

Volume and Topology

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- G/K symmetric space of noncompact type
e.g. $\mathbf{H}^3 = SL(2, \mathbf{C}) / SU(2)$.
- $\Gamma \subset G$ lattice

$$M^n = \Gamma \backslash G/K$$

Volume is a topological invariant of locally symmetric spaces:

- Mostow Rigidity
 $n = \dim(M) \geq 3$: homeomorphism $\Rightarrow$ isometry
- Chern-Gauß-Bonnet
 $n = \dim(M)$ even: $Vol(M) \sim \chi(M)$

Topological definitions of volume?

Change of volume under topological operations, Cut and Paste?

Simplicial Volume - Cut and Paste

Simplicial Volume

M^n compact, orientable manifold,
 $[M, \partial M] \in H_n(M, \partial M; \mathbf{R})$ its fundamental class

Definition (Gromov)

$$\| M \| = \inf \left\{ \sum_{i=1}^r |a_i| : \sum_{i=1}^r a_i \sigma_i \text{ represents } [M, \partial M] \right\}.$$

Example (Gromov)

If $f : M \rightarrow N$ has degree d , then

$$\| N \| \leq \frac{1}{d} \| M \|.$$

Example (Gromov)

$$\| S^n \| = 0, \| T^n \| = 0.$$

Proportionality Principle I

Theorem (Gromov)

If M is a hyperbolic n -manifold of finite volume, then

$$\text{Vol}(M) = V_n \| M \| .$$

V_n := volume of regular ideal simplex in hyperbolic space $\mathbf{H}^n$
= supremum of volume of straight simplices in $\mathbf{H}^n$.

Application to rigidity:

Gromov used the simplicial volume to give a topological proof of Mostow rigidity (homotopy equivalence implies isometry) for hyperbolic manifolds.

Proportionality principle II

Theorem (Gromov's proportionality principle)

There is a constant $c_{\tilde{M}}$ for each simply connected Riemannian manifold $\tilde{M}$ such that

$$\text{Vol}(M) = c_{\tilde{M}} \| M \|$$

whenever $\tilde{M}$ is the universal cover of a closed Riemannian manifold M .

For hyperbolic manifolds, $c_{\mathbf{H}^n} = V_n$. More generally, $c_{G/K}$ can be computed in continuous cohomology of G . (Bucher)

Theorem (Löh-Sauer, S.Kim-Kuessner)

Gromov's proportionality principle holds for finite-volume, pinched negatively curved manifolds.

Gromov's Proof of $\text{Vol}(M) = V_n \|M\|$

M closed hyperbolic manifold.

- Proof of $\|M\| \geq \frac{\text{Vol}(M)}{V_n}$:
each (fundamental) cycle can be straightened, straight simplices have volume $\leq V_n$.
- Proof of $\|M\| \leq \frac{\text{Vol}(M)}{V_n}$:
there exist fundamental cycles approximating the Haar measure on $\{\text{regular ideal simplices}\} \simeq \text{Isom}(\mathbf{H}^n)$. (In particular all involved simplices have volume nearly V_n .)

Similar argument for M cusped.

Lemma (Rigidity of efficient fundamental cycles: Jungreis, K.)

If $n \geq 3$ and M has not the same cusps as the figure 8 knot complement, then this is the only construction of efficient fundamental cycles.

For a codimension 1 submanifold $A \subset M$ we denote M_A the manifold obtained by cutting M along A .

Theorem (Gromov 1982)

If $\pi_1 A$ is amenable and injects into $\pi_1 M$, then

$$\| M_A, \partial M_A \| = \| M, \partial M \| .$$

For example abelian groups are amenable.

Not applicable to hyperbolic 3-manifolds: no incompressible tori.

Observation

If $\Sigma \subset M$ is totally geodesic and M^τ obtained from M by cutting and pasting via an isometry $\tau : \Sigma \rightarrow \Sigma$, then $\text{vol}(M^\tau) = \text{vol}(M)$.

Obvious geometric fact, since $\text{Vol}(M) = \text{Vol}(M - \Sigma)$.

Topological proofs? No proof via simplicial volume, but simple proof via group homology (in the setting of generalized mutations).

Theorem (Jungreis 1997 for M closed, K. 2003 for M cusped)

If $\Sigma \subset M$ is totally geodesic in a finite-volume hyperbolic n -manifold, $n \geq 3$, then

$$\| M_\Sigma, \partial M_\Sigma \| > \| M, \partial M \| .$$

Theorem (Frigerio-Pagliantini 2010)

Inequality is sharp: $\frac{\| M_\Sigma, \partial M_\Sigma \|}{\| M, \partial M \|} - 1$ is bounded in terms of $\frac{\text{Vol}(\Sigma)}{\text{Vol}(M)}$.

