ON MEASURE HOMOLOGY OF MILDLY WILD SPACES

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ABSTRACT. We prove injectivity of the canonical map from singular homology to measure homology for certain "mildly wild" spaces, that is, certain spaces not having the homotopy type of a CW-complex, but having countable fundamental groups.

Measure homology $\mathcal{H}_*(X)$, also called Milnor-Thurston homology, of a space X is a variant of the usually studied singular homology groups $H_*(X; \mathbb{R})$. While the latter are defined as the homology theory of the chain complex of finite linear combinations of singular simplices with the canonical boundary operator, measure homology uses the chain complex of quasicompactly determined signed measures of bounded variation on the space of singular simplices $map(\Delta^k, X)$, again with the canonical boundary operator (see Section 1.1).

Singular chains (with real coefficients) can be considered as finite sums of (real multiples of) Dirac measures, so there is a canonical homomorphism

$$\iota_* \colon H_*(X; \mathbb{R}) \to \mathcal{H}_*(X).$$

It was proven in [30] and [11] that measure homology satisfies the Eilenberg-Steenrod axioms and thus that ι_* is an isomorphism whenever X is a CW-complex.

An example from [25] shows that ι_* is not always injective. However, the example constructed there is in some sense an artificial one: it relies on the existence of non-measurable sets and ultimately on the axiom of choice. So one may ask whether for more natural spaces one can still prove injectivity of ι_* as it holds for CW-complexes.

The picture below shows the convergent arcs space CA. It is formed by one arc l_{∞} and a sequence of arcs $(l_n)_{n \in \mathbb{N}}$ with the same endpoints as l_{∞} and pointwise converging to l_{∞} . Although the arcs provide a natural cell decomposition, CA is not a CW-complex because its topology is not the weak topology from the cell decomposition: the union $\bigcup_{n \in \mathbb{N}} l_n$ is not a closed subset.



It was shown in [30, Section 6] that $H_1(CA; \mathbb{R}) \to \mathcal{H}_1(CA)$ is not surjective. On the other hand, [24, Theorem 2.8] computes the measure homology

of CA and the proof implies in particular that $H_*(CA; \mathbb{R}) \to \mathcal{H}_*(CA)$ is injective.

The convergent arcs space is a "mildly wild" space in the sense that it is semi-locally simply connected and that it has countable fundamental group. When we collapse the arc l_{∞} to a point, then we obtain the Hawaiian earring (pictured below to the left), which is not semi-locally simply connected connected and which has uncountable (and very complicated) fundamental group. This is an example of a "really wild" space. More generally, one could consider shrinking wedges of manifolds: the Hawaiian earring is the shrinking wedge of circles, and the Barratt-Milnor sphere (pictured below to the right) is the shrinking wedge of spheres. The Barratt-Milnor sphere is semi-locally simply connected and has countable (actually trivial) fundamental group, however its higher homotopy groups are not countable.



Although ultimately we would like to say something about injectivity of ι_* for "really wild" spaces of uncountable fundamental group like the Hawaiian earring, in this paper we will pursue a more modest goal: we will prove injectivity of the canonical homomorphism for two classes of "mildly wild" spaces, i.e., spaces which have countable fundamental group and thus a fortiori are semi-locally simply connected. So our results do not apply to the Hawaiian earring, but they apply to various generalizations of the convergent arc space.

The proofs of our two cases are independent and will use different methods.

The first result is the following.

Theorem 0.1. Let X be a topological space, which is T_1 , second countable, has countable fundamental group and admits a contractible generalized universal covering space \widetilde{X} in the sense of [5].

Then the kernel of $\iota_* \colon H_*(X;\mathbb{R}) \to \mathcal{H}_*(X)$ is contained in the zero-norm subspace with respect to the Gromov norm on $H_*(X;\mathbb{R})$.

In particular, if for some k the Gromov norm on k-th homology is an actual norm, that is $||x|| \neq 0$ for all $x \in H_k(X; \mathbb{R}) \setminus \{0\}$, then $\iota_k \colon H_k(X; \mathbb{R}) \to \mathcal{H}_k(X)$ is injective.

We will recall the definition of the Gromov norm in Section 2.2. The assumption on non-vanishing of the Gromov norm seems to be a more severe restriction than the others. Therefore the following result may be more useful **Theorem 0.2.** Let X be a topological space with a basepoint $x_0 \in X$. Assume that X is a countable union

$$X = \bigcup_{n \in \mathbb{N}} Y_n$$

of aspherical, closed subsets $Y_n, n \in \mathbb{N}$ which have the structure of finite CW-complexes, such that $x_0 \in Y_n$ for all n, and such that all intersections

$$Y_I := \bigcap_{i \in I} Y_i, I \subset \mathbb{N}$$

are sub-CW-complexes of each Y_i with $i \in I$. Assume that i) no $x \in X$ is a limit of a sequence $x_{\nu} \in \bigcap_{n \in I_{\nu}} Y_n$ for pairwise distinct index sets $I_{\nu} \subset \mathbb{N}$,

ii) for all $I \subset \mathbb{N}$ with $|I| \geq 2$, there are CW-neighborhoods U_I of Y_I which are open subsets of X, and

iii) the inclusions induce an isomorphism $\bigoplus_{n \in \mathbb{N}} H_k(Y_n) = H_k(X)$.

Then $\iota_* \colon H_k(X; \mathbb{R}) \to \mathcal{H}_k(X)$ is injective.

The notion of CW-neighborhood will be explained in Section 4.2. Openness of CW-neighborhoods, as well as the condition $H_*(X) = \bigoplus_n H_*(Y_n)$, rules out examples like the Hawaiian earring.

An example of spaces to which we apply these results are the convergent Y-spaces defined in Definition 2.8, which are not CW-complexes and which are sort of a generalization of the convergent arcs space CA. They are constructed by gluing countably many copies of a CW-complex along finitely many points such that all the copies Y_n of Y accumulate at one copy Y_{∞} .

This kind of example would satisfy the assumptions of Theorem 0.1 (in particular that about non-vanishing of the Gromov norm) only under additional assumptions (e.g., when Y is a negatively curved manifold, see Section 2.8). However we will see in Section 4.4 that it satisfies all assumptions of Theorem 0.2 whenever Y is an aspherical, compact, smooth manifold, and we thus get injectivity of ι_* for any convergent Y-space X.

In [31], the third author will exhibit an example of a space X, which is a countable union of CW-complexes, but with no point x_0 in the intersection of all of them, such that ι_* is not injective. (This is also the first such example which does not rely on existence of non-measurable sets.)

A technical device in the proof of Theorem 0.2 is that we can reduce the problem of computing measure homology (and the dual notion of measurable cohomology) to computing it for the subcomplex of simplices with all vertices in a given basepoint (Lemma 3.2 and Corollary 3.4). We think that this result should be of independent interest. To prove this result we are imposing a condition that X can be covered by finitely many Borel sets of compact closure contractible in X. This may look like a technical condition, but it may not be avoidable as Example 3.3 shows.

When applied to CW-complexes our argument is similar but simpler than the one in [19] which did not restrict to simplices with vertices in a basepoint and therefore needed a larger effort to prove the technical [19, Lemma A.1] on existence of a measurable section. Our argument, together with countability of the fundamental group, actually also provides such a measurable section from pointed simplices in X to pointed simplices in its (generalized) universal covering.

For the proof of Theorem 0.1 we show that (under the made assumptions) the action of the deck transformation group on a generalized universal covering space has a Borel-measurable fundamental domain. This might be of independent interest, here we use it to show in Section 2.6 that (in the case of countable fundamental groups) the homomorphism from measurable bounded cohomology to bounded cohomology is an isometric isomorphism.

We remark that the reader interested in Theorem 0.2 may skip Section 2 and just read Section 3 and Section 4.

Conventions: spaces of simplices will be equipped with the compact-open topology and "measurable" will always mean Borel-measurable with respect to that topology. "Measures" will always mean signed measures, i.e., differences of two non-negative measures. A "G-module" will always mean a Banach space V which is a module over the group ring $\mathbb{Z}G$ and such that $||gv|| \leq ||v||$ for all $g \in G, v \in V$.

1. Preliminaries

1.1. **Measure homology.** Let us start with recalling the definition of measure homology (or Milnor-Thurston-homology) from [30, Definition 1.8].

Definition 1.1. For a topological space X and $k \in \mathbb{N}$ we denote its set of singular k-simplices, i.e., of continuous maps from the standard simplex Δ^k to X, by $map(\Delta^k, X)$. We equip $map(\Delta^k, X)$ with the compact-opentopology and the corresponding σ -algebra of Borel sets.

Definition 1.2. For a topological space X and $k \in \mathbb{N}$ let

 $\mathcal{C}_k(X) = \left\{ \mu \mid \mu \text{ is a compactly determined measure on } map(\Delta^k, X), \parallel \mu \parallel < \infty \right\}.$

Here, a compactly determined measure is one that vanishes on any measurable subset of the complement of some (not necessarily measurable) compact set. (We follow the convention that a compact set need not be Hausdorff but satisfies the Heine-Borel covering property. Such sets are sometimes called quasicompact, therefore the definition in [30] speaks of quasicompactly determined measures.) The variation of a signed measure is $\|\mu\| := \max_A \mu(A) - \min_B \mu(B)$, where the maximum resp. minimum are taken over all measurable sets.

It is proved in [30, Corollary 2.9] that the canonical boundary operator $\partial = \sum_{i=0}^{k} \partial_i$ extends to an operator $d_k \colon \mathcal{C}_k(X) \to \mathcal{C}_{k-1}(X)$. Then one defines measure homology as

$$\mathcal{H}_k(X) = ker(d_k)/im(d_{k-1}).$$

1.2. Generalized universal covering spaces.

Definition 1.3. ([5, Section 1.1]) A generalized universal covering space of a path-connected topological space X is a topological space \widetilde{X} with a continuous surjection $p: \widetilde{X} \to X$ such that

(i) X is locally path-connected and simply-connected,

(ii) if Y is path-connected and locally path-connected, then every pointed

continuous map $f: (Y, y) \to (X, x)$ with $f_*(\pi_1(Y, y)) = 1$ admits unique pointed liftings, that is, for each $\tilde{x} \in p^{-1}(x)$ there is a unique pointed continuous map $g: (Y, y) \to (\tilde{X}, \tilde{x})$ with $p \circ g = f$.

