

Proportionality principle for simplicial volume

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Slides: <http://newton.kias.re.kr/~kuessner/jeonju.pdf>

Part 0: Overview of results

Simplicial Volume

The simplicial volume is a topological invariant of closed, orientable manifolds, defined as the l^1 -norm of the fundamental class:

$$\| M \| = \inf \left\{ \sum_{i=1}^r |a_i| : \sum_{i=1}^r a_i \sigma_i \text{ represents } [M] \right\},$$

where $[M] \in H_n(M; \mathbb{R}) \simeq \mathbb{R}$ is the fundamental class.

Variants of the simplicial volume I

For an open, orientable manifold define

Definition

$$\| M \| = \inf \left\{ \sum_{i=1}^{\infty} |a_i| : \sum_{i=1}^{\infty} a_i \sigma_i \text{ represents } [M] \right\},$$

where $[M] \in H_n^{lf}(M; \mathbb{R}) \simeq \mathbb{R}$ is the fundamental class in the homology of locally finite chains.

Proportionality Principle - closed case

Gromov: If M and N are **closed** Riemannian manifolds with isometric universal covering, then

$$\frac{\| M \|}{\text{Vol}(M)} = \frac{\| N \|}{\text{Vol}(N)}.$$

Bucher, Frigerio: The proportionality constant can be described in terms of continuous cohomology.

Proportionality principle - open case

The proportionality principle can not hold for arbitrary open manifolds.

For example $\|M\| > 0$ for many (all?) locally symmetric spaces of noncompact type of $\mathbb{Q}$ -rank ≤ 1 but $\|M\| = 0$ for locally symmetric spaces of noncompact type of $\mathbb{Q}$ -rank ≥ 3 .

However a symmetric space $\tilde{M} = G/K$ of $\mathbb{R}$ -rank 1 may cover locally symmetric spaces $\Gamma \backslash G/K$ of different $\mathbb{Q}$ -ranks:

$$\mathbb{R} - \text{rank}(G/K) \geq \mathbb{Q} - \text{rank}(\Gamma \backslash G/K).$$

Variants of the simplicial volume II

In this talk Riemannian manifolds are always assumed to be complete. For a Riemannian metric g define

Definition

$$\| M \|_{Lip} = \inf \left\{ \sum_{i=1}^{\infty} |a_i| : \sum_{i=1}^{\infty} a_i \sigma_i \text{ represents } [M]_{Lip} \right\},$$

where $[M] \in H_n^{lf, Lip}(M; \mathbb{R})$ is the fundamental class in the homology of locally finite **Lipschitz** chains (i.e., chains of simplices with a uniform Lipschitz constant).

(A priori this may depend on the Riemannian metric. For closed manifolds $\| M \|_{Lip}$ does not depend on the Riemannian metric.)

Proportionality principle - open case

The proportionality principle may hold for the Lipschitz simplicial volume. Löh-Sauer have proven that the proportionality principle for Lipschitz simplicial volume holds for nonpositively curved Riemannian manifolds (of finite volume).

Proportionality principle - open case

In joint work with Sungwoon Kim we prove $\|M\|_{Lip} = \|M\|$ for pinched negatively curved Riemannian manifolds (of finite volume), thus the proportionality principle for simplicial volume holds for this class of manifolds.

The proof works for all manifolds with amenable boundary and $\|M\|_{Lip} < \infty$, it uses multicomplexes and bounded cohomology.

Part I: Introduction to Simplicial Volume

- G/K symmetric space of noncompact type
e.g. $H^3 = SL(2, \mathbf{C})/SU(2)$.
- $\Gamma \subset G$ lattice

$$M^n = \Gamma \backslash G/K$$

Volume is a topological invariant of locally symmetric spaces:

- **Mostow Rigidity** $n = \dim(M) \geq 3$:
homeomorphism $\Rightarrow$ isometry
- **Chern-Gauß-Bonnet** $n = \dim(M)$ even: $Vol(M) \sim \chi(M)$

Topological definitions of volume?

