

REPRESENTATIONS OF SURFACE GROUPS

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Chapter 1

Introduction

The purpose of this lecture series is to describe the representation variety of the fundamental group $\pi_1(S)$ of a surface S with values in a Lie group G which is defined by

$$\mathrm{Rep}(\pi_1(S), G) := \mathrm{Hom}(\pi_1(S), G)/G,$$

where the action of G on $\mathrm{Hom}(\pi_1(S), G)$ is given by conjugation. Historically, the motivation for the study of $\mathrm{Rep}(\pi_1(S), G)$ emerged from Teichmüller theory which is the study of hyperbolic structures on a surface S . If $G = P\mathrm{SL}(2, \mathbb{R})$ then hyperbolic structures on S are described by a certain subset of $\mathrm{Rep}(\pi_1(S), G)$. Another classical example is $\mathrm{Rep}(\pi_1(S), \mathbb{T}^1)$, which is isomorphic to the quotient $H^1(S, \mathbb{R})/H^1(S, \mathbb{Z})$ and has various applications in physics. Here we will discuss $\mathrm{Rep}(\pi_1(S), G)$ for general Lie groups G and give various interpretations of this space, e.g. in terms of differential geometry (flat connections) or a combinatorial description (using ribbon graphs). The latter description will produce for us a natural topology on the representation variety, which turns a big open subset of $\mathrm{Rep}(\pi_1(S), G)$, which we call the very regular part, into a smooth manifold. This manifold admits a symplectic structure and thus smooth functions on it can be interpreted as observables in the sense of classical mechanics. We will discuss two important classes of such observables, namely Wilson loops and spin network observables. The Poisson brackets describing their dynamics admit a nice combinatorial description in terms of the Goldman algebra. While most of the aspects in our discussion work for general Lie groups G , some aspects have to be treated separately for compact and non-compact groups G respectively. For example, counting connected components of the representation variety can be reduced to counting elements of the fundamental group when G is compact, while in the non-compact case more complicated methods have to be applied. This will lead us - in the special case of $G = \mathrm{SL}(2, \mathbb{R})$ - to a discussion of the Milnor-Wood inequality. Also, the Liouville volume of the very regular part with respect to the symplectic form

depends crucially on the point whether G is compact or non-compact: In the non-compact case it is usually infinite infinite, while in the compact case it is always finite. In the latter case we will in fact compute the volume using a formula of Witten.

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Contents

1	Introduction	1
2	Surfaces	6
2.1	Surfaces as 2-dimensional manifolds	6
2.1.1	Basic definitions	6
2.1.2	Surfaces with boundary	7
2.1.3	Gluing surfaces	9
2.2	Surfaces as combinatorial objects	11
2.2.1	Ribbon graphs	11
2.2.2	Classification of compact oriented surfaces I: Existence	18
2.3	The fundamental group of a surface	22
2.3.1	The fundamental group of a topological space	22
2.3.2	Cayley graph and presentation complex	25
2.3.3	Classification of compact oriented surfaces II: Uniqueness	27
2.4	Combinatorial versions of the fundamental group	28
2.4.1	A combinatorial description of $\pi_1(S)$ using a ribbon graph	28
2.4.2	A combinatorial description of the fundamental group using covers	29
2.5	Cohomology of Surfaces	31
2.5.1	De Rham cohomology	31
2.5.2	Ribbon graphs and cohomology	35
2.5.3	The intersection form	39
3	Vector bundles and connections	42
3.1	Vector bundles	42
3.1.1	Definitions	42
3.1.2	A combinatorial interpretation through cocycles	43
3.2	Vector bundles with structures	46
3.3	Connections	48
3.3.1	Koszul connection	48
3.3.2	Holonomy along a path	51

3.3.3	Holonomy and linear differential equations	52
3.3.4	Curvature	53
3.3.5	Flat connections	53
3.3.6	G -connections	54
3.3.7	The holonomy map of a flat connection	55
3.4	Combinatorial versions of connections	56
3.4.1	Discrete connections on ribbon graphs	56
3.4.2	Local systems	57
3.5	Four models of the representation variety	58
3.5.1	Flat connections and local systems	58
3.5.2	Local systems and representations	59
3.5.3	Smooth and discrete connections	60
4	Twisted Cohomology	62
4.1	De Rham version of twisted cohomology	62
4.1.1	Motivation: variation of connection	62
4.1.2	De Rham version of the twisted cohomology	65
4.2	A combinatorial version	69
4.2.1	The combinatorial complex	69
4.2.2	The isomorphism Theorem	70
4.2.3	Duality	73
4.3	Torsion	76
4.3.1	Determinants	76
4.3.2	An isomorphism between determinants	77
4.3.3	The torsion of a metric complex	78
4.3.4	The torsion of a flat metric connexion	78
4.3.5	Torsion and symplectic complexes	80
4.3.6	Parallel metric on bundle	82
5	Moduli spaces	84
5.1	Topology and smooth structure	84
5.2	Proof of the main theorem	87
5.2.1	The Zariski closure of a group	88
5.2.2	More on the holonomy map	91
5.2.3	The Zariski dense part is Zariski open and non-empty	93
5.2.4	The action of G^V on $\mathcal{M}^{zd}(S, G)$ is proper	95
5.2.5	The moduli space $\mathcal{M}^{zd}(S, G)$ and its tangent space	97
5.2.6	The complementary of $\mathcal{D}_F^{zd}(S, G)$	98
5.3	Connected components of the moduli space	98
5.3.1	Connected components I: the compact case	98
5.3.2	Connected Components II : an invariant	100

5.3.3	Connected components III: Milnor-Wood inequality . . .	102
5.3.4	Connected components IV: Goldman's work	105
6	Symplectic structure	106
6.1	Universal connections and smooth structures	106
6.1.1	Lifting lemmas	106
6.1.2	Universal connections	107
6.2	The symplectic form	109
6.3	Classical mechanics and observables	110
6.4	Special observables I: Wilson loops and the Goldman algebra . .	111
6.5	Special observables II: Spin networks	116
6.5.1	An observable on the space of all flat connections	117
6.5.2	Poisson bracket of spin network observables	118
6.6	Volumes of moduli spaces	120
6.6.1	Witten's formula	120
6.6.2	Volume and the disintegrated measure	121
6.6.3	Characters and the disintegrated measure	123
7	Three manifolds and integrality	126
7.1	Integrality	126
7.1.1	The Cartan form	127
7.1.2	Integrating the Cartan form	127
7.2	Handlebodies and Lagrangians	131
7.2.1	Submanifolds of the moduli space	132
7.2.2	Isotropic submanifold	132
7.2.3	Chern-Simons invariants	133
7.2.4	Chern Simons action difference and the symplectic form	135

Chapter 2

Surfaces

2.1 Surfaces as 2-dimensional manifolds

2.1.1 Basic definitions

In this section, we will recall the point of view of differential geometers on surfaces.

Definition 2.1.1 *A surface is a connected two-dimensional smooth manifold.*

We assume that the reader is familiar with the definition of a manifold. Nevertheless, in order to fix ideas and notation, let us briefly discuss the definition above. We assume first that S is a metrisable topological space, in particular Hausdorff. A two-dimensional *chart* for such a topological space S is a pair (U, φ) where $U \subset S$ is an open subset and φ is a map from U to $\mathbb{R}^2$, where $V = \varphi(U)$ is open in $\mathbb{R}^2$ and φ is a homeomorphism onto its image. The map φ is called the *coordinate* of the chart. A collection of charts $\{(U_i, \varphi_i) \mid i \in I\}$ is called an *atlas* for S if $S = \bigcup U_i$. An atlas is called *smooth* or C^∞ (we use these two expressions synonymously) if the *coordinate changes*

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \rightarrow \varphi_i(U_i \cap U_j)$$

are smooth functions for all $i, j \in I$. The reason to introduce a smooth atlas is the desire to be able to talk about smooth functions on S . If smooth functions exists, we definitely want the coordinates φ_i and there inverses φ_i^{-1} to be smooth.