A generalized universal covering space, if it exists, is in one-to-one correspondence with the homotopy classes of paths in X which emanate from a fixed $x_0 \in X$. (For more details see [5, Section 2].)

A generalized universal covering is a Serre fibration, thus one has $\pi_k \widetilde{X} \cong \pi_k X$ for $k \ge 2$, see [5, Section 1.2]. Moreover the deck transformation group of $p: \widetilde{X} \to X$ is isomorphic to the fundamental group $\pi_1 X$, and it acts freely and transitively on each fiber, see [5, Proposition 2.14].

For our arguments, the most important property of the generalized universal covering space will be that the lifts of a singular simplex $\sigma: \Delta^k \to X$ form exactly a *G*-orbit of singular simplices in \widetilde{X} , where $G \cong \pi_1(X, x_0)$ is the deck transformation group. Moreover the lifts of the simplices with all vertices in $x_0 \in X$ are exactly the simplices with vertices in $G\tilde{x}_0$, for a preimage $\tilde{x}_0 \in \widetilde{X}$ of x_0 .

1.3. Relatively injective modules and bounded cohomology.

Definition 1.4. For a topological space X we let

$$C_b^k(X) := B(map(\Delta^k, X), \mathbb{R}) = \left\{ f \colon map(\Delta^k, X) \to \mathbb{R} \mid f \text{ is bounded} \right\}$$

be the vector space of bounded cochains. It is a Banach space with the norm $||f|| = \sup \{|f(\sigma)|: \sigma \in map(\Delta^k, X)\}$. The usual coboundary operator

$$\delta_k f(\sigma) = \sum_{i=0}^k (-1)^i f(\partial_i \sigma)$$

makes $C_b^*(X)$ a cochain complex and its cohomology is denoted by $H_b^*(X)$ and called the bounded cohomology of X.

If X comes with an action of a group G, then $C_b^k(X)$ becomes a G-module via the induced action. In particular, if $\widetilde{X} \to X$ is a generalized universal covering space and $G \cong \pi_1(X, x_0)$ its group of deck transformations, then $C_b^k(\widetilde{X})$ is naturally understood as a G-module and this will always be meant when we refer to $C_b^k(\widetilde{X})$ as a G-module. For readers familiar with [22] we want to mention that, although $\pi_1(X, x_0)$ can be topologized as a nondiscrete topological group acting continuously on \widetilde{X} , this is not what we are going to do and we rather consider G as a discrete group. In particular, for the proof of Lemma 1.8 it will be sufficient to consider the module B(G, V)of bounded functions rather than the module of continuous, bounded functions and so we will not need the general results on continuous bounded cohomology from [22] but only the results on bounded cohomology from [14].

It is often useful to compute bounded cohomology via other resolutions. The general setting for this to work are strong resolutions by relatively injective modules.

Definition 1.5. Let G be a topological group. A G-module U is called relatively injective if any diagram of the form



can be completed. Here $i: V_1 \to V_2$ is an injective morphism of *G*-modules, $\sigma: V_2 \to V_1$ is a bounded (not necessarily *G*-equivariant) linear operator with $\sigma \circ i = id$ and $\|\sigma\| \leq 1$, α is a *G*-morphism, and we want β to be a *G*-morphism with $\beta \circ i = \alpha$ and $\|\beta\| \leq \|\sigma\|$.

Definition 1.6. A strong resolution of a G-module U is an exact sequence of G-modules and G-morphisms

$$0 \longrightarrow U \xrightarrow{\delta_{-1}} U_0 \xrightarrow{\delta_0} U_1 \xrightarrow{\delta_1} U_2 \xrightarrow{\delta_2} \cdots$$

for which there exists a sequence of linear (not necessarily *G*-equivariant) operators $\kappa_n : U_n \to U_{n-1}$ such that $\delta_{n-1}\kappa_n + \kappa_{n+1}\delta_n = id$ and $\|\kappa_n\| \leq 1$ for all $n \geq 0$ and $\kappa_0\delta_{-1} = id$.

According to [22, Lemma 7.2.6] the trivial G-module \mathbb{R} has a strong resolution by relatively injective G-modules, and any two such resolutions are chain homotopy equivalent. In particular the cohomology of the G-invariants of the resolution does not depend on the chosen resolution. This cohomology is, by definition, the continuous bounded cohomology of G, denoted by $H_{cb}^*(G)$. As said, we only consider the bounded cohomology $H_b^*(G)$ defined by equipping G with the discrete topology. We will need the following two facts, which can be found for example in [14] or in the more general setting of continuous bounded cohomology in [22].

Lemma 1.7. i) ([14, Lemma 3.2.2]) For any Banach space V, the G-module B(G, V) of bounded functions with values in V is relatively injective.

ii) ([14, Lemma 3.3.2]) Let

 $0 \to U \to U_1 \to U_2 \to \dots$

be a strong resolution of the G-module U and

$$0 \to V \to V_1 \to V_2 \to \dots$$

be a complex of relatively injective G-modules, then any G-morphism $U \rightarrow V$ can be extended to a G-morphism of complexes and any two such extensions are G-chain homotopic.

The following lemma is well-known for CW-complexes and more generally for semi-locally simply connected spaces, and we are going to show that the same proof also works for spaces that admit a generalized universal covering space in the sense of Section 1.2.

Lemma 1.8. Let $\widetilde{X} \to X$ be a generalized universal covering space and G its group of deck transformations. Then

$$0 \to \mathbb{R} \to C^0_b(\widetilde{X}) \to C^1_b(\widetilde{X}) \to C^2_b(\widetilde{X}) \to \dots$$

is a strong resolution by relatively injective G-modules. In particular one has an isometric isomorphism $H_b^*(X) = H_{cb}^*(G)$. **Proof:** We will prove this by copying the argument in the proof of [14, Theorem 4.1].

By Lemma 1.7i), B(G, V) is relatively injective for each Banach space V. By the axiom of choice there exists a set $F \subset \widetilde{X}$ meeting each G-orbit exactly once. Let $map((\Delta^k, v_0), (\widetilde{X}, F))$ be the set of those singular simplices which send the first vertex of the standard simplex to F. We make $B^k(\widetilde{X}, F) := B(map((\Delta^k, v_0), (\widetilde{X}, F)), \mathbb{R})$ a Banach space by equipping it with the sup-norm. Then there is an obvious isomorphism

$$C_b^k(\widetilde{X}) = B(G, B^k(\widetilde{X}, F))$$

and thus $C_b^k(\widetilde{X})$ is a relatively injective *G*-module.

By simple connectivity of \widetilde{X} and [14, Theorem 2.4] there is a contracting algebraic homotopy for $C_b^*(\widetilde{X})$. Hence we have a strong resolution.

2. Measurable bounded cohomology - proof of Theorem 1

2.1. **Definitions.** In the previous section we defined bounded cohomology, now we are going to define measurable bounded cohomology.

Definition 2.1. Let X be a topological space and again $map(\Delta^k, X)$ equipped with the compact-open-topology and the corresponding σ -algebra of Borelmeasurable sets. We let

$$\mathcal{C}_b^k(X) = \left\{ f \colon map(\Delta^k, X) \to \mathbb{R} \mid f \text{ is Borel measurable and bounded} \right\}$$

be the measurable bounded cochains.

The usual coboundary operator makes $\mathcal{C}_b^k(X)$ into a cochain complex and its cohomology is denoted by $\mathcal{H}_b^*(X)$, see [19, Section 3.4]. The inclusion ι induces a homomorphism

$$\iota^* \colon \mathcal{H}^*_b(X) \to H^*_b(X;\mathbb{R})$$

from the measurable bounded cohomology to the bounded cohomology.

2.2. Connecting the Gromov norm to measurable bounded cohomology. The following arguments are well-known, cf. [19, Section 3]. We will need them for the proof of Theorem 0.1.

For a topological space X there is an l^1 -norm on its singular chain complex $C_*(X; \mathbb{R})$ defined by $\|\sum_{i=1}^r a_i \sigma_i\|_1 = \sum_{i=1}^r |a_i|$. The Gromov norm on homology $H_*(X; \mathbb{R})$ is defined as $\|\alpha\| = \inf \{\|z\|_1 \colon [z] = \alpha\}$, i.e., one takes the infimum of the l^1 -norm over all cycles z representing the homology class α . We denote $NH_k(X) = \{\alpha \in H_k(X; \mathbb{R}) \colon \|\alpha\| = 0\}$.

Lemma 2.2. Let X be a topological space and $k \in \mathbb{N}$. If

$$\iota^* \colon \mathcal{H}^k_b(X) \to H^k_b(X;\mathbb{R})$$

is an epimorphism, then

$$ker(\iota_* \colon H_k(X; \mathbb{R}) \to \mathcal{H}_k(X)) \subset NH_k(X).$$

Proof: Assume there is some $\alpha \in H_k(X; \mathbb{R})$ with $\|\alpha\| \neq 0$ and $\iota_*(\alpha) = 0$. By [10, Section 1.1] the l^1 -norm on $H_k(X; \mathbb{R})$ is dual to the norm on $H_b^k(X)$, which for $\phi \in H_b^k(X)$ is defined as infimum of $\|f\|$ over all bounded cocycles f representing ϕ . In particular, there is some $\phi \in H_b^k(X)$ with $\langle \phi, \alpha \rangle = 1$. By assumption there is some $\psi \in \mathcal{H}_b^k(X)$ with $\iota^* \psi = \phi$. Then

$$1 = \langle \phi, \alpha \rangle = \langle \iota^* \psi, \alpha \rangle = \langle \psi, \iota_* \alpha \rangle = 0,$$

yielding a contradiction.

2.3. Construction of a measurable fundamental domain. The following Lemma 2.3 will be used in this paper for Proposition 2.7 in Section 2.6, though we think that it might be of independent interest.