Change of volume under topological operations, Cut and Paste?

Simplicial Volume

M^n compact, orientable manifold,
 $[M, \partial M] \in H_n(M, \partial M; \mathbb{R})$ its fundamental class

Definition (Gromov)

$$\| M \| = \inf \left\{ \sum_{i=1}^r |a_i| : \sum_{i=1}^r a_i \sigma_i \text{ represents } [M, \partial M] \right\}.$$

Example

*If there is a map $f : M \rightarrow N$ of degree d , then $d \| N \| \leq \| M \|$.
In particular, if there is a map $f : M \rightarrow M$ of degree $d \geq 2$, then $\| M \| = 0$.*

Example

$$\| S^n \| = 0, \| T^n \| = 0.$$

Example (Gromov-Thurston)

$\text{Int}(M) = M \setminus \partial M$ hyperbolic of finite volume, then

$$\text{Vol}(M) = V_n \| M \|.$$

$V_n :=$ volume of regular ideal simplex in H^n .

Let $\beta \in H^n(M, \partial M; \mathbb{R})$ be the unique cohomology class with

$$\langle \beta, [M, \partial M] \rangle = 1.$$

Then

$$\| M \| = \frac{1}{\| \beta \|_\infty}$$

with $\| \beta \|_\infty = \inf \{ \| f \|_\infty : f \text{ represents } \beta \}$ and

$\| f \|_\infty := \sup \{ | f(\sigma) | : \sigma \text{ singular simplex} \}.$

In particular $\| M, \partial M \| = 0$ if and only if β has no bounded cocycle representative.

Let $C_b^*(M, \partial M) = \{f : C_*(M, \partial M) \rightarrow \mathbb{R} : \|f\|_\infty < \infty\}$ be the complex of bounded cochains.

Its cohomology is the bounded cohomology $H_b^*(M, \partial M)$, equipped with a (finite) semi-norm.

In general, $H_b^*(M, \partial M) \rightarrow H^*(M, \partial M; \mathbb{R})$ need not be injective or surjective.

- $\|M, \partial M\| \neq 0$ if and only if $H_b^n(M, \partial M) \rightarrow H^n(M, \partial M; \mathbb{R})$ is surjective.

Theorem (Gromov)

Let $\Gamma = \pi_1 M$. The classifying map $M \rightarrow B\Gamma$ induces an isometric isomorphism $H_b^*(B\Gamma) \rightarrow H_b^*(M)$.

In particular, $H_b^*(M)$ depends only on $\pi_1 M$. The proof uses amenability of higher homotopy groups.

A group Γ is amenable if it admits a Γ -invariant mean, that is a linear functional on bounded functions

$$A : B(\Gamma) \rightarrow \mathbb{R}$$

with

$$A(\gamma f) = A(f) \quad \forall \gamma \in \Gamma$$

$$\inf(f) \leq A(f) \leq \sup(f).$$

Examples:

- finite groups $A(f) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} f(g)$
- solvable groups

Lemma (Gromov)

If $p : X_1 \rightarrow X_2$ is a normal covering with amenable deck group, then averaging bounded cochains defines a left-inverse to $p^* : H_b^*(X_2) \rightarrow H_b^*(X_1)$. In particular, p^* is an injective isometry.

Theorem (Gromov)

If $\pi_1 M$ is amenable, then $H_b^*(M) = 0$ for $* \geq 1$.

Corollary (Gromov)

If $\pi_1 M$ is amenable, then $\|M\| = 0$.

Theorem (Gromov)

If $f : (M, \partial M) \rightarrow (N, \partial N)$ induces injections of path-components and - for each path-component - surjections of fundamental groups with amenable kernel, then

$$\| M \| = \| N \| .$$

Proof by Gromov using multicomplexes, alternative proof by Ivanov (for connected boundary) using resolutions.

Gromov's Proof of $\text{Vol}(M) = V_n \|M\|$

M closed hyperbolic manifold.

