

BOREL CLASS AND CARTAN INVOLUTION

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ABSTRACT. Let $\rho: \pi_1 M^3 \rightarrow PGL(n, \mathbb{C})$ be a representation of a 3-manifold group to the projective linear group. In this note we prove that the Borel class of ρ is preserved up to sign under Cartan involution.

For representations to $PGL(3, \mathbb{C})$ this is implied by the more general result of E. Falbel and Q. Wang from [5], however our proof appears to be much shorter for that special case.

1. INTRODUCTION

Representations of fundamental groups of (ideally triangulated) 3-manifolds to $PGL(3, \mathbb{C})$ are determined by so-called decorations, i.e., equivariant maps from the 0-skeleton of the universal covering to $Flag(\mathbb{C}P^2)$. Bergeron, Falbel and Guilloux introduced in [2] invariants of 4-tuples of flags, which for a given decoration determine an element $\beta(T)$ in the pre-Bloch group $\mathcal{P}(\mathbb{C})$ for each tetrahedron T . Summing up the $\beta(T)$ over all simplices in the triangulation defines an element in the Bloch group, whose Bloch-Wigner dilogarithm divided by 4 is said to be the volume of the representation.

In [5], Falbel and Wang considered the effect of the involution $A \rightarrow (A^T)^{-1}$ (or on the level of decorations of going from flags to their duals) on these invariants. Using ingenious computations in the pre-Bloch group they expressed in [5, Proposition 3.6] the difference of the values of $\beta(T)$, for a decoration and its dual, as the sum of the Fock-Goncharov triple ratios associated to the induced decorations of the four faces of the simplex T . When summing up over all 3-simplices, faces occurring twice with opposite orientations yield canceling difference terms. In particular, when the boundary consists of cusps only, the authors obtain that the element in the Bloch group (and hence the volume) is the same for the decoration and its dual. This proves that the involution $A \rightarrow (A^T)^{-1}$ of $PGL(3, \mathbb{C})$ preserves the volume and the Cartan involution $A \rightarrow (\overline{A}^T)^{-1}$ multiplies it by -1 .

In the introduction of [5] the authors point out that this implies the same invariance properties for the Borel class and they express the belief that an analogous invariance under duality should hold for representations to $PGL(n, \mathbb{C})$ with $n \geq 4$, and that also for non-toroidal boundary one should then have an exact coboundary term relating the two cocycle representatives coming from the decoration and its dual.

In this paper we prove the first part of this conjecture with a short argument which in particular also gives a much shorter proof for the case $n = 3$.

What we actually prove is that the Cartan involution preserves the volume and the Chern-Simons invariant of representations of fundamental groups up to a sign depending on

dimension. It follows from the van Est theorem that all invariants coming from the continuous cohomology group $H_c^3(PGL(n, \mathbb{C}))$ must be multiples of each other, so the invariance of the volume implies the invariance of the Borel class and also of the volume defined in [2].

Our proof via secondary characteristic classes does not yield information on the pre-Bloch group elements associated to the individual simplices. So it remains open whether, as conjectured in [5], also for $n \geq 4$ a simplex-wise formula analogous to [5, Proposition 3.6] relates the two values of $\beta(T)$ for a decoration and its dual.

In Section 5 we include a proof that the volume of [2] agrees up to multiplication by $-\frac{1}{4}$ with the volume defined via Cheeger-Chern-Simons classes. This seems not to have been known (see [6, Remark 9.19]), although the proof is actually a straightforward application of the methods introduced in [6].

2. PRELIMINARIES

Let M be a compact, orientable, $(2k-1)$ -dimensional manifold, possibly with boundary ∂M . Let $\rho: \pi_1 M \rightarrow SL(n, \mathbb{C})$ be a boundary-unipotent representation of its fundamental group, where boundary-unipotent means that the image of the fundamental group of ∂M can be conjugated into the group $N(n, \mathbb{C}) \subset SL(n, \mathbb{C})$ of upper triangular matrices with 1's on the diagonal.

Let $SL(\mathbb{C}) = \bigcup_n SL(n, \mathbb{C})$ and $N(\mathbb{C}) = \bigcup_n N(n, \mathbb{C})$ be the increasing unions with respect to the canonical embeddings as block matrices. It is well known that the image of the relative fundamental class in group homology

$$(B\rho)_*[M, \partial M] \in H_{2k-1}(BSL(\mathbb{C})^\delta, BN(\mathbb{C})^\delta; \mathbb{Z})$$

has a preimage in $H_{2k-1}(BSL(\mathbb{C})^\delta; \mathbb{Z})$, which we will denote by the same name.

