

PARAMETRISING ANOSOV REPRESENTATIONS TO $PGL(3, \mathbb{C})$

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ABSTRACT. We give a local parametrisation for the space of Anosov representations from a genus g surface group to $PGL(3, \mathbb{R})$ or $PGL(3, \mathbb{C})$ as an open subset of a $(16g - 16)$ -dimensional (real or complex) non-Hausdorff manifold.

1. INTRODUCTION

The aim of this paper is to give a (local) parametrisation of the space of Anosov representations up to conjugacy from a surface group to $PGL(3, \mathbb{R})$ or $PGL(3, \mathbb{C})$, generalizing the parametrisation of Hitchin representations to $PGL(n, \mathbb{R})$ in [1], which in turn was motivated by the parametrisation of positive representations of punctured surface groups in [4]. While the space of Hitchin representations to $PGL(3, \mathbb{R})$ turned out to be $(\mathbb{R}_{>0})^{16g-16}$, the space of all Anosov representations is known to be non-Hausdorff, so the parametrisation will necessarily invoke a non-Hausdorff parameter space. The Hausdorffification will yield a parametrisation of the character variety.

Let us start with the basic definitions.

1.1. Space of representations up to conjugacy. It is well-known that the space of representations $Hom(\Gamma, G)$ from a finitely generated group to a semisimple algebraic Lie group is a (not necessarily irreducible) algebraic variety and that its quotient $Hom(\Gamma, G)/G$ under the action by conjugation is in general not Hausdorff and in particular not an algebraic variety. Therefore one often considers the character variety, which is the GIT-quotient $Hom(\Gamma, G)//G$.

In this paper we will parametrise an open subset of $Hom(\pi_1 \Sigma_g, PGL(3, \mathbb{C}))/PGL(3, \mathbb{C})$ and hence also of $Hom(\pi_1 \Sigma_g, PGL(3, \mathbb{C}))/PGL(3, \mathbb{C})$ for surface groups $\pi_1 \Sigma_g$, i.e., fundamental groups of closed, orientable surfaces of genus $g \geq 2$.

1.2. Anosov representations. Recall that a flag in $\mathbb{CP}^2$ is a pair (p, l) , where p and l are a point and a line in $\mathbb{CP}^2$, i.e., the equivalence class of a line and a plane in $\mathbb{C}^3$, and $p \in l$. (We will usually talk of lines and planes in $\mathbb{C}^3$ when we actually mean their equivalence classes in $\mathbb{CP}^2$, and we will also not distinguish lines and planes from their equivalence classes in our notation.) Let $X := Flag(\mathbb{CP}^2)$ be the variety of flags. The space of non-collinear triples of points in $\mathbb{CP}^2$ embeds into $X \times X$ via

$$(u, v, w) \rightarrow ((u, (uv)^\perp), (w, (vw)^\perp)).$$

The tangent space of the image at any point y is naturally identified with $T_y(X \times X) = T_y X \oplus T_y X$, the two direct summands give vector bundles E^+ and E^- over X .

Definition 1. Let Σ_g be the closed, oriented surface of genus $g \geq 2$, equipped with a hyperbolic structure identifying the universal covering space $\widetilde{\Sigma}_g$ to the hyperbolic plane H^2 . A representation $\rho: \pi_1 \Sigma_g \rightarrow PGL(3, \mathbb{C})$ is an Anosov representation if there exist continuous, ρ -equivariant maps

$$\xi^\pm: \partial_\infty H^2 \rightarrow Flag(\mathbb{CP}^2)$$

such that

i) $\xi^+(\eta)$ and $\xi^-(\eta)$ are transverse in the sense of Definition 2 for each $\eta \in \partial_\infty H^2$, so ξ^+ and ξ^- combine to a map

$$\tilde{\sigma}: T^1 H^2 \rightarrow Frame(\mathbb{CP}^2).$$

ii) The lifted geodesic flow on $\tilde{\sigma}^*E^+$ resp. $\tilde{\sigma}^*E^-$ is dilating resp. contracting.

It is well-known ([7, Theorem 6.57]) that Anosov representations satisfy the following condition:

iii) $\xi^+(\eta)$ and $\xi^+(\eta')$ are transverse for any $\eta \neq \eta' \in \partial_\infty H^2$.

What we will do in this paper is to parametrise all representations which satisfy condition iii) and not necessarily ii). It is well-known that the set of Anosov representations is an open subset of the set of representation variety and in particular of the variety of representations satisfying iii). So we actually obtain a local parametrisation around any Anosov representation.

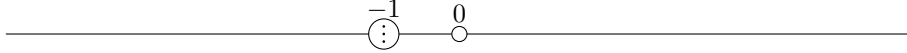
1.3. Parameter spaces. Let Σ_g be the closed, oriented surface of genus $g \geq 2$, equipped with a hyperbolic structure and an ideal triangulation. The reader may remember that Bonahon-Dreyer in [1] parametrised the variety of Hitchin representations as an open subset of the $(16g - 16)$ -dimensional solution set of some system of equations in

$$\mathbb{R}_{>0}^{\#triangles} \times \mathbb{R}_{>0}^{2\#edges}$$

for the given ideal triangulation.

Unlike the Hitchin component, the variety of all Anosov representations is not a Hausdorff space, so we will need some non-Hausdorff manifold as set of parameters.

Let T be the non-Hausdorff space which is $\mathbb{C}_{\neq 0}$ with three non-separable points in the place of -1 . (Formally T is defined as the quotient of 3 copies of $\mathbb{C}_{\neq 0}$ after identifying the 3 copies of $\mathbb{C}_{\neq 0, -1}$.) Moreover, let us denote $T_{\mathbb{R}}$ the non-Hausdorff space which is $\mathbb{R}_{\neq 0}$ with three non-separable points in the place of -1 , i.e., the quotient of 3 copies of $\mathbb{R}_{\neq 0}$ after identifying the 3 copies of $\mathbb{R}_{\neq 0, -1}$. The picture below shows $T_{\mathbb{R}}$, the circles in the picture below represent the missing point 0 and the three copies of -1 , respectively.



The space T will serve to parametrise triples of pairwise transverse flags.

Notation. Throughout the paper we will use self-explaining notations such as $\mathbb{R}_{\neq 0}$ for $\{x \in \mathbb{R} : x \neq 0\}$, $\mathbb{R}_{\neq 0, -1}$ for $\{x \in \mathbb{R} : x \neq 0, -1\}$ and $\mathbb{R}_{>0}$ for $\{x \in \mathbb{R} : x > 0\}$. T and $T_{\mathbb{R}}$ will always mean the non-Hausdorff spaces defined above. The three non-separable points of these spaces, which correspond to the occurrences of -1 in the three copies of $\mathbb{C}_{\neq 0}$, will be denoted by p, l, t . The namesaking is intended to reflect their parametrising of point-nongeneric, line-nongeneric and totally nongeneric triples of flags, see Section 3.

1.4. Result. Denote

$$P' = T^{\#triangles} \times (\mathbb{C}_{\neq 0})^{2\#edges}.$$

Here, by edges we mean not only edges of ideal triangles but also closed edges that are approached by ideal triangles. It is easy to see that P' is a $(22g - 22)$ -dimensional complex, non-Hausdorff manifold.

Let $\{l, p, t\} \in T_{\mathbb{R}}$ be the three non-Hausdorff points of T . We define an equivalence relation on P' as follows. - For any pair of adjacent triangles (which by the choice of our triangulation have necessarily all three edges in common) we identify

$$(\dots, t, t, \dots) \times (\dots, x_1, \dots, x_6, \dots) \sim (\dots, t, t, \dots) \times (\dots, kx_1, \frac{1}{k}x_2, kx_3, \frac{1}{k}x_4, kx_5, \frac{1}{k}x_6, \dots)$$

$$\forall k \in \mathbb{C}_{\neq 0}, (x_1, \dots, x_6) \in \mathbb{C}_{\neq 0}^6.$$

Here $(x_1, \dots, x_6)$ are the coordinates corresponding to the three (common) edges of the ideal triangles.

- For any two pairs of adjacent triangles which approach the same closed edge we identify

$$\begin{aligned} & (\dots, t, t, t, t, \dots) \times (\dots, x_1, \dots, x_6, y_1, y_2, x'_1, x'_2, x'_3, x'_4, x'_5, x'_6, \dots) \sim \\ & (\dots, t, t, t, t, \dots) \times (\dots, kx_1, \frac{1}{k}x_2, kx_3, \frac{1}{k}x_4, kx_5, \frac{1}{k}x_6, ly_1, \frac{1}{l}y_2, k'x'_1, \frac{1}{k'}x'_2, k'x'_3, \frac{1}{k'}x'_4, k'x'_5, \frac{1}{k'}x'_6, \dots) \end{aligned}$$

$$\forall k, l, k' \in \mathbb{C}_{\neq 0}, (x_1, \dots, x_6, y_1, y_2, x'_1, \dots, x'_6) \in \mathbb{C}_{\neq 0}^{14}.$$

Let $\mathcal{P} \subset P$ be the set of solutions of the "Closed Leaf Equalities" and "Closed Leaf Inequalities", see Section 3. It is a $(16g - 16)$ -dimensional complex, non-Hausdorff manifold. Let $\mathcal{R} \subset P / \sim$ be the image of $\mathcal{P}$ in $P / \sim$.

We claim that the set of Anosov representations to $SL(3, \mathbb{C})$ can be parametrised as an open subset of $\mathcal{R}$. Moreover, the Anosov representations to $SL(3, \mathbb{R})$ correspond to the real locus, i.e., to the intersection with the quotient of

$$P'_{\mathbb{R}} = T_{\mathbb{R}}^{\#triangles} \times (\mathbb{R}_{\neq 0})^{2\#edges}.$$

We emphasize that this is only a local parametrisation, i.e we get full parameters in some neighborhood of any Anosov representations. Globally, we do not know which parameters do actually correspond to an Anosov representation.

1.5. Dependence of triangulations and choice of transverse arcs. Because any closed surface can be given an ideal triangulation by decomposing it into pairs of pants and then decomposing each pair of pants into two ideal tetrahedra, we will in this paper only consider ideal triangulations of that kind. It is plausible that everything should also work for arbitrary ideal triangulations, but some of the non-generic cases in the proof of Lemma 1 would require cumbersome computations when considering arbitrary triangulations, which we did not want to perform. Moreover as in [1] our construction will need to have fixed a transverse arc for each closed edge of the ideal triangulation. There is no canonical choice of such an arc, so the parameters become somewhat uncanonical, which one should keep in mind when wanting to compare them to other coordinates. In this paper we will use the following semi-canonical choice for the choice of transverse arcs: for each pair of pants in the given decomposition we fix one ideal triangle in the pull-back triangulation, and we choose all transverse arcs such that the endpoints (of their lifts to the universal covering) lie in those triangles. This convention will greatly simplify explicit computations of examples.

1.6. Approach. The approach is in principle as in [1]. We assume to have fixed an ideal triangulation Υ of our surface Σ_g and a transverse arc for each closed edge of the ideal triangulation. Consider an Anosov representation ρ with boundary map ξ .