- Proof of $\|M\| \geq \frac{\text{Vol}(M)}{V_n}$:
each (fundamental) cycle can be straightened, straight simplices have volume $\leq V_n$.
- Proof of $\|M\| \leq \frac{\text{Vol}(M)}{V_n}$:
there exist fundamental cycles approximating the Haar measure on $\{\text{regular ideal simplices}\} \simeq \text{Isom}(H^n)$. (In particular all involved simplices have volume nearly V_n .)

Similar argument for M cusped.

Remark (Rigidity of efficient fundamental cycles: Jungreis, K.)

If $n \geq 3$ and M has not the same cusps as the figure 8 knot complement, then this is the only construction of efficient fundamental cycles.

On Mostow rigidity

Gromov used the simplicial volume to give a topological proof of Mostow rigidity (homotopy equivalence implies isometry) in the constant curvature case:

- homotopy equivalences f preserve $\| M \|$
- hence simplices of maximal volume must be sent to simplices of maximal volume
- reflection group of regular ideal simplex (= simplex of maximal volume) has dense orbits on $\partial_\infty H^n$
- hence $\partial_\infty \tilde{f}$ is conformal, f is homotopic to isometry

Rigidity of efficient fundamental cycles implies a priori: $\tilde{f}$ even preserves the equidistribution on regular ideal simplices.

For a codimension 1 submanifold $A \subset M$ we denote M_A the manifold obtained by cutting M along A .

Theorem (Gromov 1982)

If $\pi_1 A$ is amenable and injects into $\pi_1 M$, then

$$\| M_A, \partial M_A \| = \| M, \partial M \| .$$

For example abelian groups are amenable.

Not applicable to hyperbolic 3-manifolds: no incompressible tori.

Cut and Paste II

Observation

If $\Sigma \subset M$ is totally geodesic and M^τ obtained from M by cutting and pasting via an isometry $\tau : \Sigma \rightarrow \Sigma$, then $\text{vol}(M^\tau) = \text{vol}(M)$.

Obvious geometric fact, since $\text{Vol}(M) = \text{Vol}(M - \Sigma)$.

Topological proofs? No proof via simplicial volume, but simple proof via group homology (in the setting of generalized mutations).

Theorem (Jungreis 1997 for M closed, K. 2003 for M cusped)

If $\Sigma \subset M$ is totally geodesic in a finite-volume hyperbolic n -manifold, $n \geq 3$, then

$$\| M_\Sigma, \partial M_\Sigma \| > \| M, \partial M \| .$$

Theorem (Frigerio-Pagliantini 2010)

Inequality is sharp: $\frac{\| M_\Sigma, \partial M_\Sigma \|}{\| M, \partial M \|} - 1$ is bounded in terms of $\frac{\text{Vol}(\Sigma)}{\text{Vol}(M)}$.

Part II: Proportionality Principle (Closed Manifolds)

- **Measure homology** $\mathcal{H}_*(M)$, defined using signed measures (of compact determination set and finite total variation) on the space of simplices (rather finite sums only)
- $H_*(M) \rightarrow \mathcal{H}_*(M)$ is an isomorphism (Zastrow, Hansen) and an isometry of l^1 -norms (Löh).
- Gromov: smearing simplices over $\pi_1 N \backslash \text{Isom}(\tilde{M})$ yields a norm-preserving homomorphism $H_n(M) \rightarrow \mathcal{H}_n(N)$ that corresponds to multiplication by $\frac{\text{vol}(M)}{\text{vol}(N)}$.

Hence $\|N\| \leq \frac{\text{Vol}(N)}{\text{Vol}(M)} \|M\|$ and (by exchanging M and N) vice versa.

M Riemannian manifold, $dvol$ its volume form, then integration over simplices defines a (not necessarily bounded) cohomology class $[dvol] \in H^n(M; \mathbb{R})$. Clearly:

$$\frac{\|M\|}{Vol(M)} = \frac{1}{\|[dvol]\|_\infty}$$

Thus the proportionality principle amounts to show that $\|[dvol]\|_\infty$ depends only on $\tilde{M}$.