Let $\hat{c}_k \in H^{2k-1}(BSL(\mathbb{C})^\delta, \mathbb{C}/\mathbb{Z})$ be the universal Cheeger-Chern-Simons class. Then

$$\langle \hat{c}_k, (B\rho)_*[M, \partial M] \rangle \in \mathbb{C}/\mathbb{Z}$$

is the Cheeger-Chern-Simons invariant of the flat bundle with monodromy ρ . It has become common, at least in the context of 3-manifolds, to denote its real and imaginary part as the Chern-Simons invariant and the volume of the representation, respectively. See [6, Section 2].

We remark that this approach implies that volume and Chern-Simons invariant are constant on components of the character variety of boundary-unipotent representations, because they can be computed from the fundamental class in the discrete subset $H_{2k-1}(BSL(\mathbb{C})^\delta; \mathbb{Z}) \subset H_{2k-1}(BSL(\mathbb{C})^\delta; \mathbb{R})$ which can not vary continuously. This local constantness is hard to see from other definitions, compare [1] and [7] for other proofs in the case of $\partial M = \emptyset$.

3. INVARIANCE OF CHEEGER-CHERN-SIMONS-CLASSES

Theorem 1. *Let M be a compact, orientable $(2k-1)$ -manifold. Let $E_\rho \rightarrow M$ be a trivializable, flat $SL(n, \mathbb{C})$ -bundle with monodromy $\rho: \pi_1 M \rightarrow SL(n, \mathbb{C})$, let $\rho^\vee: \pi_1 M \rightarrow SL(n, \mathbb{C})$ be the dual of ρ under Cartan involution. Then*

$$vol(\rho^\vee) = (-1)^{k+1} vol(\rho), \quad CS(\rho^\vee) = (-1)^k CS(\rho)$$

Proof: Recall (e.g., from [1]) that $\hat{c}^k(E_\rho)$ can be computed as

$$\hat{c}^k(E_\rho) = \frac{1}{(2\pi)^k} \int_M s^*(A_k P_k(\theta)) \in \mathbb{C}/\mathbb{Z}$$

for the flat connection $\theta \in \omega^1(E_\rho, \mathfrak{sl}(n, \mathbb{C}))$, with the constant $A_k = (-1)^{k-1} \frac{k!(k-1)!}{2^{k-1}(2k-1)!}$ and the invariant polynomial $P_k(A) = \text{Tr}(A^k)$, and with a section $s: M \rightarrow E_\rho$ which exists by trivializability of the bundle. (The *mod* $\mathbb{Z}$ -indeterminacy comes from the possible choice of different trivializations.)

For any group homomorphism $\phi: SL(n, \mathbb{C}) \rightarrow SL(n, \mathbb{C})$ we obtain a trivializable, flat bundle with the "same" section $s: M \rightarrow E_{\phi \circ \rho}$, the monodromy $\phi \circ \rho$ and flat connection $\phi_*\theta$, where $\phi_* = D_1\phi$ denotes the induced homomorphism of Lie algebras. Thus we have

$$\hat{c}^k(E_{\phi \circ \rho}) = \frac{1}{(2\pi)^k} \int_M s^*(A_k P_k(\phi_*\theta)).$$

The Cartan involution is given by $\phi(A) = (\overline{A}^T)^{-1}$ and its differential at $\mathbb{1}$ is

$$\phi_*(X) = -\overline{X}^T.$$

Clearly

$$\text{Tr}((-\overline{X}^T)^k) = (-1)^k \overline{\text{Tr}(X^k)},$$

which implies

$$P_k(\phi_*\theta) = (-1)^k \overline{P_k(\theta)},$$

from which the claim follows because volume and Chern-Simons invariant are defined by the imaginary and real part of $\hat{c}_2$, respectively. QED

Corollary 1. *For a compact, orientable 3-manifold M and a representation $\rho: \pi_1 M \rightarrow SL(n, \mathbb{C})$ we have $\text{vol}(\rho^\vee) = -\text{vol}(\rho)$, $CS(\rho^\vee) = CS(\rho)$.*

Proof: $SL(n, \mathbb{C})$ -bundles over 3-manifolds are trivializable, thus Theorem 1 applies. QED

4. THE BOREL CLASS

The van Est theorem implies $H_c^3(PGL(n, \mathbb{C}); \mathbb{R}) \cong H_{dR}^3(SU(n)) \cong \mathbb{R}$. The generator of the image of $H_c^3(PGL(n, \mathbb{C}); \mathbb{Z})$ is called the Borel class b_3 . For representations ρ of 3-manifold groups, their Borel invariant is defined as $\langle \rho^* b_3, [M] \rangle$. See [3] for an explicit description of the Borel invariant.

