ON MEASURE HOMOLOGY OF MILDLY WILD SPACES

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Abstract. We prove injectivity of the canonical map from singular homology to measure homology for certain "mildly wild" spaces, that is, certain spaces not having the homotopy type of a CW-complex, but having countable fundamental groups.

Measure homology $\mathcal{H}_*(X)$, also called Milnor-Thurston homology, of a space X is a variant of the usually studied singular homology groups $H_*(X;\mathbb{R})$. While the latter are defined as the homology theory of the chain complex of finite linear combinations of singular simplices with the canonical boundary operator, measure homology uses the chain complex of quasicompactly determined signed measures of bounded variation on the space of singular simplices $map(\Delta^k, X)$, again with the canonical boundary operator (see Section 1.1).

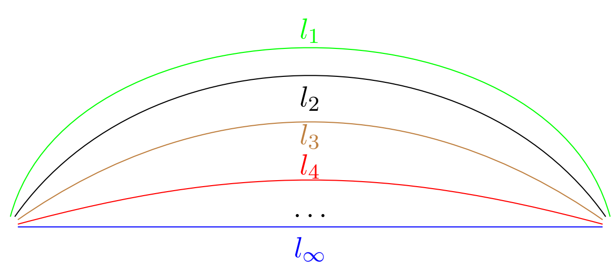
Singular chains (with real coefficients) can be considered as finite sums of (real multiples of) Dirac measures, so there is a canonical homomorphism

$$\iota_*\colon H_*(X;\mathbb{R}) \to \mathcal{H}_*(X).$$

It was proven in [27] and [10] that measure homology satisfies the Eilenberg-Steenrod axioms and thus that ι_* is an isomorphism whenever X is a CW-complex.

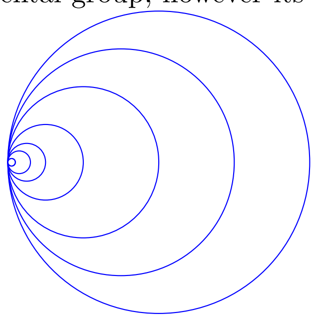
An example from [23] shows that ι_* is not always injective. However, the example constructed there is in some sense an artificial one: it relies on the existence of non-measurable sets and ultimately on the axiom of choice. So one may ask whether for more natural spaces one can still prove injectivity as it holds for CW-complexes.

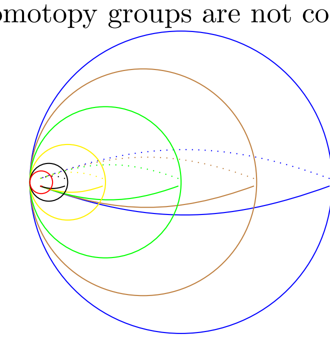
The picture below shows the convergent arcs space CA. It is formed by one arc l_∞ and a sequence of arcs $(l_n)_{n\in\mathbb{N}}$ with the same endpoints as l_∞ and pointwise converging to l_∞. Although the arcs provide a natural cell decomposition, CA is not a CW-complex because its topology is not the weak topology from the cell decomposition: the union $\bigcup_{n\in\mathbb{N}} l_n$ is not a closed subset.



It was shown in [27, Section 6] that $H_1(CA) \to \mathcal{H}_1(CA)$ is not surjective. On the other hand, [22, Theorem 2.8] computes the measure homology of CA and the proof implies in particular that $H_*(CA) \to \mathcal{H}_*(CA)$ is injective.

The convergent arcs space is a "mildly wild" space in the sense that it is semi-locally simply connected and that it has countable fundamental group. When we collapse the arc l_∞ to a point, then we obtain the Hawaiian earring (pictured below to the left), which is not semi-locally simply connected connected and which has uncountable (and very complicated) fundamental group. This is an example of a "really wild" space. More generally, one could consider shrinking wedges of manifolds: the Hawaiian earring is the shrinking wedge of circles, and the Barratt-Milnor sphere (pictured below to the right) is the shrinking wedge of spheres. The Barratt-Milnor sphere is semi-locally simply connected and has countable (actually trivial) fundamental group, however its higher homotopy groups are not countable.





Although ultimately we would like to say something about injectivity of ι_* for "really wild" spaces of uncountable fundamental group like the Hawaiian earring, in this paper we will pursue a more modest goal: we will prove injectivity of the canonical homomorphism for two classes of "mildly wild" spaces, i.e., spaces which have countable fundamental group and thus a fortiori are semi-locally simply connected. So our results do not apply to the Hawaiian earring, but they apply to various generalizations of the convergent arc space.

The proofs of our two cases are independent and will use different methods.

The first result is the following.

Theorem 0.1. *Let X be a topological space, which is T_1, second countable, has countable fundamental group and admits a contractible generalized universal covering space $\widetilde{X}$ in the sense of [7].*

Then the kernel of $\iota_*\colon H_*(X;\mathbb{R}) \to \mathcal{H}_*(X)$ is contained in the zero-norm subspace with respect to the Gromov norm on $H_*(X;\mathbb{R})$.

In particular, if for some k the Gromov norm on k-th homology is an actual norm, that is $\|x\| \neq 0$ for all $x \neq 0$, then $\iota_*\colon H_*(X;\mathbb{R}) \to \mathcal{H}_*(X)$ is injective.

We will recall the definition of the Gromov norm in Section 2.2. The assumption on non-vanishing of the Gromov norm seems to be a more severe restriction than the others. may be more useful.

Theorem 0.2. *Let X be an aspherical topological space with a basepoint $x_0 \in X$. Assume that*

- the fundamental group $\pi_1(X, x_0)$ is countable,

- for all $k \geq 1$, the homotopy classes (rel. vertices) of k-simplices with vertices in x_0 are Borel sets in $map(\Delta^k, X)$,

- X can be covered by finitely many Borel sets whose closures are compact and contractible in X.

Then $\iota_*\colon H_*(X;\mathbb{R}) \to \mathcal{H}_*(X)$ is injective.

An example of spaces to which we apply these results is the following class of spaces which are not CW-complexes and are sort of a generalization of the convergent arcs space CA. We call a metric space X *a convergent Y-space* for a pointed metric space (Y, y_0) if it is a union

$$X = (\bigcup_{n\in\mathbb{N}} Y_n) \cup Y_\infty$$

with $Y_\infty = Y$ and for each $n \in N$ there is a pointed homeomorphism $f_n\colon (Y_\infty, y_0) \to (Y_n, y_0)$ with $\lim_{n\to\infty} d(f_n(y), y) = 0$ for all $y \in Y$ and such that for all but finitely many $y \in Y$ one has $d(f_n(y), y) > 0$ and $d(f_n(y), f_m(y)) > 0$ for all n, m.

This kind of example would satisfy the assumptions of Theorem 0.1 (in particular that about non-vanishing of the Gromov norm) only under additional assumptions (e.g., when Y is a negatively curved manifold, see Section 2.8). However we will see in Section 4.5 that it satisfies all assumptions of Theorem 0.2 whenever Y is an aspherical smooth manifold, and we thus get injectivity of ι_* for any convergent Y-space X.

Technically, the main new idea for the proof of Theorem 0.2 is that we can reduce the problem of computing measure homology (and the dual notion of measurable cohomology) to computing it for the subcomplex of simplices with all vertices in a given basepoint (Lemma 3.2 and Corollary 3.4).

When applied to CW-complexes our argument is similar but simpler than the one in [17] which did not restrict to simplices with vertices in a basepoint and therefore needed a larger effort to prove the technical [17, Lemma A.1] on existence of a measurable section. Our argument, together with countability of the fundamental group, actually also provides such a measurable section from pointed simplices in X to pointed simplices in its (generalized) universal covering.

For the proof of Theorem 0.1 we show that (under the made assumptions) the action of the deck transformation group on a generalized universal covering space has a Borel-measurable fundamental domain. This might be of independent interest, here we use it to show in Section 2.6 that (in the case of countable fundamental groups) the homomorphism from measurable bounded cohomology to bounded cohomology is an isometric isomorphism.

In Theorem 0.2 we are imposing a condition that X can be covered by finitely many Borel sets of compact closure contractible in X. This may look like a technical condition, but it can actually not be avoided as an example discussed in Section 4.4 shows.

We remark that the reader interested in Theorem 0.2 may skip Section 2 and just read Section 3 and Section 4.

Conventions: spaces of simplices will be equipped with the compact-open topology and "measurable" will always mean Borel-measurable with respect to that topology. "Measures" will always mean signed measures, i.e., differences of two non-negative measures. A "G-module" will always mean a Banach space V which is a module over the group ring $\mathbb{Z}G$ and such that $\parallel gv \parallel \leq \parallel v \parallel$ for all $g \in G, v \in V$.

1. Preliminaries

1.1. Measure homology.

Let us start with recalling the definition of measure homology (or Milnor-Thurston-homology) from [27, Definition 1.8].

Definition 1.1. For a topological space X and $k \in \mathbb{N}$ we denote its set of singular k-simplices, i.e., of continuous maps from the standard simplex Δ^k to X, by $map(\Delta^k, X)$. We equip $map(\Delta^k, X)$ with the compact-open-topology and the corresponding σ-algebra of Borel sets.

Definition 1.2. For a topological space X and $k \in \mathbb{N}$ let

$$\mathcal{C}_k(X) = \left\{ \mu \mid \mu \text{ is a compactly determined measure on } map(\Delta^k, X), \parallel \mu \parallel < \infty \right\}.$$

Here, a compactly determined measure is one that vanishes on any measurable subset of the complement of some (not necessarily measurable) compact set. (We follow the convention that a compact set need not be Hausdorff but satisfies the Heine-Borel covering property. Such sets are sometimes called quasicompact, therefore the definition in [27] speaks of quasicompactly determined measures.) The variation of a signed measure is $\|\mu\| := \max_A \mu(A) - \min_B \mu(B)$, where the maximum resp. minimum are taken over all measurable sets.

It is proved in [27, Corollary 2.9] that the canonical boundary operator $\partial = \sum_{i=0}^k \partial_i$ extends to an operator $d_k : \mathcal{C}_k(X) \to \mathcal{C}_{k-1}(X)$. Then one defines measure homology as

$$\mathcal{H}_k(X) = ker(d_k)/im(d_{k-1}).$$

1.2. Generalized universal covering spaces.

Definition 1.3. ([7, Section 1.1]) A generalized universal covering space of a path-connected topological space X is a topological space $\widetilde{X}$ with a continuous surjection $p \colon \widetilde{X} \to X$ such that
(i) $\widetilde{X}$ is locally path-connected and simply-connected,
(ii) if Y is path-connected and locally path-connected, then every pointed continuous map $f \colon (Y, y) \to (X, x)$ with $f_*(\pi_1(Y, y)) = 1$ admits unique pointed liftings, that is, for each $\tilde{x} \in p^{-1}(x)$ there is a unique pointed continuous map $g \colon (Y, y) \to (\widetilde{X}, \tilde{x})$ with $p \circ g = f$.