So let us just pretend that they *are* smooth.

If the notion of smooth functions is any good, then given a smooth function $\varphi : S \rightarrow \mathbb{R}$ also the composition

$$\varphi_i^{-1} \circ \varphi : \varphi_i(U_i) \rightarrow \mathbb{R}$$

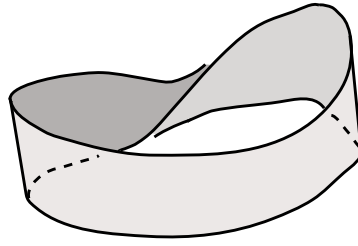


Figure 2.1: The Möbius strip

should be smooth. But these are just functions from subsets of $\mathbb{R}^2$ to $\mathbb{R}$ and for such functions we know what smoothness is supposed to be. Thus we can just define a function φ to be smooth if all the above functions are smooth. In fact, it suffices to check this for only one particular chart around each point as we were clever enough to demand the coordinate changes to be smooth themselves. Thus we may talk about smooth functions on a surface like we may talk about smooth functions on $\mathbb{R}^2$.

Obvious examples of surfaces are given by connected open subsets of $\mathbb{R}^2$. Indeed, if $U \subset \mathbb{R}^2$ and $i : U \rightarrow \mathbb{R}^2$ is the inclusion map then $\{(U, i)\}$ is a smooth atlas for U . Geometric intuition tells us that all these examples are oriented, while the Möbius strip (Figure 2.1) is not. One way to formalise this is via the derivatives of the coordinate changes:

Definition 2.1.2 *Let S be a surface with atlas $\{(U_i, \varphi_i) \mid i \in I\}$. S is called oriented if the Jacobians*

$$\text{Jac}(\varphi_i, \varphi_j) := \det(d(\varphi_i \circ \varphi_j^{-1}))$$

are positive for any $i, j \in I$.

2.1.2 Surfaces with boundary

In order to study surfaces it is often useful to cut a surface into pieces along a curve or to glue two such pieces in order to obtain a new surface. The pieces occurring in such geometric operations are not surfaces in the sense of Definition 2.1.1. Thus we need a more general definition in order to deal with

these pieces as well. For this we need another model space: Let

$$H^+ := \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$$

be the closed upper half plane. This has a boundary

$$\partial H^+ := \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$$

when considered as a subset of $\mathbb{R}^2$. Given a metrisable space S , a two-dimensional *chart with boundary* is a pair (U, φ) where U is an open subset of S and $\varphi : U \rightarrow V$ is a homeomorphism onto an open subset $V \subset H^+$, which is again called the coordinate of the chart. The subset

$$\partial U := \varphi^{-1}(\varphi(U) \cap \partial H^+) \subset U$$

will be called the boundary of U . In order to define a smooth atlas of charts with boundary we need to define a notion of smoothness for functions between open subsets of H^+ .

Definition 2.1.3 *Let V_1, V_2 be open subsets of H^+ . A function $f : V_1 \rightarrow V_2$ is called smooth if there exists an open subset $\tilde{V}_1$ of $\mathbb{R}^2$ with $\tilde{V}_1 \cap H^+ = V_1$. and a smooth function $\tilde{f} : \tilde{V}_1 \rightarrow \mathbb{R}^2$ such that $f = \tilde{f}|_{V_1}$.*

An atlas of charts with boundary $\{(U_i, \varphi_i) \mid i \in I\}$ will be called smooth if the coordinate changes

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \rightarrow \varphi_i(U_i \cap U_j)$$

are smooth in the sense of definition 2.1.3. Then we may define:

Definition 2.1.4 *A surface with boundary is a metrisable space S together with a smooth atlas of charts with boundary.*

Similar as before we have the notion of smooth functions on a manifold with boundary. In order to define the boundary of a surface with boundary one needs to prove the following lemma:

Lemma 2.1.5 *Let S be a surface with boundary and let $(U_1, \varphi_1), (U_2, \varphi_2)$ be charts with boundary of S . Suppose $x \in \partial U_1 \cap U_2$, i.e. x is a boundary point of U_1 . Then $x \in \partial U_2$ is a boundary point of U_2 .*

Exercise 2.1.6 Deduce Lemma 2.1.5 from the implicit function theorem.

In fact boundary points are not only preserved by smooth maps but even by continuous maps. To see this, however, requires some algebraic topology. By Lemma 2.1.5 the notion of being a boundary point of a chart does not depend on the particular chart. Thus we may define:

Definition 2.1.7 *Let S be a surface with boundary. A point $x \in S$ is called a boundary point of S if $x \in \partial U$ for some (and hence any) chart (U, φ) containing it. The set of boundary points of S is denoted ∂S .*

In order to glue surfaces along their boundaries we need to know how they look like in a neighbourhood of a boundary component. This is described by the following lemma:

Lemma 2.1.8 *Let M be a surface with boundary ∂M and $c \subset \partial M$ a connected component. Then there exists a neighbourhood U of c in M and a diffeomorphism $\psi : U \rightarrow V$ onto a subset $V \subset \mathbb{R}^2$ of the form $V \cong c \times [0, 1)$ mapping c onto $c \times \{0\}$.*

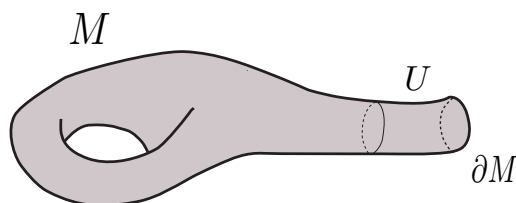


Figure 2.2: A collar neighbourhood

The neighbourhood U will be called a *collar neighbourhood* of c . (Sometimes this is also called a tubular neighbourhood.) The proof of Lemma 2.1.8 is easy if one uses Riemannian geometry and in particular exponential charts. There does not seem to be an elementary proof though, and as we do not want to get too far into Riemannian geometry here, we omit the proof.

2.1.3 Gluing surfaces

Given two surfaces with boundary S_1, S_2 and diffeomorphic connected components c_1 of ∂S_1 and c_2 of ∂S_2 respectively, everybody intuitively knows how to

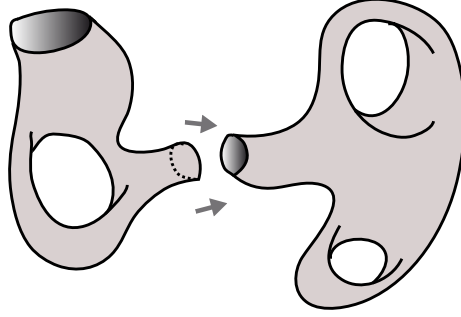


Figure 2.3: Gluing two surfaces

glue them together. This is depicted in Figure 2.3.