Given an ideal triangle of Υ with vertices v_0, v_1, v_2 , we look at the triple of flags $(\xi(v_0), \xi(v_1), \xi(v_2))$. As explained in Section 3 below it determines an element of T . (In the case of Hitchin representation this would be the triple ratio and would belong to $\mathbb{R}_{>0}$.)

Given a non-closed edge of the ideal triangulation, it bounds two ideal triangles with vertices v_0, v_1, v_2, v_3 . We associate two numbers (x_1, x_2) to this edge. If the two triple ratios of the adjacent triangles are not both equal to $t \in T$, then the two numbers will be the double ratios that [1] considered in the case of a Hitchin representations.

Given a closed edge g and its transverse arc k we get two triples of flags as described in [1, Section 3.5]: let x, y be the endpoints of a lift $\tilde{g}$, let $\tilde{T}, \tilde{T}'$ be the triangles containing the lift $\tilde{k}$ which intersects $\tilde{g}$, let z resp. z' be the vertices of $\tilde{T}$ resp. $\tilde{T}'$ which are farthest away from $\tilde{g}$, then we consider the triples of flags associated to x, y, z and x, y, z' . Again, we associate two numbers (D_1, D_2) .

1.7. Coordinates. We will work in the same setting as [1] and will therefore be somewhat brief about points that have been explained there in more detail. Let S be the closed, orientable surface of genus $g \geq 2$. Consider an ideal triangulation arising from a decomposition of S into pairs of pants by cutting each pair of pants along three bi-infinite curves spiraling around the boundary components to obtain three infinite triangles. So we have $3(g - 1)$ closed curves of the pair-of-pants decomposition, together with the $6(g - 1)$ spiraling bi-infinite curves, and S is decomposed into $4(g - 1)$ ideal triangles. As in [1, Section 3.1] we also fix an orientation for each edge and for each closed edge a transverse arc that meets no other closed edge.

An Anosov representation $\rho: \pi_1 S \rightarrow PGL(3, \mathbb{C})$ comes with an equivariant boundary map $\xi: \partial_\infty H^2 \rightarrow Flag(\mathbb{CP}^2)$ which depends continuously on ρ . If ρ is an Anosov representation to $PGL(3, \mathbb{R})$, then ξ has image in $Flag(\mathbb{RP}^2)$.

Associated to ξ and the ideal triangulation we have the following invariants.

For each triangle T_i of the ideal triangulation and each of its vertices we have the invariant

$$\tau(T_i, v_i) \in T.$$

For each (closed or non-closed) non-closed edge we have the invariants

$$D_1, D_2 \in \mathbb{C} \setminus \{0\}.$$

The invariants have to satisfy the closed leaf equalities

$$L_i^{right} = L_i^{left}, i = 1, 2$$

and the closed leaf inequalities

$$L_i^{right} \neq 1, i = 1, 2$$

for each closed edge E_i .

Let $\mathcal{P}$ be the set of solutions of the closed leaf equalities and inequalities and $\mathcal{R}$ the quotient under the equivalence relation $\sim$.

Our main theorem will be Theorem 2: *Every element of $\mathcal{R}$ corresponds to a unique representation $\rho: \pi_1 S \rightarrow PGL(3, \mathbb{C})$. The representation depends continuously on the parameters and every Anosov representation is obtained this way. Elements of $\mathcal{R}_{\mathbb{R}}$ correspond to Anosov representations to $PGL(3, \mathbb{R})$.*

1.8. Example: genus 2. The genus 2 surface is obtained by glueing two pairs of pants together along their three boundary curves. The pairs of pants are decomposed into two ideal triangles which are glued along their three edges. For the first pair of pants we will denote by X, Y the triple ratios of the triangles and by $D_1, D_2, E_1, E_2, F_1, F_2$ the double ratios of the edges arranged as in the following picture. For the second pair of pants we denote the corresponding parameters by $X', Y', D'_1, D'_2, E'_1, E'_2, F'_1, F'_2$. Moreover we have the double ratios $G_1, G_2, H_1, H_2, I_1, I_2$ associated to the three boundary curves of the pairs of pants. Then, for each of the two pairs of pants separately, we have the inequalities from the proof of Corollary 2, and we have the closed leaf equalities and inequalities which in this case take the following form.

$$F'_1 D'_2 X' Y' = F_1 D_2 X Y \neq 1$$

$$F'_2 D'_1 = F_2 D_1 \neq 1$$

$$F'_1 E'_2 X' Y' = F_1 E_2 X Y \neq 1$$

$$F'_2 E'_1 = F_2 E_1 \neq 1$$

$$D'_2 E'_1 X' Y' = D_2 E_1 X Y \neq 1$$

$$D'_1 E'_2 = D_1 E_2 \neq 1$$

The set of Anosov representations is parametrised by an open subset of $\mathcal{P}$.

2. RECONSTRUCTING REPRESENTATIONS FROM COORDINATES EXPLICITLY

Positive representations can be reconstructed from their Fock-Goncharov coordinates. For positive representations of punctured surface groups this is explained nicely in [3], the reader may also consult Section 4.3 of Palesi's very readable survey paper [10]. Essentially the same approach works for positive representations of closed surface groups, the only difference being that one has to take care of the closed edges that necessarily appear in ideal triangulations of closed surfaces. We will discuss this issue in the course of this section.

A difference between positive and arbitrary Anosov representations is that in some exceptional cases the double ratios are not well-defined and that actually in some cases (namely for completely reducible representations) the representation is not determined by the boundary map. For this reason we will not in all cases define the parameters via the boundary map.

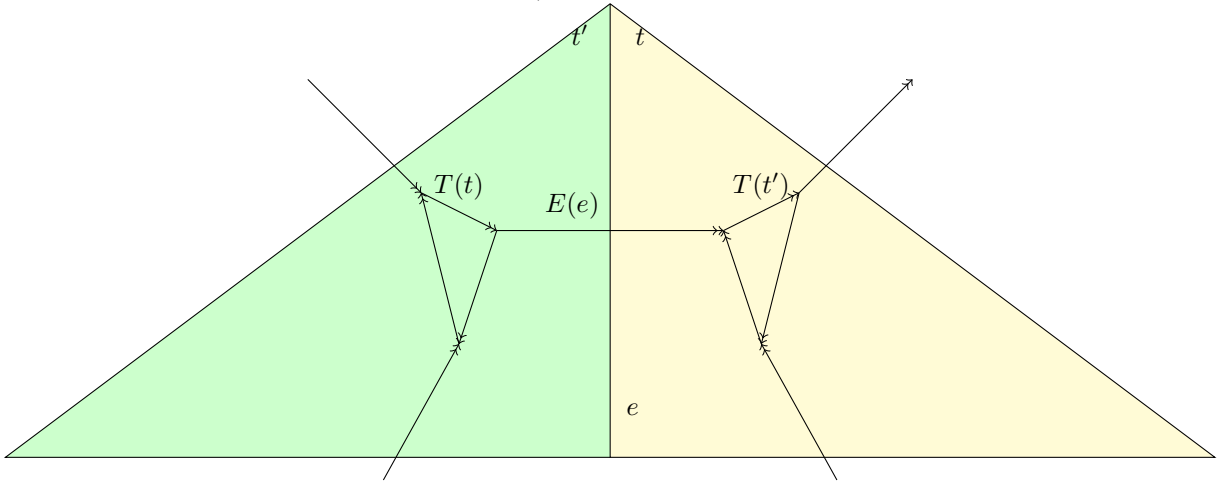
In fact we will have as parameters an element of T for each triangle and two elements of $\mathbb{C}_{\neq 0}$ for each edge. In most cases these parameters will be triple ratios and double ratios as in [1]. The exceptional case is when two adjacent triangles are mapped by the boundary map to totally degenerate triples of flags as defined in Section 3. In this case the two numbers associated to their common edge do not have an interpretation in terms of the boundary map.

Let us now describe how the representation is reconstructed from its parameters. (This essentially follows [3], just that in some cases the parameters are no double ratios.)

Let us first remark that any closed loop on an ideally triangulated surface can be homotoped to lie in the dual graph of the triangulation. So to define the representation it is sufficient to associate matrices to each arc of the ideal triangulation in a way that every 0-homotopic loop is sent to the identity.

Unlike [3] and [10] (which discussed punctured surfaces) we will actually work not with the dual graph itself, but with a certain contraction. As discussed in the introduction we will (following [1]) fix one transverse arc for each closed edge. This transverse arc will intersect an infinite number of non-closed edges on both sides of the closed edge. To each of these non-closed edges of the triangulation we have of course an edge of the dual graph. We will then consider the graph which - for each closed edge of the ideal triangulation - contains one transverse edge, to which the before-mentioned infinitely many edges of the dual graph have been contracted. So the graph we are considering is actually a finite graph.

The picture below shows the intersection of (the contraction of) the dual graph with two adjacent ideal triangles. As in [10] we will call $E(e)$ the matrix associated to an arc traversing an edge and $T(t)$ the matrix associated to any of the three arcs inside an ideal triangle. (Note that the arcs are oriented and the inverse arc is sent to the inverse matrix.)



Morally, the idea of the computation is of course the same as in [10]: if t is a triangle with flags A, B, C at its vertices, then the matrix $T(t)$ performs the cyclic permutation $(A, B, C) \rightarrow (B, C, A)$, and if (A, B, C, D) is a quadruple of flags with A, C associated to the vertices of the edge e , then $E(e)$ sends A to C , C to A and the point of B to the point of D ,

However, in this paper we will take the following as a definition of how to associate a representation to the given parameters and in the course of the paper we will prove that this gives a continuous parametrisation of the space of Anosov representations. For generic representations it will coincide with the parametrisation via boundary maps.

For $T(t)$ we obtain in the generic case with triple ratio X (as in [10])) $T(t) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ X & 1+X & 1 \end{pmatrix}$,
for the point-nongeneric case (as defined in Section 3) $T(t) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$, for the line-nongeneric
case $T(t) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$ and for the totally non-generic case $T(t) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$.

For $E(e)$, if the numbers associate to the edge e are D_1, D_2 , then we define $E(e) = \begin{pmatrix} 0 & 0 & \frac{1}{D_1} \\ 0 & -1 & 0 \\ D_2 & 0 & 0 \end{pmatrix}$.
Finally, we have to assign a matrix to those edges intersecting a closed edge of the ideal triangulation. To such an edge f we have assigned two real numbers D_1, D_2 and we assign to such an edge again the
matrix $E(f) = \begin{pmatrix} 0 & 0 & \frac{1}{D_1} \\ 0 & -1 & 0 \\ D_2 & 0 & 0 \end{pmatrix}$.

3. TRIPLES OF PAIRWISE TRANSVERSE FLAGS

3.1. Pairs of transverse flags. Throughout this paper “a flag” will always mean a flag in $\mathbb{CP}^2$ and we will consider the action of $PGL(3, \mathbb{C})$ on (tuples of) flags in $\mathbb{CP}^2$.