Consider $M = \Gamma \backslash G / K$ loc.symm.space of noncomp.type
 $G = \text{Isom}(\tilde{M})$ group of isometries of the universal covering
 $H_c^*(G)$ continuous group cohomology

- Let $v_n \in H_c^n(G)$ be the volume class, represented by the cocycle

$$v_n(g_1, \dots, g_n) = \text{vol}(\text{str}(\tilde{x}, g_1\tilde{x}, \dots, g_n\tilde{x}))$$

for $\tilde{x} \in G/K$, $\text{str}(\dots)$ denoting the straight simplex.

Remark: for closed $M = \Gamma \backslash G / K$ and $j : M \simeq B\Gamma \rightarrow BG$:

$$\langle v_n, j_*[M] \rangle = \text{vol}(M).$$

Proportionality principle for closed locally symmetric spaces (Bucher)

$$M = \Gamma \backslash G / K$$

$$H_{bc}^n(G) \rightarrow H_b^n(\Gamma) = H_b^n(M)$$

- is an injective isometry (inverse given by integrating Γ -invariant cochains over compact fundamental domain for $\Gamma \backslash G$)
- sends volume class to $[dvol]$
Thus $\| [dvol] \|_\infty = \| [v_n] \|_\infty$ independent of Γ

Only known explicit values of $\| [v_n] \|_\infty$:

- Gromov-Thurston:

$\tilde{M} = H^n$ Hyperbolic space :

$$\| [v_n] \|_\infty = V_n$$

volume of regular ideal simplex, e.g. $V_2 = \pi$

- Bucher:

$$\tilde{M} = H^2 \times H^2 :$$

$$\| [v_4] \|_\infty = \frac{2}{3}\pi^2$$

Proportionality constant for closed manifolds (Bucher, Frigerio)

Let M be a Riemannian manifold. Then $G = \text{Isom}^+(\tilde{M})$ is a (possibly 0-dimensional) Lie group and $\Gamma := \pi_1 M$ is a cocompact lattice in G .

Want to have an injective isometry

$$H^n(C_b^*(\tilde{M})^G) \rightarrow H^n(C_b^*(\tilde{M})^\Gamma) = H_b^n(M).$$

The construction of a left-inverse isometry by integrating over a fundamental domain of $\Gamma \subset G$ works if one considers only continuous cochains on the space of smooth simplices - which however yields the usual cohomology. (Bucher, Frigerio)

Thus $\| [dvol] \|_\infty = \| [dvol]_{\tilde{M}} \|_\infty$ independently of Γ .

Proportionality constant for cusped $\mathbb{R}$ -rank 1 manifolds (K.)

Assume $\text{int}(M)$ **locally symmetric space of $\mathbb{R}$ -rank 1, finite volume**. Denote $DCone(M, \partial M)$ the compactification of $\text{int}(M)$ by adding cusps, let $[dvol] \in H^n(DCone(M, \partial M); \mathbb{R})$ be the class of the cocycle defined by integrating the volume form (declaring cusps to have measure 0).

Using Gromov's theory of multicomplexes we can show that $H_b^*(DCone(M, \partial M)) \rightarrow H_b^*(M)$ is an isometry.

$$H_{bc}^*(G) \rightarrow H_b^*(\Gamma) = H_b^*(M) \rightarrow H_b^*(DCone(M, \partial M))$$

sends $[v_n]$ to $[dvol]$, hence $\frac{\|M, \partial M\|}{\text{Vol}(\text{Int}(M))} = \| [v_n] \|_\infty$.