A generalized universal covering space, if it exists, is in one-to-one correspondence with the homotopy classes of paths in X which emanate from a fixed $x_0 \in X$. (For more details see [7, Section 2].)

A generalized universal covering is a Serre fibration, thus one has $\pi_k \widetilde{X} \cong \pi_k X$ for $k \geq 2$, see [7, Section 1.2]. Moreover the deck transformation group of $p \colon \widetilde{X} \to X$ is isomorphic to the fundamental group $\pi_1 X$, and it acts freely and transitively on each fiber, see [7, Proposition 2.14].

For our arguments, the most important property of the generalized universal covering space will be that the lifts of a singular simplex $\sigma \colon \Delta^k \to X$

form exactly a G-orbit of singular simplices in $\widetilde{X}$, where $G \cong \pi_1(X, x_0)$ is the deck transformation group. Moreover the lifts of the simplices with all vertices in $x_0 \in X$ are exactly the simplices with vertices in $G\tilde{x}_0$, for a preimage $\tilde{x}_0 \in \widetilde{X}$ of x_0.

1.3. Relatively injective modules and bounded cohomology.

Definition 1.4. For a topological space X we let

$$C_b^k(X) := B(map(\Delta^k, X), \mathbb{R}) = \left\{ f\colon map(\Delta^k, X) \to \mathbb{R} \mid f \text{ is bounded} \right\}$$

be the vector space of bounded cochains. It is a Banach space with the norm $\|f\| = sup\left\{ |f(\sigma)|\colon \sigma \in map(\Delta^k, X) \right\}$. The usual coboundary operator

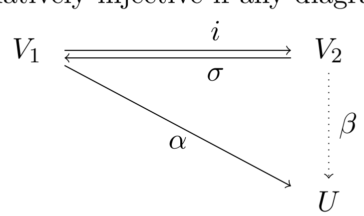
$$\delta_k f(\sigma) = \sum_{i=0}^{k} (-1)^i f(\partial_i \sigma)$$

makes $C_b^*(X)$ a cochain complex and its cohomology is denoted by $H_b^*(X)$ and called the bounded cohomology of X.

If X comes with an action of a group G, then $C_b^k(X)$ becomes a G-module via the induced action. In particular, if $\widetilde{X} \to X$ is a generalized universal covering space and $G \cong \pi_1(X, x_0)$ its group of deck transformations, then $C_b^k(\widetilde{X})$ is naturally understood as a G-module and this will always be meant when we refer to $C_b^k(\widetilde{X})$ as a G-module. For readers familiar with [20] we want to mention that, although $\pi_1(X, x_0)$ can be topologized as a non-discrete topological group acting continuously on $\widetilde{X}$, this is not what we are going to do and we rather consider G as a discrete group. In particular, for the proof of Lemma 1.8 it will be sufficient to consider the module $B(G, V)$ of bounded functions rather than the module of continuous, bounded functions and so we will not need the general results on continuous bounded cohomology from [20] but only the results on bounded cohomology from [13].

It is often useful to compute bounded cohomology via other resolutions. The general setting for this to work are strong resolutions by relatively injective modules.

Definition 1.5. Let G be a topological group. A G-module U is called relatively injective if any diagram of the form



can be completed. Here $i\colon V_1 \to V_2$ is an injective morphism of G-modules, $\sigma\colon V_2 \to V_1$ is a bounded (not necessarily G-equivariant) linear operator with $\sigma \circ i = id$ and $\|\sigma\| \leq 1$, α is a G-morphism, and we want β to be a G-morphism with $\beta \circ i = \alpha$ and $\|\beta\| \leq \|\sigma\|$.

Definition 1.6. A strong resolution of a G-module U is an exact sequence of G-modules and G-morphisms

$$0 \longrightarrow U \xrightarrow{\delta_{-1}} U_0 \xrightarrow{\delta_0} U_1 \xrightarrow{\delta_1} U_2 \xrightarrow{\delta_2} \cdots$$

for which there exists a sequence of linear (not necessarily G-equivariant) operators $\kappa_n \colon U_n \to U_{n-1}$ such that $\delta_{n-1}\kappa_n + \kappa_{n+1}\delta_n = id$ and $\|\kappa_n\| \leq 1$ for all $n \geq 0$ and $\kappa_0\delta_{-1} = id$.

According to [20, Lemma 7.2.6] the trivial G-module $\mathbb{R}$ has a strong resolution by relatively injective G-modules, and any two such resolutions are chain homotopy equivalent. In particular the cohomology of the G-invariants of the resolution does not depend on the chosen resolution. This cohomology is, by definition, the continuous bounded cohomology of G, denoted by $H^*_{cb}(G)$. As said, we only consider the bounded cohomology $H^*_b(G)$ defined by equipping G with the discrete topology. We will need the following two facts, which can be found for example in [13] or in the more general setting of continuous bounded cohomology in [20].

Lemma 1.7. *i) ([13, Lemma 3.2.2]) For any Banach space V, the G-module $B(G,V)$ of bounded functions with values in V is relatively injective.*

ii) ([13, Lemma 3.3.2]) Let

$$0 \to U \to U_1 \to U_2 \to \ldots$$

be a strong resolution of the G-module U and

$$0 \to V \to V_1 \to V_2 \to \ldots$$

be a complex of relatively injective G-modules, then any G-morphism $U \to V$ can be extended to a G-morphism of complexes and any two such extensions are G-chain homotopic.

The following lemma is well-known for CW-complexes and more generally for semi-locally simply connected spaces, and we are going to show that the same proof also works for spaces that admit a generalized universal covering space in the sense of Section 1.2.

Lemma 1.8. *Let $\widetilde{X} \to X$ be a generalized universal covering space and G its group of deck transformations. Then*

$$0 \to \mathbb{R} \to C^0_b(\widetilde{X}) \to C^1_b(\widetilde{X}) \to C^2_b(\widetilde{X}) \to \ldots$$

is a strong resolution by relatively injective G-modules. In particular one has an isometric isomorphism $H^*_b(X) = H^*_{cb}(G)$.

Proof: We will prove this by copying the argument in the proof of [13, Theorem 4.1].

By Lemma 1.7i), the G-module of bounded functions $B(G,V)$ is relatively injective for each Banach space V.

By the axiom of choice there exists a set $F \subset \widetilde{X}$ meeting each G-orbit exactly once. Let $map((\Delta^k, v_0), (\widetilde{X}, F))$ be the set of those singular simplices which send the first vertex of the standard simplex to F. We make $B^k(\widetilde{X}, F) := B(map((\Delta^k, v_0), (\widetilde{X}, F)), \mathbb{R})$ a Banach space by equipping it with the sup-norm. Then there is an obvious isomorphism

$$C^k_b(\widetilde{X}) = B(G, B^k(\widetilde{X}, F))$$

and thus $C_b^k(\widetilde{X})$ is a relatively injective G-module.

By simple connectivity of $\widetilde{X}$ and [13, Theorem 2.4] there is a contracting algebraic homotopy for $C_b^*(\widetilde{X})$. Hence we have a strong resolution.

■

2. Measurable bounded cohomology - proof of Theorem 1

2.1. Definitions.

Definition 2.1. Let X be a topological space and again $map(\Delta^k, X)$ equipped with the compact-open-topology and the corresponding σ-algebra of Borel-measurable sets. We let

$$\mathcal{C}_b^k(X) = \left\{ f\colon map(\Delta^k, X) \to \mathbb{R} \mid f \text{ is Borel measurable and bounded} \right\}$$

be the measurable bounded cochains.

The usual coboundary operator makes $\mathcal{C}_b^k(X)$ into a cochain complex and its cohomology is denoted by $\mathcal{H}_b^*(X)$, see [17, Section 3.4]. The inclusion ι induces a homomorphism

$$\iota^*\colon \mathcal{H}_b^*(X) \to H_b^*(X)$$

from the measurable bounded cohomology to the bounded cohomology.

2.2. Connecting the Gromov norm to measurable bounded cohomology.

The following arguments are well-known, cf. [17, Section 3]. We will need them for the proof of Theorem 0.1.

For a topological space X there is an l^1-norm on its singular chain complex $C_*(X;\mathbb{R})$ defined by $\|\sum_{i=1}^r a_i\sigma_i\|_1 = \sum_{i=1}^r |a_i|$. The Gromov norm on homology $H_*(X;\mathbb{R})$ is defined as $\|\alpha\| = \inf\{\|z\|_1\colon [z]=\alpha\}$, i.e., one takes the infimum of the l^1-norm over all cycles z representing the homology class α. We denote $NH_k(X) = \{\alpha \in H_k(X;\mathbb{R})\colon \|\alpha\| = 0\}$.

Lemma 2.2. *Let X be a topological space and $k \in \mathbb{N}$. If*

$$\iota^*\colon \mathcal{H}_b^k(X) \to H_b^k(X)$$

is an epimorphism, then

$$ker(\iota_*\colon H_k(X;\mathbb{R}) \to \mathcal{H}_k(X)) \subset NH_k(X).$$

Proof: Assume there is some $\alpha \in H_k(X;\mathbb{R})$ with $\|\alpha\| \neq 0$ and $\iota_*(\alpha) = 0$. By [9, Section 1.1] the l^1-norm on $H_k(X;\mathbb{R})$ is dual to the norm on $H_b^k(X)$, which for $\phi \in H_b^k(X)$ is defined as infimum of $\|f\|$ over all bounded cocycles f representing ϕ. In particular, there is some $\phi \in H_b^k(X)$ with $\langle \phi, \alpha \rangle = 1$. By assumption there is some $\psi \in \mathcal{H}_b^k(X)$ with $\iota^*\psi = \phi$. Then

$$1 = \langle \phi, \alpha \rangle = \langle \iota^*\psi, \alpha \rangle = \langle \psi, \iota_*\alpha \rangle = 0,$$

yielding a contradiction. ■

2.3. **Construction of a measurable fundamental domain.** The following Lemma 2.3 will be used in this paper for Proposition 2.7 in Section 2.6, though we think that it might be of independent interest.

Properly discontinuous group actions have a measurable fundamental domain, see [3, Chapter 7, Par. 2, Ex. 12]. However, the action of the group of deck transformations on a generalized universal covering space is in general not properly discontinuous. We are going to show that (under weak assumptions) one can nevertheless adapt the argument and obtain a measurable fundamental domain.

