

Fundamental classes and hyperbolic volume

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Volume and Topology

- G/K symmetric space of noncompact type
e.g. $H^3 = SL(2, \mathbf{C})/SU(2)$.
- $\Gamma \subset G$ lattice

$$M^n = \Gamma \backslash G/K$$

Volume is a topological invariant of locally symmetric spaces:

- **Mostow Rigidity** $n = \dim(M) \geq 3$:
homeomorphism $\Rightarrow$ isometry
- **Chern-Gauß-Bonnet** $n = \dim(M)$ even: $Vol(M) \sim \chi(M)$

Topological definitions of volume?

Change of volume under topological operations, Cut and Paste?

Simplicial Volume

M^n compact, orientable manifold,
 $[M, \partial M] \in H_n(M, \partial M; \mathbf{R})$ its fundamental class

Definition

$$\| M \| = \inf \left\{ \sum_{i=1}^r |a_i| : \sum_{i=1}^r a_i \sigma_i \text{ represents } [M, \partial M] \right\}.$$

Example

$$\| S^n \| = 0, \| T^n \| = 0.$$

Example

For all G/K exists $c_{G/K}$ such that $M = \Gamma \backslash G/K$ implies
 $\text{Vol}(M) = c_{G/K} \| M \|.$

$c_{H^n} = V_n :=$ volume of regular ideal simplex in H^n .

For a codimension 1 submanifold $A \subset M$ we denote M_A the manifold obtained by cutting M along A .

Theorem (Gromov 1982)

If $\pi_1 A$ is amenable and injects into $\pi_1 M$, then

$$\| M_A, \partial M_A \| = \| M, \partial M \| .$$

For example abelian groups are amenable.

Not applicable to hyperbolic 3-manifolds: no incompressible tori.

Observation

If $\Sigma \subset M$ is totally geodesic and M^τ obtained from M by cutting and pasting via an isometry $\tau : \Sigma \rightarrow \Sigma$, then $\text{vol}(M^\tau) = \text{vol}(M)$.

Obvious geometric fact, since $\text{Vol}(M) = \text{Vol}(M - \Sigma)$.

Topological proofs? No proof via simplicial volume, but simple proof via group homology (in the setting of generalized mutations).

Theorem (Jungreis 1997 for M closed, K. 2003 for M cusped)

If $\Sigma \subset M$ is totally geodesic in a finite-volume hyperbolic n -manifold, $n \geq 3$, then

$$\| M_\Sigma, \partial M_\Sigma \| > \| M, \partial M \| .$$

Theorem (Frigerio-Pagliantini 2010)

Inequality is sharp: $\frac{\| M_\Sigma, \partial M_\Sigma \|}{\| M, \partial M \|} - 1$ is bounded in terms of $\frac{\text{Vol}(\Sigma)}{\text{Vol}(M)}$.

Group Homology and volume cocycle

- $\Gamma \backslash G / K$ is an Eilenberg-MacLane space $B\Gamma$.

$$j : M = \Gamma \backslash G / K \simeq B\Gamma \rightarrow BG$$

M closed: $j_* [M] \in H_n(BG; \mathbf{R})$ determines $\text{vol}(M)$.

- Let $v_n \in H_c^n(G)$ be the volume class, represented by

$$v_n(g_1, \dots, g_n) = \text{vol}(\text{str}(\tilde{x}, g_1\tilde{x}, \dots, g_n\tilde{x})).$$

Then $\langle v_n, j_* [M] \rangle = \text{vol}(M)$ if M closed.

Remark: $H_3(SL(2, \mathbf{C})) / \text{Torsion} \cong \mathcal{B}(\mathbf{C})$.

If M is a closed hyperbolic 3-manifold, then the isomorphism maps

$j_* [M]$ to the Bloch invariant $\beta(M)$. (Neumann-Yang)

$\beta(M)$ can be computed using ideal triangulations and SnapPy.

Cut and Paste III: Generalized Mutation

Recall: $\Sigma \subset M$ codim 1 submanifold, $\tau : \Sigma \rightarrow \Sigma$ diffeo, then M^τ is the manifold obtained by cutting M along Σ and regluing via τ .

Definition

$M_1 = \Gamma_1 \backslash G/K$, $M_2 = \Gamma_2 \backslash G/K$ locally symmetric spaces,
Assume $\Sigma \subset M_1$, and $M_2 = M_1^\tau$ for some finite order diffeomorphism $\tau : \Sigma \rightarrow \Sigma$.