Part III: Proportionality Principles (Open Manifolds)

$$\tilde{M} = H^2 \times H^2 \times H^2$$

- If $\Sigma_1, \Sigma_2, \Sigma_3$ are finite-volume noncompact hyperbolic surfaces, then $\| \Sigma_1 \times \Sigma_2 \times \Sigma_3 \| = 0$. (Gromov)
- If M is a Hilbert modular variety, i.e. finitely covered by $(\sigma_1, \sigma_2, \sigma_3)(SL_2(O_F)) \backslash H^2 \times H^2 \times H^2$ for O_F the ring of integers in a real cubic field F and $\sigma_1, \sigma_2, \sigma_3 : F \rightarrow \mathbb{R}$ its embeddings, then $\| M \| > 0$. (Löh-Sauer)

Approach to the proportionality principle

Löh-Sauer have proved

$$\frac{\|M\|_{Lip}}{Vol(M)} = \frac{\|N\|_{Lip}}{Vol(N)}$$

for nonpositively curved manifolds of finite volume.

We prove

$$\|M\|_{Lip} = \|V, \partial V\| = \|M\|$$

if M is pinched negatively curved of finite volume and V a compact manifold with boundary s.t. $M = V - \partial V$.

(The proof uses amenability of $\pi_1 \partial V$ and so does not extend to the case of nonpositive curvature.)

For an open manifold M

$$\| M \| = \inf_{\alpha \in [M]^{l^1}} \sup \left\{ \frac{1}{\| f \|_{\infty}} : \langle f, \alpha \rangle = 1 \right\}$$

$$\| M \|_{Lip} = \inf_{\alpha \in [M]_{Lip}^{l^1}} \sup \left\{ \frac{1}{\| f \|_{\infty}} : \langle f, \alpha \rangle = 1 \right\}$$

Here, $[M]^{l^1} \subset H_n^{l^1}(M)$ denotes the set of l^1 -homology classes that map to $[M]$, and $[M]_{Lip}^{l^1} \subset [M]^{l^1}$ the subset representable by Lipschitz chains.

(Note that $H_*^{l^1}(M) \rightarrow H_*^{lf}(M)$ need not be injective.)

Theorem (Sungwoon Kim-T.K.)

If the Riemannian manifold $M \neq \mathbb{R}^1$ is the interior of a compact manifold V with $\pi_1 \partial V$ amenable, and if $\|M\|_{Lip} < \infty$, then

$$\|M\| = \|M\|_{Lip} = \|V, \partial V\|,$$

in particular $\|M\|_{Lip}$ does not depend on the Riemannian metric.

Remark: The assumption $\|M\|_{Lip} < \infty$ is necessary, as the example of the hyperbolic plane shows. This assumption is always satisfied for **pinched negatively curved** manifolds **of finite volume**.

Proof assuming $H_b^*(V, \partial V) = H_b^*(V)$

Assuming that $H_b^*(V, \partial V) \rightarrow H_b^*(V)$ is an isometry, the same is true for

$$i^* : H_b^*(M, M - K) \rightarrow H_b^*(M)$$

where the compact set $K \subset M$ is chosen such that $M \setminus K \simeq \partial V \times (0, \infty)$.

One can then derive that - for any class $\alpha \in [M]^{l^1}$ -

$$\{f \in H_b^n(M) : \langle f, \alpha \rangle = 1\} =$$

$$\{i^*\beta : \beta \in H_b^n(M, M - K), \langle \beta, [M, M - K] \rangle = 1\}.$$

But the right-hand side does not depend on the chosen α and hence $\sup \left\{ \frac{1}{\|f\|_\infty} : \langle f, \alpha \rangle = 1 \right\}$ does not depend on $\alpha \in [M]^{l^1}$.

In particular, assuming $\|M\|_{Lip} < \infty$, we may choose $\alpha \in [M]_{Lip}^{l^1}$ and obtain $\|M\| = \|M\|_{Lip}$.

Part IV: Multicomplexes and bounded cohomology

Multicomplexes are like simplicial complexes except that there may be many simplices with the same set of vertices. (But vertices of a simplex are still assumed to be pairwise distinct.)

Gromov's theory of multicomplexes

X topological space:

- $\hat{K}(X)$ multicomplex consisting of all points of X and one simplex in each homotopy class rel. boundary.