The imaginary part of the Cheeger-Chern-Simons class is a continuous $\mathbb{R}$ -valued cohomology class and thus a nontrivial multiple of the Borel class. Hence Corollary 1 implies that the Borel class is preserved under the involution $A \rightarrow (A^T)^{-1}$, which has been conjectured in the introduction of [5].

5. THE $PGL(3, \mathbb{C})$ CASE

One of the results in [5] is the equality $\text{vol}(\rho^\vee) = -\text{vol}(\rho)$ for the volume of representations defined in [2]. We claim that the the volume from [2] agrees with $-\frac{1}{4}$ times the imaginary part of the Cheeger-Chern-Simons class and thus this result is a special case of Theorem 1. (Using the same reasoning as in Section 4 it would actually be enough to apply the van Est theorem to recover that result from Theorem 1, but we think that the precise relation between the two volumes should be of independent interest.)

We remark that this relation of volumes seems not to appear in the literature, see [6, Remark 9.19], although its proof is actually a rather straightforward application of the results in [6].

Theorem 2. *For a boundary-parabolic representation $\rho: \pi_1 M^3 \rightarrow PGL(3, \mathbb{C})$ of a 3-manifold group, its volume $\text{vol}(\rho)$ as defined in [2] agrees with $-\frac{1}{4}$ times the imaginary part of the Cheeger-Chern-Simons invariant $\hat{c}_2$.*

Proof: According to the main result of [6], specialized to $G = PGL(3, \mathbb{C})$, the imaginary part of the Cheeger-Chern-Simons invariant can be computed as follows. Fix a generalized ideal triangulation of M , let L be the corresponding generalized triangulation of the space obtained from the universal covering $\widetilde{M}$ by collapsing each boundary component to a point, and let L_0 be its 0-skeleton. Let $N \subset G$ be the group of upper triangular matrices with 1's on the diagonal and let $\bar{\xi}: L_0 \rightarrow G/N$ be a ρ -equivariant map, which can for example be defined by mapping one vertex in each orbit arbitrarily and then extending equivariantly. (Stabilizers of points in L_0 are fundamental groups of boundary components, thus boundary-parabolicity guarantees that extending equivariantly yields a well-defined map to G/N .) Then for each simplex of K the images under $\bar{\xi}$ of the vertices of its lifts to L are a $\rho(\pi_1 M)$ -orbit of 4-tuples $(g_0 N, g_1 N, g_2 N, g_3 N)$. Then, following [6, Definition 5.1], for any 4-tuple $t = (t_0, t_1, t_2, t_3)$ of nonnegative integers with $t_0 + t_1 + t_2 + t_3 = 3$ one considers its ptolemy coordinate $c_t = \det(\cup_{i=0}^3 \{g_i\}_{t_i})$, where $\{g_i\}_{t_i}$ means the ordered set of the first t_i column vectors of g_i and $\cup_{i=0}^3 \{g_i\}_{t_i}$ means the matrix whose ordered column set is composed by the $\{g_i\}_{t_i}$. (This is well-defined for each simplex of K , i.e., does not depend on the lift to L .) With this ptolemy coordinates one computes, following [6, Section 5.3] an element

$$\hat{c}(\rho) =$$

$$\begin{aligned} & (\log(c_{2001}) + \log(c_{1110}) - \log(c_{2010}) - \log(c_{1101}), \log(c_{2100}) + \log(c_{1011}) - \log(c_{2010}) - \log(c_{1101}) \\ & + (\log(c_{1101}) + \log(c_{0210}) - \log(c_{1110}) - \log(c_{0201}), \log(c_{1200}) + \log(c_{0111}) - \log(c_{1110}) - \log(c_{0201}) \\ & + (\log(c_{1011}) + \log(c_{0120}) - \log(c_{1020}) - \log(c_{0111}), \log(c_{1110}) + \log(c_{0021}) - \log(c_{1020}) - \log(c_{0111}) \\ & + (\log(c_{1002}) + \log(c_{0111}) - \log(c_{1011}) - \log(c_{0102}), \log(c_{1101}) + \log(c_{0012}) - \log(c_{1011}) - \log(c_{0102}) \in \widehat{\mathcal{P}}(\mathbb{C}) \end{aligned}$$

in the extended Bloch group $\widehat{\mathcal{P}}(\mathbb{C})$ associated to the simplex. Summing over all simplices of the triangulation one obtains an element $\rho_*[M, \partial M] \in \widehat{\mathcal{B}}(\mathbb{C}) \subset \widehat{\mathcal{P}}(\mathbb{C})$, such that application of the extended Roger's dilogarithm $R: \widehat{\mathcal{P}}(\mathbb{C}) \rightarrow \mathbb{C}/4\pi^2\mathbb{Z}$ to this element gives the Cheeger-Chern-Simons invariant $\hat{c}_2$. See [6, Theorem 8.3].