Lemma 2.3. *Let X be a second-countable T_1-space and assume that there is only an at most countable set of points, at which X is not semi-locally simply connected. If there exists a generalized universal covering space $p\colon \widetilde{X} \to X$, then the action of the deck transformation group $\Gamma \cong \pi_1(X, x_0)$ on $\widetilde{X}$ has a Borel-measurable fundamental domain.*

Proof:

Let N be the countably many points where X is not semi-locally simply connected. To any $x \in X \setminus N$ and each $\widetilde{x} \in p^{-1}(x)$ there is an open neighborhood $\widetilde{U_{\tilde{x}}} \subset \widetilde{X}$ such that the restriction of p to that neighborhood is injective. (Namely one can take a neighborhood $V_x \subset X$ satisfying $im(\pi_1(V_x, x) \to \pi_1(X, x)) = 0$ and a connected component $\widetilde{U_{\tilde{x}}}$ of its preimage $p^{-1}(V_x)$. Note that this does not necessarily surject onto V_x. The intersection of $\widetilde{U_{\tilde{x}}}$ with any Γ-orbit has at most one element.)

For $x \in X$ choose some $\tilde{x} \in p^{-1}(x)$ and let $U_x = p(\widetilde{U_{\tilde{x}}}) \subset X$ be the image of $\widetilde{U_{\tilde{x}}}$. Second-countable spaces have the Lindelöf property and hence there is a countable family of U_x that covers X.

With these preparations we define a measurable fundamental domain as follows. Let $\{U_1, U_2, U_3, \ldots\}$ be an enumeration of the countable family of U_x's and $\left\{\widetilde{U_1}, \widetilde{U_2}, \widetilde{U_3}, \ldots\right\}$ the corresponding subsets of $\widetilde{X}$. Then

$$W_1 = \widetilde{U_1}$$
$$W_2 = \widetilde{U_2} \cap (\widetilde{X} \setminus \Gamma\widetilde{U_1})$$
$$W_3 = \widetilde{U_3} \cap (\widetilde{X} \setminus (\Gamma\widetilde{U_1} \cup \Gamma\widetilde{U_2}))$$
$$\ldots$$

are all Borel-measurable. (One should pay attention that we are using the possibly uncountable unions $\Gamma\widetilde{U}_i$, however these are unions of open sets and so no problem arises.) So

$$\bigcup_{n \in \mathbb{N}} W_n$$

is a measurable set, and one easily checks that it contains exactly one point from each Γ-orbit not meeting $p^{-1}(N)$. Adding one point of each of the countably many Γ-orbits in $p^{-1}(N)$ we obtain a fundamental domain.

We claim that the T_1-property for X implies the T_1-property for $\widetilde{X}$. Namely, as pointed out in [7, Lemma 2.10, Lemma 2.11] a space possessing a generalized universal covering space must be homotopically Hausdorff and then two points lying on the same fibre of p can be even separated in the T_2-sense. For two points not lying on the same fibre of p, the analogue of the arguments contained in these lemmas for the T_2-case (taking complete preimages of neighbourhoods with corresponding separation properties), gives for our assumption that the points can be at least separated in the T_1-sense, giving the claim. Finally, the T_1-property for $\widetilde{X}$ implies that points are closed and their countable union is a Borel set, so that the constructed fundamental domain is Borel-measurable. ■

Remark. A more explicit construction of the fundamental domain may exist for spaces that satisfy a condition of negative curvature, that for the generalized universal covering space amounts to a global CAT(0)-condition. For such spaces we may hope that we can connect each point via the shortest geodesic to a base point, making a choice if there should be different geodesics of the same length. Then the domain covered by the lift of at least one of the combing path starting at one lift of the base point will form a fundamental domain. In a purely topological context, path systems that satisfy similar properties as CAT(0)-geodesics and could be used for analogous constructions, have been axiomatically described and introduced in [1],[2] under the name "arc-smooth systems". Actually, in our context, when adapting these conditions (that can only be satisfied for a kind of covering space) to the base space, we would be happy with a bit less. Instead of having one uniquely defined path between any two points, it would suffice to have for each point one uniquely defined path connecting to some base point, usually continuously depending on the other endpoint, but for a non-contractible base space there must be border-zones where this continuity-condition cannot be satisfied; such path-systems are sometimes called a "combing". In our case we would need a combing that is prefix-closed, i.e. each path starting on the trace of another combing path c or crossing the trace of another combing path c would have to follow the same trace as the path c to the base point. With one combing path starting in each point of the space, then the set covered by the lift of at least one of the combing paths, starting at one lift of the base-point, will form a fundamental domain, and for a sensible choice of the border-zones there is a chance that the result will be a measurable set.

2.4. Measurable coning construction.

Definition 2.4. Let $(\widetilde{X}, x_0)$ be a pointed topological space. It is said to have a measurable (resp. continuous) coning construction if there is a sequence of Borel-measurable (resp. continuous) maps

$$L_i : map(\Delta^i, \widetilde{X}) \to map(\Delta^{i+1}, \widetilde{X})$$

such that the 0-th vertex of $L_i(\sigma)$ is x_0 and

$$\partial_0 L_i(\sigma) = \sigma$$

$$\partial_k L_i(\sigma) = L_{i-1}(\partial_{k-1}\sigma), k = 1, \ldots, i+1$$

for each $\sigma \in map(\Delta^i, \widetilde{X})$, where by $\partial_k\colon map(\Delta^{i+1}, X) \to map(\Delta^i, X)$ for $k = 0, \ldots, i+1$ we mean the face map omitting the k-th vertex.

Lemma 2.5. *A topological space $\widetilde{X}$ has a continuous coning construction if it is contractible.*

Proof: Assume $\widetilde{X}$ is contractible. Then there is an $x_0 \in \widetilde{X}$ and a continuous map $H\colon \widetilde{X} \times [0,1] \to \widetilde{X}$ with $H(x,0) = x, H(x,1) = x_0$ for all $x \in \widetilde{X}$. For a singular i-simplex

$$\sigma\colon \Delta^i \to \widetilde{X}$$

the map

$$h\colon \Delta^i \times [0,1] \to \widetilde{X}$$

$$(x,t) \to H(\sigma(x),t)$$

factors over the canonical projection

$$\Delta^i \times [0,1] \to \Delta^{i+1},$$

which collapses $\Delta^i \times \{1\}$ to the 0-th vertex of Δ^{i+1}. So the map h defines a singular $(i+1)$-simplex

$$L_i(\sigma)\colon \Delta^{i+1} \to X$$

and it is easy to check that this assignment has the desired properties. ■

2.5. **Resolution by measurable bounded cochains.** The following Lemma 2.6 will be a main ingredient in the proof of Theorem 0.1. Its proof is essentially copied from [13, Theorem 2.4], which proves the analogous result for (non-measurable) bounded cohomology.

Lemma 2.6. *Let $\widetilde{X} \to X$ be a generalized universal covering space and G its group of deck transformations. Assume that $\widetilde{X}$ is contractible. Then*

$$0 \to \mathbb{R} \to \mathcal{C}_b^0(\widetilde{X}) \to \mathcal{C}_b^1(\widetilde{X}) \to \mathcal{C}_b^2(\widetilde{X}) \to \dots$$

is a strong resolution by G-modules.

Proof:

By Lemma 2.5 we have a measurable (even continuous) coning construction. Dualizing Definition 2.4 via

$$(\kappa^i(f))(\sigma) := f(L_i(\sigma))$$

yields homomorphisms

$$\kappa^i\colon \mathcal{C}_b^i(\widetilde{X}) \to \mathcal{C}_b^{i-1}(\widetilde{X})$$

such that

$$\delta\kappa + \kappa\delta = id.$$

■

2.6. ι^* **is an isomorphism.**

Proposition 2.7. *Under the assumptions of Lemma 2.6, if one has a measurable fundamental domain for the action of G and if moreover G is countable, then*

$$\iota^*\colon \mathcal{H}_b^*(X) \to H_b^*(X)$$

is an isometric isomorphism.

Proof: We know that G-modules of the form $B(G,V)$ (for a Banach space V) are relatively injective, see Lemma 1.7i). Measurability of the fundamental domain F and countability of G imply that we have an isomorphism

$$\mathcal{C}_b^k(\widetilde{X}) = B(G, \mathcal{B}^k(\widetilde{X}, F))$$

for

$$\mathcal{B}^k(\widetilde{X}, F) = \left\{ f\colon map((\Delta^k, v_0), (\widetilde{X}, F)) \to \mathbb{R} \mid f \text{ is Borel measurable and bounded} \right\}$$

and thus relative injectivity of $\mathcal{C}_b^k(\widetilde{X})$. So

$$0 \to \mathbb{R} \to \mathcal{C}_b^0(\widetilde{X}) \to \mathcal{C}_b^1(\widetilde{X}) \to \mathcal{C}_b^2(\widetilde{X}) \to \dots$$

is a strong resolution by relatively injective modules and the claim follows in view of Lemma 1.7. ■

Remark. If G is not countable, then $\mathcal{C}_b^k(\widetilde{X})$ is a proper subset of $B(G, \mathcal{B}^k(\widetilde{X}, F))$ and we do not know whether it is relatively injective.

2.7. **Proof of Theorem 0.1.** The proof of Theorem 0.1 now follows from Lemma 2.2, Lemma 2.3 and Proposition 2.7.

2.8. **Example.**

Definition 2.8. Let (Y, y_0) be a pointed metric space. We call a metric space X a convergent Y-space if it is a union

$$X = \bigcup_{n \in \mathbb{N}} Y_n \cup Y_\infty$$

with $Y_\infty = Y$ and for each $n \in N$ there is a pointed homeomorphism

$$f_n\colon (Y_\infty, y_0) \to (Y_n, y_0)$$

such that

$$\lim_{n \to \infty} d(f_n(y), y) = 0$$

for all $y \in Y$ and such that for all but finitely many $y \in Y$ one has $d(f_n(y), y) > 0$ and $d(f_n(y), f_m(y)) > 0$ for all n, m.

Lemma 2.9. *If Y has the homotopy type of a countable CW-complex, then the homotopy groups of a convergent Y-space are countable.*

Proof:

We use the well-known fact that the homotopy groups of a countable CW-complex are countable, see [18, Theorem IV.6.1]. Although the convergent Y-space X is not locally path-connected, hence not a a CW-complex, one can find a locally path-connected space X^{lpc} with the same homotopy groups, as indicated in [15, Section 2.1]. Let $\mathcal{O} = \{V \subset X \text{ open}\}$ be the topology of X. For an open set $V \in \mathcal{O}$ and $x \in V$ let $U(V,x)$ be the path component of V containing x. The sets $U(V,x)$ for varying x and V form the basis of a topology $\mathcal{O}^{lpc}$ on the set X. We denote the so-defined topological space by X^{lpc}. The identity map

$$id\colon X^{lpc} \to X$$

is continuous but in general not open. According to [15, Corollary 2.5] it induces isomorphisms

$$\pi_k(X^{lpc}) \cong \pi_k(X)$$

for all k. Under the assumptions of Lemma 2.9, X^{lpc} is a countable CW-complex, thus its homotopy groups are countable, and so are those of X. ■

Lemma 2.10. *If Y is aspherical and has the homotopy type of a countable CW-complex, then a convergent Y-space is aspherical.*

Proof: By the proof of Lemma 2.9 we know that $\pi_k(X^{lpc}) \cong \pi_k(X)$. Thus it suffices to prove asphericity for CW-complexes that are obtained by identifying finite subsets of countably many aspherical CW-complexes. Since the image of a sphere can only intersect finitely many cells of X^{lpc} it actually suffices to prove this for a union of finitely many aspherical CW-complexes along finite subsets.