(Bounded) simplicial (co)homology of $\hat{K}(X)$ is isometrically isomorphic to (bounded) singular (co)homology of X .

- $K(X) = \hat{K}(X) / \sim$, where simplices are identified if they have the same 1-skeleton.

Realizes classifying map $X \rightarrow B\pi_1 X$, induces isometric isomorphism of bounded cohomology because higher homotopy groups are amenable.

Quotienting out amenable subsets (after Gromov)

Let $A \subset X$ have amenable fundamental group. We would like to show that $i^* : H_b^*(X, A) \rightarrow H_b^*(X)$ is an isometry.

- By Gromov, if $\pi_1 X_1 \rightarrow \pi_1 X_2$ is surjective with amenable kernel, then $H_b^*(X_2) \rightarrow H_b^*(X_1)$ is an isometric isomorphism.

This does not apply to $X \rightarrow X/A$ because $\ker(\pi_1 X \rightarrow \pi_1 X/A)$ is the subgroup normally generated by $\pi_1 A$, usually not amenable.

- However, when we consider $K(X)$ rather than X , there is a (large) amenable group acting on $K(X)$:

Quotienting out amenable subsets II

$\Pi_A(X)$ = the group of tuples of homotopy classes of paths in A with the obvious composition (concatenation and/or disjoint union).

Here, homotopy classes are taken in X , and we only consider tuples of paths with pairwise distinct startpoints and pairwise distinct endpoints. Moreover, tuples of 0-homotopic paths are identified with the empty set. (Thus the empty set becomes the neutral element in the group $\Pi_A(X)$.)

Quotienting out amenable subsets III

$\Pi_A(X)$ acts on the 1-skeleton of $K(X)$ by concatenation. As simplices in $K(X)$ are determined by their 1-skeleton, it acts on $K(X)$.

$G := \Pi_A(X)$ is an extension by $\bigoplus_{x \in A} \text{im}(\pi_1(A, x) \rightarrow \pi_1(X, x))$ of the group of locally finite permutations, hence **amenable**.

Thus - after averaging - bounded cochains on $K(X)$ can be assumed to be **G -invariant**.

Theorem (S.Kim-Kuessner,
Bucher-Burger-Frigerio-Iozzi-Pagliantini-Pozzetti)

If (X, A) is a pair of spaces with $\pi_1 A$ amenable (for all path-components), then

$$i^* : H_b^*(X, A) \rightarrow H_b^*(X)$$

is an isometric isomorphism in degrees ≥ 2 .

Remark: It follows from $H_b^*(A) = 0$ and the long exact sequence that i^* is an isomorphism. The new fact is that it is an isometry.

Proof using multicomplexes

By the above theory of multicomplexes, we can reduce to show that $H^*(C_b^*(K(X), K(A))^G) \rightarrow H^*(C_b^*(K(X))^G)$ is an isometry, for the action of $G = \Pi_A(X)$ on $(K(X), K(A))$.
It turns out that - in degrees $* \geq 1$ -

$$C_*(X)/G \rightarrow C_*(X, A)/G$$

is a bijection, even on the level of chain complexes. Namely, for each $\sigma \in C_*(A)$ with $* \geq 1$ there exists $g \in G$ with $g\sigma = -\sigma$, thus $\sigma \equiv 0 \mod G$.

An isometric chain isomorphism in degrees $* \geq 1$ clearly provides an isometric isomorphism of bounded cohomology in degrees $* \geq 2$. QED

Proportionality constant for cusped $\mathbb{R}$ -rank 1 manifolds

Remark: a similar (but different) argument shows that

$$H_b^*(DCone(M, \partial M)) \rightarrow H_b^*(M)$$

is an isometry in degrees ≥ 2 and thus proves

$$\| [dvol] \|_\infty = \| [v_n] \|_\infty$$

for $\mathbb{R}$ -rank 1 locally symmetric spaces of finite volume.