Then M_2 is obtained from M_1 by generalized mutation, if there is some $A \in G$ with

$$\tau_*(\sigma) = A\sigma A^{-1}$$

for all $\sigma \in \pi_1 \Sigma$.

Toy Example: $H \subset G$, $\Sigma = \Gamma \backslash H / (K \cap H)$ totally geodesic,
 $\tau : \Sigma \rightarrow \Sigma$ isometry.

Then τ_* is realised by conjugation with some $A \in H$.

Example

Mutation of knots:

If $M = S^3 - K$ is a knot complement and K' the mutant of K , then $S^3 - K' = M^\tau$ for a 4-punctured sphere $\Sigma \subset M$ and a hyperelliptic involution $\tau : \Sigma \rightarrow \Sigma$.

Ruberman (1989) proved:

- If $M = S^3 - K$ is a hyperbolic knot complement, then τ is a generalized mutation, M^τ is hyperbolic and $\text{vol}(M^\tau) = \text{vol}(M)$.

Proof uses heavy analytical machinery (existence and uniqueness results for minimal surfaces).

Remark: Hyperbolicity of M^τ can be proved topologically using the Maskit combination theorem.

K.: alternative proof of $\text{vol}(M^\tau) = \text{vol}(M)$ using group homology; proof extends to locally symmetric spaces.

Theorem (K. 2011)

If M_1 and M_2 are locally symmetric spaces,

- either closed,

- or cusped and $\text{rank}(G/K) = 1$,

*such that $M_2 = M_1^\tau$ is a **generalized mutation***

(for some $\Sigma \subset M$ and some finite order diffeomorphism

$\tau : \Sigma \rightarrow \Sigma$),

then

$$\text{vol}(M^\tau) = \text{vol}(M).$$

Cut and Paste VI: Proof of Volume Equality - closed case

- Volume of M is determined by $j_* [M] \in H_n(BG; \mathbf{Q})$, therefore it suffices to prove

$$j_* [M] = j_* [M^\tau] \in H_n(BG; \mathbf{Q}).$$

- Consider the space X obtained by identifying M and M^τ along $M - nbh(\Sigma)$, it contains an embedded copy of the mapping torus T^τ . Clearly

$$[M] - [M^\tau] = [T^\tau] \in H_n(X).$$

- Definition of generalized mutation implies:
 $\pi_1 M \rightarrow G$ extends to a well-defined homomorphism
 $\pi_1 X \rightarrow G$.

$$j_* [M] - j_* [M^\tau] = j_* [T^\tau] \in H_n(BG; \mathbf{Q})$$

- Using $\tau^m = id$ prove $j_* [T^\tau] = 0 \in H_n(BG; \mathbf{Q})$.
(Not true with $\mathbf{Z}$ -coefficients, except for classical mutation by Neumann.)

Manifolds with boundary

M^n manifold with boundary, $\text{int}(M) \cong \Gamma \backslash G/K$.

- G/K symmetric space, define

$$BG^{comp} = BG \bigcup_{c \in \partial_\infty G/K} \text{Cone}(BG),$$

the volume cocycle $v_n \in H^n(BG)$ extends to $\overline{v}_n \in H^n(BG^{comp})$.

If $\text{rank}(G/K) = 1$ and $\text{int}(M) \cong \Gamma \backslash G/K$ for a lattice $\Gamma \subset G$, then

$$j_*[M, \partial M] \in H_n(BG^{comp})$$

can be defined such that

$$\langle \overline{v}_n, j_*[M, \partial M] \rangle = \text{vol}(M).$$

In particular, $\text{vol}(M)$ depends only on $j_*[M, \partial M] \in H_n(BG^{comp})$.

Cut and Paste VII: Proof of volume equality - cusped case

Recall: X obtained by identifying M and M^τ along $M - nbh(\Sigma)$, contains embedded copy of the mapping torus T^τ . For generalized mutations $\rho : \pi_1 M \rightarrow G$ extends to $\rho^X : \pi_1 X \rightarrow G$.

Let $t \in \pi_1 \partial X$ be the extending element in $\pi_1 \partial T^\tau \subset \pi_1 \partial X$.

$\rho^X(t) \in G$ is not parabolic (therefore $B\rho^X$ does not extend to $B(\pi_1 X)^{comp}$), but elliptic: t lifts to 1 in some finite covering $\hat{X}$.