We sketch a formalisation of this procedure now. Let notation be as above and let $\varphi : c_1 \rightarrow c_2$ be a diffeomorphism. We define a topological space

$$S_1 \cup_{\varphi} S_2$$

called the *gluing* of S_1 and S_2 along φ as follows: On the disjoint union $S_1 \amalg S_2$ we consider the equivalence relation generated by

$$x \sim y \Leftrightarrow y = \varphi(x)$$

for $x \in c_1, y \in c_2$. Then $S_1 \cup_{\varphi} S_2$ is the quotient of $S_1 \amalg S_2$ by this equivalence relation.

Exercise 2.1.9 Show that the space $S_1 \cup_{\varphi} S_2$ is metrisable.

An atlas for $S_1 \cup_{\varphi} S_2$ is now constructed as the union of three subsets: Firstly, we take a smooth atlas $\{(U_i, \varphi_i) \mid i \in I\}$ for $S_1 \setminus c_1$ and a smooth atlas $\{(U_j, \varphi_j) \mid j \in J\}$ for $S_2 \setminus c_2$. Denote by $i_1 : S_1 \hookrightarrow S_1 \cup_{\varphi} S_2$, $i_2 : S_2 \hookrightarrow S_1 \cup_{\varphi} S_2$ the canonical inclusions. Then

$$\{(i_1(U_i), \varphi_i \circ i_1^{-1}) \mid i \in I\} \cup \{(i_2(U_j), \varphi_j \circ i_2^{-1}) \mid j \in J\}$$

is an atlas for the complement of the gluing curve in $S_1 \cup_{\varphi} S_2$. It remains to construct a chart around this gluing curve which is compatible with the charts above. For this we use Lemma 2.1.8: Let U_1, U_2 be collar neighbourhoods of c_1, c_2 , ψ_1 a diffeomorphism from U_1 onto $c_1 \times (-1, 0]$ (mapping c_1 to $c_1 \times \{0\}$)

and ψ_2 a diffeomorphism from U_2 onto $c_2 \times [0, 1)$ (mapping c_2 to $c_2 \times \{0\}$). We consider the open subset $O := U_1 \cup_\varphi U_2$ of $S_1 \cup_\varphi S_2$ and fix an embedding $\iota : c_2 \times (-1, 1) \rightarrow \mathbb{R}^2$. This is always possible as c_2 is either an interval or a circle. We define coordinates for O by

$$\psi : O \rightarrow \mathbb{R}^2, \quad x \mapsto \begin{cases} \iota \circ (\varphi, id) \circ \psi_1(x) & \text{if } x \in U_1 \\ \iota \circ \psi_2(x) & \text{if } x \in U_2. \end{cases}$$

The one has:

Proposition 2.1.10 *$S_1 \cup_\varphi S_2$ is a surface with boundary with smooth atlas given by*

$$\{(i_1(U_i), \varphi_i \circ i_1^{-1}) \mid i \in I\} \cup \{(i_2(U_j), \varphi_j \circ i_2^{-1}) \mid j \in J\} \cup \{O, \psi\}.$$

Exercise 2.1.11 Prove Proposition 2.1.10.

Exercise 2.1.12 Let S_1, S_2 be oriented surfaces with boundary and c_1, c_2 be connected components of the respective boundaries. Then c_1, c_2 carry an induced orientation. Suppose now that $\varphi : c_1 \rightarrow c_2$ is an orientation reversing diffeomorphism. Then there is a unique orientation of $S_1 \cup_\varphi S_2$ compatible with the orientations of S_1 and S_2

So far the only examples of surfaces we have seen are open subsets of $\mathbb{R}^2$. Proposition 2.1.10 allows us to construct many other examples of surfaces. For example the closed unit disc in $\mathbb{R}^2$ is a surface with boundary and gluing two copies of this along the unit circle gives rise to the two-sphere S^2 . In order to study such gluings more systematically we shall introduce some combinatorial notions.

2.2 Surfaces as combinatorial objects

2.2.1 Ribbon graphs

A graph is just a collection of points (*vertices*) which are joined by some lines (*edges*) as in picture 2.4. One may or may not equip the edges with a preferred orientation, which leads to the notion of an oriented graph. The following definition provides a formal description of the objects in the picture:

Definition 2.2.1 *An oriented graph Γ is a triple $\Gamma = (V, E, \varphi)$, where V and E are sets whose elements are called vertices and edges respectively and*

$$\varphi : E \rightarrow V \times V, \quad e \mapsto (e_-, e_+)$$

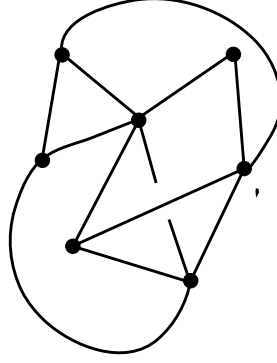


Figure 2.4: A graph

is a map. The vertex e_- is called the origin of e while the vertex e_+ is called the terminus of e .

A graph is a pair (Γ, I) , where $\Gamma = (V, E, \varphi)$ is an oriented graph and $I : E \rightarrow E$, $e \mapsto \bar{e}$ is a fixed point free involution on E satisfying

$$\bar{e}_+ = e_-, \quad \bar{e}_- = e_+.$$

A pair $(e, \bar{e})$ in E^2 is called a geometric edge of (Γ, I) .

The geometric realisation $|\Gamma|_I$ of a graph (Γ, I) is the topological space

$$|\Gamma|_I = E \times [0, 1] / \sim,$$

where $\sim$ is the equivalence relation generated by the following relations:

- $(e, t) \sim (\bar{e}, 1 - t)$.
- If $e, f \in E$ with $e_- = f_-$ then $(e, 0) \sim (f, 0)$.
- If $e, f \in E$ with $e_+ = f_+$ then $(e, 1) \sim (f, 1)$.

The geometric realisation of a finite graph cannot always be drawn on a plane with non intersecting edges. However, one can draw a graph in the plane, as if it were the projection of a graph in space, so that edges appears to cross over or under as in Figure 2.4.

We will be interested with graph with some extra structure. For this we need the notion of a *cyclic ordering* on a finite set S . By this we shall mean

a bijection $s : S \rightarrow S$ such that for any $x \in S$ the orbit $\{s^n(x)\}$ is S . Then given $x \in S$ we will call $s(x)$ the (cyclic) *successor* of x and $s^{-1}(x)$ the (cyclic) *predecessor* of x .

Definition 2.2.2 Let (Γ, I) be a finite graph and for $v \in V$ denote by $E_v := \{e \in E \mid e_- = v\}$ the set of edges starting from v . This is called the *star* of v . Suppose we are given a cyclic ordering s_v on each of the sets E_v . Then $(\Gamma, I, \{s_v\}_{v \in V})$ is called a *ribbon graph*.

Any planar graph is a ribbon graph. More precisely, given an embedding of a graph¹ into $\mathbb{R}^2$ the orientation of $\mathbb{R}^2$ induces a cyclic ordering on each of the sets E_v . (We may take a small circle around each vertex intersecting each edge only once. Then the circle obtains an orientation from $\mathbb{R}^2$ and this defines the ordering.)

It is also convenient to represent a graph on the plane as the projection of a graph in space – so again with edges that may cross, over or under – so that the cyclic ordering of the edges coincide with the cyclic ordering arising from the orientation in the plane as in Figure 2.5.)