Properly discontinuous group actions have a measurable fundamental domain, see [1, Chapter 7, Par. 2, Ex. 12]. However, the action of the group of deck transformations on a generalized universal covering space is in general not properly discontinuous. We are going to show that (under weak assumptions) one can nevertheless adapt the argument and obtain a measurable fundamental domain.

Lemma 2.3. Let X be a second-countable T_1 -space and assume that there is only an at most countable set of points, at which X is not semi-locally simply connected. If there exists a generalized universal covering space $p: \widetilde{X} \to X$, then the action of the deck transformation group $\Gamma \cong \pi_1(X, x_0)$ on \widetilde{X} has a Borel-measurable fundamental domain.

Proof:

Let N be the countably many points where X is not semi-locally simply connected. To any $x \in X \setminus N$ and each $\tilde{x} \in p^{-1}(x)$ there is an open neighborhood $\widetilde{U}_{\tilde{x}} \subset \widetilde{X}$ such that the restriction of p to that neighborhood is injective. (Namely one can take a neighborhood $V_x \subset X$ satisfying $im(\pi_1(V_x, x) \to \pi_1(X, x)) = 0$ and a connected component $\widetilde{U}_{\tilde{x}}$ of its preimage $p^{-1}(V_x)$. Note that this does not necessarily surject onto V_x . The intersection of $\widetilde{U}_{\tilde{x}}$ with any Γ -orbit has at most one element.)

For $x \in X$ choose some $\tilde{x} \in p^{-1}(x)$ and let $U_x = p(\widetilde{U_x}) \subset X$ be the image of $\widetilde{U_x}$. Second-countable spaces have the Lindelöf property and hence there is a countable family of U_x that covers X.

With these preparations we define a measurable fundamental domain as follows. Let $\{U_1, U_2, U_3, \ldots\}$ be an enumeration of the countable family of U_x 's and $\{\widetilde{U_1}, \widetilde{U_2}, \widetilde{U_3}, \ldots\}$ the corresponding subsets of \widetilde{X} . Then

$$W_1 = \widetilde{U_1}$$
$$W_2 = \widetilde{U_2} \cap (\widetilde{X} \setminus \Gamma \widetilde{U_1})$$
$$W_3 = \widetilde{U_3} \cap (\widetilde{X} \setminus (\Gamma \widetilde{U_1} \cup \Gamma \widetilde{U_2}))$$

are all Borel-measurable. (One should pay attention that we are using the possibly uncountable unions $\Gamma \tilde{U}_i$, however these are unions of open sets and

so no problem arises.) So

$$\bigcup_{n\in\mathbb{N}}W_n$$

is a measurable set, and one easily checks that it contains exactly one point from each Γ -orbit not meeting $p^{-1}(N)$. Adding one point of each of the countably many Γ -orbits in $p^{-1}(N)$ we obtain a fundamental domain.

We claim that the T_1 -property for X implies the T_1 -property for \widetilde{X} . Namely, as pointed out in [5, Lemma 2.10, Lemma 2.11] a space possessing a generalized universal covering space must be homotopically Hausdorff and then two points lying on the same fibre of p can be even separated in the T_2 -sense. For two points not lying on the same fibre of p, the analogue of the arguments contained in these lemmas for the T_2 -case (taking complete preimages of neighbourhoods with corresponding separation properties), gives for our assumption that the points can be at least separated in the T_1 -sense, giving the claim. Finally, the T_1 -property for \widetilde{X} implies that points are closed and their countable union is a Borel set, so that the constructed fundamental domain is Borel-measurable.

Remark. A more explicit construction of the fundamental domain may exist for spaces that satisfy a condition of negative curvature, that for the generalized universal covering space amounts to a global CAT(0)-condition. For such spaces we may hope that we can connect each point via the shortest geodesic to a base point, making a choice if there should be different geodesics of the same length. Then the domain covered by the lift of at least one of the chosen geodesics starting at one lift of the base point will form a fundamental domain. In a purely topological context, path systems that satisfy similar properties as CAT(0)-geodesics and could be used for analogous constructions, have been axiomatically described and introduced in [7],[8] under the name "arc-smooth systems". Actually, in our context, when adapting these conditions (that can only be satisfied for a kind of covering space) to the base space, we would be happy with a bit less. Instead of having one uniquely defined path between any two points, it would suffice to have for each point one uniquely defined path connecting to some base point, usually continuously depending on the other endpoint, but for a non-contractible base space there must be border-zones where this continuity-condition cannot be satisfied; such path-systems are sometimes called a "combing". In our case we would need a combing that is prefixclosed, i.e. each path starting on the trace of another combing path c or crossing the trace of another combing path c would have to follow the same trace as the path c to the base point. With one combing path starting in each point of the space, then the set covered by the lift of at least one of the combing paths, starting at one lift of the base-point, will form a fundamental domain, and for a sensible choice of the border-zones there is a chance that the result will be a measurable set.

2.4. Measurable coning construction. The following construction will later be applied to the (generalized) universal covering \widetilde{X} of a topological space X.

Definition 2.4. Let (\tilde{X}, x_0) be a pointed topological space. It is said to have a measurable (resp. continuous) coning construction if there is a sequence of Borel-measurable (resp. continuous) maps

$$L_i: map(\Delta^i, \widetilde{X}) \to map(\Delta^{i+1}, \widetilde{X})$$

such that for each $\sigma \in map(\Delta^i, \widetilde{X})$ the 0-th vertex of $L_i(\sigma)$ is x_0 and

$$\partial_0 L_i(\sigma) = \sigma$$

$$\partial_k L_i(\sigma) = L_{i-1}(\partial_{k-1}\sigma)$$
 for $k = 1, \dots, i+1$,

where by $\partial_k : map(\Delta^{i+1}, X) \to map(\Delta^i, X)$ for $k = 0, \ldots, i+1$ we mean the face map omitting the k-th vertex.

Lemma 2.5. A topological space \widetilde{X} has a continuous coning construction if it is contractible.

Proof: Assume \widetilde{X} is contractible. Then there is an $x_0 \in \widetilde{X}$ and a continuous map $H: \widetilde{X} \times [0,1] \to \widetilde{X}$ with $H(x,0) = x, H(x,1) = x_0$ for all $x \in \widetilde{X}$. For a singular *i*-simplex

$$\sigma\colon \Delta^i \to X$$

the map

$$h: \Delta^i \times [0,1] \to \widetilde{X}$$
$$(x,t) \to H(\sigma(x),t)$$

factors over the canonical projection

$$\Delta^i \times [0,1] \to \Delta^{i+1},$$

which collapses $\Delta^i \times \{1\}$ to the 0-th vertex of Δ^{i+1} . So the map h defines a singular (i+1)-simplex

$$L_i(\sigma) \colon \Delta^{i+1} \to X$$

and it is easy to check that this assignment has the desired properties.

2.5. Resolution by measurable bounded cochains. The following Lemma 2.6 will be a main ingredient in the proof of Theorem 0.1. Its proof is essentially copied from [14, Theorem 2.4], which proves the analogous result for (non-measurable) bounded cohomology.

Lemma 2.6. Let $\widetilde{X} \to X$ be a generalized universal covering space and G its group of deck transformations. Assume that \widetilde{X} is contractible. Then

$$0 \to \mathbb{R} \to \mathcal{C}^0_b(\widetilde{X}) \to \mathcal{C}^1_b(\widetilde{X}) \to \mathcal{C}^2_b(\widetilde{X}) \to \dots$$

is a strong resolution by G-modules, where the maps in the resolution are $\delta_{-1} \colon \mathbb{R} \to \mathcal{C}^0_b(\widetilde{X})$ sending real numbers to constant functions, and for $i \ge 0$ the coboundary operator $\delta_i \colon \mathcal{C}^i_b(\widetilde{X}) \to \mathcal{C}^{i+1}_b(\widetilde{X})$ from Definition 1.4.

Proof:

By Lemma 2.5 we have a measurable (even continuous) coning construction for a fixed base point $x_0 \in \widetilde{X}$. Dualizing Definition 2.4 via

$$\kappa^{i}(f))(\sigma) := f(L_{i-1}(\sigma))$$

for $i \ge 1$ yields homomorphisms

$$\kappa^i \colon \mathcal{C}^i_b(\widetilde{X}) \to \mathcal{C}^{i-1}_b(\widetilde{X})$$

for $i \geq 1$ such that

$$\delta_{i-1}\kappa^i + \kappa^{i+1}\delta_i = id$$

for all $i \geq 0$, where for i = 0 we define $\kappa^0 \colon \mathcal{C}^0_b(\widetilde{X}) \to \mathbb{R}$ by sending f to $f(x_0)$ for the chosen base point x_0 .

Because L_i sends each simplex to another simplex, we clearly have $\|\kappa^i\| \leq 1$ for all i.

2.6. ι^* is an isomorphism.

Proposition 2.7. Under the assumptions of Lemma 2.6, if one has a measurable fundamental domain for the action of G, and if moreover G is countable, then

$$\iota^* \colon \mathcal{H}^*_b(X) \to H^*_b(X)$$

is an isometric isomorphism.

Proof: We know that G-modules of the form B(G, V) (for a Banach space V) are relatively injective, see Lemma 1.7i). Measurability of the fundamental domain F and countability of G imply that we have an isomorphism

$$\mathcal{C}_b^k(\widetilde{X}) = B(G, \mathcal{B}^k(\widetilde{X}, F))$$

for

$$\mathcal{B}^{k}(\widetilde{X},F) = \left\{ f \colon map((\Delta^{k},v_{0}),(\widetilde{X},F)) \to \mathbb{R} \mid f \text{ is Borel measurable and bounded} \right\}$$

and thus relative injectivity of $\mathcal{C}_b^k(\widetilde{X})$. So

$$0 \to \mathbb{R} \to \mathcal{C}^0_b(\widetilde{X}) \to \mathcal{C}^1_b(\widetilde{X}) \to \mathcal{C}^2_b(\widetilde{X}) \to \dots$$

is a strong resolution by relatively injective modules and the claim follows in view of Lemma 1.7.

Remark. If G is not countable, then $\mathcal{C}_b^k(\widetilde{X})$ is a proper subset of $B(G, \mathcal{B}^k(\widetilde{X}, F))$ and we do not know whether it is relatively injective.

