this case there exists some vertex which is an extremum of the foliations on all of $\tau_0, \ldots, \tau_n$ (except for the simplex which does not contain this vertex, of course).
First note that there are $n+1$ simplices with (at least) $2n+2$ extrema, but only $n+2$ vertices. If v is not an extremum of all $\tau_0, \ldots, \tau_n$, then some $v_k \neq v$ must be the extremum of at least two different simplices. W.l.o.g. we may assume that v_0 is an extremum of τ_1, τ_2.

We claim that this implies that
- either v_0 **is an extremum of** $\tau_1, \ldots, \tau_n$
- or v_0 belongs to the same leaf as $v, v_3, \ldots, v_n$ and v_1, v_2 are extrema of all simplices (where they occur).

For $n = 2$, there is nothing to prove, so we will restrict to the assumption $n \geq 3$.
To prove the claim by contradiction, assume that v_0 is not an extremum of τ_3. Then we have:

$$f_3(v_k) > f_3(v_0) > f_3(v_l)$$

for some v_k, v_l. Since we are assuming $n \geq 3$, at least one of the following three cases holds:
- there exists $v_m \neq v_k$ with $f_3(v_k) \geq f_3(v_m) > f_3(v_0) > f_3(v_l)$,
- or there exists $v_m \neq v_l$ with $f_3(v_k) > f_3(v_0) > f_3(v_m) \geq f_3(v_l)$,
- or $f_3(v_k) > f_3(v_0) = \ldots = f_3(v_j) > f_3(v_l)$.

Note that the first two cases are equivalent after replacing f_3 with $-f_3$, so it suffices to consider one of them.
Consider case 1, i.e., $f_3(v_k) \geq f_3(v_m) > f_3(v_0) > f_3(v_l)$. If $l \neq 1$ and $m \neq 1$, then we have $v_m > v_0 > v_l$ on $\tau_1 \cap \tau_3$, i.e., v_0 is not an extremum of $\tau_1 \cap \tau_3$, contradicting the fact that it is an extremum of τ_1. If $l \neq 1$ and $k \neq 1$, then we have $v_k > v_0 > v_l$ on $\tau_1 \cap \tau_3$, i.e., v_0 is not an extremum of $\tau_1 \cap \tau_3$, contradicting the fact that it is an extremum of τ_1. Since k and m can not be both equal to 1, we have derived a contradiction if $l \neq 1$.
If $l = 1$ and $m \neq 2$, then $v_m > v_0 > v_1$ in $\tau_2 \cap \tau_3$, i.e., v_0 is not an extremum of $\tau_2 \cap \tau_3$, contradicting the fact that it is an extremum of τ_2. If $l = 1$ and $k \neq 2$, then $v_k > v_0 > v_1$ in $\tau_2 \cap \tau_3$, i.e., v_0 is not an extremum of $\tau_2 \cap \tau_3$, contradicting the fact that it is an extremum of τ_2. Thus we have derived a contradiction also if $l = 1$. This finishes case 1 (and the equivalent case 2.)

In case 3, we conclude that all vertices except v_k, v_l and possibly v_3 belong to the same leaf. If $k, l \neq 1$, we get a contradiction because v_0 would not be an extremum of $\tau_1 \cap \tau_3$. If $k, l \neq 2$, we get a contradiction because v_0 would not be an extremum of $\tau_2 \cap \tau_3$. There remains the cases $k = 1, l = 2$ resp. $l = 1, k = 2$ which need some more care. If $k = 1, l = 2$, then

$$f_3(v_1) > f_3(v_0) = f_3(v) > f_3(v_2).$$

We distinguish the two possibilities that v_3 belongs to the same leaf as $v, v_0, v_4, \ldots, v_n$ or not.
Consider first the case that v_3 does not belong to the same leaf as v and v_0. Consider the subtetrahedron T_0 of the simplex τ_0 which is spanned by the four vertices v_1, v_2, v_3, v. Since v is not an extremum of $T_0 \cap \tau_3$ but is an extremum of $T_0 \cap \tau_1$, and since v_1, v_2 are extrema of $T_0 \cap \tau_3$, we necessarily have either