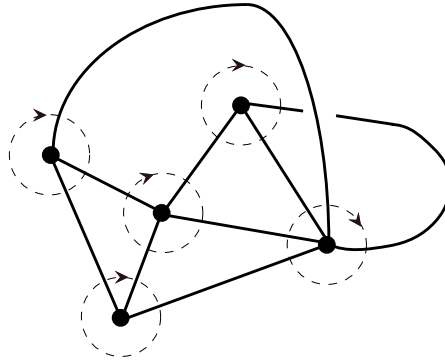


Figure 2.5: Cyclic ordering of edges in a planar graph

Exercise 2.2.3 Show that any embedding of a graph into an oriented surface gives rise to a cyclic orientation on the sets E_v .

We will now go the other way round and construct surfaces starting from ribbon graphs. In a first step we will embed ribbon graphs into open (i.e.

¹When we talk about an embedding of a graph into a topological space we actually mean an embedding of the geometric realisation of that graph. This slight abuse of language will be used throughout.

non-compact) surfaces, even though we remarked it is not always possible to do so on the plane.

Lemma 2.2.4 *Every ribbon graph can be embedded into an oriented surface such that its cyclic orderings are induced from the orientation of the surface.*

PROOF: We construct a suitable surface S by gluing certain elementary pieces: for every vertex $v \in V$ define a subset $U_v \subset \mathbb{R}^2$ with $|E_v|$ boundary components labelled by the elements of E_v of the form depicted in Figure 2.6 (Only the straight lines of the boundary are part of the surface.) We do the labelling in the following way: We start with an arbitrary edge $e \in E_v$. Then the next boundary component in the counterclockwise sense is labelled by $s_v(e)$ and so on.

Similarly we attach to each edge e a strip V_e as depicted in Figure 2.7 whose two boundary components (again only the straight lines are part of the surface) are labelled by e_+ and e_- . Now for each $e \in E$ we glue the connected component of ∂V_e labelled by v with the connected component of ∂U_v labelled by e (if v is the origin of e) or $\bar{e}$ (if v is the terminus). If we do this in an orientation reversing way then by Exercise 2.1.12 we obtain an oriented surface. At each vertex v the orientation on the corresponding star is preserved under the gluing process. Thus, by construction the orientation is compatible with s_v . Q.E.D.

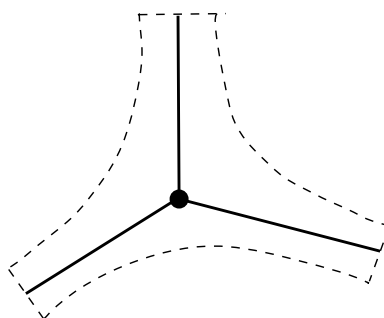


Figure 2.6: A surface associated with a vertex

The open surface constructed from a ribbon graph as in the proof of Lemma 2.2.4 is called the associated *ribbon surface* of that graph. In order to embed

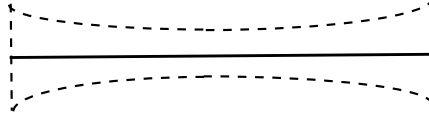


Figure 2.7: A surface associated with an edge

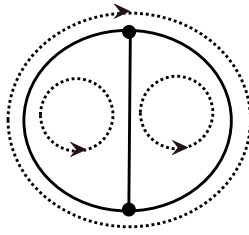


Figure 2.8: A graph with three faces

ribbon graphs into *compact* surfaces one has to close the holes in the ribbon surface. For a graph as depicted in Figure 2.8 this is easy: We see how to glue discs inside the holes so that we obtain a sphere. Observe that the boundaries of these faces can be read off from the picture as follows: Start with any edge e and follow this edge to the next vertex. Then take the successor of $\bar{e}$ and continue until you get a closed curve. Then this curve is supposed to be the boundary of a disc in the surface we want to construct.

This procedure works also in more complicated situations as depicted in Figure 2.9 and we formalise it in the following definition:

Definition 2.2.5 *Let $(\Gamma, I, \{s_v\}_{v \in V})$ be a ribbon graph. Then a face is an equivalence class – up to cyclic permutation – of n -tuples $(e_1, \dots, e_n)$ of edges*

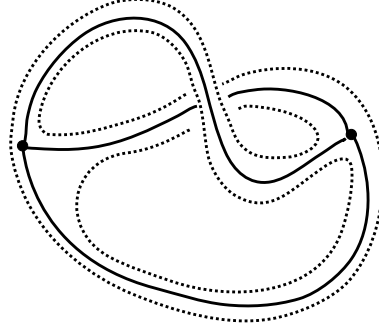


Figure 2.9: A graph with one face

such that $e_p^+ = e_{p+1}^-$ and $s_{e_p^+}(\bar{e}_p) = e_{p+1}$ for all $1 \leq p \leq n$ (and addition modulo n).

Note that every oriented edge is contained in a unique face.

Definition 2.2.6 Let us call an embedding of a graph Γ into a surface S filling if the complement

$$S \setminus |\Gamma| = \coprod_{f \in F} D_f$$

of the (geometric realisation of the) graph is a disjoint union of subsets D_f , where f runs over some finite index set F and each D_f is homeomorphic to a disc.

Then we have:

Proposition 2.2.7 Every ribbon graph Γ has a filling embedding into a compact oriented surface S . The connected components of $S \setminus |\Gamma|$ are in bijection with the faces of Γ .

PROOF: The edges of a face define a circle in the ribbon surface and we can glue a disc along this circle. Doing this for all faces with the correct orientation gives rise to a compact oriented surface. Q.E.D.

Exercise 2.2.8 What surfaces do you obtain by performing the above construction from the ribbon graphs depicted by Figures 2.8 and 2.9 respectively?

The surface obtained from a ribbon graph by Proposition 2.2.7 is in fact unique in a very strong sense. To make this precise, we will need the following basic fact from point-set topology:

Lemma 2.2.9 [CLUTCHING LEMMA] *Let $X = U \cup V$ be a decomposition of a topological space X into closed subsets U, V . If $f_1 : U \rightarrow Y$ and $f_2 : V \rightarrow Y$ are continuous maps from U and V into some topological space Y with $f_1|_{U \cap V} = f_2|_{U \cap V}$ then the induced map $f : X \rightarrow Y$ is continuous.*

Using this we can show the following weak uniqueness result:

Proposition 2.2.10 *Let $\Gamma \subset S$, $\Gamma' \subset S'$ be filling ribbon graphs of compact oriented surfaces and let $\varphi : \Gamma \rightarrow \Gamma'$ be an isomorphism of ribbon graphs. Then φ induces an orientation-preserving homeomorphism $\varphi : |\Gamma| \rightarrow |\Gamma'|$ of geometric realisations and this extends to a homeomorphism $S \rightarrow S'$.*

PROOF: It is easy to see that φ extends to the geometric realisation and further to the closure of the associated ribbon surfaces. Let

$$S \setminus |\Gamma| = \coprod_{f \in F} D_f, \quad S' \setminus |\Gamma'| = \coprod_{f \in F} D'_f.$$

Let S_Γ and $S_{\Gamma'}$ be the ribbon surfaces associated with $|\Gamma|, |\Gamma'|$. Then

$$S = \overline{S_\Gamma} \cup \left(\coprod_{f \in F} \overline{d_f} \right),$$

where d_f is a disc slightly smaller than D_f and

$$\overline{S_\Gamma} \cap \left(\coprod_{f \in F} \overline{d_f} \right) = \coprod \partial \overline{d_f}$$

is a union of circles. The same statement is true for S' . By Lemma 2.2.9 it then suffices to construct for each $f \in F$ a homeomorphism $\overline{d_f} \rightarrow \overline{d'_f}$ which agrees on the boundary $\partial \overline{d_f}$ with the extension of φ to $\overline{S_\Gamma}$. But if $\psi : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ is any given homeomorphism of circles then there is an obvious way to extend it to the corresponding discs. Indeed, every element x of the disc D may be written in polar coordinates as

$$x = re^{i\theta}$$

for some $r \in [0, 1]$ and some $e^{i\theta} \in \mathbb{T}^1$. But then we may simply define $\psi(re^{i\theta}) := r\varphi(e^{i\theta})$ to obtain the desired homeomorphism. Q.E.D.