First consider the one-point union $Y_1 \vee Y_2$ of two path-connected, aspherical CW-complexes. Asphericity of Y_1 and Y_2 implies that $Y_1 \times Y_2$ is aspherical. The homotopy faser of the inclusion $Y_1 \vee Y_2 \to Y_1 \times Y_2$ is the union of $PY_1 \times \Omega Y_2$ and $\Omega Y_1 \times PY_2$ along their intersection $\Omega Y_1 \times \Omega Y_2$. (Here PY means the path space and ΩY the loop space.) For CW-complexes Y_1, Y_2 it is known that this homotopy fiber is homotopy equivalent to the join $\Omega Y_1 * \Omega Y_2$, cf. the proof of [8, Theorem 2.2]. If Y_1, Y_2 are aspherical, then ΩY_1 and ΩY_2 have the homotopy type of discrete spaces, hence the join $\Omega Y_1 \times \Omega Y_2$ has the homotopy type of a wedge of circles. In particular, the homotopy fiber of $Y_1 \vee Y_2 \to Y_1 \times Y_2$ is aspherical. This implies by the long exact sequence of homotopy groups that $Y_1 \vee Y_2$ is aspherical.

Next, if we identify two vertices in the same path component of a CW-complex Y, then the resulting CW-complex is homotopy-equivalent to the one-point union $Y \vee S^1$. Since S^1 is aspherical, we obtain asphericity of $Y \vee S^1$ from asphericity of Y. Finally, by induction we can extend asphericity to the CW-complex obtained by identifying finite subsets. ■

Lemma 2.11. *If Y is semi-locally simply connected and first-countable, then any convergent Y-space has a generalized universal covering.*

Proof: The convergent Y-space X is semi-locally simpy connected, but not locally path-connected. X^{lpc} is semi-locally simply connected and locally path-connected, thus it has a (classical) universal covering $\widetilde{X^{lpc}}$. We claim that $\widetilde{X^{lpc}}$ is a generalized universal covering of X.

According to [7, Proposition 5.1] (and the characterization of generalized universal coverings from [7, Section 1]) for a first-countable space it suffices to check the path lifting property for $\widetilde{X^{lpc}} \to X$. But any path in X lifts to a unique path in X^{lpc} (see [15, Corollary 2.5]), and thus (for a given lift of the initial point) to a unique path in $\widetilde{X^{lpc}}$.

■

Let us show how Theorem 0.1 can be applied at least to a special class of convergent Y-spaces.

Corollary 2.12. *If Y is a compact Riemannian manifold of negative sectional curvature, and X is a convergent Y-space with $d(f_n(y), y) > 0$ for all but one $y \in Y$, then*

$$\iota_k \colon H_k(X; \mathbb{R}) \to \mathcal{H}_k(X)$$

is injective in degrees $k \geq 2$.

Proof: The assumptions of Theorem 0.1 are satisfied, so it suffices to prove nontriviality of the Gromov norm in degrees $k \geq 2$.

By a well-known argument from [9, Section 1.1], nontriviality of the Gromov norm is implied if we have surjectivity of $H_b^k(X) \to H^k(X)$ in degrees $k \geq 2$. Namely, for a homology class α let β be the dual cohomology class with $\langle \beta, \alpha \rangle = 1$. Then surjectivity of $H_b^k(X) \to H^k(X)$ implies $\|\beta\| < \infty$, and from $1 \leq \|\beta\| \|\alpha\|$ we obtain $\|\alpha\| > 0$.

So we have to prove surjectivity of $H_b^k(X) \to H^k(X)$ in degrees $k \geq 2$. Let $x_0 \in X$ be the (by assumption) only one point along which the Y_n got identified with Y_∞. We call a 1-simplex in X straight if its intersections with x_0 decompose it into 1-simplices that are straight in one of the Y_n. It is well-known that 1-simplices in nonpositively curved manifolds are homotopic (rel. vertices) to a unique straight 1-simplex. This implies that 1-simplices in X are homotopic (rel. vertices and intersections with the x_i) to a unique straight 1-simplex in X.

Higher-dimensional straight simplices are then defined by succesively taking straight cones over straight subsimplices as in [9, Section 1.2]. Every simplex in Y_n is homotopic rel. vertices to a unique straight simplex. This implies that every simplex σ in X is homotopic rel. (vertices and intersections with the x_0) to a unique straight simplex $str(\sigma)$ in X. In particular we can straighten any cycle c recursively by straightening its k-skeleton for $k = 1, \ldots, dim(c)$. Dually this yields that any cocycle c is cohomologous to the "straightened" cocycle $c \circ str$.

The volume of straight simplices (of dimension ≥ 2) in negatively curved manifolds is uniformly bounded (see [9, Section 1.2] or [12, Proposition 1]). From the proof of [16, Lemma 5] we know that for every simplex in X its straightening has at most one "central simplex" (in the terminology of [16]) and that all other parts of the straightened simplex are degenerate. In particular, the volume of $str(\sigma)$ equals the volume of the "central simplex", which lies in one of the Y_n and therefore satisfies the above upper bound on the volume.

This implies that $c \circ str$ is a bounded cocycle for any cocycle c in degree ≥ 2. In particular $H_b^k(X) \to H^k(X)$ is surjective in degrees $k \geq 2$. ■

In Section 4.5 we will use Theorem 0.2 to obtain a more general result.

3. Reduction to simplices with all vertices in the basepoint

3.1. **Eilenberg's argument.** For a topological space X with basepoint x_0 we denote by $C_*(X)$ the complex of singular simplices, i.e., the chain complex whose k-th group is the free abelian group generated by $S_k(X) = map(\Delta^k, X)$ with the usual boundary operator and by $C_*^{x_0}(X) \subset C_*(X)$ the subcomplex generated by

$$S_k^{x_0}(X) = \left\{ \sigma \colon \Delta^k \to X \mid \sigma(v_0) = \sigma(v_1) = \ldots = \sigma(v_k) = x_0 \right\},$$

where $v_0, v_1, \ldots, v_k$ denote the vertices of the standard simplex Δ^k.

It is a classical result of Eilenberg (Corollary 31.2 in [5]) that for path-connected X the inclusion

$$\iota\colon C_*^{x_0}(X) \to C_*(X)$$

is a chain homotopy equivalence. It is well-known that this dualizes to give chain homotopy equivalences also in cohomology and bounded cohomology. In this section we are going to show that (under a suitable assumption) the argument also yields chain homotopy equivalences for measure homology and measurable bounded cohomology.

Let us start with recalling Eilenberg's argument (which in [5] is given in a more general setting).

Lemma 3.1. *For each path-connected space, there is a chain map $\eta_*\colon C_*(X) \to C_*^{x_0}(X)$ such that $\eta\iota = id$ and a chain homotopy $s_*\colon C_*(X) \to C_{*+1}(X)$ such that*

$$\partial s + s\partial = \iota\eta - id.$$

Proof: For $x \in S_0(X)$ we have to define $\eta_0(x) = x_0$.

Because X is path-connected we have a 1-simplex $s_0(x)\colon \Delta^1 \to X$ with

$$\partial_0 s_0(x) = x_0, \partial_1 s_0(x) = x$$

for each $x \in X$. Let us fix a choice of $s_0(x)$ for each x.

Now we define η_* and s_* by induction on the dimension of simplices. Suppose they are already defined for all simplices in $S_{k-1}(X)$ and let $\sigma \in S_k(X)$. By induction hypothesis we have $\eta_{k-1}(\partial\sigma) \in C_{k-1}^{x_0}(X)$ and $s_{k-1}(\partial\sigma) \in C_k(X)$ such that

$$\eta_{k-1}(\partial\sigma) - \partial\sigma = \partial s_{k-1}(\partial\sigma) + s_{k-2}\partial(\partial\sigma) = \partial s_{k-1}(\partial\sigma).$$

We will inductively prove the slightly stronger statement that s_k is of the form $s_k = s_k^0 + \ldots + s_k^k$ and that the maps $s_k^0, \ldots, s_k^k$ can be defined through some map $F\colon \Delta^k \times [0,1] \to X$ via the canonical subdivision

$$\Delta^k \times [0,1] = \Delta_0 \cup \ldots \cup \Delta_k$$

as the restrictions of F to $\Delta_0, \ldots, \Delta_k$.

So consider $\Delta^k \times [0,1]$. We can use σ to define a continuous map $\Delta^k \times \{0\} \to X$ and by the above inductive hypothesis we have

$$s_{k-1}(\partial\sigma) = (s_{k-1}^0 + \ldots + s_{k-1}^{k-1})(\partial\sigma)$$

defined through a continuous map $\partial\Delta^k \times [0,1] \to X$. These two maps agree on $\partial\Delta^k \times \{0\}$, so they define a continuous map

$$Q\colon \Delta^k \times \{0\} \cup \partial\Delta^k \times [0,1] \to X.$$

It is easy to construct a continuous map

$$P\colon \Delta^k \times [0,1] \to \Delta^k \times \{0\} \cup \partial\Delta^k \times [0,1]$$