Simplicial Volume and Laminations

Agol's inequality for compact surfaces in 3-manifolds

Theorem (Agol-Storm-Thurston 2005, strengthening Agol 1999)

If M is a hyperbolic 3-manifold containing a π_1 -injective surface F , then

$$\text{Vol}(M) \geq \frac{V_{\text{oct}}}{4} \parallel \partial \text{Guts}(\overline{M - F}) \parallel$$

with $V_{\text{oct}} = 3.66 \dots$ the volume of a regular ideal octahedron in $\mathbf{H}^3$.

The proof uses analytical methods (Perelman's entropy estimate for the Ricci flow, work of Bray and Miao on the Penrose conjecture) and does not seem to generalize to laminations. Using the original topological approach from Agol we give a generalization to laminations and to higher-dimensional hyperbolic manifolds.

Tight laminations

Corollary (Agol 1999 for surfaces, K. 2007 for laminations)

If M is a hyperbolic 3-manifold and $\mathcal{F}$ a tight lamination (the leaf space is Hausdorff), then

$$\text{Vol}(M) \geq -2V_3 \chi(\text{Guts}(\overline{M - \mathcal{F}})),$$

where $V_3 = 1,014\dots$ is the volume of a regular ideal simplex in $\mathbf{H}^3$.

As an application, the Weeks manifold W admits no tight laminations: Calegari-Dunfield (2003) have proved that W admits no tight lamination with empty guts, on the other hand non-empty guts would imply $\text{Vol}(W) \geq 2V_3$ contradicting

$$\text{Vol}(W) = 0.94\dots$$

(Together with results of Tao Li this implies that the Weeks manifold has no essential laminations at all.)

Proof of Theorem for case of geodesic hypersurfaces

Let F be a geodesic hypersurface in M .

Let $\sum_{i=1}^r a_i \sigma_i$ represent $[M]$. We can assume all σ_i normal to F .

We want to prove $\sum_{i=1}^r |a_i| \geq \frac{2}{n+1} \|F\|$.

$[M] \cap F = [F]$, would like: each σ_i intersects F at most $n+1$ times.

$Q = \overline{M - F}$ is acylindrical.

Whenever a normal simplex meets F more than $n+1$ times, then two intersecting triangles must have a parallel edge, i.e. cut out a square. If this square were mapped to a cylinder, then acylindricity implies that the cylinder degenerates after homotopy, hence can be removed without changing the homology class, thus the number of intersections can be reduced.

The proof of $\sum_{i=1}^r |a_i| \geq \frac{2}{n+1} \|F\|$ proceeds by defining a straightening which produces the maximally possible number of cylinders.

Fundamental class in group homology

Group Homology and volume cocycle

$\Gamma \backslash G/K$ locally symmetric space

How to define an invariant in group homology $H_*(BG)$?

- $\Gamma \backslash G/K$ is an Eilenberg-MacLane space $B\Gamma$.

$$j : M = \Gamma \backslash G/K \simeq B\Gamma \rightarrow BG$$

M closed: $j_*[M] \in H_n(BG; \mathbf{R})$ determines $\text{vol}(M)$.

- Let $v_n \in H^n(BG; \mathbf{R})$ be the volume class, represented by

$$v_n(g_1, \dots, g_n) = \text{vol}(\text{str}(\tilde{x}, g_1\tilde{x}, \dots, g_n\tilde{x}))$$

for $\tilde{x} \in G/K$, $\text{str}(\dots)$ denoting the straight simplex.

Then for closed M :

$$\langle v_n, j_*[M] \rangle = \text{vol}(M).$$

If M_1, M_2 are locally symmetric spaces covered by G/K , then

$$\langle v_n, j_* [M] \rangle = \text{vol}(M)$$

implies that M_1, M_2 have the same volume if

$$j_* [M_1] = j_* [M_2] \in H_*(BG).$$

We are going to apply this to generalized mutations.

Cut and Paste III: Generalized Mutation

Recall: $\Sigma \subset M$ codim 1 submanifold, $\tau : \Sigma \rightarrow \Sigma$ diffeo, then M^τ is the manifold obtained by cutting M along Σ and regluing via τ .

Definition

$M_1 = \Gamma_1 \backslash G/K$, $M_2 = \Gamma_2 \backslash G/K$ locally symmetric spaces,

Assume $\Sigma \subset M_1$, and $M_2 = M_1^\tau$ for some finite order diffeomorphism $\tau : \Sigma \rightarrow \Sigma$.

Then M_2 is obtained from M_1 by generalized mutation, if there is some $A \in G$ with

$$\tau_*(\sigma) = A\sigma A^{-1}$$

for all $\sigma \in \pi_1 \Sigma$.