Combining this with Proposition 2.2.7 we obtain:

Corollary 2.2.11 *For any ribbon graph Γ there exists a unique compact oriented surface S_Γ (up to homeomorphism) such that Γ can be embedded as a filling ribbon graph into S_Γ .*

Corollary 2.2.11 will enable us to classify surfaces up to homeomorphism. A classification up to diffeomorphism is possible along the same lines, but it would require a stronger form of the corollary. Namely, one would have to prove that any homeomorphism of surfaces is homotopic to a diffeomorphism and thus a ribbon graph determines the surface even up to diffeomorphism. This is much harder to prove and we will not pursue a classification up to diffeomorphism here.

Corollary 2.2.11 allows us to construct surfaces starting from ribbon graphs. There is a converse to this given by the following proposition:

Proposition 2.2.12 *Every compact oriented surface admits a filling ribbon graph.*

We will not prove Proposition 2.2.12 here. The problem of constructing a filling ribbon graph is equivalent to the probably more familiar problem of constructing a triangulation of a given surface. Indeed the set of edges of the triangulation will yield a filling ribbon graph. There are several standard methods in order to produce a triangulation (and hence a filling ribbon graph), but any of them uses some basic results from either Riemannian geometry or Morse theory, which we do not want to discuss here.

2.2.2 Classification of compact oriented surfaces I: Existence

By Corollary 2.2.11 a convenient description of compact surfaces is given by their ribbon graphs. We consider a family Γ_g , $g \geq 1$ of ribbon graphs given as follows: Take g copies of the graph shown in Figure 2.10 and glue them together as shown in Figure 2.11 to produce the *graph with g petals*.

The graph Γ_1 has one face and with some imagination one sees that the associated surface $S_1 := S_{\Gamma_1}$ is a torus. Similarly, each copy of Γ_1 in Γ_g gives rise to a torus with a hole and the holes of two consecutive tori are glued together. Thus, after some gymnastics, $S_g := S_{\Gamma_g}$ is the surface of a handle body with g handles.

There is another description of S_g which turns out to be useful: Let D_g be the disc in S_g corresponding to the unique face of Γ_g . Then S_g is obtained

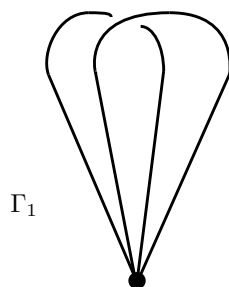


Figure 2.10: The graph Γ_1

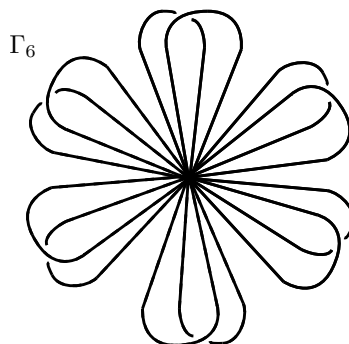


Figure 2.11: The graph Γ_g for $g = 6$

from gluing the boundary of D_g . Each edge of Γ_g occurs precisely once in the boundary of D_g . More precisely, let a_i, b_i be the two edges of the i -th copy of Γ_1 in Γ_g . Then the boundary of D_g is given by the series of edges

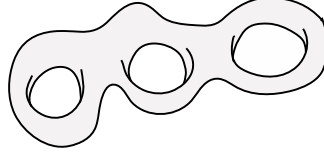
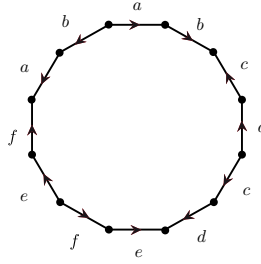
$$a_1, b_1, a_1^{-1}, b_1^{-1}, \dots, a_g, b_g, a_g^{-1}, b_g^{-1}.$$

For $g = 3$ this is depicted in Figure 2.13.

It is convenient to define $S_0 := S^2$, the two-sphere. Then we can state one-half of the classification as follows:

Theorem 2.2.13 *Every compact oriented surface S is homeomorphic to one of the surfaces S_g for $g \geq 0$.*

PROOF: By Proposition 2.2.12 we may choose a filling ribbon graph Γ for S .

Figure 2.12: The surface S_3 Figure 2.13: Constructing S_3 by gluing a disc

If the graph has no edges, then S must be the two-sphere S^2 . Thus we may assume that Γ has at least one edge. We will now deform the graph Γ into a graph Γ_g without changing its filling property. The theorem will then follow from Corollary 2.2.11. Suppose firstly, that Γ has more than one face. Then there exists a geometric edge $(e, \bar{e})$ such that e and $\bar{e}$ are contained in different faces. Let Γ' be the graph obtained from Γ by deleting e and $\bar{e}$. Then Γ' is still filling for S but has one face less 2.14. Iterating this process we end up with a filling graph with only one face. Thus we may assume that Γ has only one face.

Secondly, we reduce the number of vertices. Suppose that e is an edge joining the two vertices e_+ and e_- . Replace $\Gamma = (V, E, \varphi)$ by $\Gamma' = (V', E', \varphi')$ where $V' = (V \setminus \{e_+, e_-\}) \cup \{v_e\}$ and $E' = E \setminus \{e, \bar{e}\}$ and $\varphi' = (\sigma, \sigma) \circ \varphi|_{E'}$ where $\sigma : V \rightarrow V'$ is given by

$$\sigma(e_+) = \sigma(e_-) = v_e, \quad \sigma(f) = f \ (f \notin \{e_+, e_-\}).$$

Again this does not change the fact that Γ is filling and reduces the number of vertices by one without increasing the number of faces. (See Figure 2.15.) Hence we may assume that Γ has only one vertex and one face. This means that S is obtained by gluing the boundary of a polygon. The sides of the polygon are labelled by the edges of Γ and by definition of face every edge

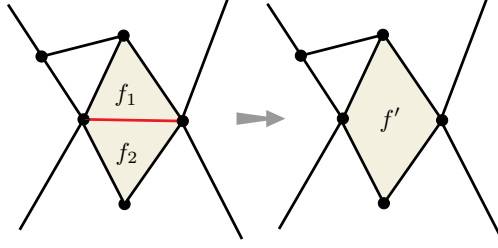


Figure 2.14: Reducing the number of faces

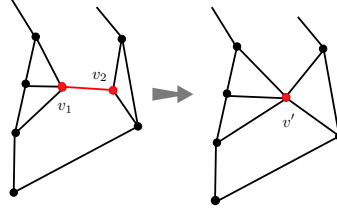


Figure 2.15: Reducing the number of vertices

occurs precisely one. (The gluing is given by identifying e and $\bar{e}$ with reversed orientation.)