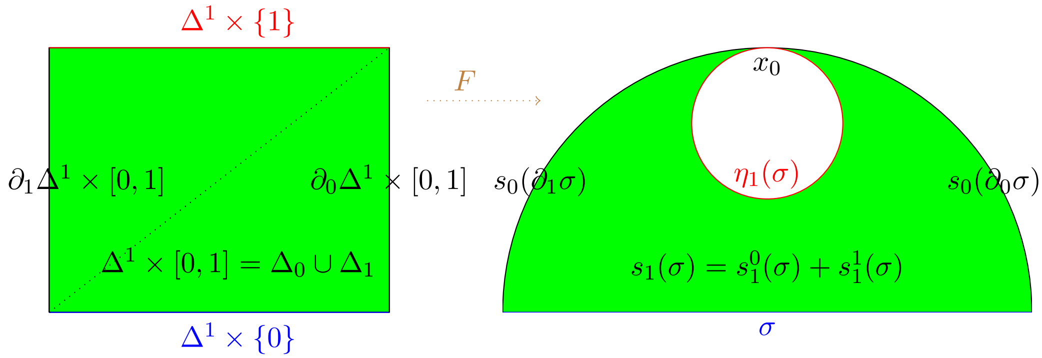
which is the identity map on $\Delta^k \times \{0\} \cup \partial\Delta^k \times [0,1]$. We can compose P with the before-defined map Q to obtain a continuous map

$$F\colon \Delta^k \times [0,1] \to X$$

that on $\Delta^k \times \{0\} \cup \partial\Delta^k \times [0,1]$ agrees with Q. We use the canonical triangulation of $\Delta^k \times [0,1]$ into $k+1$ simplices to consider F as a formal sum of $k+1$ simplices, which we denote by $s_k^0(\sigma), \ldots, s_k^k(\sigma)$. We obtain thus an element

$$s_k(\sigma) := s_k^0(\sigma) + \ldots + s_k^k(\sigma) \in C_{k+1}(X).$$

In particular $F|_{\Delta^k \times \{1\}}$ defines $\eta_k(\sigma) \in C_k(X)$ which actually belongs to $C_k^{x_0}(X)$ because all vertices are in x_0. It is then clear by construction that the equality $\partial s_k(\sigma) + s_{k-1}(\partial\sigma) = \eta_k(\sigma) - \sigma$ holds. ■



The figure visualizes the construction of the map F in case of $k = 1$. It in particular shows that the two vertices (endpoints of the simplex $\Delta^1 \times \{1\}$) are taken under F to the same point $x_0 \in X$.

3.2. **Pointed measure homology.** We now want to argue that an analogous result as in Lemma 3.1 holds for measure homology, i.e., that (under suitable assumptions) the inclusion

$$\iota\colon \mathcal{C}_*^{x_0}(X) \to \mathcal{C}_*(X)$$

is a chain homotopy equivalence. Here $\mathcal{C}_*^{x_0}(X) \subset \mathcal{C}_*(X)$ means the subcomplex consisiting of those signed measures (of quasicompact determination set and bounded variation) which vanish on each measurable subset of the complement of $S_*^{x_0}(X)$.

Lemma 3.2. *If X is a path-connected space that has a finite covering*

$$X = \cup_{i=1}^n U_i$$

such that

- $U_1, \ldots, U_n$ are Borel-measurable sets

- the closures $\overline{U}_1, \ldots, \overline{U}_n$ are contractible in X and compact,

then for any $x_0 \in X$ there is a chain map $\eta_*\colon \mathcal{C}_*(X) \to \mathcal{C}_*^{x_0}(X)$ such that $\eta\iota = id$, and a chain homotopy $s_*\colon \mathcal{C}_*(X) \to \mathcal{C}_{*+1}(X)$ such that

$$ds + sd = \iota\eta - id.$$

Proof: The natural approach to proving this statement would be to define η and s as in the proof of Lemma 3.1. One would have to check then that signed measures of compact determination set and bounded variation are mapped to signed measures of compact determination set and bounded variation.

It is clear that a so-constructed η_k does not increase the variation and that s_k multiplies the variation by at most $k+1$, so the second condition on boundedness of the variation will be satisfied.

To satisfy the first condition on compactness of the determination set it would be sufficient that η_k and the maps $s_k^0, \ldots, s_k^k$ from the proof of Lemma 3.1 could be defined via some continuous maps on $map(\Delta^k, X)$, because then compact determination sets of simplices would be mapped to compact sets. In general it will not be possible to define such a continuous map. It would be possible if X were contractible. It is still possible on subsets that are contractible in X and our argument will make use of this fact.

Let $\overline{U} \subset X$ be contractible in X. Then there is some continuous map $H : \overline{U} \times [0,1] \to X$ with $H(x, 0) = x$ and $H(x, 1) = x_0$ for all $x \in \overline{U}$. Define

$$s_0 \colon map(\Delta^0, \overline{U}) \to map(\Delta^1, X)$$

by

$$s_0(x)(t) = H(x, t)$$

upon identification $\Delta^1 = [0, 1]$. Continuity of H and compactness of $[0, 1]$ imply that s_0 is continuous.

Now consider the by assumption existing covering $X = \cup_{i \in I} U_i$ by finitely many Borel sets whose closures are compact and contractible in X. (I is a finite index set.) W.l.o.g. we can assume that the U_i are disjoint. Indeed, if they were not, we could replace U_i by $V_i = U_i \setminus \cup_{j=1}^{i-1} U_j$ for $i \geq 2$. The closures $\overline{V}_i$ are subsets of $\overline{U}_i$ and hence again compact and contractible in X (although not necessarily in $\overline{U}_i$), and of course the V_i are again Borel sets.

For each ordered $(k+1)$-tuple $(i_0, \ldots, i_k)$ of (not necessarily distinct) elements of the index set I we let $S_{i_0,\ldots,i_k}$ be the set of singular simplices with 0-th vertex in U_{i_0}, 1-st vertex in U_{i_1}, ..., k-th vertex in U_{i_k} and we consider its closure $\overline{S}_{i_0,\ldots,i_k}$ which is contained in the set of singular simplices with 0-th vertex in $\overline{U}_{i_0}$, 1-st vertex in $\overline{U}_{i_1}$, ..., k-th vertex in $\overline{U}_{i_k}$.

By the above we have defined η_0 and s_0 on $S_0 = U_0, \ldots, S_k = U_k$ (i.e., on all of X), such that the restriction to each S_i extends continuously to $\overline{S}_i$. Now we assume by induction that for all k-tuples $(i_0, \ldots, i_{k-1})$ we already have maps

$$\eta_{k-1} \colon S_{i_0,\ldots,i_{k-1}} \to map(\Delta^{k-1}, X)$$

and

$$s_{k-1}^0, \ldots, s_{k-1}^{k-1} \colon S_{i_0,\ldots,i_{k-1}} \to map(\Delta^k, X)$$

with the desired properties and which all extend continuously to $\overline{S}_{i_0,\ldots,i_{k-1}}$. We claim that η_k and $s_k^0, \ldots, s_k^k$ (defined as in the proof of Lemma 3.1) are again continuous maps on $\overline{S}_{i_0,\ldots,i_k}$ for each $(k+1)$-tuple $(i_0, \ldots, i_k)$.

This is seen as follows. Continuity of $s_{k-1}^0, \ldots, s_{k-1}^{k-1}$ implies that the map

$$\overline{S}_{i_0,\ldots,i_{k-1}} \to map(\Delta^k \times \{0\} \cup \partial\Delta^k \times [0,1], X)$$

which sends $\sigma \colon \Delta^k \to X$ to the "union" of $\sigma \times \{0\}$ and $s_{k-1}^0(\partial_j \sigma), \ldots, s_{k-1}^{k-1}(\partial_j \sigma), j = 0, \ldots, k$, is continuous. Moreover, precomposition with the uniformly continuous map $P \colon \Delta^k \times [0,1] \to \Delta^k \times \{0\} \bigcup \partial\Delta^k \times [0,1]$ from the proof of Lemma 3.1 defines a continuous map

$$map(\Delta^k \times \{0\} \bigcup \partial\Delta^k \times [0,1], X) \to map(\Delta^k \times [0,1], X),$$

so we obtain a continuous map

$$\Phi\colon map(\Delta^k, X) \to map(\Delta^k \times [0,1]\,, X).$$

Since $\eta_k(\sigma)$ and $s_k^0(\sigma), \ldots, s_k^k(\sigma)$ are all defined by restricting $\Phi(\sigma)$ to subsets of $\Delta^k \times [0,1]$, they also depend continuously on σ.

So we have proved that η_k and $s_k^0, \ldots, s_k^k$ (defined on $S_{i_0,\ldots,i_k}$) can be extended continuously to $\overline{S}_{i_0,\ldots,i_k}$ (although this extension on $\overline{S}_{i_0,\ldots,i_k} \setminus S_{i_0,\ldots,i_k}$ of course does not have to agree with the actual definition of η_k and $s_k^0, \ldots, s_k^k$ coming from some other $S_{j_0,\ldots,j_k}$). Since all the $S_{i_0,\ldots,i_k}$ are pairwise disjoint, this allows a (not continuous but measurable) definition of η_k and $s_k^0, \ldots, s_k^k$ on

$$map(\Delta^k, X) = \bigcup_{(i_0,\ldots,i_k)} S_{i_0,\ldots,i_k}.$$

For any compact subset $K \subset map(\Delta^k, X)$ we obtain that the image of $K \cap S_{i_0,\ldots,i_k}$ under η or $s_0, \ldots, s_k$ is contained in the image of $K \cap \overline{S}_{i_0,\ldots,i_k}$ under some continuous extension of η_k or $s_k^0, \ldots, s_k^k$ and thus is contained in a compact set. Hence the image of $K \cap S_{i_0,\ldots,i_k}$ has compact closure. So the image of K under any of η and $s_0, \ldots, s_k$ is a finite union of (subsets of) compact sets, hence has compact closure.

In particular, because the image of a determination set under any map is a determination set for the push-forward measure, η_k and $s_k^0, \ldots, s_k^k$ map measures of compact determination set to measures of compact determination set. ∎

Corollary 3.3. *Under the assumptions of Lemma 3.2 every measure cycle is homologous to a measure cycle with determination set contained in $S_*^{x_0}(X)$.*

Recall that we have defined bounded cohomology in Definition 1.4 and measurable bounded cohomology in Definition 2.1. Similarly one defines measurable cohomology, see Definition 4.1 below. Let us denote by $H_{b,x_0}^*(X)$, $\mathcal{H}_{x_0}^*(X)$ and $\mathcal{H}_{b,x_0}^*(X)$ the cohomology groups of the complexes of bounded, measurable resp. bounded measurable functions from $C_*^{x_0}(X)$ to $\mathbb{R}$. Using [17, Section 3.4] there is a well-defined pairing between $\mathcal{H}_{b,x_0}^*(X)$ and $\mathcal{C}_*^{x_0}(X)$.

Corollary 3.4. *Under the assumptions of Lemma 3.2, the canonical restriction induces isomorphisms*

$$H_b^*(X) \to H_{b,x_0}^*(X)$$

$$\mathcal{H}_b^*(X) \to \mathcal{H}_{b,x_0}^*(X)$$

$$\mathcal{H}^*(X) \to \mathcal{H}_{x_0}^*(X)$$

Proof: The above constructed maps η and s are bounded in the sense that η_k sends a simplex to a simplex and s_k sends a k-dimensional simplex to a formal sum of (at most) $k{+}1$ $(k{+}1)$-dimensional simplices. This implies that η^* and s^* send bounded cochains to bounded cochains. Moreover η_k and s_k are continuous on each of the finitely many disjoint Borel sets $S_{i_0\ldots i_k}$, so they are Borel-measurable on $map(\Delta^k, X)$ and hence η^* and s^* send measurable cochains to measurable cochains. ∎

3.3. **Examples.** Let us conclude with some examples fulfilling or not fulfilling the assumptions of Lemma 3.2:

Example. *CW-complexes*
Any compact manifold or finite CW-complex can be covered by finitely many measurable sets with contractible, compact closures. Thus the assumptions of Lemma 3.2 are satisfied.