$$f_0(v_1) > f_0(v) > f_0(v_3) > f_0(v_2)$$

or

$$f_0(v_1) > f_0(v) > f_0(v_2) > f_0(v_3)$$

(possibly after replacing f_0 with $-f_0$).
In both cases we have $v_1 > v > v_3$ on $T_0 \cap \tau_2$, hence necessarily

$$f_2(v_1) > f_2(v_0) = f_2(v) > f_2(v_3)$$

(possibly after replacing f_2 by $-f_2$) because v_0 and v belong to the same leaf. But this contradicts the assumption that v_0 is an extremum of f_2.
Thus we are left with the case that all vertices except v_1 and v_2 belong to the same leaf. Recall that v_1 and v_2 are isolated extrema of f_3. Let $m \neq 1, 2, 3$. Looking at $\tau_m \cap \tau_3$, we note that v_1 and v_2 are extrema of $\tau_m \cap \tau_3$. Since $f_m(v_3) = f_m(v_i)$ for $i \neq 1, 2$, this impllies

$$f_m(v_1) > f_m(v) = f_m(v_0) = f_m(v_4) = \ldots f_m(v_n) > f_m(v_2).$$

Finally, for $m = 1, 2$ we trivially have (possibly after replacing f_1 by $-f_1$ or f_2 by $-f_2$):

$$f_2(v_1) > f_2(v) = f_2(v_0) = f_2(v_3) = \ldots = f_2(v_n),$$

$$f_1(v) = f_1(v_0) = f_1(v_3) = \ldots = f_1(v_n) > f_1(v_2).$$

Hence, v_1 and v_2 are extrema of all τ_i.
If $k = 2, l = 1$, the same argument works.

So, we now can assume that some vertex, say v_0, is an extremum of all affine mappings (except f_0, where it does not occur). Replacing some f_i by $-f_i$ if necessary, we have that v_0 is a maximum of $f_1, \ldots, f_n$. We <u>claim</u> that this

implies that the
assumptions of corollary 7 are satisfied for $\partial_{n+1}\sigma_{deg}(\tau_0,\ldots,\tau_n)$.

First, since v_0 is a maximum of $f_1,\ldots,f_n$ we have that

$$f_j(u) > f_j(w) \Longleftrightarrow f_i(u) > f_i(w), f_j(u) = f_j(w) \Longleftrightarrow f_i(u) = f_i(w)$$

whenever $i,j \neq 0$ and $u,w \in \{v_1,\ldots,v_n,v\} - \{v_i,v_j\}$. Indeed, if not, we would get a contradiction by looking at the induced foliation of the triangle $\Delta(v_0uw)$ in $\tau_i \cap \tau_j$. (Hence we have checked the assumptions a,b,c of corollary 1 except for f_0.)

In particular, there exists some $u \in \{v_1,\ldots,v_n,v\}$ such that $f_i(u) \geq f_i(w)$ for all $1 \leq i \leq n$ and $w \in \{v_1,\ldots,v_n,v\} - \{u,v_i\}$. This vertex u is then a maximum of $\tau_0 \cap \tau_i$ for all $1 \leq i \leq n$ except possibly for $\tau_0 \cap \tau_j$ if $u = v_j$.

For the rest of the proof we have to distinguish the cases $n \geq 3$ and $n = 2$. Consider $n \geq 3$. In this case, u being a maximum of $\tau_0 \cap \tau_i$ for all $1 \leq i \leq n$ implies that u is a maximum of τ_0. Then we may argue as before: $f_0(w) > f_0(w')$ is equivalent to $f_i(w) > f_i(w')$ for any i because, if not, we would get a contradiction in $\Delta(uww') \subset \tau_0 \cap \tau_i$. Hence we get conditions a,b,d for corollary 7.