If there are no edges left, then S is the two-sphere S^2 and we are done. Thus assume there are still edges left. We will call a pair of geometric edges $((a, \bar{a}), (b, \bar{b}))$ *linked* if their relative position is as in Figure 2.16. We claim that any geometric edge is linked to at least one other geometric edge. Indeed, from the ribbon graph point of view, geometric edge which is not linked to any other edge would produce an additional face contradicting our assumption that there is only one face. Thus we are left to prove the following claim:

Given a linked pair $(a, \bar{a}), (b, \bar{b})$ of geometric edges there is a way to rearrange the labelling of the polygon without changing the resulting quotient space such that $a, b, \bar{a}, \bar{b}$ appears as a subsequence and no subsequence of this type is destroyed during this process.

Indeed, if the claim is true then we can relabel the edges of the polygon in such a way that we obtain one of the graphs Γ_g . For the proof of the claim we give an explicit construction which consists of adding and deleting certain

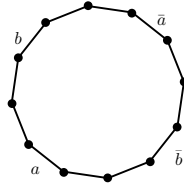


Figure 2.16: Linked Edges

edges of the graph. Such a procedure is depicted in Figure 2.17. The figure has to be read as follows. The main horizontal step is as follows, we add an edge which is dotted in the picture. We therefore obtain two faces which have different colours in the picture. Then we erase in the graph the green edge, which has the effect on the polygon to glue together these two green lines in the red one. Then we repeat this procedure two times as is depicted in the picture. We observe that not consecutive subsequence of the form $c, d, \bar{c}, \bar{d}$ have been destroyed in this construction. The resulting surface (which is homeomorphic to the one before) is thus brought into a new position such that all edges lie on the boundary. In the final picture we have created an additional subsequence of the form $a, b, \bar{a}, \bar{b}$ which proves the claim and finishes the proof of the theorem. Q.E.D.

2.3 The fundamental group of a surface

2.3.1 The fundamental group of a topological space

The result obtained in Theorem 2.2.13 is not yet a complete classification of compact oriented surfaces as we do not know yet whether some of the surfaces S_g for distinct g are homeomorphic. In order that this does not happen, to prove this we need an invariant that distinguishes the surfaces S_g from each other.

This invariant is the *fundamental group* – a central object of these series of lectures – and we briefly recall its definition:

Definition 2.3.1 *Let X, Y be a topological spaces.*

- (i) *A (parametrised) loop in X based at $x_0 \in X$ is a continuous map $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) = \gamma(1) = x_0$. Denote by $\Omega(X, x_0)$ the set of loops in X based at x_0 .*

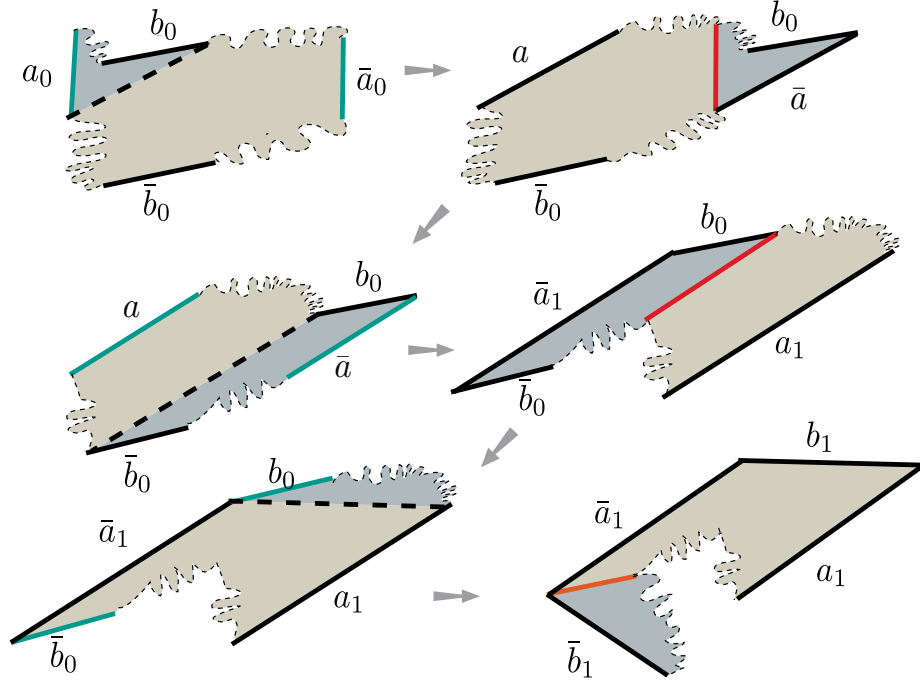


Figure 2.17: Moving linked edges into generic position

(ii) The composition of two based loops $\gamma_0, \gamma_1 \in \Omega(X, x_0)$ is defined to be

$$(\gamma_0 * \gamma_1)(t) := \begin{cases} \gamma_0(2t), & 0 \leq t \leq \frac{1}{2} \\ \gamma_1(2t - 1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

(This is continuous by Lemma 2.2.9.)

(iii) Let $f_0, f_1 : Y \rightarrow X$ be continuous maps which agree on a subset $A \subset Y$. Then f_0 and f_1 are called homotopic relative A , denoted $f_0 \simeq f_1 \text{ (rel } A)$ if there exists a map $H : [0, 1] \times Y \rightarrow X$ with $f_0(y) = H(0, y)$, $f_1(y) = H(1, y)$ and $H(s, a) = f_0(a) = f_1(a)$ for all $a \in A$. The map H is then called a homotopy relative A . A space X is called contractible if $\text{id} : X \rightarrow X$ is homotopic to a constant map $X \rightarrow X, x \mapsto x_0$ for some $x_0 \in X$.

(iv) Two based loops $\gamma_0, \gamma_1 \in \Omega(X, x_0)$ are called homotopic if they are homotopic relative $\{0, 1\}$. (Notice the abuse of language. If two loops are homotopic in the sense of (iii) we will call them freely homotopic.) The set of homotopy classes of loops based at x_0 is denoted $\pi_1(X, x_0)$.

Exercise 2.3.2 (i) Show that composition descends to a well-defined map $*$: $\pi_1(X, x_0) \times \pi_1(X, x_0) \rightarrow \pi_1(X, x_0)$.

(ii) Show that with this multiplication $(\pi_1(X, x_0), *)$ is a group.

Definition 2.3.3 The group $(\pi_1(X, x_0), *)$ is called the fundamental group of X at x_0 .

isomorphism

Exercise 2.3.4 Let $x_0, x_1 \in X$ and let $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) = x_0$, $\gamma(1) = x_1$. Show that

$$\pi_1(X, x_0) \rightarrow \pi_1(X, x_1), \quad [\alpha] \mapsto [\gamma * \alpha * \gamma^-]$$

is an isomorphism, where composition of paths is defined as composition of loops above and $\gamma^-(t) := \gamma(1 - t)$. Conclude that the isomorphism type of the fundamental group of a arcwise connected space X does not depend on the base point.

Definition 2.3.5 An arcwise connected space X is called simply-connected if $\pi_1(X, x_0)$ is trivial for some (and hence any) $x_0 \in X$.

In certain cases the fundamental group can be computed directly:

Exercise 2.3.6 (i) Show that any contractible space (in particular the disc) is simply-connected.

(ii) Show that if T is a topological space, $S \subset T$, D is the unit disc and $\mathbb{T}^1$ its boundary and $\varphi : \mathbb{T}^1 \rightarrow S$ is a homeomorphism, then there is a surjective group homomorphism $\pi_1(T) \rightarrow \pi_1(T \cup_\varphi D)$.

(iii) Using (i) and (ii), show that the 2-sphere is simply-connected.