Example. *Hawaiian earring*
The Hawaiian earring is the shrinking wedge of circles pictured in the introduction, that is, it can be written in the form

$$HE = \bigcup_{n=1}^{\infty} C_n \subset \mathbb{R}^2,$$

where $C_n \subset \mathbb{R}^2$ is the circle with center $(0, \frac{1}{n})$ and radius $\frac{1}{n}$. Let $A_n^{\pm}$ be the intersection of C_n with the closure of the upper resp. lower half-plane, and let $A^{\pm} = \bigcup_{n=1}^{\infty} A_n^{\infty}$. Then

$$HE = A^+ \cup A^-$$

is a covering by two measurable sets with contractible, compact closures. Thus the assumptions of Lemma 3.2 are satisfied for the Hawaiian earring.

Example. *Warsaw circle*
The Warsaw circle is a closed subset $W \subset \mathbb{R}^2$, which is the union of the graph of the function

$$y = \sin(\frac{1}{x})$$

for $0 < x \leq 1$, the segment

$$Y = \{(x, y) \colon x = 0, -1 \leq y \leq 1\},$$

and a curve connecting these two parts to get a path-connected space.

This space can be covered by finitely many contractible, measurable, relatively compact sets. The easiest way to do this is to use the decomposition

$$W = Y \cup Y^c$$

into Y and its complement. However the closure of Y^c is all of W, which is known to be not contractible.

On the other hand, W can be covered by countably many contractible, compact sets. For this one has to decompose the graph of $y = \sin(\frac{1}{x})$ into its segments for $\frac{1}{n+1} \leq x \leq \frac{1}{n}$ with n running through all natural numbers, and then add Y and the connecting curve as two more contractible, compact sets to the decomposition.

These two decompositions show that in Lemma 3.2 the assumption on having contractible closures and the assumption on finiteness of the covering can not be relaxed by just assuming contractibility of the relatively compact sets themselves or by countability of the covering, respectively. Indeed for the Warsaw circle W, the second author proved in [21, Theorem 4] that $H_0(\mathcal{W})$ is uncountable-dimensional, while of course $H_0^{x_0}(\mathcal{W}) \simeq \mathbb{R}$.

The Warsaw circle does however not provide a counterexample to the conclusion of Theorem 0.2 in view of $H_0(W) = \mathbb{R}$ and $H_n(W) = 0$ for all $n > 0$.

Example. *A space with non-injective canonical homomorphism*

The space from [23, Section 5], which we will discuss in Section 4.4 below, can be covered by two contractible sets, but they are not Borel-measurable.

This space satisfies all other assumptions from Lemma 3.2, but not the finiteness of a contractible, measurable covering.

4. Proof of Theorem 0.2

4.1. **Measurable cohomology.** For what follows it will be useful to consider measurable cohomology rather than measurable bounded cohomology.

Definition 4.1. Let X be a topological space and again $map(\Delta^k, X)$ equipped with the compact-open-topology and the corresponding σ-algebra of Borel-measurable sets. We let

$$\mathcal{C}^k(X) = \left\{ f\colon map(\Delta^k, X) \to \mathbb{R} \mid f \text{ is Borel measurable} \right\}$$

be the measurable cochains on $map(\Delta^k, X)$.

The usual coboundary operator makes $\mathcal{C}^*(X)$ into a cochain complex, we denote its cohomology by $\mathcal{H}^*(X)$. The inclusion ι induces a homomorphism

$$\iota^*\colon \mathcal{H}^*(X) \to H^*(X)$$

from measurable cohomology to singular cohomology.

Lemma 4.2. *Let X be a topological space and $k \in \mathbb{N}$. If*

$$\iota^*\colon \mathcal{H}^k(X) \to H^k(X)$$

is an epimorphism, then

$$\iota_*\colon H_k(X;\mathbb{R}) \to \mathcal{H}_k(X)$$

is injective.

Proof: Assume there is some $\alpha \in H_k(X;\mathbb{R})$ with $\iota_*\alpha = 0$. Thus there is some $\mu \in \mathcal{C}_{k+1}(X)$ with $\iota_* z = d\mu$.

There is some cohomology class $\beta \in H^k(X;\mathbb{R})$ with $\beta(\alpha) = 1$, in particular if a cycle z represents α and a cocycle f represents β, then $f(z) = 1$.

On the other hand, surjectivity of ι^* implies that β can be represented by some measurable cocycle f.

Recall that measure homology is defined using measure chains with compact determination set and bounded variation. Thus, if f were bounded, then we could just use $\delta f = 0$ (and integration of bounded functions over compact sets) to compute

$$f(\iota^* z) = f(d\mu) = \int f(\sigma) d\mu(\sigma) = \int f(\partial\sigma)\mu(\sigma) = \int (\delta f)(\sigma)\mu(\sigma) = 0,$$

contradicting $f(z) = 1$. There is however no reason for f to be bounded.

Let us decompose $f = f^+ - f^-$ as a difference of two nonnegative functions. We want to prove $f^+(d\mu) - f^-(d\mu) = 0$.

We know from $f(z) = 1$ that f can be integrated over $d\mu$, however we can not assume f to be integrable over μ. We therefore consider the two exhaustions

$$map(\Delta^k, X) = \bigcup_{n=1}^{\infty} K_n^+ = \bigcup_{n=1}^{\infty} K_n^-$$

with

$$K_n^+ = \left\{\sigma \in map(\Delta^k, X) \colon sup_{0\le i\le k} f^+(\partial_i\sigma) \le n\right\},$$

$$K_n^- = \left\{\sigma \in map(\Delta^k, X) \colon sup_{0\le i\le k} f^-(\partial_i\sigma) \le n\right\}.$$

Let us denote $\mu_n^+ := \chi_{K_n^+}\mu, \mu_n^- := \chi_{K_n^-}\mu$, where $\chi_{K_n^\pm}$ means the characteristic function of $K_n^\pm$. By construction, f^+ is bounded on the determination set of $d\mu_n^+$, and hence can be integrated against this signed measure. The same way f^- can be integrated against $d\mu_n^-$.

Our assumption $f(d\mu) = f(z) = 1$ implies that $f^\pm$ is integrable with respect to $d\mu$ and hence also with respect to all $d\mu_n^\pm$. Let us denote $A_n = \int f^+(d\mu_n^+)$ and $B_n = \int f^-(d\mu_n^-)$. Then we can apply the argument before (using $\delta f = 0$ and integration of bounded functions over compact sets) to obtain

$$A_n - B_n = 0$$

$A_n = \int f^+(d\mu_n^+)$ is an increasing sequence which is bounded by $\int f^+(d\mu)$. Hence the limit $A = \lim_{n\to\infty} A_n$ exists. Similarly, $B = \lim_{n\to\infty} B_n$ exists. From $A_n - B_n = 0$ we have $A - B = 0$.

We are going to prove that $A = \int f^+(d\mu)$ and $B = \int f^-(d\mu)$, which then yields our wanted equality.

Integrability of f over $\partial\mu$ implies

$$\lim_{n\to\infty} \|d\mu - d\mu_n^+\| = 0$$

because of

$$n\|d\mu - d\mu_n^+\| \le \int f^+|d\mu - d\mu_n^+| \le \int f^\pm|d\mu|.$$

The same argument with f^- instead of f^+ shows $\lim_{n\to\infty} \|d\mu - d\mu_n^-\| = 0$. In particular we have convergence

$$\lim_{n\to\infty} \int s(d\mu_n^\pm) = \int s(d\mu)$$

for every step function s.

From the definition of the Lebesgue integral this implies

$$\int f^+(d\mu) = sup_{s^+\le f^+} \int s^+(d\mu) = sup_{s^+\le f^+} \lim_{n\to\infty} \int s^+(d\mu_n^+) \le \lim_{n\to\infty} \int f^+(d\mu_n^+)$$

and together with the trivial inequality $\int f^+(d\mu_n^+) \le \int f^+(d\mu)$ we get the equality

$$\lim_{n\to\infty} \int f^+(d\mu_n^+) = \int f^+(d\mu).$$

The same argument shows $\lim_{n\to\infty} \int f^-(d\mu_n^-) = \int f^-(d\mu)$.

We conclude

$$0 = \lim_{n\to\infty} A_n - B_n = A - B = \int f(d\mu),$$

which contradicts $f(z) = 1$. ■

4.2. **A simplicial construction: straightening.** Recall that for a topological space X and a point $x_0 \in X$ we denote

$$S_k^{x_0}(X) = \left\{ \sigma \colon \Delta^k \to X \mid \sigma(v_0) = \sigma(v_1) = \ldots = \sigma(v_k) = x_0 \right\},$$

where $v_0, v_1, \ldots, v_k$ denote the vertices of the standard simplex Δ^k. Two simplices $\sigma_0, \sigma_1 \in S_k^{x_0}(X)$ are said to be homotopic rel. boundary if there exists a continuous map $F \colon \Delta^k \times [0,1]$ with $F(x,0) = \sigma_0(x), F(x,1) = \sigma_1(x)$ for all $x \in \Delta^k$ and $F(x,t) = x_0$ for all $x \in \partial\Delta^k, t \in [0,1]$.

Let us denote by $C_*^{x_0}(X) \subset C_*(X)$ for some fixed $x_0 \in X$ the subcomplex generated by $S_*^{x_0}(X)$. In the following lemma we define a topological analogue of straightening. (Similar constructions in somewhat different settings can be found in [19, Theorem 9.5] and [4, Proposition 3.1].)

Lemma 4.3. *Let X be a topological space and $x_0 \in X$.*

i) There is a subset $S_*^{str}(X) \subset S_*^{x_0}(X)$ such that $S_k^{str}(X)$ contains one k-simplex in each homotopy class rel. boundary of simplices with all boundary faces in $S_{k-1}^{str}(X)$.

ii) There is a chain map

$$str \colon C_*^{x_0}(X) \to C_*^{x_0}(X)$$

which is chain homotopic to the identity and whose image lies in the chain complex $C_*^{str}(X) \subset C_*^{x_0}(X)$ spanned by the simplices in $S_*^{str}(X)$.

iii) If X is aspherical, then str is constant on homotopy classes rel. vertices.