Since we are interested only in its imaginary part we can use the relation between the extended Roger's dilogarithm and the Bloch-Wigner dilogarithm: the imaginary part of the extended Rogers' dilogarithm of an element in $\widehat{\mathcal{P}}(\mathbb{C})$ is the Bloch-Wigner dilogarithm of its image in $\mathcal{P}(\mathbb{C})$. For the above element $\hat{c}(\rho) \in \widehat{\mathcal{P}}(\mathbb{C})$ its image in $\mathcal{P}(\mathbb{C})$ is

$$(1) \quad c(\rho) = \left[\frac{c_{2001}c_{1110}}{c_{2010}c_{1101}} \right] + \left[\frac{c_{1101}c_{0210}}{c_{1110}c_{0201}} \right] + \left[\frac{c_{1011}c_{0120}}{c_{1020}c_{0111}} \right] + \left[\frac{c_{1002}c_{0111}}{c_{1011}c_{0102}} \right] \in \mathcal{P}(\mathbb{C}).$$

One obtains the volume associated to the simplex of K by applying the Bloch-Wigner dilogarithm $D: \mathcal{P}(\mathbb{C}) \rightarrow \mathbb{R}$ to $c(\rho)$, and the volume of ρ by summing these volumes over all simplices of the triangulation.

From $\bar{\xi}: L_0 \rightarrow G/N$ one obtains a ρ -equivariant map $\xi: L_0 \rightarrow \text{Flag}(\mathbb{CP}^2)$ via the obvious projection $G/N \rightarrow G/B = \text{Flag}(\mathbb{CP}^2)$ with $B \subset G$ the group of all upper triangular matrices. For given ξ , the volume of the associated flag structure is according to [2, Section

3.6] defined by summing over all simplices of the triangulation $\frac{1}{4}$ times the Bloch-Wigner dilogarithms of

$$(2) \quad [z_{12}] + [z_{21}] + [z_{34}] + [z_{43}] \in \mathcal{P}(\mathbb{C}),$$

where the numbers in brackets are associated to a 4-tuple $([x_i], [f_i])_{1 \leq i \leq 4}$ of flags as follows.

To define z_{ij} define k and l such that the permutation $(1, 2, 3, 4) \rightarrow (i, j, k, l)$ is even. The lines through x_i form a projective line, so one has a well-defined cross ratio of four lines through x_i . Then z_{ij} is defined as the cross ratio of the lines $\ker(f_i), (x_i x_j), (x_i x_k), (x_i x_l)$.

According to [2, Lemma 2.7] the value of z_{ij} can be computed by the formula

$$z_{ij} = \frac{f_i(x_k) \det(x_i, x_j, x_l)}{f_i(x_l) \det(x_i, x_j, x_k)}.$$

Now consider the map $G/N \rightarrow G/B = \text{Flag}(\mathbb{CP}^2)$. Denote (e_1, e_2, e_3) the standard basis of $\mathbb{C}^3$ and $(x_0, f_0) = ([e_1], [(e_3, \cdot)])$ the standard flag in $\mathbb{CP}^2$. For $g_i \in G/N$ we denote $(x_i, f_i) = (g_i(x_0), g_i(f_0))$ the corresponding element in $G/B = \text{Flag}(\mathbb{CP}^2)$.

An elementary linear algebra computation shows

$$\det(\{g_i\}_1 \{g_j\}_1 \{g_k\}_1) = \det(x_i, x_j, x_k)$$

and

$$\det(\{g_i\}_2 \{g_j\}_1) = f_i(x_j)$$

for all i, j, k . Thus, for example,

$$\left[\frac{c_{2001}c_{1110}}{c_{2010}c_{1101}} \right] = \frac{f_1(x_4) \det(x_1, x_2, x_3)}{f_1(x_3) \det(x_1, x_2, x_4)} = \left(\frac{f_1(x_3) \det(x_1, x_2, x_4)}{f_1(x_4) \det(x_1, x_2, x_3)} \right)^{-1}$$

and hence

$$D\left(\left[\frac{c_{2001}c_{1110}}{c_{2010}c_{1101}}\right]\right) = -D\left(\frac{f_1(x_3) \det(x_1, x_2, x_4)}{f_1(x_4) \det(x_1, x_2, x_3)}\right)$$

because the Bloch-Wigner dilogarithm satisfies the symmetry $D(z) = -D(\frac{1}{z})$. The same computation shows that also the Bloch-Wigner dilogarithms of the second, third, fourth summand in (1) are the negatives of the Bloch-Wigner dilogarithms of the second, third, fourth summand in (2). So for each simplex its contribution to the volume from [2] is $-\frac{1}{4}$ times its contribution to the volume from [6]. Summing over the simplices of the triangulation one obtains the claimed equality.

QED

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