Recall that by a flag in $\mathbb{CP}^2$ we mean a pair (p, l) , where p is a point in $\mathbb{CP}^2$ and l is a line containing that point. We will write lines in $\mathbb{CP}^2$ in the form $v^\perp$ with respect to the standard hermitian metric on $\mathbb{C}^3$.

Definition 2. Two flags (p_1, l_1) and (p_2, l_2) are transverse if $p_1 \notin l_2$ and $p_2 \notin l_1$.

The following lemma is elementary and well-known.

Lemma 1. *i) A pair of transverse flags is in the $PGL(3, \mathbb{C})$ -orbit of $\{(e_1, e_3^\perp), (e_3, e_1^\perp)\}$.*

ii) The stabilizer of a pair of transverse flags is conjugate to the subgroup of diagonal matrices.

3.2. Triple ratio and non-genericities. The classification of generic triples of flags appears at many places in the literature, but we will also need the non-generic case. In the following lemma the first case corresponds to generic triples and the number X is the triple ratio in that case. In the non-generic cases the triple ratio is always -1 . (In [4, Section 2.3] it is erroneously claimed that the triple ratio would be $X = 1$ in the third case, but it is easily checked to be $X = -1$.)

Lemma 2. A triple $\{(p_1, l_1), (p_2, l_2), (p_3, l_3)\}$ of pairwise transverse flags is in the $PGL(3, \mathbb{C})$ -orbit of
- either

$$\{(e_1, e_3^\perp), (e_3, e_1^\perp), (e_1 - e_2 + e_3, (e_1 + (1 + X)e_2 + Xe_3)^\perp)\}$$

for a unique $X \in \mathbb{R} \setminus \{0, -1\}$,

- or

$$\{(e_1, e_3^\perp), (e_3, e_1^\perp), (e_1 + e_3, (e_1 + e_2 - e_3)^\perp)\},$$

- or

$$\{(e_1, e_3^\perp), (e_3, e_1^\perp), (e_1 + e_2 + e_3, (e_1 - e_3)^\perp)\},$$

- or

$$\{(e_1, e_3^\perp), (e_3, e_1^\perp), (e_1 + e_3, (e_1 - e_3)^\perp)\}$$

Proof: The generic case is well-known to correspond to the first possibility, see [3, Section 9] (for the case of real flags, but the complex case is of course analogous).

In the non-generic case we first apply Lemma 1 to get $(p_1, l_1) = (e_1, e_3^\perp)$, $(p_2, l_2) = (e_3, e_1^\perp)$ and then we either have $p_3 \in e_1 \oplus e_3$ or $l_3 \cap (e_1^\perp \cap e_3^\perp) \neq \emptyset$ or both.

If $p_3 \in e_1 \oplus e_3$, then we have after application of a suitable diagonal matrix that $p_3 = e_1 + e_3$ and then $p_3 \in l_3$ implies that l_3 is either $e_2^\perp$ or of the form $(e_1 + Ze_2 - e_3)^\perp$. But $l_3 = e_2^\perp$ would contradict the transversality assumptions $p_1 \notin l_3, p_2 \notin l_3$, so we must be in the second case $l_3 = (e_1 + Ze_2 - e_3)^\perp$. For

$Z = 0$ we are in the fourth case. If $Z \neq 0$, then upon application of a diagonal matrix $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \frac{1}{\lambda^2} & 0 \\ 0 & 0 & \lambda \end{pmatrix}$,

which does not affect $p_3 = e_1 + e_3$, we can assume $Z = 1$.

Similarly, if $l_3 \cap (e_1^\perp \cap e_3^\perp) \neq \emptyset$, then after application of a suitable diagonal matrix we have $l_3 = (e_1 - e_3)^\perp$, so either $p_3 = e_2$ (which would contradict transversality) or $p_3 = e_1 + X = Ze_2 + e_3$. For $Z = 0$ we are in the fourth case, and for $Z \neq 0$ we can upon application of a diagonal matrix $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \frac{1}{\lambda^2} & 0 \\ 0 & 0 & \lambda \end{pmatrix}$, which does not affect $l_3 = (e_1 - e_3)^\perp$, assume $Z = 1$. QED

We will henceforth refer to the four cases as the generic, the point-nongeneric, the line-nongeneric and the totally nongeneric case. In the generic case the parameter X agrees with the triple ratio, which is defined in [1, Section 1.2] as

$$X = \frac{l_1 \wedge p_3 \ l_2 \wedge p_1 \ l_3 \wedge p_2}{p_1 \wedge l_3 \ p_2 \wedge l_1 \ p_3 \wedge l_2}.$$

We note that the triple ratio is also defined in the non-generic cases and always takes the value -1 for non-generic triples. In particular, other as in the generic case, there are three distinct equivalence classes of non-generic triples that have this same triple ratio.

3.3. Parametrising triples of pairwise transverse flags. Let us denote by T the topological space obtained from three copies of $\mathbb{C}_{<0}$ by identifying all copies of $\mathbb{C}_{\neq 0} \setminus \{-1\}$.

We will denote the three non-identified copies of -1 in T by p, l and t . In other words, we have

$$T = (\mathbb{C} \setminus \{0, -1\}) \cup \{p, l, t\}$$

and there is a continuous map

$$proj: T \rightarrow \mathbb{C} \setminus \{0\}$$

which is a homeomorphism of $T \setminus \{p, l, t\}$ onto $\mathbb{C} \setminus \{0, -1\}$, and is a local homeomorphism at each of the points p, l, t , i.e., a homeomorphism from a neighborhood of any of them to a neighborhood to -1 . (The points p, l, t , however, can not be separated by open neighborhoods.)

T contains the subspace $T_{\mathbb{R}}$ consisting of two connected components, one being homeomorphic to the open interval $\mathbb{R}_{>0}$ and one obtained from three copies of $\mathbb{R}_{<0}$ by identifying all copies of $\mathbb{R}_{<0} \setminus \{-1\}$.

Corollary 1. *The space of equivalence classes of pairwise transverse triples of flags, as a subspace of*

$$(Flag(\mathbb{CP}^2) \times Flag(\mathbb{CP}^2) \times Flag(\mathbb{CP}^2))/PGL(3, \mathbb{C}),$$

is homeomorphic to T .

The subspace of equivalence classes of pairwise transverse triples of real flags, as a subspace of

$$(Flag(\mathbb{RP}^2) \times Flag(\mathbb{RP}^2) \times Flag(\mathbb{RP}^2))/PGL(3, \mathbb{R}),$$

is homeomorphic to $T_{\mathbb{R}}$.

Proof: Let $Trip$ be the space of equivalence classes of triples and

$$Trip_{gen} \subset Trip$$

be the subsets of classes of generic triples. By Lemma 2 we have bijections $F: Trip \rightarrow T$ and $F^{-1}: T \rightarrow Trip$.

It is well known (and easy to prove) that F and F^{-1} induce homeomorphisms between $Trip_{pos}$ (the space of generic triples with positive real triple ratio) and $\mathbb{R}_{>0}$, and the same argument actually provides homeomorphisms between $Trip_{gen}$ and $\mathbb{C} \setminus \{0, -1\}$. We have to show that the extensions of F resp. F^{-1} to the non-Hausdorff points are still continuous.

We note that the totally nongeneric triple is in the closure of the $PGL(3, \mathbb{C})$ -orbits of both the point- and line-nongeneric triples. Hence these three equivalence classes can not be separated from each other in $Trip$.

Continuity of F in the non-Hausdorff points is a consequence of the fact that the triple ratio by its definition above is a continuous function on $Trip$ and so a sequence converging to a non-generic triple must have triple ratios converging to -1 , so its image under F must converge to p, l and t . (Keep in mind that the non-Hausdorffness of T implies that a sequence converging to one of p, l, t must converge to all of them.)

It remains to prove continuity of F^{-1} in the non-Hausdorff points of T and for this one has to show that for a sequence converging to p, l, t there is a sequence of generic flags converging to, accordingly, a point-nongeneric, line-nongeneric or totally nongeneric flag.

So, let x_n be a sequence in $\mathbb{C} \setminus \{0, -1\} \subset T$ converging to p, l, t . This means in particular that in $\mathbb{C} \setminus \{0\}$ it converges to -1 . Then we see that the sequence of triples

$$\{(e_1, e_3^\perp), (e_3, e_1^\perp), (e_1 - e_2 + e_3, (e_1 + (1 + x_n)e_2 + x_ne_3)^\perp)\}$$

converges to a line-nongeneric triple, thus the sequence of equivalence classes in $Flag$ converges to $F^{-1}(l)$. We know that any triple with triple ratio $x_n \neq -1$ belongs to the same class, so

$$\left\{ (e_1, e_3^\perp), (e_3, e_1^\perp), (e_1 + (1 + \frac{1}{x_n})e_2 + \frac{1}{x_n}e_3, (e_1 - e_2 + e_3)^\perp) \right\}$$

represents exactly the same sequence in $Trip$ but it converges to $F^{-1}(p)$. Finally we can consider for example

$$\left\{ (e_1, e_3^\perp), (e_3, e_1^\perp), ((1 + x_n + \frac{1}{x_n})e_1 - (1 + \frac{1}{x_n})e_2 + \frac{1 + x_n + \frac{1}{x_n}}{1 + \frac{1}{x_n}}e_3, (\frac{1}{1 + x_n}e_1 + (1 + x_n)e_2 + e_3)^\perp) \right\}$$

and this again represents the same sequence in $Trip$ (because one computes the triple ratio to be x_n) while it converges to $F^{-1}(t)$. QED

The naming of p, l and t is chosen as is because we will be thinking of them as representing the point-nongeneric, line-nongeneric and totally nongeneric triple.

3.4. Stabilizers and dualities.

Lemma 3. *A pairwise transverse triple of flags has trivial stabilizer in $PGL(3, \mathbb{C})$ if and only if it is not totally non-generic.*

For a totally non-generic, but pairwise transverse triple of flags the stabilizer is conjugate to

$$\left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \frac{1}{\lambda^2} & 0 \\ 0 & 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{R}^* \right\}.$$

Proof: By Lemma 1 the stabilizer of a transverse pair of flags is conjugate to the group of diagonal matrices. Using the normal form from Lemma 2 one checks easily that in the generic as well as in the point- and line-nongeneric case, there is no diagonal matrix except the identity preserving the third flag, and that in the totally non-generic case the only diagonal matrices that preserve the third flag are as above. QED

The duality $\mathrm{Flag}(\mathbb{CP}^2) \rightarrow \mathrm{Flag}(\mathbb{CP}^2)$ is defined by fixing the hermitian metric on $\mathbb{C}^3$ and sending e_i to $e_i^\perp$ and $e_i^\perp$ to e_i for $i = 1, 2, 3$. (One may use any other scalar product and get a different duality.) The following lemma is straightforward.

Lemma 4. *The duality $\mathrm{Flag}(\mathbb{CP}^2) \rightarrow \mathrm{Flag}(\mathbb{CP}^2)$ sends*

- *a generic triple of triple ratio X to a generic triple of triple ratio $\frac{1}{X}$,*
- *a point-nongeneric triple to a line-nongeneric triple,*
- *a line-nongeneric triple to a point-nongeneric triple,*
- *a totally nongeneric triple to a totally nongeneric triple.*

3.5. Realisability. For further reference let us state the following reformulation of Lemma 2, which will serve as the analogue of [1, Lemma 19] in our setting.