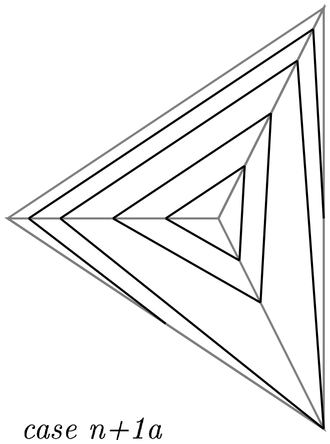
It remains to check condition c of corollary 7.
Trivially, $f_i(v_p) < f_i(v_r)$ and $f_j(v_r) < f_j(v_s)$ together imply $f_k(v_j) > f_k(v_p)$ except possibly if $j = p$ or $i = s$. If $j = p$, then we have to check that $f_i(v_j) < f_i(v_r), f_j(v_r) < f_j(v_s)$ and $f_k(v_s) < f_k(v_j)$ can not happen simultaneously. However, these three inequalities would lead, looking at the triangle $\Delta(v_jv_rv_s)$ with the induced foliation, to the contradiction $v_j < v_r < v_s < v_j$. To be precise, choose some $q \neq i,j,r$ (this is possible if $n \geq 3$), then we have $f_q(v_j) < f_q(v_r) < f_q(v_s) < f_q(v_j)$, getting a contradiction.
If $i = s$, an analogous argument works. This finishes the proof for $n \geq 3$.

We are left with the case $n = 2$.
The conditions a,b,c of corollary 7 are empty for $n = 2$, we want to check condition d.
Case 1: $u = v$. (This will be the step in the proof which uses that we are working with strongly transverse simplices rather then just transverse ones.) We claim that v is an extremum of τ_0. If not, then either the leaf through v would intersect K in a trivalent graph (if τ_0 is nondegenerate) as in the picture of *case n* above, which is of course impossible for the leaf being a manifold, or τ_0 would be degenerate. We claim that in the latter case, τ_0 could not be strongly transverse. Indeed, looking at a small foliation chart around v, we observe that the leaf space of this chart (which is just an open intervall) is decomposed by the leaf through v into two components, and that the image of τ_0 in the leaf

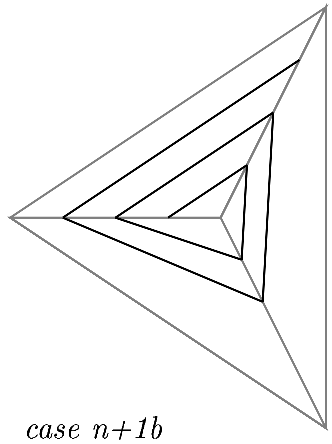
space is completely contained in the closure of one of these components. This means that the map from τ_0 to the leaf space is not a submersion at $\tau_0^{-1}(v)$, contradicting the strong transversality of τ_0. Thus, v is an extremum of τ_0. Together with $f_1(v_0) \geq f_1(v) \geq f_1(v_2)$ and $f_2(v_0) \geq f_2(v) \geq f_2(v_1)$ this implies that condition d holds true.
Case 2: $u = v_1$. This means $f_2(v_0) \geq f_2(v_1) \geq f_2(v)$. We know that v_0 is a maximum of f_1. Thus one possibilty for f_1 is $f_1(v_0) \geq f_1(v) \geq f_1(v_2)$. Then, if v were an extremum of f_0, we would be in *case n* pictured at the beginning of the proof (with degenerate τ_1). Clearly, τ_1 would not be strongly transverse, giving a contradiction. Thus, v must not be an extremum of f_0, from which condition d of corollary 7 easily follows. The other possibility for f_1 is $f_1(v_0) \geq f_1(v_2) \geq f_1(v)$. But then it is immediate that τ_0 is not strongly transverse, hence this possibility can not happen.
The case $u = v_2$ works the same way.

We have checked the assumptions of corollary 7. Thus we may apply corollary 7 to get an affine mapping $f : \partial_{n+1}\sigma_{deg}(\tau_0, \ldots, \tau_n) \to \mathbb{R}$. This shows transversality. (We note that we have used the strong transversality of $\tau_0, \ldots, \tau_n$ only in one step, namely in the special discussion to handle the case $n = 2$. Thus, for $n \geq 3$, we get part a) of theorem 14.)

We continue with the proof for $K_{st}^{\mathcal{F}}$, i.e. we wish to prove that $\partial_{n+1}\sigma_{deg}(\tau_0, \ldots, \tau_n)$ is even strongly transverse. We assume that $\tau_0, \ldots, \tau_n$ are strongly transverse. Thus, the only points where strong transversality of $\partial_{n+1}\sigma_{deg}(\tau_0, \ldots, \tau_n)$ may fail are points in some intersection $\tau_i \cap \tau_j$, where necessarily this intersection must belong to the same leaf. However, looking at the foliation chart in a neighborhood of v we would get that the leaf space is decomposed by the leaf through v into two components,and that any other simplex as τ_i and τ_j can therefore not be strongly transverse at v, giving a contradiction. (One may look at the picture for *case n* for understanding.) Thus, $\partial_{n+1}\sigma_{deg}(\tau_0, \ldots, \tau_n)$ is automatically strongly transverse.



case n+1a



case n+1b

B: v is an extremum of $f_0, \dots, f_n$.