Proof: We recursively construct a subcomplex $C_*^{str}(X) \subset C_*^{x_0}(X)$, whose simplices we call the "straight simplices", and a map $str \colon C_*^{x_0}(X) \to C_*^{str}(X)$, which we call the straightening map.

The 0-skeleton of $C_*^{str}(X)$ consists of the one vertex x_0. For the 1-skeleton of $C_*^{str}(X)$ we choose one 1-simplex in each homotopy classe (rel. vertices) of 1-simplices in $C_*^{x_0}(X)$, and we define str on 1-simplices by sending each of them to the unique straight simplices in its homotopy class. For each 1-simplex σ we fix a homotopy (rel. vertices) between σ and $str(\sigma)$.

Assume now that for some $k > 1$ we have already defined $S_{*\leq k-1}^{str}(X)$ and $str \colon C_{*\leq k-1}^{x_0}(X) \to C_{*\leq k-1}^{str}(X)$. For $S_k^{str}(X)$ we choose one k-simplex with all boundary faces in $S_{k-1}^{str}(X)$ inside each homotopy class (rel. boundary) of k-simplices in $S_k^{x_0}(X)$ with all boundary faces in $S_{k-1}^{str}(X)$.

For a simplex $\sigma \in C_k^{x_0}(X)$ we can assume by induction that we have defined $str(\partial\sigma)$ and that we have a homotopy between $\partial\sigma$ and $str(\partial\sigma)$. By the cofibration property of the inclusion $\partial\Delta^k \to \Delta^k$ this homotopy extends to a homotopy of σ keeping vertices fixed. Let σ' be the result of this homotopy. Among simplices with boundary $str(\partial\sigma)$ we have in the homotopy class (rel. boundary) of σ' exactly one simplex in $S_k^{str}(X)$. Define this simplex to be $str(\sigma)$. By construction we have a homotopy from σ to $str(\sigma)$ whose restriction to $\partial\sigma$ is a reparametrisation of the homotopy from $\partial\sigma$ to $str(\partial\sigma)$ given by recursion. This family of compatible homotopies yields the wanted chain homotopy between id and str.

For (iii), if σ and σ' are homotopic rel. vertices, then the 1-skeleta of $str(\sigma)$ and $str(\sigma')$ agree. Assuming inductively that for some $k \geq 2$ the

$(k-1)$-skeleta of $str(\sigma)$ and $str(\sigma')$ agree, we get from asphericity of X that the k-skeleta of both simplices are homotopic rel. boundary, and thus the k-skeleta must be equal by definition of str. Proof by induction on k yields that $str(\sigma) = str(\sigma')$.

■

Of course str need in general not be continuous or measurable.

4.3. **Proof of injectivity.** In this section we prove Theorem 0.2 which says that for an aspherical space X which has countable fundamental groups, Borel-measurable homotopy classes of simplices, and can be covered by finitely many Borel sets whose closures are compact and contractible in X, the canonical map

$$\iota_*\colon H_*(X) \to \mathcal{H}_*(X)$$

is injective.

Proof: Recall that we denote by $H^*_{x_0}$ and $\mathcal{H}^*_{x_0}$ the cohomology of the complex of cochains resp. measurable cochains on $C_*^{x_0}(X)$. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{H}^*(X) & \longrightarrow & H^*(X) \\ \downarrow & & \downarrow \\ \mathcal{H}^*_{x_0}(X) & \longrightarrow & H^*_{x_0}(X) \end{array}$$

By assumption, X can be covered by finitely many Borel sets whose closures are compact and contractible in X. Thus the assumptions of Lemma 3.2 are satisfied and we know by Corollary 3.4 that the vertical arrows are isomorphisms, so it is enough to show that $\mathcal{H}^*_{x_0}(X) \to H^*_{x_0}(X)$ is an epimorphism for some $x_0 \in X$.

Let

$$str\colon C_*^{x_0}(X) \to C_*^{x_0}(X)$$

be the chain map from Lemma 4.3 and let

$$str^*\colon C^*_{x_0}(X) \to C^*_{x_0}(X)$$

be the dual map on cochains.

We know by part (iii) of Lemma 4.3 that str is constant on homotopy classes rel. vertices. So the level sets of str are unions of homotopy classes.

We claim that the image of str are only countably many singular k-simplices with vertices in x_0 as a consequence of the assumption on countability of the fundamental group $\pi_1(X, x_0)$. This is obvious for $k = 1$ and we prove the general case by induction.

Namely, for a k-simplex σ its 0-th face $\partial_0\sigma$ must be in one of the (by the inductive hypothesis) countably many homotopy classes. Fixing one such homotopy class and one k-simplex σ with $\partial_0\sigma$ in this fixed class, we then get a bijection between on the one hand homotopy classes of k-simplices σ' with $\partial_0\sigma'$ in the fixed class, and on the other hand pointed homotopy classes of maps $S^k \to X$: we can homotope $\partial_0\sigma'$ such that it agrees with $\partial_0\sigma$, this homotopy can be extended to all of σ' (because $\partial_0\Delta^k \to \Delta^k$ is a cofibration) and after that homotopy σ' can be glued along its 0-th face to σ

to obtain a map $S^k \to X$. This operation is compatible with homotopies and distinct homotopy classes would be sent to distinct elements of $\pi_k(X, x_0)$. Asphericity of X then implies that there is only one straight k-simplex σ with $\partial_0\sigma$ in the given homotopy class.

By the inductive hypothesis there are only countably many straight $(k-1)$-simplices that can occur as $\partial_0\sigma$, hence there are only countably many straight k-simplices possible for σ. This proves the inductive step.

So there are countably many homotopy classes, and measurability of $str^*(f) = f \circ str$ for any $f \in C^*_{x_0}(X)$ follows directly from the assumption that homotopy classes are measurable subsets of $C^*_{x_0}(X)$. Since str sends arbitrary cochains to measurable cochains, it defines a chain map

$$str^*\colon C^*_{x_0}(X) \to \mathcal{C}^*_{x_0}(X).$$

On the other hand, $str \circ \iota\colon C^{x_0}_*(X) \to C^{x_0}_*(X)$ is chain homotopic to the identity and thus the induced map

$$(str \circ \iota)^*\colon H^*_{x_0}(X) \to H^*_{x_0}(X)$$

is the identity. In particular ι^* is surjective.

From Lemma 4.2 we conclude injectivity of ι_*.

■

We remark that the proof in particular applies to CW-complexes and then gives a simpler proof than the one given in [17]. The point of the simplification is that using the results from Section 3 we could reduce the problem to the simpler situation of simplices having all their vertices in the basepoint.

In fact, in the above setting we can also prove the existence of a measurable section, which was the main technical lemma in [17]. Namely, let $G \simeq \pi_1(X, x_0)$ be the (by assumption countable) deck transformation group of the generalized universal covering $p\colon \widetilde{X} \to X$. The lift of x_0 to $\widetilde{X}$ is a G-orbit $G\tilde{x}_0$ for some $\tilde{x}_0$, see Section 1.2. The lift of a homotopy class (rel. x_0) of n-simplices is a homotopy class inside $S^{\gamma_0\tilde{x}_0,\ldots,\gamma_n\tilde{x}_0}_*(\widetilde{X})$, by which we mean the set of simplices mapping their i-th vertex to $\gamma_i\tilde{x}_0$ for $i = 0, \ldots, n$. Clearly the projection

$$p\colon S^{\gamma_0\tilde{x}_0,\ldots,\gamma_n\tilde{x}_0}_*(\widetilde{X}) \to S^{x_0}_*(X)$$

maps a homotopy class homeomorphically onto its image, which is also a homotopy class. In particular, the restriction of p to any homotopy class has a continuous right-inverse s defined on the image of that homotopy class. Thus we get a right-inverse s defined on each of the homotopy classes downstairs. Since the homotopy classes of simplices (rel. vertices) are Borel sets by assumption, and there are only countably many homotopy classes, the so-defined s yields a measurable map

$$s\colon S^{x_0}_*(X) \to S^{G\tilde{x}_0}_*(\widetilde{X})$$

right-inverse to p.

4.4. **A counterexample.** In Theorem 0.2 we are imposing a condition that X can be covered by finitely many Borel sets of compact closure contractible in X. (It is needed to apply the results of Section 3, i.e., to reduce the problem to simplices with all vertices in the basepoint.) The condition may look like a technical one, but it can actually not be avoided as the following example shows. Let Z be the space constructed in [23, Section 5]. There are two points $z_0, z_1 \in Z$ such that

$$[z_1] - [z_0] \in ker(H_0(Z) \to \mathcal{H}_0(Z)),$$

see [23, Theorem 5.7].

To get examples in any degree n, let F be a closed, orientable manifold of dimension $n \geq 1$ and consider $F \times Z$ with the "boundary" manifolds $F_0 = F \times \{z_0\}$ and $F_1 = F \times \{z_1\}$. Then

$$[F_1] - [F_0] \in ker(H_n(F \times Z) \to \mathcal{H}_n(F \times Z)).$$

To get a path-connected example, let X be the space obtained by gluing one arc with the end points to the two different path components of $F \times Z$. This does not change the homology in degrees ≥ 2 and thus one has for $n \geq 2$:

$$[F_1] - [F_0] \in ker(H_n(X) \to \mathcal{H}_n(X)).$$

The reason why Theorem 0.2 does not apply is exactly that X can not be covered by finitely many contractible Borel sets. (Though it can be covered by two contractible non-Borel sets.)

4.5. **Example: convergent Y-spaces.** Recall from Section 2.8 the definition of convergent Y-space for a pointed metric space (Y, y_0) as a union

$$X = \bigcup_{n \in \mathbb{N}} Y_n \cup Y_\infty,$$

where $Y_\infty = Y$ and one has pointed homeomorphisms

$$f_n : (Y_\infty, y_0) \to (Y_n, y_0)$$

such that for the metric d_X on X one has

$$\lim_{n \to \infty} d_X(f_n(y), y) = 0$$

for all $y \in Y$, and such that for all but finitely many $y \in Y$ one has $d_X(f_n(y), y) > 0$ and $d_X(f_n(y), f_m(y)) > 0$ for all n, m.

For $Y = [0, 1], y_0 = 0, f_n(1) \equiv 1$ and $f_m(y) > f_n(y)$ whenever $m < n, 0 < y < 1$ this yields the convergent arc space drawn in the introduction.

Such a convergent Y-space space will usually not be a CW-complex even if Y is: although X inherits a cell decomposition from that of Y, the accumulation property $\lim_{n\to\infty} d(f_n(y), y) = 0$ implies that X does not have the weak topology with respect to that cell decomposition.

In the following lemma and in the corollary below we consider spaces homeomorphic to a simplicial complex. This is for example true for each smooth manifold, see [26].