Lemma 5. *Given $t \in T$ and an ideal triangle $\tilde{\Delta}$ there exists a map $\xi: \partial_\infty \tilde{\Delta} \rightarrow \mathrm{Flag}(\mathbb{CP}^2)$ such that the triple ratio of the three vertices is t . It is unique up to composition with $\mathrm{PGL}(3, \mathbb{C})$.*

Lemma 6. *Given an ideal triangle $\tilde{\Delta}$ with ideal vertices v_0, v_1, v_2 , let r be the map sending v_0 to v_1 , v_1 to v_2 and v_2 to v_0 . Then*

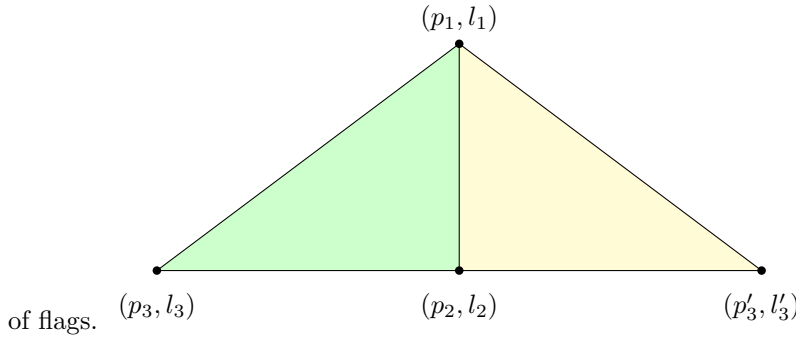
$$\xi \circ r = T(t) \circ \xi$$

for the matrix $T(t)$ defined in Section 2.

Proof: Explicit computation. QED

4. QUADRUPELS OF PAIRWISE TRANSVERSE FLAGS

In our construction we will reconstruct representations from triple ratios and from certain parameters D_1, D_2 which in the generic case correspond to the double ratios of Fock-Goncharov. In the non-generic case the parameters will not have such a geometric interpretation, in fact there will be cases where they are not determined by the boundary map and only the product $D = D_1 D_2$ can be given an interpretation in terms of flags, but we will nonetheless start with discussing the relation of the parameters to quadruples



4.1. Quadruples of pairwise transverse flags.

Lemma 7. *Every 4-tuple*

$$((p_1, l_1), (p_2, l_2), (p_3, l_3), (p'_3, l'_3))$$

of pairwise transverse flags is in the $PGL(3, \mathbb{C})$ -orbit of a 4-tuple of the form

$$(p_1, l_1) = (e_1, e_3^\perp),$$

$$(p_2, l_2) = (e_3, e_1^\perp),$$

(p_3, l_3) is of the form of (one of the four cases) of Lemma 2,

(p'_3, l'_3) is the product of a diagonal matrix $\text{diag}(\lambda_1, 1, \lambda_3)$ with

a flag of the form of (one of the four cases) of Lemma 2.

If not both triples are totally non-generic, then the diagonal matrix $\text{diag}(\lambda_1, 1, \lambda_3)$ is uniquely determined.

If both of the triples are totally non-generic, then the diagonal matrix $\text{diag}(\lambda_1, 1, \lambda_3)$ is uniquely determined only up to multiplication by any diagonal matrix of the form $\text{diag}(\lambda, 1, \lambda)$ with $\lambda \neq 0$.

Proof: Given two triples of flags

$$\{(p_1, l_1), (p_2, l_2), (p_3, l_3)\}$$

and

$$\{(p_1, l_1), (p_2, l_2), (p'_3, l'_3)\}$$

we can by Lemma 1 upon multiplication by some matrix in $PGL(3, \mathbb{C})$ assume

$$(p_1, l_1) = (e_1, e_3^\perp), (p_2, l_2) = (e_3, e_1^\perp)$$

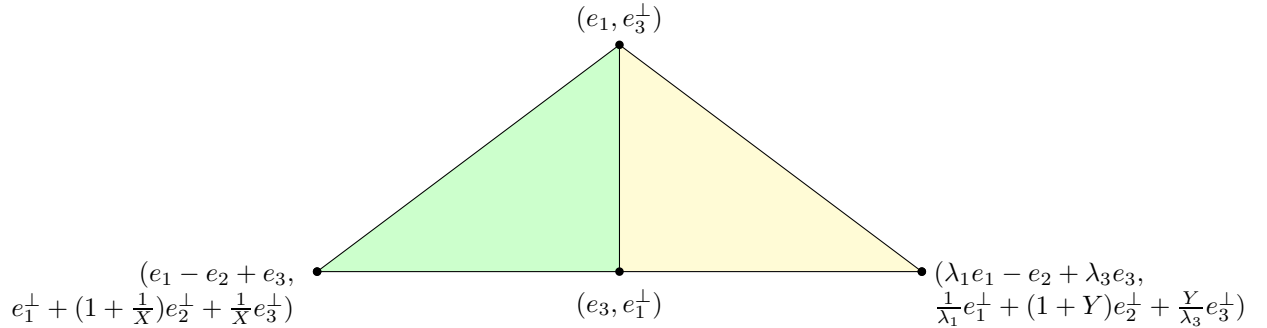
and we can also assume that (p_3, l_3) is of the form in (one of the four cases of) Lemma 2 and that (p'_3, l'_3) is obtained by multiplication of a diagonal matrix

$$\text{diag}(\lambda_1, \lambda_2, \lambda_3)$$

with a flag in one of the normal forms from Lemma 2. (The latter fact is a consequence of Lemma 3.)

Since an element in $PGL(3, \mathbb{C})$ is determined upon multiplication by a scalar matrix, we may w.l.o.g. assume that $\lambda_2 = 1$.

The picture below shows the standard form for a generic quadruple of flags, with the two generic triples having triple ratios X and Y , and with λ_1 and λ_3 defined as above for $\lambda_2 = 1$.



The uniqueness statement for $\text{diag}(\lambda_1, 1, \lambda_3)$ follows from Lemma 1 and Lemma 3.

QED

4.2. Double ratios. The notation in the following definition is chosen such that in the generic case it is consistent with the notation in [1].

Definition 3. For a pairwise transverse quadruple of flags $\{(p_1, l_1), (p_2, l_2), (p_3, l_3), (p'_3, l'_3)\}$ we denote

$$D = \frac{\lambda_1}{\lambda_3}$$

and moreover, if not both triples $\{(p_1, l_1), (p_2, l_2), (p_3, l_3)\}$ and $\{(p_1, l_1), (p_2, l_2), (p'_3, l'_3)\}$ are totally non-generic, we denote

$$D_1 = -\lambda_1, D_2 = -\frac{1}{\lambda_3}.$$

Note that by the uniqueness part of Lemma 7 λ_1 and λ_3 are uniquely defined if not both triples are totally non-generic. If both triples are totally non-generic, then they are defined up to multiplication with some $\lambda \neq 0$, so their quotient is still uniquely defined,

In the case of positive representations the parameters D_1 and D_2 are exactly the double ratios from [1]. (This is shown by a straightforward calculation.) Clearly $D = D_1 D_2$ whenever both triples are not totally non-generic.

We note that we can define D by

$$D = \frac{l_2 \wedge p'_3}{l_2 \wedge p_3} \frac{l_1 \wedge p_3}{l_1 \wedge p'_3}$$

which is well-defined (and neither 0 nor ∞) even in the non-generic case, while the usual definitions (see [1, Section 1.4])

$$D_1 = -\frac{p_1 \wedge p_2 \wedge p_3}{p_1 \wedge p_2 \wedge p'_3} \frac{l_2 \wedge p'_3}{l_2 \wedge p_3}, D_2 = -\frac{p_1 \wedge p_2 \wedge p'_3}{p_1 \wedge p_2 \wedge p_3} \frac{l_1 \wedge p_3}{l_1 \wedge p'_3}$$

yield $\frac{0}{0}$ if both triangles are point-nongeneric or totally non-generic. (They yield 0 or ∞ if this is the case for only one of them.)

We remark that in the case that one triple is point-nongeneric and the other either generic or point-nongeneric, one may still define D_1 and D_2 by going to the dual configuration and using the above formulas. This will however not work, if one triple is point-nongeneric and the other line-nongeneric, and of course also not if one of the triples is totally non-generic. This is - besides Proposition 1 - another reason why in these cases we will have to define the parameters without making reference to the boundary map.

4.3. Realisability. The following lemma is the analogue of [1, Lemma 20], however its proof is slightly different because we are using another (though in the generic case equivalent) definition of double ratios.

Lemma 8. *Let $\tilde{T}_j$ and $\tilde{T}_k$ be two adjacent ideal triangles separated by an infinite leaf $\tilde{g}_i$. Given two elements $X_j, X_k \in T$ and*

- *either, if $X_j \neq t$ or $X_k \neq t$, two non-zero real numbers D_1, D_2*
- *or, if $X_j = t, X_k = t$, one non-zero real number D ,*

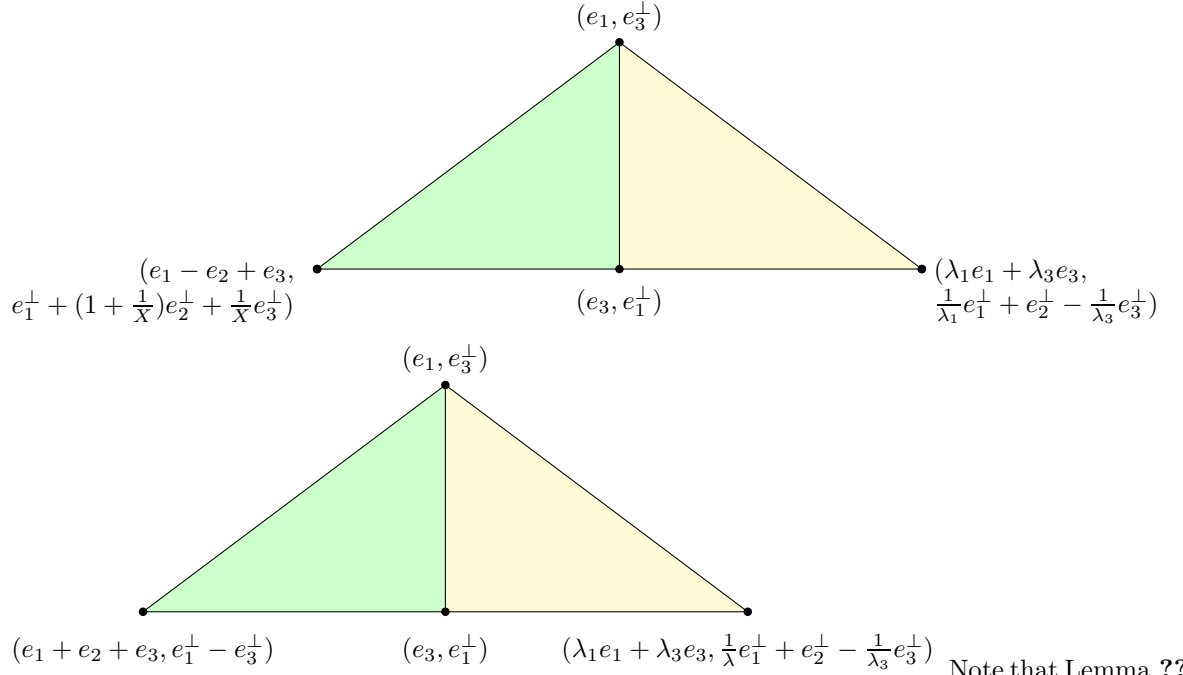
then there is a map

$$\xi: \partial_\infty(\tilde{T}_j \cup \tilde{T}_k) \rightarrow \mathrm{Flag}(\mathbb{R}P^2)$$

such that the vertices of $\tilde{T}_j$ and $\tilde{T}_k$ are sent to triples of flags representing X_j and X_k , respectively, and the double ratios associated to $\tilde{g}_i$ are D_1, D_2 or D (according to the case distinction above). A map with this properties is unique up to composition with $\mathrm{PGL}(3, \mathbb{C})$.