We assume that $f_i : \tau_i \to \mathbb{R}, i = 0, \dots, n$ are affine mappings such that the common vertex of the n+1 n-simplices is an isolated extremum of $f_0, \dots, f_n$, as pictured above. (We will see that a picture as in case n+1b actually can not happen.) Since $K := \tau_0 \cup \dots \cup \tau_n$ is contractible, we may lift it to the universal cover $\widetilde{M}$, where it inherits the same foliation. Denote the image of the lift by $\widetilde{K}$.
Let $\widetilde{\mathcal{F}}$ be the pull-back of $\mathcal{F}$ to $\widetilde{M}$. Clearly, $\pi_1\left(\widetilde{M}\right) \to \pi_1\left(\widetilde{M}/\widetilde{\mathcal{F}}\right)$ is surjective (any loop can be lifted). In particular, $\pi_1\left(\widetilde{M}/\widetilde{\mathcal{F}}\right) = 0$, i.e., the leaf space of $\widetilde{\mathcal{F}}$ is simply connected. This implies, since $\dim\left(\widetilde{M}/\widetilde{\mathcal{F}}\right) = 1$, that every point in the leaf space of $\widetilde{\mathcal{F}}$ separates, hence every leaf $\widetilde{F}$ of $\widetilde{\mathcal{F}}$ separates.
This means in particular that no connected component of $\widetilde{K} \cap \widetilde{F}$ can intersect an edge of some τ_i twice, that is, a situation as in the picture 'case n+1b' below does not happen. Namely, if $\widetilde{K} \cap \widetilde{F}$ intersected an edge twice, this edge would connect points in the two path-components of $\widetilde{M} - \widetilde{F}$, giving the contradiction that $\widetilde{M} - \widetilde{F}$ were path-connected.

As a consequence we have that the intersections $\widetilde{F} \cap \tau_i$, which are n+1 $n-1$-dimensional simplices, fit together (as in the picture of case n+1a) in a standard way, that is:
- close to the common vertex v they fit together to $n-1$-dimensional PL-spheres,
- there may be leaves which do not intersect all simplices, but they fit together to proper subsets of $n-1$-dimensional PL-spheres.
In other words, the foliation of K is precisely the foliation that one would obtain if K were to be embedded into the $\mathbb{R}^n$, foliated by horizontal planes, such that the embedding is linear on each τ_i.

We are assuming that $\tilde{\pi} : \widetilde{M} \to \widetilde{M}/\widetilde{\mathcal{F}}$ is a locally trivial fibration. $\widetilde{M}/\widetilde{\mathcal{F}}$ is a simply connected 1-manifold, hence contractible. Thus $\tilde{\pi}$ must be a trivial fibration, i.e.,

$$\widetilde{M} \simeq \left(\widetilde{M}/\tilde{\mathcal{F}}\right) \times \left(\tilde{F}\right)$$

is a product. This product structure makes the construction of the n-simplex with transversal $n+1$-th boundary face kind of obvious. Namely, using this product structure, the foliation of $\tau_0 \cup \ldots \cup \tau_n$ corresponds to a continuous family of PL-$n-1$-spheres, at least until the first vertex $\neq v$ is reached, in F. After the first vertex the family continues as a family of proper subsets of spheres. The picture is, as mentioned before, the same that one would get by embedding K piecewise linearly into the standard foliation.
Observe that all spheres represent trivial elements in $\pi_{n-1}\left(\tilde{F}\right)$, since the starting point of the continuous family has been a sphere mapped to a point. This means that this family of spheres can be extended to a continuous family of n-balls in the respective leaves.

If $v_0, \ldots, v_n$ belong to the same leaf F_0, then this continuous family of balls allows us to define a continuous mapping σ from the standard $n+1$-simplex to $\widetilde{M}$ such that $\partial_{n+1}\sigma$ is contained in the leaf F_0, thus is strongly transverse.