2.7. **Proof of Theorem 0.1.** The proof of Theorem 0.1 now follows from Lemma 2.2, Lemma 2.3 and Proposition 2.7.

2.8. Example.

Definition 2.8. Let Y be a metric space. We call a metric space X a convergent Y-space if it is a union

$$X = \bigcup_{n \in \mathbb{N}} Y_n \cup Y_\infty$$

with $Y_{\infty} = Y$ and for each $n \in N$ there is a homeomorphism

$$f_n: Y_\infty \to Y_n$$

such that there are finitely many points $y \in Y$ such that

$$f_n(y) = y \ \forall \ n \in \mathbb{N}$$

and for all other points y one has $f_n(y) \neq y$ and $f_n(y) \neq f_m(y)$ for all n, m, but

$$\lim_{n \to \infty} d(f_n(y), y) = 0$$

Lemma 2.9. If Y has the homotopy type of a countable CW-complex, then the homotopy groups of a convergent Y-space are countable.

Proof:

We use the well-known fact that the homotopy groups of a countable CWcomplex are countable, see [20, Theorem IV.6.1]. Although the convergent Y-space X is not locally path-connected, hence not a a CW-complex, one can find a locally path-connected space X^{lpc} with the same homotopy groups, as indicated in [16, Section 2.1]. Let $\mathcal{O} = \{V \subset X \text{ open}\}$ be the topology of X. For an open set $V \in \mathcal{O}$ and $x \in V$ let U(V, x) be the path component of V containing x. The sets U(V, x) for varying x and V form the basis of a topology \mathcal{O}^{lpc} on the set X. We denote the so-defined topological space by X^{lpc} . The identity map

$$id: X^{lpc} \to X$$

is continuous but in general not open. According to [16, Corollary 2.5] it induces isomorphisms

$$\pi_k(X^{lpc}) \cong \pi_k(X)$$

for all k. Under the assumptions of Lemma 2.9, X^{lpc} is a countable CW-complex, thus its homotopy groups are countable, and so are those of X.

Lemma 2.10. If Y is aspherical and has the homotopy type of a countable CW-complex, then a convergent Y-space is aspherical.

Proof: By the proof of Lemma 2.9 we know that $\pi_k(X^{lpc}) \cong \pi_k(X)$. Thus it suffices to prove asphericity for CW-complexes that are obtained by identifying finite subsets of countably many aspherical CW-complexes. Since the image of a sphere can only intersect finitely many cells of X^{lpc} it actually suffices to prove this for a union of finitely many aspherical CWcomplexes along finite subsets.

First consider the one-point union $Y_1 \vee Y_2$ of two path-connected, aspherical CW-complexes. There is a well-known construction (see [12, Prop. 4.64]), which to every map $f: A \to B$ associates a fibration $p: E_f \to B$ and a homotopy equivalence $A \to E_f$. Namely,

$$E_f = \left\{ (a, \gamma) \in A \times B^{[0,1]} \colon \gamma(0) = f(a) \right\}$$

and $p(a, \gamma) = \gamma(1)$. The fiber of p is called the homotopy fiber of f. In our setting, we see that the homotopy fiber of the inclusion

$$Y_1 \lor Y_2 \to Y_1 \times Y_2$$

is the union of $PY_1 \times \Omega Y_2$ and $\Omega Y_1 \times PY_2$ along their intersection $\Omega Y_1 \times \Omega Y_2$. (Here PY means the path space and ΩY the loop space.) For CW-complexes Y_1, Y_2 it is known that there is a weak homotopy equivalence w from the join $\Omega Y_1 * \Omega Y_2$ to the homotopy fiber of the inclusion

$$Y_1 \lor Y_2 \to Y_1 \times Y_2,$$

cf. the final paragraph of the proof of [9, Theorem 2.2]. If Y_1, Y_2 are aspherical, i.e., $\pi_k(Y_1) = \pi_k(Y_2) = 0$ for $k \ge 2$, then

$$\pi_k(\Omega Y_1) = \pi_{k+1}(Y_1) = 0$$
 and $\pi_k(\Omega Y_2) = \pi_{k+1}(Y_2) = 0$ for $k \ge 1$,

i.e., ΩY_1 and ΩY_2 are weakly homotopy equivalent to discrete spaces. The loop space of a countable CW-complex has the homotopy type of a CWcomplex by a theorem of Milnor, see [6, Corollary 5.3.7]. Thus a weak homotopy equivalence is actually a homotopy equivalence by Whitehead's theorem. So ΩY_1 and ΩY_2 have the homotopy type of discrete spaces, hence the join $\Omega Y_1 * \Omega Y_2$ has the homotopy type of a wedge of circles. In particular, $\Omega Y_1 * \Omega Y_2$ is aspherical and the weak homotopy equivalence w yields that also the homotopy fiber of

$$Y_1 \lor Y_2 \to Y_1 \times Y_2$$

is aspherical. Moreover, asphericity of Y_1 and Y_2 implies that $Y_1 \times Y_2$ is aspherical. This implies by the long exact sequence of homotopy groups

 $\ldots \to \pi_k(homotopy\ fiber) \to \pi_k(Y_1 \lor Y_2) \to \pi_k(Y_1 \times Y_2) \to \ldots$

that also $Y_1 \vee Y_2$ is aspherical.

Next, if we identify two vertices in the same path component of a CW-complex Y, then the resulting CW-complex is homotopy-equivalent to the one-point union $Y \vee S^1$. Since S^1 is aspherical, we obtain asphericity of $Y \vee S^1$ from asphericity of Y. Finally, by induction we can extend asphericity to the CW-complex obtained by identifying finite subsets.

Lemma 2.11. If Y is semi-locally simply connected and first-countable, then any convergent Y-space has a generalized universal covering.

Proof: The convergent Y-space X is semi-locally simply connected, but not locally path-connected. X^{lpc} is semi-locally simply connected and locally path-connected, thus it has a (classical) universal covering $\widetilde{X^{lpc}}$. We claim that $\widetilde{X^{lpc}}$ is a generalized universal covering of X.

According to [5, Proposition 5.1] (and the characterization of generalized universal coverings from [5, Section 1]) for a first-countable space it suffices to check the path lifting property for $\widetilde{X^{lpc}} \to X$. But any path in X lifts to a unique path in X^{lpc} (see [16, Corollary 2.5]), and thus (for a given lift of the initial point) to a unique path in $\widetilde{X^{lpc}}$.

Let us show how Theorem 0.1 can be applied at least to a special class of convergent Y-spaces.

Corollary 2.12. If Y is a compact Riemannian manifold of negative sectional curvature, and X is a convergent Y-space with $d(f_n(y), y) > 0$ for all but one $y \in Y$, then

$$\iota_k \colon H_k(X;\mathbb{R}) \to \mathcal{H}_k(X)$$

is injective in degrees $k \geq 2$.

Proof: The assumptions of Theorem 0.1 are satisfied, so it suffices to prove nontriviality of the Gromov norm in degrees $k \ge 2$.

By a well-known argument from [10, Section 1.1], nontriviality of the Gromov norm is implied if we have surjectivity of $H_b^k(X) \to H^k(X;\mathbb{R})$ in degrees $k \geq 2$. Namely, for a homology class α let β be a cohomology class with $\langle \beta, \alpha \rangle = 1$. Then surjectivity of $H_b^k(X) \to H^k(X;\mathbb{R})$ implies $\|\beta\| < \infty$, and from $1 \leq \|\beta\| \|\alpha\|$ we obtain $\|\alpha\| > 0$.

So we have to prove surjectivity of $H_b^k(X) \to H^k(X; \mathbb{R})$ in degrees $k \geq 2$. Let $x_0 \in X$ be the (by assumption) only one point along which the Y_n got identified with Y_{∞} . We call a 1-simplex in X straight if its intersections with x_0 decompose it into 1-simplices that are straight in one of the Y_n . It is well-known that 1-simplices in nonpositively curved manifolds are homotopic (rel. vertices) to a unique straight 1-simplex. This implies that 1-simplices in X are homotopic (rel. vertices and intersections with the x_i) to a unique straight 1-simplex in X.

Higher-dimensional straight simplices are then defined by succesively taking straight cones over straight subsimplices as in [10, Section 1.2]. Every simplex in Y_n is homotopic rel. vertices to a unique straight simplex. This implies that every simplex σ in X is homotopic rel. (vertices and intersections with the x_0) to a unique straight simplex $str(\sigma)$ in X. In particular we can straighten any cycle c recursively by straightening its k-skeleton for $k = 1, \ldots, dim(c)$. Dually this yields that any cocycle c is cohomologous to the "straightened" cocycle $c \circ str$.

The volume of straight simplices (of dimension ≥ 2) in negatively curved *n*-manifolds is uniformly bounded (see [10, Section 1.2] or [13, Proposition 1]) by a constant V(n, K) depending on the negative upper curvature bound K (which exists because the manifold is compact). From the proof of [17, Lemma 5] we know that for every simplex in X its straightening has at most one "central simplex" (in the terminology of [17]) and that all other parts of the straightened simplex are degenerate. In particular, the volume of $str(\sigma)$ equals the volume of the "central simplex", which lies in one of the Y_n and therefore satisfies the above upper bound on the volume.

This implies by [10, Section 1.2, Theorem (C)] that

$$||c||_{\infty} \le V(n, K) comass(\omega)$$

for a differential form ω representing c. Thus $c \circ str$ is a bounded cocycle for any cocycle c in degree ≥ 2 . In particular, $H_b^k(X) \to H^k(X; \mathbb{R})$ is surjective in degrees $k \geq 2$.

In Section 4.4 we will use Theorem 0.2 to obtain a more general result.

3. Reduction to simplices with all vertices in the basepoint

3.1. Eilenberg's argument. For a topological space X with basepoint x_0 we denote by $C_*(X)$ the complex of singular simplices, i.e., the chain complex whose k-th group is the free abelian group generated by $S_k(X) = map(\Delta^k, X)$ with the usual boundary operator and by $C_*^{x_0}(X) \subset C_*(X)$ the subcomplex generated by

$$S_k^{x_0}(X) = \left\{ \sigma \colon \Delta^k \to X \mid \sigma(v_0) = \sigma(v_1) = \ldots = \sigma(v_k) = x_0 \right\},\$$

where v_0, v_1, \ldots, v_k denote the vertices of the standard simplex Δ^k .