Proof: This is proved by an explicit computation. Using the duality from Lemma 4 it suffices to consider the following seven cases: both triples are generic, one triple is generic and the other line-nongeneric, one triple is generic and the other totally non-generic, both triples are line-nongeneric, one triple is point-nongeneric and the other line-nongeneric, one triple is line-nongeneric and the other totally non-generic, both triples are totally nongeneric.

The result for the case that both triples are generic is shown in the picture before Lemma ???. The following two pictures show the results in the cases that one triple is generic and the other line-nongeneric, and that one triple is point-generic and the other line-nongeneric.



Note that Lemma ??

and the uniqueness part of Lemma 7 imply that these configurations are unique up to composition with an element of $PGL(3, \mathbb{C})$.

For the remaining four cases one obtains, always assuming $(p_1, l_1) = (e_1, e_3^\perp)$, $(p_2, l_2) = (e_3, e_1^\perp)$:

- if one triple is generic and the other totally non-generic, then

$$(p_3, l_3) = (e_1 - e_2 + e_3, e_1^\perp + (1 + \frac{1}{X})e_2^\perp + \frac{1}{X}e_3^\perp), (p'_3, l'_3) = (\lambda_1 e_1 + \lambda_3 e_3, \frac{1}{\lambda_1}e_1^\perp - \frac{1}{\lambda_3}e_3^\perp)$$

- if both triples are line-nongeneric, then

$$(p_3, l_3) = (e_1 + e_2 + e_3, e_1^\perp - e_3^\perp), (p'_3, l'_3) = (\lambda_1 e_1 + e_2 + \lambda_3 e_3, \frac{1}{\lambda_1}e_1^\perp - \frac{1}{\lambda_3}e_3^\perp)$$

- if one triple is line-nongeneric and the other totally non-generic, then

$$(p_3, l_3) = (e_1 + e_2 + e_3, e_1^\perp - e_3^\perp), (p'_3, l'_3) = (\lambda_1 e_1 + \lambda_3 e_3, \frac{1}{\lambda_1}e_1^\perp - \frac{1}{\lambda_3}e_3^\perp)$$

- if both triples are totally non-generic, then

$$(p_3, l_3) = (e_1 + e_3, e_1^\perp - e_3^\perp), (p'_3, l'_3) = (\lambda_1 e_1 + \lambda_3 e_3, \frac{1}{\lambda_1}e_1^\perp - \frac{1}{\lambda_3}e_3^\perp).$$

QED

Lemma 9. *Given two adjacent ideal triangles $\tilde{\Delta}_1, \tilde{\Delta}_2$ with ideal vertices v_0, v_1, v_2, v_3 with v_1, v_2 the vertices of the common edge, let r be the map sending v_0 to v_3 , v_1 to v_2 and v_2 to v_1 . Then*

$$\xi \circ r = E(D_1, D_2) \circ \xi$$

for the matrix $E(D_1, D_2)$ defined in Section 2.

Proof: Explicit computation.

QED

4.4. Parametrising quadruples of pairwise transverse flags. Recall that in Section 3.3 we defined $T = \mathbb{C}_{\neq 0, -1} \cup \{p, l, t\}$ to be the space obtained from three copies of $\mathbb{C}_{\neq 0}$ by identifying all copies of $\mathbb{C}_{\neq 0, -1}$. Let us denote

$$Q' := T^2 \times \mathbb{C}_{\neq 0}^2$$

and let us denote by Q the space

$$Q := Q' / \sim,$$

where the equivalence relation $\sim$ identifies

$$(t, t, D_1, \frac{D}{D_1}) \sim (t, t, D'_1, \frac{D}{D'_1})$$

for any

$$D, D_1, D'_1 \in \mathbb{R}_{\neq 0}$$

and otherwise only equal elements are equivalent.

Corollary 2. *The space of pairwise transverse quadruples of flags*

$$\{(p_1, l_1), (p_2, l_2), (p_3, l_3), (p'_3, l'_3)\}$$

as a subspace of

$$(\mathrm{Flag}(\mathbb{CP}^2) \times \mathrm{Flag}(\mathbb{CP}^2) \times \mathrm{Flag}(\mathbb{CP}^2) \times \mathrm{Flag}(\mathbb{CP}^2)) / \mathrm{PGL}(3, \mathbb{C})$$

is homeomorphic to an open subset $U \subset Q$.

Proof: Q is defined such that one has taken into account that for $x = y = t$ only the product $D = D_1 D_2$ is well-defined. Given that it is clear that the map which to each quadruple of flags associates its triple ratios and double ratios as elements of Q is continuous. The computations in the proof of Lemma 2 show that also conversely the triple and double ratios continuously determine the quadruple of flags.

Recall that we have parametrised pairwise transverse triples of flags in Corollary 1. Here we have now two such triples of flags and the parameter space would be Q if we forgot the additional condition that also (p_3, l_3) and (p'_3, l'_3) have to be transverse to each other. This additional condition yields an open subset of Q , namely one checks that this condition is equivalent to the two inequalities

$$D_1 + 1 + \frac{1}{X} + \frac{1}{D_2 X} \neq 0,$$

$$\frac{1}{D_1} + 1 + Y + D_2 Y \neq 0.$$

QED

4.5. Stabilizers.

Lemma 10. *The stabilizer of a pairwise transverse 4-tuple of flags is trivial except if both triangles are totally non-generic. It is conjugate to*

$$\left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \frac{1}{\lambda^2} & 0 \\ 0 & 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C}_{\neq 0} \right\}$$

if both triples are totally non-generic.

Proof: This is immediate from Lemma 3.

QED

5. COMPLETELY REDUCIBLE REPRESENTATIONS

A representation is called completely reducible if it is the direct sum of two smaller representations, in our case of a 2-dimensional and a 1-dimensional representation. (Since we are looking at representations to $PGL(3, \mathbb{C})$, we can assume that the 1-dimensional representation is identical 1.) Completely reducible representation will play a singular role in the representation variety, thus we devote them an own section.

For a completely reducible representation the boundary map sends all triangles to totally non-generic flags, i.e., all triple ratios are $X = t$. The double ratios D_1, D_2 are not well-defined but their product D is.

According to [9, Theorem 1.2] one obtains all representations with the same boundary map via multiplication by some conjugate of $diag(\lambda, \frac{1}{\lambda^2}, \lambda)$ for some homomorphism $\lambda: \pi_1 \Sigma_g \rightarrow \mathbb{C} \setminus \{0\}$. In particular, completely reducible representations will not be parametrised by their boundary map alone, but one would need some additional parameters to parametrise them. Therefore, completely reducible representations will be treated separately in some of our arguments later.

We will need two propositions about completely reducible representations. The first is proved in [9] for representations to $PGL(3, \mathbb{R})$, but literally the same proof works for representations to $PGL(3, \mathbb{C})$.

Proposition 1. ([9, Theorem 1.2]) *If $\rho_1, \rho_2: \pi_1 \Sigma_g \rightarrow PGL(3, \mathbb{C})$ are Anosov representations with the same boundary map $\xi: \partial_\infty \pi_1(\Sigma_g) \rightarrow \text{Flag}(\mathbb{CP}^2)$, then both factor over some completely reducible representation $PGL(2, \mathbb{C}) \rightarrow PGL(3, \mathbb{C})$ and ρ_2 is obtained from ρ_1 by left multiplication with some conjugate of*

$$\begin{pmatrix} \lambda(\gamma) & 0 & 0 \\ 0 & \frac{1}{\lambda(\gamma)^2} & 0 \\ 0 & 0 & \lambda(\gamma) \end{pmatrix}$$

for some homomorphism $\lambda: \pi_1 \Sigma_g \rightarrow \mathbb{C} \setminus \{0\}$.

The second proposition follows from arguments very close to those used in [9], we include it here for completeness.

Proposition 2. *An Anosov representation $\rho: \pi_1 \Sigma_g \rightarrow PGL(3, \mathbb{C})$ is completely reducible if and only if its boundary map sends all ideal triangles to totally degenerate triples of flags.*

Proof: If ρ is completely reducible, then there is a $\rho(\pi_1 \Sigma_g)$ -invariant 2-dimensional subspace $V \subset \mathbb{C}^3$. The image of the boundary map ξ lies then in $\text{Flag}(PV) \subset \text{Flag}(\mathbb{CP}^2)$. In particular, for the vertices v of the ideal triangulation, the point components of all the $\xi(v)$ lie on the same line and the line components of all the $\xi(V)$ go through the same point. In particular, for each ideal triangle the images of its vertices form a totally degenerate triple of flags.