If $v_0, \ldots, v_n$ do not belong to the same leaf, then we get a continuous family of balls until the first vertex $\neq v$, say v_0, is reached. Let F_0 be the leaf containing v_0 and assume $v_n \notin F_0$. We may use this family of balls to homotope $\tau_0, \ldots, \tau_n$ by homotoping v (and leaving $\partial_n\tau_0, \ldots, \partial_n\tau_n$ fixed) until $v \in F_0$, preserving (strong) transversality.

After this homotopy we are in a situation that v_0 is a common extremum for the induced foliations on the homotoped simplices $\hat{\tau}_0, \ldots, \hat{\tau}_n$. We have shown in the course of the proof of case A that this implies that the assumptions of corollary 7 are satisfied for $\tau_{n+1} := \partial_{n+1}\sigma_{deg}\left(\hat{\tau}_0, \ldots, \hat{\tau}_n\right)$. Hence, by corollary 7, τ_{n+1} is transverse and, as we have seen in the proof of case A, then automatically strongly transverse since $\hat{\tau}_0, \ldots, \hat{\tau}_n$ are strongly transverse. Since $\partial_i\tau_{n+1} = \partial_n\hat{\tau}_i = \partial_n\tau_i$, this finishes the proof.

□

We are now ready to prove lemma 13.

PROOF. First note that b) holds for any foliation $\mathcal{F}$, as has been observed at the beginning of section 3.
We discuss a) and c). In the course of the proof, we will use some deep results from foliation theory. Assume that $(M, \mathcal{F})$ is a foliated 3-manifold. One

distinguishes the case that there exists a leaf homeomorphic to the 2 -sphere or not. If **some leaf is homeomorphic to** $\mathbb{S}^2$, the Reeb stability theorem implies that the foliation is a locally trivial fibration with fiber $\mathbb{S}^2$. In this case, arguments analogously to the proof of theorem 14 imply that the weak Kan property in degree 2 is fulfilled for $K_{st}^{\mathcal{F}}$. (The proof of case A did not use any assumptions on $\mathcal{F}$, and in the discussion of case B, it suffices to use that the fibers are simply connected to fill 1-spheres by 2-balls, thus getting the weak Kan property in degree 2.) To prove the weak Kan property in degree 3, for $\mathbb{S}^2$-bundles over $\mathbb{S}^1$, one observes that, even though the fibers are not 2-connected, the argument in possiblity B still works. Namely, working in the universal covering $\mathbb{S}^2 \times \mathbb{R}$, one observes that, after an identification of all fibers with a fixed fiber, we have a continuous family of 2-spheres terminating in a constant 2-sphere (coming from the lift of the fiber through v), in particular all 2-spheres arising from the intersection are 0-homotopic in their respective fibers, thus can be fibre-wise filled with 3-balls. This allows again to define a transversal 4-simplex with transversal boundary faces.

If there is **no leaf homeomorphic to** $\mathbb{S}^2$ and the foliation has **no Reeb component** (i.e., no compressible torus leaf), it follows from Palmeira's work that the induced foliation of the universal cover $\left(\widetilde{M}, \widetilde{\mathcal{F}}\right)$ is a foliation of $\widetilde{M} = \mathbb{R}^3$ by leaves homeomorphic to $\mathbb{R}^2$ such that each leaf separates $\mathbb{R}^3$ into two connected components. (Namely, if there is no Reeb component, then all leaves F are π_1-injective, by Novikov's theorem. Hence $\widetilde{F} \simeq \mathbb{R}^2$, i.e. $\widetilde{\mathcal{F}}$ is a foliation by planes. Then apply [15], cor. 3.) Then we may apply theorem 14 to get the weak Kan property in degrees 2 (for $K_{st}^{\mathcal{F}}$) and 3 (for $K^{\mathcal{F}}$ and $K_{st}^{\mathcal{F}}$).

It remains to discuss the case that $\mathcal{F}$ **has Reeb components** (which will actually not be used in the paper) and to show that $\mathcal{F}$ nevertheless satisfies the weak Kan property in degree 3, but not in degree 2.