Covering and fundamental group

In general, direct computations of π_1 using the definition are rarely possible. Usually the fundamental group of a space is computed from a realisation of the space as a nice quotient of a simply-connected one. For this we introduce the following notions:

Definition 2.3.7 Let Γ be a (discrete) group acting on a space M . Then the action is called free if it has no fixed point, i.e. $\gamma.m = m$ for some $\gamma \in \Gamma$, $m \in M$ implies $\gamma = e$. It is called properly if for any compact set $K \subset M$ the set

$$\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}$$

is finite.

Then the following is a weak version of the main theorem of covering theory:

Theorem 2.3.8 *Let Γ be a group acting freely and properly on a locally connected, contractible space M . Then for any $x \in M$ the map*

$$\Gamma \rightarrow \pi_1(\Gamma \backslash M, [x]), \quad \gamma \mapsto [c_\gamma],$$

where c_γ is the projection of a path from x to $\gamma(x)$ in M down to $\Gamma \backslash M$ and $[x] = \Gamma.x$, defines a group isomorphism. In particular,

$$\pi_1(\Gamma \backslash M, [x]) \cong \Gamma.$$

2.3.2 Cayley graph and presentation complex

Using Theorem 2.3.8 there is a way to realise any given group Γ as the fundamental group of a topological space. More precisely given a group Γ we can construct a simply-connected space $C_2(\Gamma)$ on which Γ acts freely and properly. Then the theorem implies that $\pi_1(\Gamma \backslash C_2(\Gamma), x) \cong \Gamma$ for any $x \in \Gamma \backslash C_2(\Gamma)$. For the construction of $C_2(\Gamma)$ we proceed as follows:

Start with a presentation of Γ of the form

$$\Gamma = \langle c_1, \dots, c_k \mid R_1 = \dots = R_p = e \rangle,$$

where $R_1, \dots, R_p$ are certain words on $c_1, \dots, c_k$. By adding further generators and relations if necessary we may assume that there is a fixed point free involution denoted $c_j \mapsto c_{\bar{j}}$ for $j = 1, \dots, k$ on the set of generators such that $c_{\bar{j}}$ is inverse to c_j as an element of Γ . (This means in particular, that c_j^{-1} is contained in $c_1, \dots, c_k$ for any $j = 1, \dots, k$. However, this is not enough: It might be that several distinct letters represent the same group element and in this case we want to have a prescribed inverse letter for each of them.) Let us call such a presentation admissible. For an admissible presentation we then define the *Cayley graph* of Γ with respect to the given presentation by $C(\Gamma) = (V, E)$, where the set of vertices is given by $V = \Gamma$ and the set of oriented edges is given as follows: If $\eta, \gamma \in \Gamma$ and $\eta = \gamma c_i$ then there is an oriented edge from η to γ labelled by c_i . In this case there is also a unique edge $c_{\bar{i}}$ from γ to η and there is a unique fixed point free involution of E mapping these two edges into each other and thus turning the directed graph $C(\Gamma) = (V, E)$ into a graph.

Example 2.3.9 Let F be a free group over a set X . Then F has a presentation

$$F = \langle \{x\}_{x \in X} \mid \emptyset \rangle$$

In order to bring this into the required form we add one generator x^{-1} for each $x \in X$ and obtain a new presentation

$$F = \langle \{x, x^{-1}\}_{x \in X} \mid xx^{-1} = x^{-1}x = e \rangle.$$

On the set of generators of this presentation we then have a fixed-point free involution $x \leftrightarrow x^{-1}$. The vertices of the Cayley graph are labelled by reduced words over $\{x, x^{-1}\}_{x \in X}$, i.e. words in these letters without subwords of the form (x, x^{-1}) for $x \in X$, and there is a geometric edge labelled by (x, x^{-1}) for $x \in X$ between $(x_1, \dots, x_k)$ and $(x_1, \dots, x_k, x)$, where $x_1, \dots, x_k \in X$. The corresponding Cayley graph $C(F)$ is a tree and hence its geometric realisation $|C(\Gamma)|$ is simply-connected.

In general the geometric realisation of a Cayley graph need not be simply-connected. However, it is easy to compute the fundamental group using an admissible presentation:

Exercise 2.3.10 Let $\Gamma = \langle c_1, \dots, c_k \mid R_1 = \dots = R_p = e \rangle$ be an admissible presentation of a group Γ and let F be the free group on generators $c_1, \dots, c_k$. Denote by R the subgroup of F normally generated by the relations $R_1, \dots, R_p$.

- (i) Show that F and hence R act freely and properly on the geometric realisation $|C(F)|$ of the Cayley graph of the free group F with respect to the given presentation.
- (ii) Show that $|C(\Gamma)| \cong |C(F)/R|$ and conclude that $\pi_1(|C(\Gamma)|) \cong R$.

Let R_j , $1 \leq j \leq p$ be a relation in Γ , say $R_j = c_{i_1} \dots c_{i_k}$. Then any $\gamma \in \Gamma$ satisfies

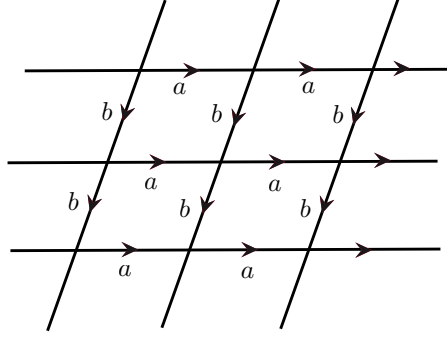
$$\gamma c_{i_1} \dots c_{i_k} = \gamma,$$

thus there exists a loop in $C(\Gamma)$ starting from γ consisting of edges labelled by $c_{i_1}, \dots, c_{i_k}$ in precisely that ordering. In the geometric realisation of $C(\Gamma)$ these form a circle and we can glue a disc along this circle to $|C(\Gamma)|$ for any such loop. The resulting space is called the *presentation 2-complex* of Γ with respect to the given presentation and denoted $C_2(\Gamma)$. The case of the group $\mathbb{Z}^2$ with the admissible presentation

$$\langle a, b, a^{-1}b^{-1} \mid aba^{-1}b^{-1} = aa^{-1} = bb^{-1} = e \rangle$$

is depicted in Figure 2.18.

Exercise 2.3.11 Using the special form of $\pi_1(C(\Gamma))$ derived in Exercise 2.3.10, prove that the image of $\pi_1(|C(\Gamma)|)$ in $\pi_1(C_2(\Gamma))$ is trivial. (Hint: It suffices to prove that the generators $gR_i g^{-1}$ of $\pi_1(|C(\Gamma)|)$ are mapped to contractible loops!) Then apply Exercise 2.3.6 to conclude that $C_2(\Gamma)$ is simply-connected.

Figure 2.18: A presentation 2-complex of $\mathbb{Z}^2$

Observe that the action by left multiplication of Γ on itself induces an action on $C(\Gamma)$, i.e. an action of the set of vertices which preserves adjacency. This in turn induces an action of Γ on the geometric realisation $|C(\Gamma)|$. This action extends to $C_2(\Gamma)$. Indeed, the action of Γ maps each loop to another loop and the corresponding homeomorphism of circles in the geometric realisation extend to the discs bounded by these circles as in the proof of Proposition 2.2.10.

Proposition 2.3.12 *The action of Γ on $C_2(\Gamma)$ is free and properly discontinuous. Thus*

$$\pi_1(\Gamma \backslash C_2(\Gamma)) \cong \Gamma.$$

Exercise 2.3.13 Prove Proposition 2.3.12.