Let $\hat{M}, \hat{M}^\tau$ be the preimages of M, M^τ in $\hat{X}$. Prove

$$j_* [\hat{M}, \partial \hat{M}] = j_* [\hat{M}^\tau, \partial \hat{M}^\tau] \in H_n(BG^{comp}; \mathbf{Q})$$

$$\implies \text{vol}(\hat{M}) = \text{vol}(\hat{M}^\tau) \implies \text{vol}(M) = \text{vol}(M^\tau).$$

$M = \Gamma \backslash G / K$ closed

$$\rho : G \rightarrow GL(N, \mathbf{C})$$

Weil rigidity: w.l.o.g. $\rho(\Gamma) \subset GL(N, \overline{\mathbf{Q}})$, hence

$$(B\rho)_* j_* [M] \in H_n(BGL(N, \overline{\mathbf{Q}})) .$$

$(B\rho)_* j_* [M]$ contains more information than the volume, for example the volume of Galois conjugates (if they are discrete).

Volume cocycle, representations and Borel class

$$\rho : G \rightarrow GL(N, \mathbf{C})$$

$$(B\rho)_* j_* [M] \in H_n(BGL(N, \overline{\mathbf{Q}})) .$$

$H_c^*(GL(N, \mathbf{C})) = \Lambda(b_1, b_3, \dots, b_{2N-1})$ generated by Borel classes

$$\rho^* b_n = c_\rho v_n, c_\rho \in \mathbf{R}$$

Hence, if M^n closed, then

$$\langle b_n, (B\rho)_* j_* [M] \rangle = c_\rho \text{vol}(M) .$$

K-theoretic invariant:

There exists a projection $H_n(BGL(N, \overline{\mathbf{Q}}); \mathbf{Q}) \rightarrow K_n(\overline{\mathbf{Q}}) \otimes \mathbf{Q}$ preserving values of Borel classes. The image of $(B\rho)_* j_* [M]$ under this projection is the Goncharov invariant $\gamma(M)$.

Nontriviality

The following is a complete list of irreducible symmetric spaces G/K of noncompact type and of fundamental representations $\rho : G \rightarrow GL(N, \mathbb{C})$ with $\rho^* b_n \neq 0$ for $n = \dim(G/K)$.

Symmetric Space	Representation
$SL_l(\mathbb{R})/SO_l, l \equiv 0, 3, 4, 7 \pmod{8}$	any fundamental representation
$SL_l(\mathbb{C})/SU_l, l \equiv 0 \pmod{2}$	any fundamental representation
$SL_{2l}(\mathbb{H})/Sp_l, l \equiv 0 \pmod{2}$	any fundamental representation
$Spin_{p,q}/(Spin_p \times Spin_q),$ $p, q \equiv 1 \pmod{2}, p \not\equiv q \pmod{4}$	any fundamental representation
$Spin_{p,q}/(Spin_p \times Spin_q),$ $p, q \equiv 1 \pmod{2}, p \equiv q \pmod{4}$	positive and negative half-spinor representations
$SO_l(\mathbb{C})/SO_l, l \equiv 3 \pmod{4}$	any fundamental representation
$Sp_l(\mathbb{C})/Sp_l, l \equiv 1 \pmod{4}$	any fundamental representation
$E_7(\mathbb{C})/E_7$	any fundamental representation

In particular, $\gamma(M) \neq 0$ whenever $M = \Gamma \backslash G/K$ is closed.

K-theoretic invariant - the cusped case

Assume again $\text{rank}(G/K) = 1$ and $\text{int}(M) \cong \Gamma \backslash G/K$ for a lattice $\Gamma \subset G$, thus $j_*[M, \partial M] \in H_n(BG^{\text{comp}})$ is defined.

Lemma (K. 2009)

If $\rho : G \rightarrow SL(N, \mathbf{C})$, then

$$(B\rho)_* j_*[M, \partial M] \in H_*(BSL(N, \mathbf{C})^{\text{comp}})$$

has a preimage

$$\overline{\gamma}(M) \in H_*(SL(N, \mathbf{C}))$$

such that $\langle b_n, \overline{\gamma}(M) \rangle = c_\rho \text{vol}(M)$.

In particular, the Goncharov invariant $\gamma(M) \in K_*(\overline{\mathbf{Q}}) \otimes \mathbf{Q}$ can be defined. We have $\gamma(M^\tau) = \gamma(M)$ if M^τ is a generalized mutation of M .