We start with discussing degree 2. We want to show that existence of a Reeb component destroys the weak Kan property (for the set of strongly transversal simplices) in degree 2. Consider the Reeb foliation of $\mathbb{D}^2 \times \mathbb{S}^1$ and let τ_0, τ_1, τ_2 be 3 triangles with a common vertex in $(0, \alpha) \in \mathbb{D}^2 \times \mathbb{S}^1$ and satisfying $\partial_0\tau_0 = \partial_0\tau_1, \partial_0\tau_2 = \partial_1\tau_0, \partial_1\tau_1 = \partial_1\tau_2$, such that $\partial_2\tau_0, \partial_2\tau_1, \partial_2\tau_2$ lie on the boundary torus $\partial\left(\mathbb{D}^2 \times \mathbb{S}^1\right)$ and their concatenation is the generator of

$$ker\left(\pi_1\left(\partial\left(\mathbb{D}^2 \times \mathbb{S}^1\right)\right) \rightarrow \pi_1\left(\mathbb{D}^2 \times \mathbb{S}^1\right)\right) \simeq \mathbb{Z}.$$

These 2-simplices may chosen to be strongly transverse to the Reeb foliation. For any 3-simplex T with $\partial_i T = \tau_i$ for $i = 0, 1, 2$ we have that $\tau_3 := \partial_3 T$ is a 2-simplex with $\partial_0\tau_3 = \partial_2\tau_0, \partial_1\tau_3 = \partial_2\tau_1, \partial_2\tau_3 = \partial_2\tau_2$. In particular, the restriction of $\mathcal{F}$ to τ_3 contains $\partial\tau_3$ as a leaf, because $\partial\left(\mathbb{D}^2 \times \mathbb{S}^1\right)$ is a leaf. I.e., $\mathcal{F}\mid_{\tau_3}$ is a foliation of a topological disk such that the boundary is a leaf. Any such foliation of a topological disk must have a singularity (because the Euler characteristic does not vanish), in particular can not be topologically conjugate to an affine foliation of the 2-simplex. Thus, $\partial_3 T$ can not be transverse, except if it were contained in the torus leaf $\partial\left(\mathbb{D}^2 \times \mathbb{S}^1\right)$. But the latter is impossible because, by construction, the union $im\left(\tau_0\right) \cup im\left(\tau_1\right) \cup im\left(\tau_2\right)$ represents a nontrivial element in

$$\pi_2\left(\mathbb{D}^2 \times \mathbb{S}^1, \partial\left(\mathbb{D}^2 \times \mathbb{S}^1\right)\right) \simeq \mathbb{Z},$$

so, if τ_3 were contained in $\partial\left(\mathbb{D}^2 \times \mathbb{S}^1\right)$, there could be no 3-simplex T with $\partial_i T = \tau_i$ for $i = 0, 1, 2, 3$.

Finally we show that any foliation of a 3-manifold satisfies the weak Kan property in degree 3. We have already seen this for foliations without Reeb components.

Let $R \neq \emptyset$ be the union of the Reeb components. Let transversal 3-simplices $\tau_1, \tau_2, \tau_3, \tau_4$ be given and let $K = im\left(\tau_1\right) \cup im\left(\tau_2\right) \cup im\left(\tau_3\right) \cup im\left(\tau_4\right)$. Let v_0 be the common vertex of $\tau_1, \ldots, \tau_4$ and let v_i be the vertex not contained in τ_i, for $i = 1, \ldots, 4$. We distinguish the following 4 cases:
a) $v_0 \in int\left(R\right), v_1, \ldots, v_4 \notin int\left(R\right)$,
b) $v_0 \notin int\left(R\right), v_1, \ldots, v_4 \in int\left(R\right)$,
c) $v_0 \notin int\left(R\right), v_i \notin int\left(R\right)$ for some $i \in \{1, \ldots, 4\}$,
d) $v_0 \in int\left(R\right), v_i \in int\left(R\right)$ for some $i \in \{1, \ldots, 4\}$.

Case a): We observe that in this case $\partial K := \partial_0\tau_1 \cup \partial_0\tau_2 \cup \partial_0\tau_3 \cup \partial_0\tau_4$ can not intersect $int\left(R\right)$. Indeed, if it did, then some edge $\left[v_i, v_j\right]$ would have to intersect $int\left(R\right)$ (because, if a leaf intersects a transversal simplex, then it must intersect its 1-skeleton). Hence there would be some subintervall $\left[w_i, w_j\right] \subset \left[v_i, v_j\right]$ contained in R, such that $w_i, w_j \in \partial R$. We note that the leaf space of the Reeb foliation of any connected component of $int\left(R\right)$ is homeomorphic to $\mathbb{R}$. Hence the image of $\left[w_i, w_j\right]$ in the leaf space has some extremal point. But at this extremal point, $\left[w_i, w_j\right]$ would not be a submersion, henceforth not stronly transverse.