It is a classical result of Eilenberg (Corollary 31.2 in [3]) that for pathconnected X the inclusion

$$\iota \colon C^{x_0}_*(X) \to C_*(X)$$

is a chain homotopy equivalence. It is well-known that this dualizes to give chain homotopy equivalences also in cohomology and bounded cohomology. In this section we are going to show that (under a suitable assumption) the argument also yields chain homotopy equivalences for measure homology and measurable bounded cohomology.

Let us start with recalling Eilenberg's argument (which in [3] is given in a more general setting).

Lemma 3.1. For each path-connected space, there is a chain map $\eta_* : C_*(X) \to C_*^{x_0}(X)$ such that $\eta_{\ell} = id$ and a chain homotopy $s_* : C_*(X) \to C_{*+1}(X)$ such that

$$\partial s + s\partial = \iota \eta - id.$$

Proof: For $x \in S_0(X)$ we have to define $\eta_0(x) = x_0$.

Because X is path-connected we have a 1-simplex $s_0(x): \Delta^1 \to X$ with

$$\partial_0 s_0(x) = x_0, \partial_1 s_0(x) = x$$

for each $x \in X$. Let us fix a choice of $s_0(x)$ for each x.

Now we define η_* and s_* by induction on the dimension of simplices. Suppose they are already defined for all simplices in $S_{k-1}(X)$ and let $\sigma \in S_k(X)$. By induction hypothesis we have $\eta_{k-1}(\partial \sigma) \in C_{k-1}^{x_0}(X)$ and $s_{k-1}(\partial \sigma) \in C_k(X)$ such that

$$\eta_{k-1}(\partial \sigma) - \partial \sigma = \partial s_{k-1}(\partial \sigma) + s_{k-2}\partial(\partial \sigma) = \partial s_{k-1}(\partial \sigma).$$

We will inductively prove the slightly stronger statement that s_k is of the form $s_k = s_k^0 + \ldots + s_k^k$ and that the maps s_k^0, \ldots, s_k^k can be defined through some map $F: \Delta^k \times [0, 1] \to X$ via the canonical subdivision

$$\Delta^k \times [0,1] = \Delta_0 \cup \ldots \cup \Delta_k$$

as the restrictions of F to $\Delta_0, \ldots, \Delta_k$.

So consider $\Delta^k \times [0, 1]$. We can use σ to define a continuous map $\Delta^k \times \{0\} \to X$ and by the above inductive hypothesis we have

$$s_{k-1}(\partial\sigma) = (s_{k-1}^0 + \ldots + s_{k-1}^{k-1})(\partial\sigma)$$

defined through a continuous map $\partial \Delta^k \times [0,1] \to X$. These two maps agree on $\partial \Delta^k \times \{0\}$, so they define a continuous map

$$Q\colon \Delta^k \times \{0\} \cup \partial \Delta^k \times [0,1] \to X.$$

It is easy to construct a continuous map

$$P \colon \Delta^k \times [0,1] \to \Delta^k \times \{0\} \cup \partial \Delta^k \times [0,1]$$

which is the identity map on $\Delta^k \times \{0\} \cup \partial \Delta^k \times [0, 1]$. We can compose P with the before-defined map Q to obtain a continuous map

$$F: \Delta^k \times [0,1] \to X$$

that on $\Delta^k \times \{0\} \cup \partial \Delta^k \times [0, 1]$ agrees with Q. We use the canonical triangulation of $\Delta^k \times [0, 1]$ into k + 1 simplices to consider F as a formal sum of k + 1 simplices, which we denote by $s_k^0(\sigma), \ldots, s_k^k(\sigma)$. We obtain thus an element

$$s_k(\sigma) := s_k^0(\sigma) + \ldots + s_k^k(\sigma) \in C_{k+1}(X).$$

In particular $F|_{\Delta^k \times \{1\}}$ defines $\eta_k(\sigma) \in C_k(X)$ which actually belongs to $C_k^{x_0}(X)$ because all vertices are in x_0 . It is then clear by construction that the equality $\partial s_k(\sigma) + s_{k-1}(\partial \sigma) = \eta_k(\sigma) - \sigma$ holds.



The figure visualizes the construction of the map F in case of k = 1. It in particular shows that the two vertices (endpoints of the simplex $\Delta^1 \times \{1\}$) are taken under F to the same point $x_0 \in X$.

3.2. Pointed measure homology. We now want to argue that an analogous result as in Lemma 3.1 holds for measure homology, i.e., that (under suitable assumptions) the inclusion

$$\iota \colon \mathcal{C}^{x_0}_*(X) \to \mathcal{C}_*(X)$$

is a chain homotopy equivalence. Here $\mathcal{C}_*^{x_0}(X) \subset \mathcal{C}_*(X)$ means the subcomplex consisiting of those signed measures (of quasicompact determination set and bounded variation) which vanish on each measurable subset of the complement of $S_*^{x_0}(X)$.

Lemma 3.2. If X is a path-connected space that has a finite covering

$$X = \bigcup_{i=1}^{n} U_i$$

such that

- U_1, \ldots, U_n are Borel-measurable sets

- the closures $\overline{U}_1, \ldots, \overline{U}_n$ are contractible in X and compact, then for any $x_0 \in X$ there is a chain map $\eta_* : \mathcal{C}_*(X) \to \mathcal{C}_*^{x_0}(X)$ such that $\eta_i = id$, and a chain homotopy $s_* : \mathcal{C}_*(X) \to \mathcal{C}_{*+1}(X)$ such that

$$ds + sd = \iota\eta - id.$$

Proof: The natural approach to proving this statement would be to define η and s as in the proof of Lemma 3.1. One would have to check then that signed measures of compact determination set and bounded variation are mapped to signed measures of compact determination set and bounded variation.

It is clear that a so-constructed η_k does not increase the variation and that s_k multiplies the variation by at most k+1, so the second condition on boundedness of the variation will be satisfied.

To satisfy the first condition on compactness of the determination set it would be sufficient that η_k and the maps s_k^0, \ldots, s_k^k from the proof of Lemma 3.1 could be defined via some continuous maps on $map(\Delta^k, X)$, because then compact determination sets of simplices would be mapped to compact sets. In general it will not be possible to define such a continuous map. It would be possible if X were contractible. It is still possible on

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subsets that are contractible in X and our argument will make use of this fact.

Let
$$U \subset X$$
 be contractible in X. Then there is some continuous map
 $H: \overline{U} \times [0,1] \to X$ with $H(x,0) = x$ and $H(x,1) = x_0$ for all $x \in \overline{U}$. Define
 $s_0: map(\Delta^0, \overline{U}) \to map(\Delta^1, X)$

by

$$s_0(x)(t) = H(x,t)$$

upon identification $\Delta^1 = [0, 1]$. Continuity of H and compactness of [0, 1]imply that s_0 is continuous.

Now consider the by assumption existing covering $X = \bigcup_{i \in I} U_i$ by finitely many Borel sets whose closures are compact and contractible in X. (I is a finite index set.) W.l.o.g. we can assume that the U_i are disjoint. Indeed, if they were not, we could replace U_i by $V_i = U_i \setminus \bigcup_{i=1}^{i-1} U_j$ for $i \ge 2$. The closures \overline{V}_i are subsets of \overline{U}_i and hence again compact and contractible in X (although not necessarily in \overline{U}_i), and of course the V_i are again Borel sets.

For each ordered (k + 1)-tuple (i_0, \ldots, i_k) of (not necessarily distinct) elements of the index set I we let $S_{i_0,...,i_k}$ be the set of singular simplices with 0-th vertex in U_{i_0} , 1-st vertex in U_{i_1} , ..., k-th vertex in U_{i_k} and we consider its closure $\overline{S}_{i_0,\ldots,i_k}$ which is contained in the set of singular simplices with 0-th vertex in \overline{U}_{i_0} , 1-st vertex in \overline{U}_{i_1} , ..., k-th vertex in \overline{U}_{i_k} .

By the above we have defined η_0 and s_0 on $S_0 = U_0, \ldots, S_k = U_k$ (i.e., on all of X), such that the restriction to each S_i extends continuously to \overline{S}_i . Now we assume by induction that for all k-tuples (i_0, \ldots, i_{k-1}) we already have maps

$$\eta_{k-1} \colon S_{i_0,\dots,i_{k-1}} \to map(\Delta^{k-1}, X)$$

and

$$s_{k-1}^0, \dots, s_{k-1}^{k-1} \colon S_{i_0,\dots,i_{k-1}} \to map(\Delta^k, X)$$

with the desired properties and which all extend continuously to $\overline{S}_{i_0,\ldots,i_{k-1}}$. We claim that η_k and s_k^0, \ldots, s_k^k (defined as in the proof of Lemma 3.1) are again continuous maps on $\overline{S}_{i_0,\ldots,i_k}$ for each (k+1)-tuple (i_0,\ldots,i_k) . This is seen as follows. Continuity of $s_{k-1}^0,\ldots,s_{k-1}^{k-1}$ implies that the map

$$\overline{S}_{i_0,\ldots,i_{k-1}} \to map(\Delta^k \times \{0\} \cup \partial \Delta^k \times [0,1], X)$$

which sends $\sigma: \Delta^k \to X$ to the "union" of $\sigma \times \{0\}$ and $s^0_{k-1}(\partial_j \sigma), \ldots, s^{k-1}_{k-1}(\partial_j \sigma), j =$ $0, \ldots, k$, is continuous. Moreover, precomposition with the uniformly continuous map $P: \Delta^k \times [0,1] \to \Delta^k \times \{0\} \bigcup \partial \Delta^k \times [0,1]$ from the proof of Lemma 3.1 defines a continuous map

$$map(\Delta^k \times \{0\} \bigcup \partial \Delta^k \times [0,1], X) \to map(\Delta^k \times [0,1], X),$$

so we obtain a continuous map

$$\Phi \colon \overline{S}_{i_0,\dots,i_{k-1}} \to map(\Delta^k \times [0,1], X).$$

Since $\eta_k(\sigma)$ and $s_k^0(\sigma), \ldots, s_k^k(\sigma)$ are all defined by restricting $\Phi(\sigma)$ to subsets of $\Delta^k \times [0, 1]$, they also depend continuously on σ .