Conversely, assume that for all triangles of the ideal triangulation the three vertices are sent to a totally degenerate triple of flags. Thus, if v_0, v_1, v_2 are the vertices of one of the ideal triangles, and if we denote $\xi(v_i) = (p_i, l_i)$ for $i = 0, 1, 2$, then p_2 must be on the line through p_0 and p_1 and l_2 goes through the intersection point of l_0 and l_1 . We claim that this implies that all $\xi(v), v \in \tilde{\Upsilon}_0$ are of the form $\xi(v) = (p_v, l_v)$ with all p_v lying on the same line l , and all l_v intersecting in the same point p . To see this. let $\lambda^{\text{closed}} \subset \Sigma_g$ be the union of closed leaves of Υ , let $\tilde{\lambda}^{\text{closed}} \subset \mathbb{H}^2$ its lift to $\tilde{\Sigma}_g \simeq \mathbb{H}^2$ and let $\tilde{U}$ be the component of $\mathbb{H}^2 \setminus \tilde{\lambda}^{\text{closed}}$ containing T . Looking at the dual tree of the ideal triangulation $\tilde{\Upsilon}|_{\tilde{U}}$ we can enumerate its triangles such that each T_k is adjacent to exactly one triangle T_j with $j < k$, see the proof of [1, Lemma 21]. Using this enumeration we see by induction that the claim is true for all vertices of triangles in $\tilde{U}$. Further this also applies to the leaves of $\tilde{\lambda}^{\text{closed}}$ adjacent to $\tilde{U}$, because one of their ideal vertices is actually a vertex of $\tilde{\Upsilon}|_{\tilde{U}}$, while the other ideal vertex is an accumulation point of vertices and so the claim holds by continuity of ξ . Next we can extend this argument to the components of $\mathbb{H}^2 \setminus \tilde{\lambda}^{\text{closed}}$ adjacent to $\tilde{U}$. Namely, the same argument shows the claim (with a priori possibly different (p, l)) for all triangles in an adjacent component. But since the (p, l) agree on both

ideal vertices of the leaf in $\tilde{\lambda}^{closed}$ along which they are adjacent, the (p, l) must actually be the same for both adjacent components, just because there is a unique line through two points and a unique point common to two lines. Finally we use the dual tree to the decomposition into components of $\mathbb{H}^2 \setminus \tilde{\lambda}^{closed}$ to enumerate these components such that each U_k is adjacent to exactly one component U_j with $j < k$, as in the proof of [1, Lemma 24], so that we can induct on the components and finally get the claim for all triangles in all components. In particular, there is a 2-dimensional subspace $V \subset \mathbb{C}^3$ such that all $\xi(v)$ belong to $Flag(PV) \subset Flag(\mathbb{C}P^2)$. Since the vertices of the lifted ideal triangulation are dense in $\partial_\infty H^2$, this implies that the image of ξ is contained in $Flag(PV)$. For $\gamma \in \pi_1 \Sigma_g$ it is known that the Anosov representation ρ has three distinct real eigenvalues and ξ sends the attracting and repelling fixed point of γ to the invariant subspaces of $\rho(\gamma)$. In particular, $Flag(PV)$ is ρ -invariant, which implies that ρ is completely reducible. QED

6. PAIRS OF PANTS

6.1. Ideal triangulation. A pair of pants H , that is an orientable surface of genus 0 with three boundary components, can be decomposed into two ideal triangles. Fixing compatible orientations for the two triangles and their three edges one can, as in Section 2, associate $T(t_1), T(t_2)$ to the triangles and $E(e_1), E(e_2), E(e_3)$ to the edges. One checks that the homotopy classes of the three boundary curves a, b, c with $abc = 1$ correspond to

$$\begin{aligned} a &= T(t_1)E(e_1)T(t_2)E(e_2)^{-1} \\ b &= E(e_2)T(t_2)E(e_2)^{-1}T(t_1) \\ c &= T(t_1)^{-1}E(e_3)T(t_2)E(e_1)^{-1}T(t_1)^{-1}. \end{aligned}$$

Not that $T(t_2)^3 = 1$ and thus indeed $abc = 1$.

6.2. Parametrising. Let us denote by P the space

$$P := T^2 \times (\mathbb{C}_{\neq 0})^6 / \sim,$$

where the equivalence relation identifies $(t, t, D_1, D_2, E_1, E_2, F_1, F_2)$ to $(t, t, kD_1, \frac{1}{k}D_2, kE_1, \frac{1}{k}E_2, kF_1, \frac{1}{k}F_2)$ for all $k \in \mathbb{C}_{\neq 0}$ and all $(D_1, D_2, E_1, E_2, F_1, F_2) \in \mathbb{C}_{\neq 0}^6$ and there are no other identifications. Here, $t \in T$ means the equivalence class of the totally degenerate triples of flags.

Lemma 11. *The space of Anosov representations of a rank 2 free group to $PGL(3, \mathbb{C})$ is homeomorphic to an open subset of P .*

Proof: We are going to construct continuous maps

$$\Psi: P \rightarrow Rep(\pi_1 H, PGL(3, \mathbb{C})), \Phi: \mathcal{A} \rightarrow P$$

such that the compositions $\Phi \circ \Psi$ and $\Psi \circ \Phi$ give the identity. Since $\mathcal{A}$ is known to be an open subset of $Rep(\pi_1 H, PGL(3, \mathbb{C}))$, this implies that it is homeomorphic to an open subset of P .

Let us denote the two coordinates of T by X, Y and the coordinates associated to the three edges by $D_1, D_2, E_1, E_2, F_1, F_2$. For any element of P we can associate a (not necessarily Anosov) representation via the explicit matrices from Section ?? (plugging them into the equations from Section 6.1). They associate a well-defined representation to each choice of the parameters and it is obvious that the representation depends continuously on the parameters. One easily checks by explicit computation (using the formulas from Section 6.1) that $(t, t, D_1, D_2, E_1, E_2, F_1, F_2)$ and $(t, t, kD_1, \frac{1}{k}D_2, kE_1, \frac{1}{k}E_2, kF_1, \frac{1}{k}F_2)$ yield the same representations, for all $k \in \mathbb{C}_{\neq 0}$ and all $(D_1, D_2, E_1, E_2, F_1, F_2) \in \mathbb{C}_{\neq 0}^6$. Thus Ψ is well-defined and continuous.

Conversely, to define Φ , let $\rho: \pi_1 H \rightarrow PGL(3, \mathbb{C})$ be an Anosov representation. Then its boundary map $\xi: \partial_\infty \tilde{H} \rightarrow Flag(\mathbb{C}^3)$ is well-defined and depends continuously on ρ .

If ξ does not map both triangles to totally degenerate triples of flags (which by Proposition 1 means that ρ is not completely reducible), then the triple ratios X, Y of the triangles and the two double ratios of each of the three edges are well-defined and depend continuously on ρ . (Note that the completely reducible representations form a closed set, so all invariants are well-defined in a neighborhood of ρ .)

If ρ is a completely reducible Anosov representation, then the triple ratios and the pairwise products $D = D_1 D_2, E = E_1 E_2, F = F_1 F_2$ of the double ratios are well-defined and depend continuously on ρ , while the double ratios themselves are not well-defined. We now use the identification $PGL(3, \mathbb{C}) = SL(3, \mathbb{C})$. Associated to the non-zero complex numbers D, E, F we have a representation $\rho: \pi_1 H \rightarrow SL(2, \mathbb{C})$ with these numbers as cross ratio invariants. According to [?], ρ_0 is Anosov and we have a homomorphism $\lambda: \pi_1 H \rightarrow \mathbb{C}^*$ such that

$$\rho = \text{diag}(\lambda, \lambda, \frac{1}{\lambda^2})(\rho_0 \otimes 1).$$

(The homomorphism λ is uniquely determined because we require ρ and ρ to have image in the matrices of determinant 1.) The representation ρ_0 to $SL(2, \mathbb{C})$ depends continuously on the cross ratios, in particular it depends continuously on ρ . Thus also the homomorphism λ depends continuously on ρ .

We claim now that from the known data $(X, Y, D, E, F$ and $\lambda)$ we can determine $D_1, D_2, E_1, E_2, F_1, F_2$ (up to the ambiguity in the definition of P) such that they depend continuously on ρ and application of Ψ yields the original representation.

It is clear that knowing $D = D_1 D_2$ it is enough to know D_1/D_2 to determine D_1 and D_2 , and likewise for the other edges. From the formulas for a, b, c in Section 6.1 one obtains

$$\frac{D_2}{D_1} \frac{E_1}{E_2} = \lambda^2(a), \frac{E_2}{E_1} \frac{F_1}{F_2} = \lambda^2(b), \frac{F_2}{F_1} \frac{D_1}{D_2} = \lambda^2(c).$$

Knowing λ and knowing $\frac{E_1}{F_2}$, these equations can be solved for $\frac{D_1}{D_2}$ and $\frac{E_1}{E_2}$ and they depend continuously on λ and the other parameters. The ambiguity in choosing $\frac{E_1}{F_2}$ corresponds exactly to the ambiguity in the definition of P , i.e., to the choice of $k \in \mathbb{C}_{\neq 0}$.

QED

7. GLUING ALONG CLOSED LOOPS

7.1. Eigenvalues of boundary loops. If $\rho: \pi_1 S \rightarrow PGL(3, \mathbb{C}) = SL(3, \mathbb{C})$ is an Anosov representation, then all $\rho(\gamma), \gamma \in \pi_1 S$ have distinct real eigenvalues $\lambda_1, \lambda_2, \lambda_3$, so we can consider the numbers

$$L_1^\rho(\gamma) = \frac{\lambda_1(\rho(\gamma))}{\lambda_2(\rho(\gamma))}, L_2^\rho(\gamma) = \frac{\lambda_2(\rho(\gamma))}{\lambda_3(\rho(\gamma))},$$

$$L^\rho(\gamma) = L_1^\rho(\gamma) L_2^\rho(\gamma) = \frac{\lambda_1(\rho(\gamma))}{\lambda_3(\rho(\gamma))}$$

The special case of the following theorem for Hitchin representation to $PGL(3, \mathbb{R})$ was proved in [1, Proposition 13] (upon taking logarithms which can be done because for Hitchin representations all invariants are positive). It is clear that literally the same proof works as long as all involved triples are generic (with not necessarily positive invariants). The point of our general proof will be to consider also non-generic triples of flags.

To prepare the statement we note that the space T defined in Section 3.3 has a canonical projection

$$\text{proj}: T \rightarrow \mathbb{C}_{\neq 0}$$

which sends all non-Hausdorff points of T to -1 . For an ideal triangle T_i we will denote $\text{proj}(\tau(T_i))$ by Υ_i . In other words, if the vertices of $\xi(T_i)$ form a generic triple of flags, then Υ_i is their usual triple ratio, and if the vertices of $\xi(T_i)$ do not form a generic triple of flags, then $\Upsilon_i = -1$.

Further, for a (closed or non-closed) edge E_j we will denote D_1^j, D_2^j its shear invariants.

Theorem 1. Consider a pairs of pants U decomposed into two ideal triangles T_1 and T_2 .

i) Assume that we have a map $\xi: \partial\tilde{U} \rightarrow \mathrm{Flag}(\mathbb{CP}^2)$ equivariant for some representation $\rho: \pi_1 U \rightarrow \mathrm{PGL}(3, \mathbb{C})$. Assume that not both triangles T_1 and T_2 are mapped to totally-nongeneric triples by ξ . (Hence the double ratios D_1, D_2 from Definition 3 are defined for each edge.)