Since ∂K does not intersect $int\left(R\right)$ and since

$$\pi_3\left(R, \partial R\right) = 0$$

we may homotope K off $int\left(R\right)$, leaving ∂K fixed. This homotopy can be made simplicial. Let $\hat{\tau}_0 \cup \hat{\tau}_1 \cup \hat{\tau}_2 \cup \hat{\tau}_3$ be the result of the homotopy. Since it is off $int\left(R\right)$ (and we have just proved that the weak Kan property holds in degree 3 for foliations without Reeb components), we do get a 4-simplex $\hat{K} \in S_*\left(M - int\left(R\right)\right)$ such that $\partial_i \hat{K} = \hat{\tau}_i$ for $i = 0, 1, 2, 3$ and such that $\partial_4 \hat{K}$ is transverse to $\mathcal{F}$. Since the Kan property is, of course, true for $S_*\left(M\right)$, we can use $\hat{K}$ and the simplicial homotopy to produce, by successive application of the

Kan property, a (not necessarily transversal) 4-simplex K with $\partial_4 K = \partial_4 \hat{K}$ and $\partial_i K = \tau_i$ for $i = 0, 1, 2, 3$. Since $\partial_4 \hat{K}$ is transverse, this finishes the proof in case a).

Case b): This is similar to case a). We observe that ∂K can not intersect $N := M - int(R)$. Indeed, if it did, we would again find some subintervall $[w_i, w_j] \subset [v_i, v_j]$ contained in N with $v_i, v_j \in \partial N$. Let $\mathcal{G} := \mathcal{F} |_N$ and $\widetilde{\mathcal{G}}$ its pull-back to the universal covering $\widetilde{N}$. We know that the leaf space of $\widetilde{\mathcal{G}}$ is simply connected. Hence the image of $[w_i, w_j]$ in the leaf space would have some extremal point, giving a contradiction.

It is well-known that the existence of a Reebless foliation implies that N is irreducible and has infinite fundamental group. Hence $\pi_3(N) = 0$. Since ∂N consists of tori, this implies $\pi_3(N, \partial N) = 0$. Thus we can homotope K into $int(R)$ and then apply the same argument as in case a). (Note that the restriction of $\mathcal{F}$ to $int(R)$ is a trivial foliation by planes, for which the weak Kan property in degrees ≥ 2 holds.)

Case c): If v_0 is not a common extremum of all $\mathcal{F} |_{\tau_i}$, we are in the situation of case A in the proof of theorem 14. We know then from the proof of that case that, without any assumptions on $\mathcal{F}$, there exists a 4-simplex K with $\partial_i K = \tau_i, i = 1, 2, 3, 4$, such that $\partial_0 K$ is transverse to $\mathcal{F}$.

So we are left to consider the case that v_0 is a common extremum of $\mathcal{F} |_{\tau_i}, i = 1, 2, 3, 4$. We claim that we can homotope K to some $\hat{K} = \hat{\tau}_1 \cup \hat{\tau}_2 \cup \hat{\tau}_3 \cup \hat{\tau}_4$, leaving ∂K fixed, such that v_0 is not a common extremum of $\mathcal{F} |_{\hat{\tau}_i}$ for all $i = 1, 2, 3, 4$. After having accomplished this homotopy, we can apply the above argument to $\hat{K}$.

