MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
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SS 2002
Sheet#11
July 2, 2002

Qualitative Theory of Ordinary Differential Equations

32. Let $U \subset \mathbb{R}^n$ be open, $0 \in U$, $f \in C^1(U, \mathbb{R}^n)$, and V a local transversal of f at 0, i.e., there exists an $n-1$-dimensional subvectorspace $H \subset \mathbb{R}^n$ such that V is an open neighborhood of 0 in H and that $f(x) \notin H$ for all $x \in V$.

    Let $\Phi : \Omega \to U$ be the flow of $\dot{x} = f(x)$ and assume $\Phi(t_0, x_0) = 0$ for some $(t_0, x_0) \in \Omega$. Show:
    a) There exists an open subset $W \subset U$ with $x_0 \in W$ and a C^1-map $\tau \in C^1(W; \mathbb{R})$ with $\tau(x_0) = t_0$ such that $\Phi(\tau(x), x) \in H$ for all $x \in W$. (Hint: Implicit function theorem.) Thus, the **Poincaré map** $P : W \to H$ defined by $P(x) := \Phi(\tau(x), x)$ is C^1.
    b) Assume in addition $x_0 = 0$ and $t_0 > 0$. Let $\Psi(.) := \Phi(\tau(0), .)$. If $\Psi'(0)$ leaves H invariant, then $P'(0)\mid_H = \Psi'(0)\mid_H$.

    (9)

33. Let $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f(x, \mu) := \begin{pmatrix} \mu^2 & -1 \\ 1 & \mu^2 \end{pmatrix} x \;+ \mid x \mid^2 x.$$

    Show:
    a) $(f(., \mu))'(0)$ has purely imaginary eigenvalues only for $\mu = 0$.
    b) $\dot{x} = f(x, \mu)$ does not have periodic solutions for any value of μ. (3)

34. Let $A := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & \sqrt{2} & 0 \end{pmatrix}$ and let $f : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ be defined by
    $f(z_1, z_2) := (A + \mu Id)(z_1, z_2) - (\mid z_1 \mid^2 z_1, \mid z_2 \mid^2 z_2)$ for $(z_1, z_2) \in (\mathbb{R}^2)^2 = \mathbb{R}^4$. Show:
    a) $(f(., \mu))'(0)$ has purely imaginary eigenvalues only for $\mu = 0$.
    b) For $\mu \leq 0$, $\dot{x} = f(x, \mu)$ does not have neither periodic solutions nor solutions remaining on a torus $\left\{(z_1, z_2) \in (\mathbb{R}^2)^2 : \mid z_1 \mid = c_1, \mid z_2 \mid = c_2\right\}$.
    c) For $\mu > 0$, $\dot{x} = f(x, \mu)$ has exactly two periodic orbits as well as orbits which are dense on the torus $T = \left\{(z_1, z_2) \in (\mathbb{R}^2)^2 : \mid z_1 \mid = \mid z_2 \mid = \sqrt{\mu}\right\}$.
    Show that T is asymptotically stable, and that the two periodic orbits are unstable. (6)

Due: Tuesday, July 9, 2002; 11:15 a.m.