So we have proved that η_k and s_k^0, \ldots, s_k^k (defined on S_{i_0,\ldots,i_k}) can be extended continuously to $\overline{S}_{i_0,\ldots,i_k}$ (although this extension on $\overline{S}_{i_0,\ldots,i_k} \setminus S_{i_0,\ldots,i_k}$ of course does not have to agree with the actual definition of η_k and s_k^0, \ldots, s_k^k coming from some other S_{j_0,\ldots,j_k}). Since all the S_{i_0,\ldots,i_k} are pairwise disjoint, this allows a (not continuous but measurable) definition of η_k and s_k^0, \ldots, s_k^k on

$$map(\Delta^k, X) = \bigcup_{(i_0, \dots, i_k)} S_{i_0, \dots, i_k}.$$

For any compact subset $K \subset map(\Delta^k, X)$ we obtain that the image of $K \cap S_{i_0,...,i_k}$ under η or $s_0,...,s_k$ is contained in the image of $K \cap \overline{S}_{i_0,...,i_k}$ under some continuous extension of η_k or s_k^0, \ldots, s_k^k and thus is contained in a compact set. Hence the image of $K \cap S_{i_0,...,i_k}$ has compact closure. So the image of K under any of η and s_0, \ldots, s_k is a finite union of (subsets of) compact sets, hence has compact closure.

In particular, because the image of a determination set under any map is a determination set for the push-forward measure, η_k and s_k^0, \ldots, s_k^k map measures of compact determination set to measures of compact determination set.

Corollary 3.3. Under the assumptions of Lemma 3.2 every measure cycle is homologous to a measure cycle with determination set contained in $S_*^{x_0}(X)$.

Recall that we have defined bounded cohomology in Definition 1.4 and measurable bounded cohomology in Definition 2.1. Similarly one defines measurable cohomology. Let us denote by $H^*_{b,x_0}(X)$, $\mathcal{H}^*_{x_0}(X)$ and $\mathcal{H}^*_{b,x_0}(X)$ the cohomology groups of the complexes of bounded, measurable resp. bounded measurable functions from $C^{x_0}_{*}(X)$ to \mathbb{R} . Using [19, Section 3.4] there is a well-defined pairing between $\mathcal{H}^*_{b,x_0}(X)$ and $\mathcal{H}^{x_0}_{*}(X)$.

Corollary 3.4. Under the assumptions of Lemma 3.2, the canonical restriction induces isomorphisms

$$H_b^*(X) \to H_{b,x_0}^*(X)$$
$$\mathcal{H}_b^*(X) \to \mathcal{H}_{b,x_0}^*(X)$$
$$\mathcal{H}^*(X) \to \mathcal{H}_{x_0}^*(X)$$

Proof: The above constructed maps η and s are bounded in the sense that η_k sends a simplex to a simplex and s_k sends a k-dimensional simplex to a formal sum of (at most) k+1 (k+1)-dimensional simplices. This implies that η^* and s^* send bounded cochains to bounded cochains. Moreover η_k and s_k are continuous on each of the finitely many disjoint Borel sets $S_{i_0...i_k}$, so they are Borel-measurable on $map(\Delta^k, X)$ and hence η^* and s^* send measurable cochains.

3.3. **Examples.** Let us conclude with some examples fulfilling or not fulfilling the assumptions of Lemma 3.2:

Example. CW-complexes

Any compact manifold or finite CW-complex can be covered by finitely many measurable sets with contractible, compact closures. Thus the assumptions of Lemma 3.2 are satisfied.

Example. Hawaiian earring

The Hawaiian earring is the shrinking wedge of circles pictured in the introduction, that is, it can be written in the form

$$HE = \bigcup_{n=1}^{\infty} C_n \subset \mathbb{R}^2,$$

where $C_n \subset \mathbb{R}^2$ is the circle with center $(0, \frac{1}{n})$ and radius $\frac{1}{n}$. Let A_n^{\pm} be the intersection of C_n with the closure of the upper resp. lower half-plane, and let $A^{\pm} = \bigcup_{n=1}^{\infty} A_n^{\infty}$. Then

$$HE = A^+ \cup A^-$$

is a covering by two measurable sets with contractible, compact closures. Thus the assumptions of Lemma 3.2 are satisfied for the Hawaiian earring.

Example. Warsaw circle

The Warsaw circle is a closed subset $W \subset \mathbb{R}^2$, which is the union of the graph of the function

$$y = \sin(\frac{1}{x})$$

for $0 < x \leq 1$, the segment

$$Y = \{(x, y) \colon x = 0, -1 \le y \le 1\},\$$

and a curve connecting these two parts to get a path-connected space.

This space can be covered by finitely many contractible, measurable, relatively compact sets. The easiest way to do this is to use the decomposition

$$W = Y \cup Y^{c}$$

into Y and its complement. However the closure of Y^c is all of W, which is known to be not contractible.

On the other hand, W can be covered by countably many contractible, compact sets. For this one has to decompose the graph of $y = \sin(\frac{1}{x})$ into its segments for $\frac{1}{n+1} \le x \le \frac{1}{n}$ with n running through all natural numbers, and then add Y and the connecting curve as two more contractible, compact sets to the decomposition.

These two decompositions show that in Lemma 3.2 the assumption on having contractible closures and the assumption on finiteness of the covering can not be relaxed by just assuming contractibility of the relatively compact sets themselves or by countability of the covering, respectively. Indeed for the Warsaw circle W, the second author proved in [23, Theorem 4] that $H_0(W)$ is uncountable-dimensional, while of course $H_0^{x_0}(W) \simeq \mathbb{R}$.

The Warsaw circle does however not provide a counterexample to the conclusion of Theorem 0.2 in view of $H_0(W; \mathbb{R}) = \mathbb{R}$ and $H_n(W; \mathbb{R}) = 0$ for all n > 0.

Example. A space with non-injective canonical homomorphism

The following space can be covered by two contractible sets, but they are not Borel-measurable.

Let Z be the space constructed in [25, Section 5]. There are two points $z_0, z_1 \in Z$ such that

$$[z_1] - [z_0] \in ker(H_0(Z; \mathbb{R}) \to \mathcal{H}_0(Z)),$$

see [25, Theorem 5.7].

To get examples in any degree n, let F be a closed, orientable manifold of dimension $n \ge 1$ and consider $F \times Z$ with the "boundary" manifolds $F_0 = F \times \{z_0\}$ and $F_1 = F \times \{z_1\}$. Then

$$[F_1] - [F_0] \in ker(H_n(F \times Z; \mathbb{R}) \to \mathcal{H}_n(F \times Z)).$$

To get a path-connected example, let X be the space obtained by gluing one arc with the end points to the two different path components of $F \times Z$. This does not change the homology in degrees ≥ 2 and thus one has for $n \geq 2$:

$$[F_1] - [F_0] \in ker(H_n(X; \mathbb{R}) \to \mathcal{H}_n(X)).$$

This space satisfies the other assumptions from Lemma 3.2, but there is no finite covering by contractible, measurable sets, though a covering by two contractible, non-measurable sets exists.

4. Proof of Theorem 2

4.1. A simplicial construction: straightening. Recall that for a topological space X and a point $x_0 \in X$ we denote

$$S_k^{x_0}(X) = \left\{ \sigma \colon \Delta^k \to X \mid \sigma(v_0) = \sigma(v_1) = \ldots = \sigma(v_k) = x_0 \right\},\$$

where v_0, v_1, \ldots, v_k denote the vertices of the standard simplex Δ^k . Two simplices $\sigma_0, \sigma_1 \in S_k^{x_0}(X)$ are said to be homotopic rel. boundary if there exists a continuous map $F: \Delta^k \times [0, 1]$ with $F(x, 0) = \sigma_0(x), F(x, 1) = \sigma_1(x)$ for all $x \in \Delta^k$ and $F(x, t) = x_0$ for all $x \in \partial \Delta^k, t \in [0, 1]$.

Let us denote by $C_*^{x_0}(X) \subset C_*(X)$ for some fixed $x_0 \in X$ the subcomplex generated by $S_*^{x_0}(X)$. In the following lemma we define a topological analogue of the well-known geometric straightening which we used in the proof of Corollary 2.12. The construction replaces geodesics and straight simplices by a somewhat arbitrary selection of simplices. (Similar constructions in somewhat different settings can be found in [21, Theorem 9.5] and [2, Proposition 3.1].)

Lemma 4.1. Let X be a topological space and $x_0 \in X$.

i) There is a subset $S^{str}_*(X) \subset S^{x_0}_*(X)$ such that $S^{str}_k(X)$ contains one k-simplex in each homotopy class rel. boundary of simplices with all boundary faces in $S^{str}_{k-1}(X)$.

ii) There is a chain map

$$str: C^{x_0}_*(X) \to C^{x_0}_*(X)$$

which is chain homotopic to the identity and whose image lies in the chain complex $C^{str}_*(X) \subset C^{x_0}_*(X)$ spanned by the simplices in $S^{str}_*(X)$. If Y is a subspace of X with $x_0 \in Y$ such that $\pi_k(Y, x_0) \to \pi_k(X, x_0)$ is injective for all $k \ge 0$, then str can be chosen to map $C^{x_0}_*(Y)$ to $C^{str}_*(Y)$.

iii) If X is aspherical, then str is constant on homotopy classes rel. vertices.

iv) If X is locally contractible, then str extends to a chain map

$$str: \mathcal{C}^{x_0}_*(X) \to C^{str}_*(X).$$

Proof: We recursively construct a subcomplex $C^{str}_*(X) \subset C^{x_0}_*(X)$, whose simplices we call the "straight simplices", and a map $str: C^{x_0}_*(X) \to C^{str}_*(X)$, which we call the straightening map. This is similar to well-known constructions, which in slightly different settings can be found in [21, Theorem 9.5] and [2, Proposition 3.1]. We recall the construction for convenience of the reader and for later reference.