Let γ be one of the boundary curves of U , and let h_1 and h_2 be the infinite edges that spiral on γ . Let v_i be the vertex of the triangle T_i in the direction of which the spiraling occurs, and define $\sigma_1(h_i) = D_1(h_i), \sigma_2(h_i) = D_2(h_i)$ if h_i is oriented towards γ and $\sigma_1(h_i) = D_2(h_i), \sigma_2(h_i) = D_1(h_i)$ if not. Then we have the following:

(a) if the spiraling is from the right and in the direction of the orientation of γ , then

$$L_1^\rho(\gamma) = \prod_{i=1}^2 \sigma_1(h_i) \Upsilon(T_i, v_i)$$

$$L_2^\rho(\gamma) = \prod_{i=1}^2 \sigma_2(h_i)$$

(b) if the spiraling is from the right and in the opposite direction of the orientation of γ , then

$$L_1^\rho(\gamma) = \prod_{i=1}^2 \sigma_2(h_i)^{-1}$$

$$L_2^\rho(\gamma) = \prod_{i=1}^2 \sigma_2(h_i)^{-1} \Upsilon(T_i, v_i)$$

(c) if the spiraling is from the left and in the direction of the orientation of γ , then

$$L_1^\rho(\gamma) = \prod_{i=1}^2 \sigma_1(h_i)^{-1} \Upsilon(T_i, v_i)^{-1}$$

$$L_2^\rho(\gamma) = \prod_{i=1}^2 \sigma_2(h_i)^{-1}$$

(d) if the spiraling is from the left and in the opposite direction of the orientation of γ , then

$$L_1^\rho(\gamma) = \prod_{i=1}^2 \sigma_1(h_i)$$

$$L_2^\rho(\gamma) = \prod_{i=1}^2 \sigma_2(h_i) \Upsilon(T_i, v_i).$$

ii) if both triangles T_1 and T_2 are mapped to totally-nongeneric triples by ξ , then we have with the notation analogous to the above the following:

(e) if either the spiraling is from the right and in the direction of the orientation of γ , or the spiraling is from the left and in the opposite direction of the orientation of γ , then

$$L^\rho(\gamma) = \prod_{i=1}^2 \sigma(h_i)$$

(f) if either the spiraling is from the right and in the opposite direction of the orientation of γ , or the spiraling is from the left and in the direction of the orientation of γ , then

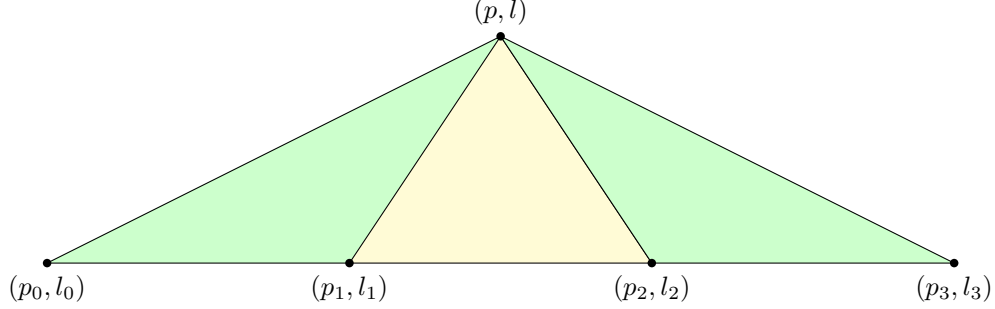
$$L^\rho(\gamma) = \prod_{i=1}^2 \sigma(h_i)^{-1}.$$

Proof:

The claims in the case that all triangles T_{j_i} are generic are already proved in [1] in the general setting of representations to $PGL(n, \mathbb{R})$. (Though in the special case of representations to $PGL(3, \mathbb{R})$ the argument could be considerably simplified by a computation completely analogous to the one that we give below.)

So let us prove part a) in the case that both triangles are generic or line-nongeneric, i.e., such that the shear invariants can be defined via the formulas.

In the universal covering we have a 2-periodic picture as follows:



Here, (p, l) is the unstable fixed point of $\rho(\gamma)$, the sequence (p_k, l_k) converges towards the stable fixed point of $\rho(\gamma)$ and we have $\rho(\gamma)(p_k, l_k) = (p_{k+2}, l_{k+2})$ for all $k \in \mathbb{Z}$.

Thus we have

$$\prod_{i=1}^2 \sigma_1(h_i) \Upsilon(T_i, v_i) = \prod_{i=1}^2 \frac{p \wedge p_{i-1} \wedge p_i}{p \wedge p_i \wedge p_{i+1}} \cdot \frac{l_i \wedge p_{i+1}}{l_i \wedge p_{i-1}} \cdot \frac{l \wedge p_{i+1}}{p \wedge l_{i+1}} \cdot \frac{l_{i+1} \wedge p_i}{p_{i+1} \wedge l_i} \cdot \frac{l_i \wedge p}{p_i \wedge l}$$

which simplifies to

$$\prod_{i=1}^2 \frac{p \wedge p_{i-1} \wedge p_i}{p \wedge p_i \wedge p_{i+1}} \cdot \frac{l_{i+1} \wedge p_i}{l_i \wedge p_{i-1}} \cdot \frac{l \wedge p_{i+1}}{p \wedge l_{i+1}} \cdot \frac{l_i \wedge p}{p_i \wedge l}$$

which is a telescope product equal to

$$\frac{p \wedge p_0 \wedge p_1}{p \wedge p_2 \wedge p_3} \cdot \frac{p_2 \wedge l_3}{p_0 \wedge l_1} \cdot \frac{l \wedge p_3}{l \wedge p_1} \cdot \frac{p \wedge l_1}{p \wedge l_3}$$

The first factor is λ_1 , the second is 1, the third $\frac{1}{\lambda_1 \lambda_2} = \frac{1}{\lambda_1}$ and the fourth is λ_1 , so the product equals λ_1 .

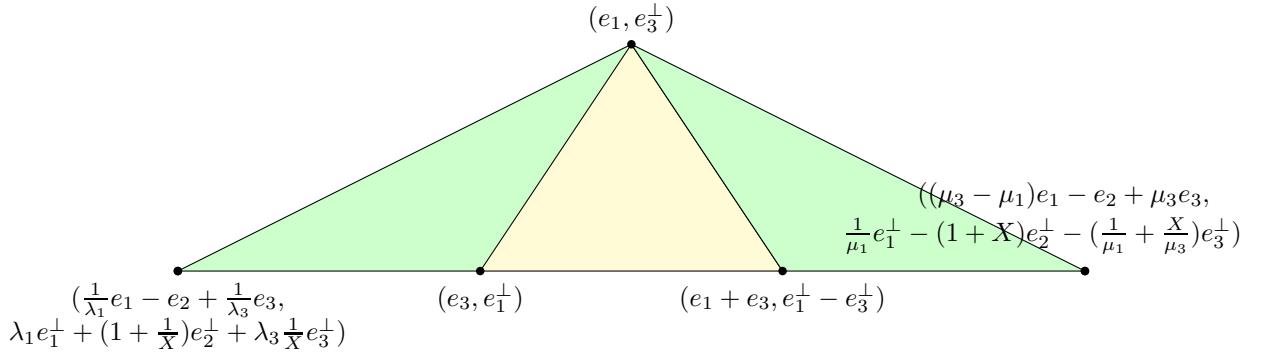
Via duality we can also handle the case that both triples are generic or point-nongeneric. Let us do the remaining cases by ad-hoc computations.

For the computations we will denote by X the triple ratio of the triple of flags associated to (p, l) , (p_0, l_0) and (p_1, l_1) , and by Y the triple ratio of the triple of flags associated to (p, l) , (p_1, l_1) and (p_2, l_2) . Note that by the 2-periodicity the triple of flags associated to (p, l) , (p_2, l_2) and (p_3, l_3) has again triple ratio X . Moreover we will denote by $\lambda_1, \lambda_3, \mu_1 \mu_3$ the numbers associated to the quadruples of flags (p, l) , (p_0, l_0) , (p_1, l_1) and (p_2, l_2) resp. (p, l) , (p_1, l_1) , (p_2, l_2) and (p_3, l_3) from which as in Subsection 4.2 the double ratios $D_1(h_1) = -\frac{1}{\lambda_1}$, $D_2(h_1) = -\lambda_3$, $D_1(h_2) = -\frac{1}{\mu_1}$, $D_2(h_2) = -\mu_3$ are computed.

For the computations we will assume that (p, l) , (p_1, l_1) and (p_2, l_2) are in the normal form from Lemma 2, which can be achieved by multiplying everything with a suitable element from $PGL(3, \mathbb{R})$. Then the flag associated to (p_3, l_3) is given in terms of μ_1, μ_3 as in Subsection 4.2 and it is also easy to compute the flag associated to (p_0, l_0) in terms of λ_1, λ_3 .

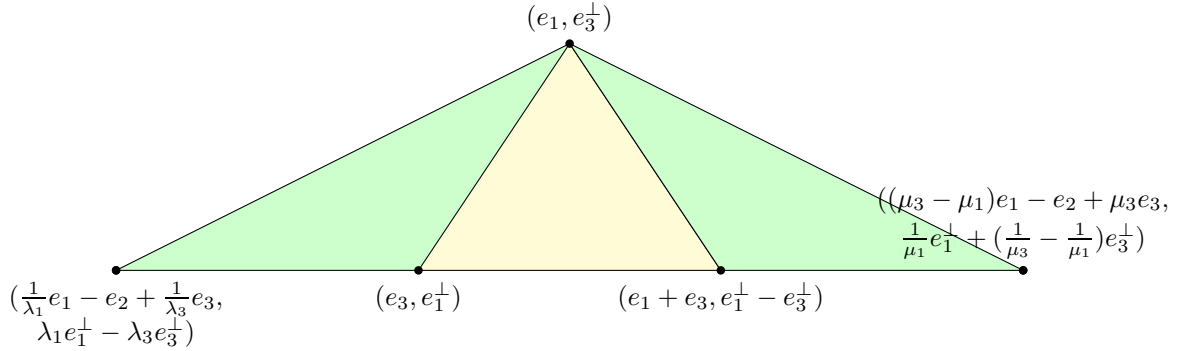
We will now discuss the remaining cases one by one.

First let us consider the case that one triple is generic and the other is totally non-generic.



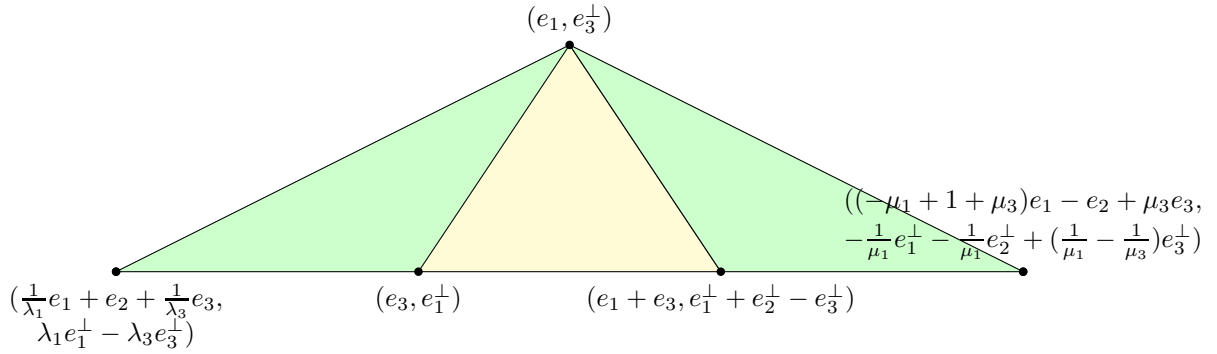
The reason why the λ_i are associated to the first vertex rather than the third are explained in ???. One computes that the unique matrix sending the first to the third triangle is $\begin{pmatrix} -\lambda_1 \mu_1 X & -\mu_1(1+X) & \lambda_3(\mu_3 - \mu_1) \\ 0 & -1 & -\lambda_3 \\ 0 & 0 & \lambda_3 \mu_3 \end{pmatrix}$.

