# Density of stable manifolds, after Plante

Thilo Kuessner

We consider an Anosov flow $\phi_t$ on a compact manifold $M$, with the associated splitting
$$TM = E^u \oplus E^s \oplus E^T.$$
The distributions
$$E^u, E^s, E^u \oplus E^T, E^s \oplus E^T$$
are integrable, we denote by
$$\mathcal{F}^{uu}, \mathcal{F}^{ss}, \mathcal{F}^u, \mathcal{F}^s$$
the corresponding foliations. For each $x \in M$, we let
$$W^{uu}(x), W^{ss}(x), W^u(x), W^s(x)$$
denote the leaf of $\mathcal{F}^{uu}, \mathcal{F}^{ss}, \mathcal{F}^u, \mathcal{F}^s$ containing $x$.

**Example 1:** Let $\phi_t$ be the geodesic flow on the unit tangent bundle $SN$ of a closed hyperbolic surface $N$. The leaves of $\mathcal{F}^{ss}$ are the orbits of the horocycle flow. A theorem of Hedlund (1937) asserts that the orbits of the horocycle flow are dense in $SN$. Furstenberg (1972) proved the much stronger statement that the orbits of the horocycle flow are uniformly distributed, i.e., not only they meet every open set $U$ in $SN$, but even they spend there (over large times) amounts of time proportional to the volume of $U$. This theorem is a very special case of the Raghunathan conjecture, which was proved by Ratner (1991), and which has far-reaching applications, especially to number theory.

**Example 2:** Let $A \in SL(2,\mathbb{Z})$ be hyperbolic, i.e., having two distinct real eigenvalues. Let $f : T^2 \rightarrow T^2$ be the Anosov diffeomorphism induced by $A$ and let $\phi_t$ the associated suspension flow on the mapping torus $M_f$ of $f$. Since the eigenvectors of $A$ have irrational slope, the leaves of $\mathcal{F}^{uu}$ and $\mathcal{F}^{ss}$ are dense in the fibers of $M_f$ (but, of course, not in $M_f$). The leaves of $\mathcal{F}^u$ and $\mathcal{F}^s$ are dense in $M_f$.

**Theorem 1** *: If $\phi_t$ is an Anosov flow on a compact connected manifold $M$, such that all points are non-wandering, then, for each $x \in M$, $W^u(x)$ and $W^s(x)$ are dense in $M$.*

**Theorem 2** *: If $\phi_t$ is an Anosov flow on a compact connected manifold $M$, such that all points are non-wandering, then*

*(i) either $W^{uu}(x)$ and $W^{ss}(x)$ are dense for each $x \in M$,*

*(ii) or there is a $C^1$-submanifold $K \subset M$ such that $\phi_t$ is the suspension flow of an Anosov diffeomorphism $f : K \to K$. (In particular, $M$ fibers over $S^1$ with fiber $K$.)*

# 1 Preliminaries

**Product neighborhood.** Each $x \in M$ has a neighborhood homeomorphic to $B^s_\delta(x) \times B^{uu}_\delta(x)$, where the homeomorphism maps

$$(y, z) \in B^s_\delta(x) \times B^{uu}_\delta(x)$$

to the unique intersection point

$$B^s_{2\delta}(z) \cap B^{uu}_{2\delta}(y).$$

**Density of periodic points.** Let $\phi_t$ be an Anosov flow on a compact manifold $M$. A theorem of Anosov asserts that the set of periodic points is dense in the set of non-wandering points. ($x$ is called non-wandering if $U \cap (\cup_{t \geq T} \phi_t U) \neq \emptyset$ for each $T \in \mathbb{R}$ and each neighborhood $U$ of $x$.) **Standing assumption:** all points $x \in M$ are non-wandering, i.e., periodic points are dense in $M$.

# 2 Proof of Theorem 1

**Theorem 3** *: If $\phi_t$ is an Anosov flow on a compact connected manifold $M$, $x \in M$, then $W^u(x)$ and $W^s(x)$ are dense in $M$.*

*Proof:* We want to show that $\overline{W^u(x)} = M$. Since $\overline{W^u(x)}$ is closed by definition, it suffices to show that it is open.

Let $z \in \overline{W^u(x)}$ and let $N$ be a *product neighborhood* of $z$ as in section 1. We want to show $N \subset \overline{W^u(x)}$. (Since $z \in \overline{W^u(x)}$ was arbitrary, this implies openness of $\overline{W^u(x)}$.) Since periodic points are dense in $M$, it suffices to show that $p \in \overline{W^u(x)}$ holds for each *periodic* point $p \in N$.