The 0-skeleton of $C^{str}_*(X)$ consists of the one vertex x_0 . For the 1-skeleton of $C^{str}_*(X)$ we choose one 1-simplex in each homotopy class (rel. vertices) of 1-simplices in $C^{x_0}_*(X)$, and we define str on 1-simplices by sending each of them to the unique straight simplices in its homotopy class. For each 1-simplex σ we fix a homotopy (rel. vertices) between σ and $str(\sigma)$.

Assume now that for some k > 1 we have already defined $S_{*\leq k-1}^{str}(X)$ and $str: C_{*\leq k-1}^{x_0}(X) \to C_{*\leq k-1}^{str}(X)$. For $S_k^{str}(X)$ we choose one k-simplex with all boundary faces in $S_{k-1}^{str}(X)$ inside each homotopy class (rel. boundary) of k-simplices in $S_k^{x_0}(X)$ with all boundary faces in $S_{k-1}^{str}(X)$.

For a simplex $\sigma \in C_k^{x_0}(X)$ we can assume by induction that we have defined $str(\partial \sigma)$ and that we have a homotopy between $\partial \sigma$ and $str(\partial \sigma)$. By the cofibration property of the inclusion $\partial \Delta^k \to \Delta^k$ this homotopy extends to a homotopy of σ keeping vertices fixed. Let σ' be the result of this homotopy. Among simplices with boundary $str(\partial \sigma)$ we have in the homotopy class (rel. boundary) of σ' exactly one simplex in $S_k^{str}(X)$. Define this simplex to be $str(\sigma)$. By construction we have a homotopy from σ to $str(\sigma)$ whose restriction to $\partial \sigma$ is a reparametrisation of the homotopy from $\partial \sigma$ to $str(\partial \sigma)$, which was given by the inductive hypothesis. This family of compatible homotopies yields the wanted chain homotopy between *id* and *str*.

For a pair (X, Y) one chooses straight simplices to be in Y whenever this is possible. The assumption on injectivity of $\pi_k(Y, x_0) \to \pi_k(X, x_0)$ implies that homotopies between simplices in Y can be chosen to remain in Y. Inductively, for a simplex σ in Y this applies (in the above recursive construction) to the homotopy between $str(\partial\sigma)$ and $\partial\sigma$, and then the extending homotopy yields a simplex in Y.

For (iii), if σ and σ' are homotopic rel. vertices, then the 1-skeleta of $str(\sigma)$ and $str(\sigma')$ agree. Assuming inductively that for some $k \geq 2$ the (k-1)-skeleta of $str(\sigma)$ and $str(\sigma')$ agree, we get from asphericity of X that the k-skeleta of both simplices are homotopic rel. boundary, and thus these k-skeleta must be equal by property (i). Proof by induction on k yields that $str(\sigma) = str(\sigma')$.

For iv), given $\mu \in \mathcal{C}^{x_0}_*(X)$, we note that for a Borel set $A \subset map(\Delta^k, X)$, its preimage $str^{-1}(A)$ is a countable union of homotopy classes, which are Borel sets by assumption. Thus we can define $str(\mu)$ by

$$str(\mu)(A) = \mu(str^{-1}(A)).$$

Its variation is bounded because

$$||str(\mu)|| = max_A \mu(str^{-1}(A)) - min_B \mu(str^{-1}(B)) \le ||\mu|| < \infty.$$

Local contractibility of X implies by [28] that $map(\Delta^k, X)$ is locally pathconnected. In fact, [28] has a more precise result about subsets of $map(\Delta^k, X)$ mapping subpolyhedra to given closed subspaces, from which we can in particular infer that the subset of $map(\Delta^k, X)$, consisting of simplices mapping all vertices to x_0 , is locally path-connected. By a theorem of Fox, path components correspond to homotopy classes. This means that every compact set $K \subset map(\Delta^k, X)$ meets only finitely many homotopy classes. So, if μ has compact determination set $K \subset map(\Delta^k, X)$, then str(K) is a finite set by (iii). Since $str(\mu)$ is determined on this finite set, it is a finite sum, i.e., an element of $C_*^{str}(X)$. Hence the image of str does indeed belong to the subspace $C_*^{str}(X)$ consisting of singular (rather than measure) chains.

Of course str need in general not be continuous or measurable.

4.2. **CW-neighborhoods.** We recall some well-known facts about CW-complexes from [12, Appendix A]. Let X be a CW-complex with k-skeleta X^k , and $A \subset X$ a sub-CW-complex.

An open neighborhood of A is defined with the help of a function ϵ that assigns a positive number $\epsilon_{\alpha} < 1$ to each cell e_{α}^{n} of X. The construction is inductive with $N_{\epsilon}^{0}(A) = A \cap X^{0}$. If a neighborhood $N_{\epsilon}^{n}(A)$ of $A \cap X^{n}$ in X^{n} is defined, then $N_{\epsilon}^{n+1}(A)$ is defined such that it contains all (n + 1)cells of A and such that for each (n + 1)-cell e_{α}^{n+1} in $X \setminus A$ its preimage under the the characteristic map $\Phi_{\alpha} \colon D^{n+1} \to X$ consists of a product $(1 - \epsilon_{\alpha}, 1] \times \Phi_{\alpha}^{-1}(N_{\epsilon}^{n}(A))$ with respect to spherical coordinates in D^{n+1} . Finally $N_{\epsilon}(A) = \bigcup_{n} N_{\epsilon}^{n}(A)$.

We are now considering the situation that X is a countable union of CW-complexes Y_n such that all intersections

$$Y_I := \bigcap_{i \in I} Y_i, I \subset \mathbb{N}$$

are sub-CW-complexes. We remark that CW-complexes are finite-dimensional and that proper intersections of CW-complexes have smaller dimensions than the intersecting complexes, thus there can be no infinite chains of intersections and every intersection contains a maximal (or in terms of complexes minimal) intersection. Hence, for an assignment ϵ as above we can construct $U := N_{\epsilon}(\bigcup_{I \subset \mathbb{N}, |I| \ge 2} Y_I) \subset X$. We assume that no subsequence of the ϵ_{α} converges to 0. For $I \subset \mathbb{N}$ with $|I| \ge 2$ we will denote $U_I \subset U$ the neighborhood of $\bigcap_{i \in I} Y_i$ in $\bigcup_{i \in I} Y_i$ and call this the CW-neighborhood. We remark that this set is not necessarily open in X, a typical example where this would not be the case is the Hawaiian earring. So we need to impose condition ii) in Theorem 0.2 as an additional assumption.

For CW-complexes, it is proved in [12, Proposition A.5.] that there is a deformation retraction $U_I \to Y_I$: during the time interval $\left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$ retract the *n*-cells of $U_I \setminus Y_I$ to the (n-1)-cells by sliding outward radial segments in the cells. For some slightly larger $\epsilon' > \epsilon$ the map can continuously be extended to the ϵ' -neighborhood while being the identity at the boundary of the ϵ' -neighborhood. Thus the deformation retractions $U_I \to Y_I$ extends to a continuous map $r: X \to X$ homotopic to the identity. r is clearly proper.

If each Y_n , but not necessarily their union X is a CW-complex, then the collection of ϵ_{α} -neighborhoods need not be an open set and r need not be continuous. Therefore in the statement of Theorem 0.1 we have added that

there exist such CW-neighborhoods, which are open sets, such that r is continuous.

Lemma 4.2. Let $X = \bigcup_{n \in \mathbb{N}}$ be a countable union of CW-complexes Y_n such that all intersections

$$Y_I := \bigcap_{i \in I} Y_i, I \subset \mathbb{N}$$

are sub-CW-complexes. Let U_I be the above-defined CW-neighborhoods for $|I| \geq 2$. Then there is a continuous, proper map $r: X \to X$, homotopic to the identity, that maps U_I to Y_I for each $I \subset \mathbb{N}$ with $|I| \geq 2$.

In what follows we will call the above-constructed U_I the CW-neighborhood of Y_I . Under a further condition we can prove the following lemma.

The reason why one needs condition i) should be apparent from the following example. Assume that the CW-complexes Y_n are connected and that for each $n \ge 2$ the intersection $Y_n \cap Y_{n+1}$ contains a path $[x_n, y_{n+1}]$ such that the sequence (y_n) converges to some $y \in Y_1$. (It is easy to construct such an example.) Inside Y_n we can connect y_n to x_n by a path. We can then define a continuous path that on shorter and shorter intervals approaching 0 agrees with the above paths and in 0 takes y. For this 1-simplex, the conclusion of Lemma 4.3 fails.

Lemma 4.3. Let the assumptions of Lemma 4.2 be satisfied and assume in addition that no $x \in X$ is a limit of a sequence $x_{\nu} \in Y_{I_{\nu}}$ for pairwise distinct $I_{\nu} \subset \mathbb{N}$. Then any (continuous) singular simplex $\sigma \colon \Delta^k \to X$ satisfies the following condition:

for each $p \in \Delta^k$, if we denote $I(p) \subset \mathbb{N}$ the maximal index set with $\sigma(p) \in Y_{I(p)}$, then there is an open neighborhood $U(p) \subset \Delta^k$, such that restriction of σ gives a well-defined and continuous map $U(p) \to U_{I(p)}$.

Proof: It suffices to check this for k = 1 because a counterexample with k > 1 would contain embedded 1-simplices contradicting the claim for k = 1.

We have a countable, closed covering $\mathcal{U}_1 = \{\sigma^{-1}(Y_i)\}$. The sets in this covering may be disconnected, so let us look at their path components. Define an equivalence relation on [0,1] by declaring $P \sim Q$ if and only if there exists a sequence $P = p_0, p_1, \ldots, p_n = Q$ such that p_i and p_{i+1} belong to the same path components of a set $A_i \in \mathcal{U}_1$, for $i = 0, \ldots, n-1$. Let \mathcal{U}_2 be the covering by equivalence classes of \sim . This is a disjoint covering by path-connected subsets of [0,1], which must be either points, closed, open or half-open intervals. However, the assumption of Lemma 4.3 rules out the possibility that any $U \in \mathcal{U}_2$ can be an open or half-open interval. So we have a covering by disjoint points and closed intervals.