The homotopy can be accomplished by an argument similar to that in case B in the proof of theorem 14. Let $N = M - int(R)$. $\mathcal{G} = \mathcal{F} |_N$ is a Reebless foliation, hence we know from the proof of case B that $\widetilde{N} \simeq \widetilde{N}/\widetilde{\mathcal{G}} \times \widetilde{G}$ for a leaf $\widetilde{G}$ of $\widetilde{\mathcal{G}}$. Let v_i be the vertex contained in N, which exists by assumption of case c). We remark that v_i must belong to the same connected component of N as v_0. (Otherwise the edge $[v_0, v_i]$ would enter and leave some Reeb component, hence could not be transverse.) Now we can copy the argument which we used in the proof of case B in theorem 14. Namely, we can again use the product structure, to get a continuous family of 3-balls until the first vertex $\neq v_0$ is reached. (We may assume w.l.o.g. that v_i is this first vertex.) Let F_0 be the leaf containing v_i. We may use the family of balls to homotope $\tau_1, \ldots, \tau_4$ by homotoping v_0 (and leaving $\partial_0 \tau_1, \ldots, \partial_0 \tau_4$ fixed) until $v_0 \in F_0$, preserving strong transversality. After this homotopy we are in a situation that v_0 is an extremum for the induced foliations on the homotoped simplices $\hat{\tau}_1, \ldots, \hat{\tau}_4$. We have shown in the course of the proof of case A of theorem 14 that this implies that the assumptions of corollary 7 are satisfied for $\tau_0 := \partial_0 \sigma_{deg}(\hat{\tau}_1, \ldots, \hat{\tau}_4)$.

Hence τ_0 is strongly transverse. Since $\partial_i\tau_0 = \partial_0\hat{\tau}_i = \partial_0\tau_i$, this finishes the proof.

Case d): If v_0 and v_i belong to the same connected component of R, then the argument is literally the same as in case c), with the roles of N and R interchanged. If $v_0 \in R_1$ and $v_i \in R_2$ for distinct connected components R_1, R_2 of R, then we can apply the same argument as in case a) to homotope K off $int\,(R_1)$. If there is some $v_j \notin int\,(R)$, then we are, after this homotopy, in the situation of case c) and can finish the proof. So it remains to discuss the discuss the case that $v_1, v_2, v_3, v_4 \in int\,(R)$.

We assume that v_0 is a common extremum of $\tau_1, \ldots, \tau_4$. (Else we are done by the proof of case A in theorem 14.) We can apply the argument of case c) to homotope v_0, leaving $\partial_0\tau_1, \ldots, \partial_0\tau_4$ fixed and preserving strong transversality, until v_0 is contained in the boundary of a Reeb component R_j with $v_j \in R_j$ for some $j \in \{1,2,3,4\}$. Since v_0 is a common extremum, it follows that all four points v_1, v_2, v_3, v_4 must belong to this Reeb component R_j. By the same argument as in case b), this implies that no edge $[v_i, v_l]$ can leave R_j, for $i, l \in \{1, \ldots, 4\}$. In particular, ∂R_j does not intersect any edge $[v_i, v_l]$ and hence, by transversality, no 2-simplex $(v_i v_l v_k)$ with $i, l, k \in \{1,2,3,4\}$. That is, these four 2-simplices are contained in $\mathrm{int}(R_j)$. Since these four 2-simplices are strongly transverse and the foliation of $int\,(R_j)$ is a trivial product foliation, we can find a strongly transversal 3-simplex, whose boundary faces are these four 2-simplices. This finishes the proof.

$\square$

Generalizations. In [1], the notion of foliated Gromov norm has been generalized to laminations and, more generally, to group actions on order trees. We briefly describe the analogous generalization of the transversal length on the fundamental group.

Let M be a manifold and $\Gamma = \pi_1 M$ its fundamental group, which acts on the universal covering $\widetilde{M}$. Let Γ act on an order tree T (see [4] for the definition of order tree) and assume that there is a Γ-equivariant map $\phi : \widetilde{M} \to T$. (Such an ϕ exists naturally for any lamination $\mathcal{F}$ of M.) According to [1], def.4.2.1, a singular i-simplex $\sigma : \Delta^i \to M$ is transverse if, for any lift $\widetilde{\sigma} : \Delta^i \to \widetilde{M}$, the image of $\phi\widetilde{\sigma}$ is a totally ordered segment of T. We say that a singular i-simplex is strongly transverse if it is transverse and the induced mapping $\phi\tilde{\sigma} : \Delta^i \to T$ is a submersion, for any lift $\tilde{\sigma}$ of σ to $\widetilde{M}$.
Then one can again define the notions of transversal length on the fundamental group and foliated Gromov norm.
It is easy to see that the proof of theorem 10 also works in this setting. However, to get useful information from the generalized theorem 10, it would be necessary to know under what conditions the simplices strongly transverse to

a given lamination (resp. group action on an order tree) satisfy the weak Kan property in degrees ≥ 2.

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