Since $z$ and $p$ belong to the same product neighborhood, there is some

$$y \in W^s(p) \cap W^u(z).$$

Since $\overline{W^u(x)}$ is $\phi_t$-invariant and $z \in \overline{W^u(x)}$, we have

$$y \in W^s(p) \cap \overline{W^u(x)}$$

and, moreover,

$$\phi_t(y) \in W^s(p) \cap \overline{W^u(x)}$$

for each $t \in \mathbb{R}$.

But $y \in W^s(p)$ implies $\lim_{t\to\infty} \phi_t(y) = p$, hence $p \in \overline{W^u(x)}$. $\square$

# 3 Proof of Theorem 2

**Theorem 4** *: If $\phi_t$ is an Anosov flow on a compact connected manifold $M$, then*
*(i) either $W^{uu}(x)$ and $W^{ss}(x)$ are dense for each $x \in M$,*
*(ii) or there is a $C^1$-submanifold $K \subset M$ such that $\phi_t$ is the suspension of an Anosov diffeomorphism $f : K \to K$. (In particular, $M$ fibers over $S^1$ with fiber $K$.)*

*Proof:*

Assume that $W^{uu}(x)$ is not dense for some $x \in M$. As a first step, we will now prove that then $\phi_t$ must be the suspension flow of some Anosov diffeomorphism $f : K \to K$, without yet proving that $K$ is a $C^1$-submanifold. (This will be done in the second step.)

**Lemma 1** *If $W^{uu}(x)$ is not dense for some $x \in M$, then $W^{uu}(p)$ is not dense for some periodic $p \in M$.*

*Proof:*

Assume by contradiction that $W^{uu}(p)$ is dense for each periodic point $p$, let $W^{uu}(x)$ be any strongly unstable manifold, then we want to show

$$W^{uu}(x) \cap B_\epsilon(y) \neq \emptyset$$

for each $y \in M$ and $\epsilon > 0$.

By the product neighborhood theorem we have some

$$N_r(p) := \cup_{z \in B_r^{uu}(p)} B_r^s(z) \supset B_\delta(p)$$

for small $\delta$. Since $M$ is compact, we can find finitely many periodic points $p_1, \ldots, p_k$ such that

$$M = \cup_{i=1}^k B_{\frac{\delta_\epsilon}{2}}(p_i).$$

Let $t_i$ denote the smallest period of $p_i$, for $i = 1, \ldots, k$.

Since $W^{uu}(p_i)$ is dense we have

$$B_T^{uu}(p_i) \cap B_{\frac{\epsilon}{2}}(y) \neq \emptyset$$

for large $T$. Obviously we can then find a common constant $c$ such that

$$\phi_s\left(B_T^{uu}(p_i)\right) \cap B_{\frac{\epsilon}{2}}(y) \neq \emptyset \text{ for } \mid s \mid < ct_i, 1 \leq i \leq k.$$

By elementary number theory, there exist natural numbers $t, n_i$ such that

$$\mid \frac{t}{t_i} - n_i \mid < c,$$

where $t$ may be chosen arbitrarily large.

We remark that $\phi_{n_i t_i}(p_i) = p_i$ and that $\phi_{n_i t_i}$ expands the strongly unstable manifolds, hence $\phi_{n_i t_i} B_{\frac{\epsilon}{2}}^{uu}(p_i) \supset B_T^{uu}(p_i)$ for large $n_i$. We choose $t$ so large that the $n_i$ are large enough to guarantee this inclusion for $i = 1, \ldots, k$. We can then conclude

$$\phi_t B_{\frac{\epsilon}{2}}^{uu}(p_i) = \phi_{t-n_i t_i} \phi_{n_i t_i} B_{\frac{\epsilon}{2}}^{uu}(p_i) \supset \phi_{t-n_i t_i} B_T^{uu}(p_i)$$

and, with $\mid t - n_i t_i \mid < ct_i$,

$$\phi_t B_{\frac{\epsilon}{2}}^{uu}(p_i) \cap B_{\frac{\epsilon}{2}}(y) \neq \emptyset.$$