Trivial Example: Σ totally geodesic, $\tau : \Sigma \rightarrow \Sigma$ isometry.

Example

Mutation of knots:

If $M = S^3 - K$ is a knot complement and K' the mutant of K , then $S^3 - K' = M^\tau$ for a 4-punctured sphere $\Sigma \subset M$ and a hyperelliptic involution $\tau : \Sigma \rightarrow \Sigma$.

Ruberman (1989) proved:

- If $M = S^3 - K$ is a hyperbolic knot complement, then τ is a generalized mutation, M^τ is hyperbolic and $\text{vol}(M^\tau) = \text{vol}(M)$.

Proof uses heavy analytical machinery (existence and uniqueness results for minimal surfaces).

Remark: Hyperbolicity of M^τ can be proved topologically using the Maskit combination theorem.

K.: alternative proof of $\text{vol}(M^\tau) = \text{vol}(M)$ using group homology; proof extends to locally symmetric spaces.

Theorem (Kuessner 2011)

If M_1 and M_2 are locally symmetric spaces,

- either closed,
- or cusped and $\text{rank}(G/K) = 1$,

such that

$$M_2 = M_1^\tau$$

is a **generalized mutation** (for some $\Sigma \subset M$ and some finite order diffeomorphism $\tau : \Sigma \rightarrow \Sigma$),

then

$$\text{vol}(M^\tau) = \text{vol}(M).$$

Cut and Paste VI: Proof of Volume Equality - closed case

- Volume of M is determined by $j_* [M] \in H_n(BG; \mathbf{Q})$, therefore it suffices to prove

$$j_* [M] = j_* [M^\tau] \in H_n(BG; \mathbf{Q}).$$

- Consider the space X obtained by identifying M and M^τ along $M - nbh(\Sigma)$, it contains an embedded copy of the mapping torus T^τ . Clearly

$$[M] - [M^\tau] = [T^\tau] \in H_n(X).$$

- Definition of generalized mutation implies:
 $\pi_1 M \rightarrow G$ extends to a well-defined homomorphism
 $\pi_1 X \rightarrow G$.

$$j_* [M] - j_* [M^\tau] = j_* [T^\tau] \in H_n(BG; \mathbf{Q})$$

- Using $\tau^m = id$ prove $j_* [T^\tau] = 0 \in H_n(BG; \mathbf{Q})$.
(Not true with $\mathbf{Z}$ -coefficients, except for classical mutation by Neumann.)

Manifolds with boundary

M^n manifold with boundary, $\text{int}(M) \cong \Gamma \backslash G/K$.

- G/K symmetric space, define

$$BG^{comp} = BG \bigcup_{c \in \partial_\infty G/K} \text{Cone}(BG),$$

the volume cocycle $v_n \in H^n(BG)$ extends to $\bar{v}_n \in H^n(BG^{comp})$.

If $\text{rank}(G/K) = 1$ and $\text{int}(M) \cong \Gamma \backslash G/K$ for a lattice $\Gamma \subset G$, then

$$j_*[M, \partial M] \in H_n(BG^{comp})$$

can be defined such that

$$\langle \bar{v}_n, j_*[M, \partial M] \rangle = \text{vol}(M).$$

In particular, $\text{vol}(M)$ depends only on $j_*[M, \partial M] \in H_n(BG^{comp})$.

Cut and Paste VII: Proof of volume equality - cusped case

Recall: X obtained by identifying M and M^τ along $M - nbh(\Sigma)$, contains embedded copy of the mapping torus T^τ . For generalized mutations $\rho : \pi_1 M \rightarrow G$ extends to $\rho^X : \pi_1 X \rightarrow G$.

Let $t \in \pi_1 \partial X$ be the extending element in $\pi_1 \partial T^\tau \subset \pi_1 \partial X$.

$\rho^X(t) \in G$ is not parabolic (therefore $B\rho^X$ does not extend to $B(\pi_1 X)^{comp}$), but elliptic: t lifts to 1 in some finite covering $\hat{X}$.

Let $\hat{M}, \hat{M}^\tau$ be the preimages of M, M^τ in $\hat{X}$. Prove

$$j_* [\hat{M}, \partial \hat{M}] = j_* [\hat{M}^\tau, \partial \hat{M}^\tau] \in H_n(BG^{comp}; \mathbf{Q})$$

$$\implies \text{vol}(\hat{M}) = \text{vol}(\hat{M}^\tau) \implies \text{vol}(M) = \text{vol}(M^\tau).$$

Essentially the same argument shows that the (generalized) rational Bloch invariants and Goncharov invariants of $\mathbf{Q}$ -rank 1 locally symmetric spaces, the complex volume of representations and the rational Bloch invariant and volume of CR structures and flag structures are preserved under generalized mutations.