Consider the quotient space $[0,1] / \sim$ with the countable covering by the disjoint sets $\sigma^{-1}(Y_i) / \sim$. Since equivalence classes of \sim are points or closed intervals, the quotient space $[0,1] / \sim$ is again homeomorphic to [0,1]. On the other hand, the sets $\sigma^{-1}(Y_i) / \sim$ are totally disconnected and thus have empty interior. So we obtain a covering of [0,1] by countably many sets of empty interior. But this contradicts Baire's category theorem.

4.3. Proof of injectivity. In this section we prove Theorem 0.2. That is, we assume that X can be covered by countably many finite, aspherical

CW-complexes $Y_n, n \in \mathbb{N}$ such that $x_0 \in Y_n$ for all n and such that all intersections $Y_I := \bigcap_{i \in I} Y_i, I \subset \mathbb{N}$ are sub-CW-complexes with open CWneighborhoods are open, further that the assumption of Lemma 4.3 holds, and that inclusions induce an isomorphism $\bigoplus_{n \in \mathbb{N}} H_k(Y_n) = H_k(X)$, and we want to prove that $\iota_* \colon H_k(X; \mathbb{R}) \to \mathcal{H}_k(X)$ is injective.

Proof: Assume that ι_* is not injective, which means that there is some $\alpha \in H_k(X, \mathbb{R})$, represented by a cocycle $z \in C_k(X, \mathbb{R})$ such that $z = d\mu$ for some measure chain $\mu \in \mathcal{C}_{k+1}(X)$.

Since all intersections $Y_I := \bigcap_{i \in I} Y_i$ with $|I| \ge 2$ are sub-CW-complexes of each $Y_i, i \in I$, they have open neighborhoods $Y_I \subset U_I \subset X$ as discussed in Section 4.2, which deformation retract on Y_I . Moreover, for |I| = 1, that is $I = \{i\}$, we define $U_I = Y_i$. We obtain a covering of X by the sets U_I .

Some of these sets (those with |I| = 1) are not open in X. However, Lemma 4.3 that for a singular simplex in X for every point in its image there exists some open neighborhood contained in some U_I . In particular after sufficiently fine subdivision any subsimplex will be contained in some U_I .

Since μ is compactly determined, there exists an $m \in \mathbb{N}$ such that the m-th barycentric subdivision $sd^m\mu$ is determined on simplices with image contained in one of the U_I . The classical construction of a chain homotopy between sd^m and the identity (on singular chains) implies that $\partial sd^m\mu = sd^m z$ is a cycle in the homology class α .

By Lemma 4.2 there is a continuous, proper map $r: X \to X$ homotopic to the identity, which sends each U_I to Y_I . Thus $r_*(sd^m\mu)$ is a measure chain, whose boundary $\partial r_*(sd^m\mu) = r_*(sd^mz)$ is a cycle in the homology class α .

Since $sd^m\mu$ is determined on simplices with image in some U_I , it follows that $r_*(sd^m\mu)$ is determined on simplices with image in some Y_I , in particular in some Y_i . So we can decompose it as a sum

$$r_*(sd^m\mu) = \sum_{i\in\mathbb{N}}\mu_i,$$

where each μ_i is a measure chain on Y_i (it is compactly determined on Y_i because Y_i is closed in X), and $\partial \mu_i = z_i$ with the possibly infinite sum $\sum_{i \in \mathbb{N}} z_i$ yielding (possibly after cancelation of infinitely many summands) a finite sum of singular chains representing the homology class α .

Because Y_i (other than X) satisfies the assumptions of Lemma 3.2 we have that each μ_i is chain homotopic to some measure chain $\overline{\mu}_i \in C^{x_0}_*(Y_i)$. Its boundary is again a singular cycle chain homotopic to z_i . Application of the straightening procedure from Lemma 4.1 yields a measure chain $str(\overline{\mu}_i) \in C^{str}_*(Y_i)$.

But CW-complexes are locally contractible (see [12, Proposition A.4]) and by part iv) of Lemma 4.1 it follows that $str(\overline{\mu}_i)$ is a singular chain. Its boundary is a singular chain $str(\overline{z}_i)$ homologous to z_i .

A priori, $\sum_{i=1}^{\infty} str(\overline{\mu}_i)$ is a countable sum of singular chains, thus not necessarily a singular chain. However we know, by assumption, that the homology is a direct sum $H_k(X) = \bigoplus H_k(Y_i)$. Therefore there is a finite set $F \subset \mathbb{N}$ such that $\sum_{i \in F} z_i$ is homologous to z. Because of $\sum_{i \in F} \partial str(\overline{\mu}_i) =$ $\sum_{i \in F} z_i$ we obtain a (finite) singular chain $\sum_{i \in F} str(\overline{\mu}_i)$ with

$$\partial(\sum_{i\in F} str(\overline{\mu}_i)) \sim z.$$

In particular $\alpha = [z] = 0 \in H_*(X; \mathbb{R})$, which proves injectivity of ι_* .

We remark that the proof in particular applies to CW-complexes and then gives a simpler proof than the one given in [19]. The point of the simplification is that using the results from Section 3 we could reduce the problem to the simpler situation of simplices having all their vertices in the basepoint.

In fact, in the above setting we can also prove the existence of a measurable section, which was the main technical lemma in [19]. Namely, let $G \simeq \pi_1(X, x_0)$ be the (by assumption countable) deck transformation group of the generalized universal covering $p: \widetilde{X} \to X$. The lift of x_0 to \widetilde{X} is a *G*-orbit $G\tilde{x}_0$ for some \tilde{x}_0 , see Section 1.2. The lift of a homotopy class (rel. x_0) of *n*-simplices is a homotopy class inside $S_*^{\gamma_0\tilde{x}_0,\ldots,\gamma_n\tilde{x}_0}(\widetilde{X})$, by which we mean the set of simplices mapping their *i*-th vertex to $\gamma_i\tilde{x}_0$ for $i = 0,\ldots,n$. Clearly the projection

$$p\colon S^{\gamma_0\tilde{x}_0,\ldots,\gamma_n\tilde{x}_0}_*(\widetilde{X}) \to S^{x_0}_*(X)$$

maps a homotopy class homeomorphically onto its image, which is also a homotopy class. In particular, the restriction of p to any homotopy class has a continuous right-inverse s defined on the image of that homotopy class. Thus we get a right-inverse s defined on each of the homotopy classes downstairs. Since the homotopy classes of simplices (rel. vertices) are Borel sets by assumption, and there are only countably many homotopy classes, the so-defined s yields a measurable map

$$s: S^{x_0}_*(X) \to S^{G\tilde{x}_0}_*(X)$$

right-inverse to p.

The following example shows that no such section can exist when the fundamental group is uncountable, like for the Hawaiian Earrings.

Lemma 4.4. Let X be a complete, separable metric space admitting a generalized universal covering $p: \widetilde{X} \to X$. Assume that X is semi-locally simply connected at x_0 and that the deck group $G \simeq \pi_1(X, x_0)$ is uncountable. Then $p_1: S_1^{G\tilde{x}_0} \to S_1^{x_0}$ admits no Borel section.

Proof: X being semi-locally simply connected at x_0 implies that $G\tilde{x}_0$ is discrete in \tilde{X} . Then one can show that the sets $S_1^{g_1\tilde{x}_0,g_2\tilde{x}_0}$ are open and closed, so that we have decomposed $S_1^{G\tilde{x}_0}$ into a disjoint union of open and closed sets.

G is uncountable, hence the cardinality of its power set is bigger than continuum. For a measurable section s and any subset $\Gamma \subset G$ let U_{Γ} be the set of 1-simplices with the same endpoints as $s(\gamma_g)$, where γ_g means the loop representing g (and $s(\gamma_g)$ may be discontinuous, but for U_{Γ} we consider only continuous maps). As a disjoint union of sets of the form $S_1^{g_1\tilde{x}_0,g_2\tilde{x}_0}$, any U_{Γ} must be open and hence have Borel-measurable preimage under s. Thus the $s^{-1}(U_{\Gamma})$ are more than continuum many distinct Borel-measurable subsets of $map(\Delta^1, X)$.

But the latter is a Polish space by [15, Theorem 4.19] and thus has only continuum many Borel sets by [26, Theorem 3.3.18]. This is a contradiction.

4.4. Example: convergent Y-spaces. Recall from Section 2.8 the Definition 2.8 of convergent Y-spaces for a metric space Y.

For Y = [0, 1] with the exactly two points 0 and 1 along which the identification is done, this yields the convergent arc space drawn in the introduction.

Such a convergent Y-space space will usually not be a CW-complex even if Y is: although X inherits a cell decomposition from that of Y, the accumulation property $\lim_{n\to\infty} d(f_n(y), y) = 0$ implies that X does not have the weak topology with respect to that cell decomposition.

The assumption on nonvanishing of the Gromov norm, that we required in Theorem 0.1 is in general not satisfied for convergent Y-spaces. For example, it does not hold for the convergent arc space, or when the Gromov norm on Y is not an actual norm, e.g., when Y is simply connected.

We mention that for Y homeomorphic to a simplicial complex (e.g. a smooth manifold, see [29]) and X a convergent Y-space, for all k, the homotopy classes (rel. vertices) of k-simplices with vertices in x_0 are Borel sets in $map(\Delta^k, X)$. Thus the assumptions of Lemma 3.2 are satisfied in this case. We will not include the rather lengthy argument because it is not needed for the proof of Corollary 4.5.

Corollary 4.5. If the aspherical metric space Y is homeomorphic to a finite CW-complex and if X is a convergent Y-space, then

$$\iota_* \colon H_k(X; \mathbb{R}) \to \mathcal{H}_k(X)$$

is injective.

Proof: For $k \ge 2$ we have $H_k(X) = \bigoplus_{i \in \mathbb{N}} H_k(Y_i)$ and can therefore apply Theorem 0.2. For k = 1 we have the same ad-hoc argument as for the convergent arcs space in [24, Section 2.2].

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