Lemma 4.4. *If Y is a simplicial complex and X is a convergent Y-space, then, for all k, the homotopy classes (rel. vertices) of k-simplices with vertices in x_0 are Borel sets in $map(\Delta^k, X)$.*

Proof: Let $x_0, \ldots, x_m \in X$ be the finitely many points along which the Y_n got identified with Y_∞. Upon subdivision we can assume that they are vertices of the simplicial complex. Thus X inherits a division into simplices, although it is not a simplicial complex: the induced weak topology is not the topology of X. (In fact, X is not even locally path-connected.)

Each point $x \in X$ is in the interior of a unique simplex of this decomposition, which one denotes $carr(x)$. If two singular simplices $\sigma_0, \sigma_1 \colon \Delta^k \to X$ have the same vertices and there is a simplicial map $f \colon \Delta^k \to X$ with $\sigma_i(p) \subset \overline{carr(f(p))}$ for $i = 0, 1$ and all $p \in \Delta^k$, then σ_0 and σ_1 are homotopic rel. vertices: one can just choose the affine-linear homotopy inside each simplex of X.

Let us first prove the following: if $\sigma_0 \colon \Delta^k \to X$ is a singular simplex which does not intersect the image of Y_∞, then the homotopy class of σ_0 (rel. vertices) contains an open neighborhood of σ_0. Namely, here the image of σ_0 belongs to the simplicial complex $X \backslash Y_\infty$ (which for the sake of this argument we assume to be further subdivided such that σ_0 sends the vertices of Δ^k to vertices of X), so we can follow the proof of the simplicial approximation theorem, cf. [25, Section 3.2] or [11, Section 2.C]. That is, upon subdividing Δ^k sufficiently often, we can assume to have a triangulation T of Δ^k and an assignment $\phi \colon T_0 \to X_0$ from the vertices of T to the vertices of X such that $\sigma_0(ost(v)) \subset ost(\phi(v))$ holds for each $v \in T_0$. Here, $ost(v)$ means the open star of the simplex v, that is, the union of the interiors of all simplices (of all dimensions) having v as a vertex. Because the open star in a simplicial complex is open, we find an open neighborhood U of σ_0 in $map(\Delta^k, X)$ such that

$$\sigma(ost(v)) \subset ost\phi((v))$$

holds for each $v \in T_0$ and each $\sigma \in U$. The so-defined simplicial map ϕ satisfies $\sigma(p) \in \overline{carr(\phi(p))}$ for all $p \in \Delta^k$ and all $\sigma \in U$. Thus any $\sigma \in U$ with the same vertices as σ_0 is homotopic to σ_0, by a homotopy which fixes the vertices (which we assumed to be sent to vertices of X).

Now we consider the general case, where σ_0 may intersect Y_∞. In this case, after subdividing sufficiently often we have again an assignment $\phi \colon T_0 \to X_0$ such that $\sigma_0(ost(v)) \subset ost(\phi(v))$ holds for each $v \in T_0$. However, in this case the open star is not open: if $\phi_(v)$ belongs to Y_∞, then any open neighborhood intersects not only Y_∞ but also Y_n for all sufficiently large n. In fact, it contains a neighborhood of $f_n(v) \in Y_n$ for all sufficiently large n. Therefore, if $\phi(v) \in Y_\infty$, to get an open neighborhood of $\phi(v)$ we have to consider the union

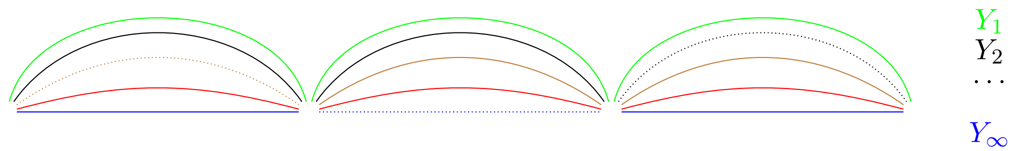
$$ost(\phi(v)) \cup \bigcup_{n \geq N} f_n(ost(\phi(v)))$$

for some $N \in \mathbb{N}$.

With the aim to obtain an open union of homotopy classes we now make the following considerations. Consider the path components $C_i, i \in I$ of $\Delta^k - \sigma_0^{-1}(\{x_0, \ldots, x_m\})$, indexed over an index set I. Let $\mathbb{N}_+ = \mathbb{N} \cup \{\infty\}$. To $i \in I$ we associate $n(i) \in \mathbb{N}_+$ if C_i is mapped to $Y_{n(i)}$.

Let $J = \{i \in I \colon n(i) = \infty\}$. For a sequence $(m_i) \in \mathbb{N}_+^J$ we denote by $f_{(m_i)}(\sigma_0) \colon \Delta^k \to X$ the singular simplex whose restriction to C_i agrees with the restriction of $f_{m_i} \circ \sigma_0$ if $i \in J$, and with σ_0 if $i \in I \setminus J$. (Recall that the map f_{m_i} maps the image of Y_∞ to the image of Y_{m_i}.)

In the picture below we have drawn the 1-simplex σ_0 densely dotted to distinguish it from the simplices of the 1-dimensional space. In this case we have three components $C_{i_1}, C_{i_2}, C_{i_3}$. We have $n(i_1) = 3, n(i_2) = \infty, n(i_3) = 2$ and thus $J = \{i_2\}$. In particular, a sequence $(m_i) \in \mathbb{N}^J$ consists of just one element m_{i_2} and the simplex $f_{(m_i)} \circ \sigma_0$ is a 1-simplex which passes through the arcs in $Y_3, Y_{m_{i_2}}, Y_2$.



Given the simplex σ_0, the index set $J \subset I$, the triangulation T of σ_0, and the assignment $\phi \colon T_0 \to X_0$, we know for any $N \in \mathbb{N}$ and any $v \in T_0$ that $ost(\phi(v)) \cup \bigcup_{n \geq N} f_n(ost(\phi(v)))$ is open. Thus we find an open neighborhood U of σ_0 in $map(\Delta^k, X)$ such that for each $\sigma \in U$ we have

$$\sigma(ost(v)) \subset ost(\phi(v)) \cup \bigcup_{n \geq N} f_n(ost(\phi(v))) \quad \forall\, v \in T_0 \text{ with } \sigma_0(v) \in Y_\infty,$$

$$\sigma(ost(v)) \subset ost(\phi(v)) \quad \forall\, v \in T_0 \text{ with } \sigma_0(v) \notin Y_\infty.$$

This implies that there is a sequence $(m_i)_i \in \mathbb{N}_+^J$ such that $\sigma(p) \in \overline{f_{(m_i)}(carr(\phi(p)))}$ for all $p \in \Delta^k$. Again by choosing the simplex-wise affine-linear homotopy (on the chosen subdivision), this implies that σ is homotopic (rel. vertices) to $f_{(m_i)}(\sigma_0)$.

In particular, we have that the union of homotopy classes

$$\bigcup_{(m_i) \colon m_i \geq N \ \forall i} \left[f_{(m_i)}(\sigma_0) \right]$$

is an open set for each natural number N. (Here we use the convention $\infty > N$ for each $N \in \mathbb{N}$.) But clearly

$$[\sigma_0] = \bigcap_{N \in \mathbb{N}} \left(\bigcup_{(m_i) \colon m_i \geq N \ \forall i} \left[f_{(m_i)}(\sigma_0) \right] \right),$$

so the homotopy class of σ_0 is a countable intersection of open sets and thus Borel-measurable.

■

The assumption on nonvanishing of the Gromov norm, that we required in Theorem 0.1 is not satisfied for arbitrary Y-spaces, in particular it does not hold for the convergent arc space, or when the Gromov norm on Y is already not an actual norm, e.g., when Y is simply connected.

However we are going to show in the proof of the following result that in some generality the conditions of Theorem 0.2 are met.

Corollary 4.5. *If the aspherical metric space Y is homeomorphic to a finite simplicial complex and if X is a convergent Y-space, then*

$$\iota_* \colon H_*(X; \mathbb{R}) \to \mathcal{H}_*(X)$$

is injective.

Proof: We have to check the assumptions of Theorem 0.2, namely that

- for all k, the homotopy classes (rel. vertices) of k-simplices with vertices in x_0 are Borel sets in $map(\Delta^k, X)$,
- the fundamental group $\pi_1(X, x_0)$ is countable and X is aspherical,
- X can be covered by finitely many Borel sets whose closures are compact and contractible in X.

The first condition is Lemma 4.4. The second condition follows from Lemma 2.9 and Lemma 2.10. The third condition is immediate from the fact that Y can be covered by finitely many simplices. Assuming (possibly after some subdivision) that the copies of one simplex in all the Y_i have at most one point in common, the union of this copies is contractible in X and has compact closure.

■

4.6. **Uncountable homotopy groups.** The proof of Theorem 0.2 made essential use of the existence of a measurable section $s\colon C_*^{x_0}(X) \to C_*^{G\tilde{x}_0}(\widetilde{X})$ which in turn was a consequence of the assumed countability of the fundamental group. The following argument will show that no such section can exist when the fundamental group is uncountable. This means that the arguments of this paper most likely can not be extended to spaces with uncountable fundamental group, like the Hawaiian Earrings.

Lemma 4.6. *Let X be a complete, separable metric space admitting a generalized universal covering $p\colon \widetilde{X} \to X$. Assume that X is semi-locally simply connected at x_0 and that the deck group $G \simeq \pi_1(X, x_0)$ is uncountable. Then $p_1\colon S_1^{G\tilde{x}_0} \to S_1^{x_0}$ admits no Borel section.*

Proof: X being semi-locally simply connected at x_0 implies that $G\tilde{x}_0$ is discrete in $\widetilde{X}$. Then one can show that the sets $S_1^{g_1\tilde{x}_0, g_2\tilde{x}_0}$ are open and closed, so that we have decomposed $S_1^{G\tilde{x}_0}$ into a disjoint union of open and closed sets.

G is uncountable, hence the cardinality of its power set is bigger than continuum. For a measurable section s and any subset $\Gamma \subset G$ let U_Γ be the set of 1-simplices with the same endpoints as $s(\gamma_g)$, where γ_g means the loop representing g (and $s(\gamma_g)$ may be discontinuous, but for U_Γ we consider only continuous maps).

As a disjoint union of sets of the form $S_1^{g_1\tilde{x}_0, g_2\tilde{x}_0}$, any U_Γ must be open and hence have Borel-measurable preimage under s. Thus the $s^{-1}(U_\Gamma)$ are more than continuum many distinct Borel-measurable subsets of $map(\Delta^1, X)$.

But the latter is a Polish space by [14, Theorem 4.19] and thus has only continuum many Borel sets by [24, Theorem 3.3.18]. This is a contradiction.

■

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