We fix this value of $t$. Since $M = \cup_{i=1}^{k} B_{\frac{\delta\epsilon}{2}}(p_i)$, we have a nonempty intersection

$$\phi_{-t}\left(W^{uu}(x)\right) \cap B_{\frac{\delta\epsilon}{2}}(p_{i_0}) \neq \emptyset$$

for some $1 \leq i_0 \leq k$. Since $\phi_t\left(B_{\frac{\epsilon}{2}}^{uu}(p_{i_0})\right) \cap B_{\frac{\epsilon}{2}}(y) \neq \emptyset$, we find a point $q_1 \in B_{\frac{\epsilon}{2}}^{uu}(p_{i_0}) \cap \phi_{-t}\left(B_{\frac{\epsilon}{2}}(y)\right)$. Since $B_{\frac{\delta\epsilon}{2}}(p_{i_0})$ is a product neighborhood, we find a point $q_2 \in \phi_{-t}\left(W^{uu}(x)\right) \cap B_{\frac{\delta\epsilon}{2}}(p_{i_0}) \cap B_{\frac{\epsilon}{2}}^{s}(q_1)$. We conclude

$$d\left(y, \phi_t(q_2)\right) \leq d\left(y, \phi_t(q_1)\right) + d\left(\phi_t(q_1), \phi_t(q_2)\right) \leq d\left(y, \phi_t(q_1)\right) + d\left(q_1, q_2\right) < \epsilon.$$

□

**Lemma 2** *Let $p$ be a periodic point such that $W^{uu}(p)$ is not dense. Then $M$ fibers over the circle with fiber $\overline{W^{uu}(p)}$ and $\phi_t$ is the suspension flow of this fibration.*

*Proof:* Let $r$ be the period of $p$. We have

$$M = \overline{W^u(p)} = \overline{\cup_{0 \leq t \leq r} \phi_t W^{uu}(p)} = \cup_{0 \leq t \leq r} \overline{\phi_t W^{uu}(p)}.$$

Let $K \subset \overline{W^{uu}(p)}$ be a minimal $\phi_r$-invariant $\mathcal{F}^{uu}$-saturated closed set. (Existence is guaranteed by Zorn's lemma.) From minimality of $K$ and denseness of $\mathcal{F}^u$-saturated sets follows easily that

$$M = \cup_{0 \leq t < s} \phi_t(K)$$

as a disjoint union, where $s > 0$ is chosen to be minimal such that $K \cap \phi_s(K) \neq \emptyset$. In other words, each $x \in M$ is of the form $x = \phi_t(k)$ for a unique $k \in K$ and a unique $0 \leq t < s$.

We define then an projection $\pi : M \to S^1 = \mathbb{R}^1 / s\mathbb{Z}^1$ by

$$\pi(\phi_t(k)) = t \bmod s.$$

One checks easily the desired properties. □

Lemmata 1 and 2 prove theorem 2 except for the claim that the fiber $\overline{W_p^{uu}}$ is a $C^1$-submanifold. This will be the aim of the remaining part of the proof.

In a product neighborhood $B_\delta^s(x) \times B_\delta^{uu}(x)$ one can use the product structure to define a projection along strong unstable manifolds. If $y$ and $z$ belong to the same strong unstable manifold (in this product neighborhood), this defines a map

$$F_{yz} : B_\epsilon^s(y) \to B_\delta^s(z)$$

for some small $\epsilon$. The foliations $\mathcal{F}^{uu}$ and $\mathcal{F}^{ss}$ are called **jointly integrable** if $F_{yz}$ maps $W^{ss}(u)$ to $W^{ss}(F_{yz}(u))$ for $u \in B_\epsilon^s(y)$. A theorem of Anosov asserts that $\mathcal{F}^{uu}$ and $\mathcal{F}^{ss}$ are jointly integrable if and only if $E^u \oplus E^s$ is the tangent field of a $C^1$-foliation. (The if-direction is trivial.)

We now finish the proof. We already have that $M$ fibers over $S^1$ with fiber $K = \overline{W^{uu}(p)}$. $K$ is also $\mathcal{F}^{ss}$-saturated. Hence $F_{yz}$ preserves fibers, implying that $\mathcal{F}^{uu}$ and $\mathcal{F}^{ss}$ are jointly integrable and we get a $C^1$ foliation tangent to $E^u \oplus E^s$. In particular, the leaf $L$ through $p$ is a $C^1$-submanifold. $L$ is compact (if not, we could find points far away in $L$ but close in $M$, i.e., $\phi_t K$ would intersect $K$ for arbitrarily small $t$ - this is impossible for a suspension flow) and contains $W^{uu}(p)$. Hence $L = \overline{W^{uu}(p)}$, i.e., $\overline{W^{uu}(p)}$ is a $C^1$-submanifold. □