The case that one triangle is line-nongeneric and the other is totally non-generic can be handled by exactly the same computations, just setting $X = -1$.



The matrix is $\begin{pmatrix} \lambda_1 \mu_1 & 0 & \lambda_3(\mu_3 - \mu_1) \\ 0 & -1 & -\lambda_3 \\ 0 & 0 & \lambda_3 \mu_3 \end{pmatrix}$. Via duality this also handles the case that one triangle is point-nongeneric and the other is totally non-generic.

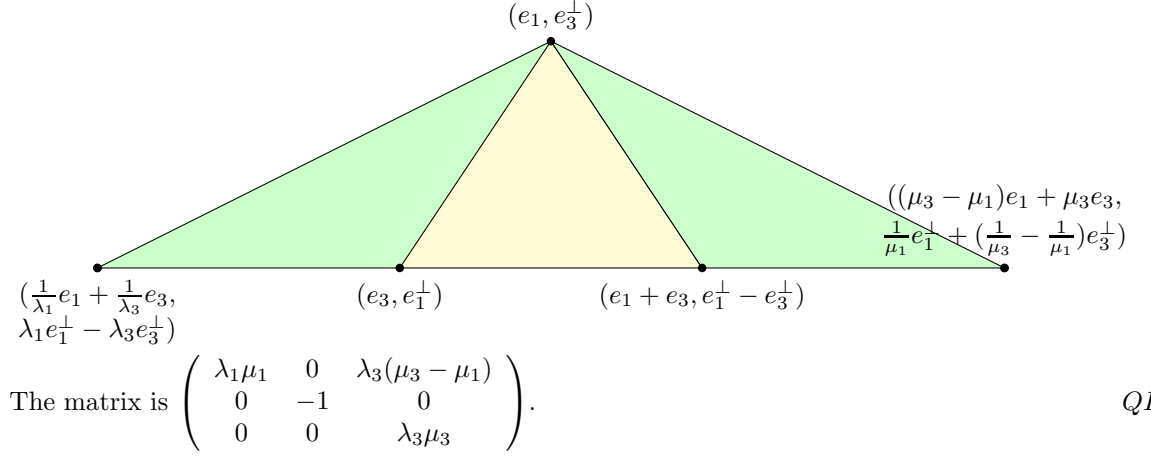
Finally we have to consider the case that one triangle is line-nongeneric and the other is point-nongeneric.



The matrix is $\begin{pmatrix} \lambda_1 \mu_1 & -1 & \lambda_3(\mu_3 - \mu_1 + 1) \\ 0 & 1 & -\lambda_3 \\ 0 & 0 & \lambda_3 \mu_3 \end{pmatrix}$.

The computations in the cases b), c) and d) are completely analogous.

The proof of (ii) is straightforward and could be done without explicit computation, but let us spell it out because we will need the computation in ???. In this case λ_1 and λ_3 are only defined upon multiplication with a common factor, and the same is true for μ_1 and μ_3 . Keeping this in mind we get the following picture.



Remark: The proof in the case that all triples are generic, line-nongeneric or point-nongeneric works in a completely analogous manner for a decomposition into more than two ideal triangles. It seems plausible that the statement for such a decomposition should still be true in the presence of totally non-generic triples, but checking this would require cumbersome computations which we did not want to perform.

7.2. Closed leaf equalities and closed leaf inequalities.

Definition 4. Let γ be a closed leaf bounding two pairs of pants (possibly the same). Let h_1, h_2 the infinite leaves and T_1, T_2 the ideal triangles that spiral on the right-hand side of γ . Let h'_1, h'_2 the infinite leaves and T'_1, T'_2 the ideal triangles that spiral on the left-hand side of γ . As in Theorem 1 let v_i be the vertex of the triangle T_i in the direction of which the spiraling occurs, and define $\sigma_1(h_i) = D_1(h_i), \sigma_2(h_i) = D_2(h_i)$ if h_i is oriented towards γ and $\sigma_1(h_i) = D_2(h_i), \sigma_2(h_i) = D_1(h_i)$ if not, and the same way also for h'_i . Then

$$L_1^{right}(\gamma) = \prod_{i=1}^2 \sigma_1(h_i) \Upsilon(T_i, v_i)$$

$$L_2^{right}(\gamma) = \prod_{i=1}^2 \sigma_2(h_i)$$

if the spiraling is in the direction of the orientation of γ , and

$$L_1^{right}(\gamma) = \prod_{i=1}^2 \sigma_2(h_i)^{-1}$$

$$L_2^{right}(\gamma) = \prod_{i=1}^2 \sigma_2(h_i)^{-1} \Upsilon(T_i, v_i)$$

if the spiraling is from the right and in the opposite direction of the orientation of γ .

Moreover,

$$L_1^{left}(\gamma) = \prod_{i=1}^2 \sigma_1(h'_i)^{-1} \Upsilon(T'_i, v_i)^{-1}$$

$$L_2^{left}(\gamma) = \prod_{i=1}^2 \sigma_2(h'_i)^{-1}$$

if the spiraling of the h'_i and T'_i is in the direction of the orientation of γ , and

$$L_1^{left}(\gamma) = \prod_{i=1}^2 \sigma_1(h'_i)$$

$$L_2^{left}(\gamma) = \prod_{i=1}^2 \sigma_2(h'_i) \Upsilon(T'_i, v_i)$$

if the spiraling of the h'_i and T'_i is in the opposite direction of the orientation of γ .

Corollary 3. *The invariants have to satisfy the closed leaf equalities*

$$L_i^{right}(\gamma) = L_i^{left}(\gamma), i = 1, 2$$

and the closed leaf inequalities

$$L_i^{right}(\gamma) \neq 1, i = 1, 2$$

for each closed edge γ .

Proof: The closed leaf equalities are immediate from Theorem 1 and the closed leaf inequalities follow because for an Anosov representation ρ and any $\gamma \neq 1$ the eigenvalues of each $\rho(\gamma)$ are pairwise distinct. *QED*

7.3. Realizability: Gluing pairs of pants together. For a closed leaf we adapt the definition from Section 4.2 via the approach from [1, Section 3.5]. That means for each closed leaf g_i of the ideal triangulation we fix a transverse arc k_i , so for each lift $\tilde{g}_i$ we have a lifted transverse arc $\tilde{k}_i$ and its endpoints are contained in ideal triangles $\tilde{T}, \tilde{T}'$ (to the left and right for the lifted orientation). Let x, y be the (positive and negative) ideal endpoints of $\tilde{g}_i$, then $\tilde{T}$ has one vertex equal to x or y , and we denote z the vertex adjacent to the two components of $\tilde{S} \setminus \tilde{T}$ not containing $\tilde{g}_i$. Similarly z' is the vertex of $\tilde{T}'$ adjacent to the two components of $\tilde{S} \setminus \tilde{T}'$ not containing $\tilde{g}_i$. With this notation we can define the double ratios of x, y, z, z' as in Section 4.2.

Lemma 12. *Let $\tilde{U}_1$ and $\tilde{U}_2$ be two adjacent components of $\tilde{S} \setminus \tilde{\lambda}^{closed}$ and $\tilde{\gamma}$ the intersection of their closures, which is the lift of a closed leaf γ . Assume we have assigned*

- two numbers $D_1, D_2 \in \mathbb{C}_{\neq 0}$ to the closed leaf γ ,
- $X_j \in T$ to each ideal triangle

and moreover to each non-closed edge $\tilde{c}_i$ with adjacent triangles $\tilde{T}_j$ and $\tilde{T}_k$ we have assigned two non-zero real numbers $D_{1,i}, D_{2,i}$.

Assume that the closed leaf equalities and the closed leaf inequalities are satisfied.

Then there is a map $\xi: \partial_\infty(\tilde{U}_1 \cup \tilde{U}_2) \rightarrow \text{Flag}(\mathbb{CP}^2)$ such that the double ratios of $\tilde{\gamma}$ are D_1, D_2 , the triple ratios of ideal triangles $\tilde{T}_j \subset \tilde{U}_1 \cup \tilde{U}_2$ are X_j , and the double ratios of each edge $\tilde{g}_i \neq \tilde{\gamma}$ are $D_{1,i}, D_{2,i}$. A map with this properties is unique up to composition with $PGL(3, \mathbb{C})$.

Proof: This is the analogue of [1, Lemma 23].

From Lemma ?? we have ρ_i -equivariant maps $\xi_i: \partial_\infty \tilde{U}_i \rightarrow \text{Flag}(\mathbb{CP}^2)$ for $i = 1, 2$. From Theorem 1 we can compute the (ratios of) the eigenvalues of $\rho_i(\gamma)$. From the closed leaf equality we see that they are independent of $i \in \{1, 2\}$ and from the closed leaf inequality we see that the ratios are different from 1. In particular, $\rho_1(\gamma)$ and $\rho_2(\gamma) \in PGL(3, \mathbb{C})$ are diagonalizable with the same eigenvalues, so there is some $A \in PGL(3, \mathbb{C})$ that conjugates $\rho_1(\gamma)$ to $\rho_2(\gamma)$. The same argument as in [1] shows that A maps

the flags $\xi_1(\tilde{\gamma}_{\pm\infty})$ to $\xi_2(\tilde{\gamma}_{\pm\infty})$. We can therefore define a map ξ by letting it agree with $A\xi_1$ and ξ_2 on $\partial_\infty\tilde{U}_1$ and $\partial_\infty\tilde{U}_2$, respectively.

This boundary map is uniquely determined up to the action of $PGL(3, \mathbb{C})$. If not all four ideal triangles are totally non-generic, then one has uniqueness of the associated representation because of freeness of the $PGL(3, \mathbb{C})$ -action on not totally degenerate triples. If all triangles are totally non-generic, then again the boundary map is uniquely determined, however the representation is only determined up to multiplication by a conjugate of $diag(\lambda, \frac{1}{\lambda^2}, \lambda)$.

QED

8. CONCLUSION

8.1. Realisability. Let $\mathcal{P}$ be the set of solutions of the closed leaf equalities and inequalities, and $\mathcal{R}$ its quotient under the equivalence relation $\sim$.

Lemma 13. *Every element of $\mathcal{P}$ is realised by some map $\xi: \partial_\infty H^2 \rightarrow Flag(\mathbb{CP}^2)$ which is equivariant for some representation $\rho: \pi_1 \Sigma_g \rightarrow PGL(3, \mathbb{C})$.*

Proof: With the preparations done so far, the proof of Theorem 2 now follows from Lemma ?? and Lemma 12 by induction on the tree dual to $\tilde{S} \setminus \tilde{\lambda}^{closed}$ exactly as in the proof of [1, Lemma 24].

As in [1, Lemma 25] we get a homomorphism for which the flag decoration is equivariant.

QED

Theorem 2. *Every element of $\mathcal{R}$ is realised by some representation $\rho: \pi_1 \Sigma_g \rightarrow PGL(3, \mathbb{C})$.*

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