2.3.3 Classification of compact oriented surfaces II: Uniqueness

We apply Proposition 2.3.12 to prove the following uniqueness part of the classification:

Theorem 2.3.14 *The fundamental group of the surface S_g is given by*

$$\pi_1(S_g) \cong \langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1} = e \rangle.$$

These groups are non-isomorphic for different choices of g .

PROOF: For $g = 0$ the theorem predicts $\pi_1(S^2) = \{e\}$ which was proved in Exercise 2.3.6. Thus let us assume $g \geq 1$ and abbreviate

$$A_g := \langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1} = e \rangle.$$

We obtain an admissible presentation of A_g by

$$A_g \cong \langle a_1, b_1, \dots, a_g, b_g, a_1^{-1}, b_1^{-1}, \dots, a_g^{-1}, b_g^{-1} \mid \prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1} = a_i a_i^{-1} = b_i b_i^{-1} = e \rangle.$$

For this presentation we may construct the associated Cayley graph and presentation 2-complex. Then $\Gamma \backslash C_2(\Gamma)$ is homeomorphic to gluing a disc whose boundary is labelled by $a_1, b_1, a_1^{-1}, b_1^{-1}, \dots, b_g^{-1}$ according to this labelling. We have seen before that this is a description of S_g and thus the identity $\pi_1(S_g) = A_g$ follows from Proposition 2.3.12. The last statement of the theorem follows from the fact that

$$\text{Hom}(A_g, \mathbb{R}) \cong \mathbb{R}^{2g}.$$

Q.E.D.

A group isomorphic to $A_g := \pi_1(S_g)$ will be called a *surface group*. By Theorem 2.2.13 and Theorem 2.3.14 we can associate to every compact oriented surface a unique non-negative integer g in such a way that $\pi_1(S) \cong A_g$. This integer is referred to as the *genus* of S . Using this terminology, we may summarise the classification by saying that any compact oriented surface is uniquely determined by its genus and conversely every non-negative integer occurs as the genus of a surface.

2.4 Combinatorial versions of the fundamental group

2.4.1 A combinatorial description of $\pi_1(S)$ using a ribbon graph

We conclude this section by a theorem on the relation between a filling ribbon graph of a surface and its fundamental group. For the first part of the classification we have used the existence of filling ribbon graphs. These ribbon graphs are not unique but they can be deformed into a filling ribbon graph of type Γ_g . Using different ribbon graphs one can compute the fundamental group of the surface in different ways. This gives rise to different presentations of surface groups.

Definition 2.4.1 [COMBINATORIAL LOOPS] *Let S be a surface and Γ a ribbon graph in S . Denote by E , V and F respectively the set of its edges, vertices and faces. Let $F(E)$ be the free group over E and fix a vertex $v_0 \in V$. Then v_0 defines a subgroup of $F(E)$ given by*

$$L_\Gamma^{v_0} := \{e_1 \cdot \dots \cdot e_k \mid e_i \in E, e_i^+ = e_{i+1}^-, e_k^+ = e_1^- = v_0\}.$$

The group $L_\Gamma^{v_0}$ is the group of combinatorial loops based at v_0

Definition 2.4.2 [RIBBON FUNDAMENTAL GROUP] *Let $R_\Gamma^{v_0}$ be the subgroup of $L_\Gamma^{v_0}$ normally generated by the products $e_f := e_1 \cdots e_k$ for faces $f = (e_1, \dots, e_k) \in F$. The ribbon fundamental group is*

$$\hat{\pi}_1(\Gamma, v_0) = L_\Gamma^{v_0} / R_\Gamma^{v_0}.$$

Observe that if Γ' is a subgraph of Γ we have a natural surjection from $\hat{\pi}_1(\Gamma, v_0)$ to $\hat{\pi}_1(\Gamma', v_0)$. Moreover if Γ is a filling graph embedded in a surface S , we have a natural map $\hat{i}$ from $\hat{\pi}_1(\Gamma, v_0)$ to $\pi_1(S, v_0)$ which is build by sending every combinatorial loop to its geometric realisation.

Theorem 2.4.3 *Let S be a surface and Γ a filling ribbon graph in S . The natural map $\hat{i}$ described in the previous definition is an isomorphism.*

The proof is essentially the same as for Theorem 2.3.14. We leave the details to the reader.

2.4.2 A combinatorial description of the fundamental group using covers

The fundamental group of a cover

Definition 2.4.4 [PATH] *Let $\mathcal{U} = \{U\}_{i \in I}$ be a cover of a space X . A path on the cover based at i_0 is an element $i_0 \dots i_p$ of the free group F_I generated by I such that $U_{i_j} \cap U_{i_{j+1}} = \emptyset$. A loop is a path $(i_0, \dots, i_p)$ so that $i_0 = i_p$. A trivial loop is a loop of the form $(j_0, \dots, j_p, I, J, K, j_p, \dots, j_0)$ where $U_I \cap U_J \cap U_K \neq \emptyset$.*

Definition 2.4.5 [FUNDAMENTAL GROUP] *Let $\mathcal{U} = \{U\}_{i \in I}$ be a cover of M . The fundamental group $\pi_1^{\mathcal{U}}(M)$ of the cover is the group of loops based at i_0 – considered as a subgroup of the free group generated by I – up to the group generated by trivial loops.*

In other words, the fundamental group of a cover $\mathcal{U} = \{U\}_{i \in I}$ be is the group of the 2-complex, whose set of vertices is the set I , set of edges is the set of intersecting open sets, and set of faces is the set of triples of intersecting open sets.

We observe that if $\mathcal{V} = \{V_j\}_{j \in J}$ is a *refinement* of $\mathcal{U}$, that is a cover such there exists a map $f : J \rightarrow I$ so that $V_j \subset U_{f(j)}$, then we have a natural map

$$\hat{f} : \pi_1^{\mathcal{V}}(S) \rightarrow \pi_1^{\mathcal{U}}(S).$$

Exercise 2.4.6 Show that the map above does not depend on the choice of the map f . Show that the map above is surjective provided the open sets U_i are connected.

We shall admit the following result which relies on Van Kampen Theorem.

Theorem 2.4.7 *Let $\{U_i\}$ be a cover such that the U_i are simply connected and $U_i \cap U_j$ are connected. For every i , let x_i be an element of U_i . For every pair (U_i, U_j) of intersecting open sets, we choose a path γ_{ij} joining x_i to x_j . We associate to every loop on the covering $i_0 \dots i_n$ where $i_n = i_0$, the loop $\gamma_{i_0 i_1} \star \dots \star \gamma_{i_{n-1} i_n}$. This construction is an isomorphism from $\pi_1^{\mathcal{U}}(M)$ to $\pi_1(M)$.*

It follows from this result that if X admits a cover by simply connected sets with connected intersections, then we have a well defined surjective map from $\pi_1(M)$ to $\pi_1^{\mathcal{U}}(M)$ for any cover $\mathcal{U}$ of X .

Covers and covering maps

We can relate covers and coverings as follows. Let $\pi : \Sigma \rightarrow S$ be a Galois cover of group G . Let $\mathcal{U}$ be a trivialising cover of S for π . Let $\bar{\mathcal{U}} = \{\pi^{-1}(U_i)\}_{i \in I}$ be the induced cover of Σ . By definition, $\pi_1^{\bar{\mathcal{U}}}(\Sigma) = \pi_1^{\mathcal{U}}(S)$. For every i , write

$$\pi^{-1}(U_i) = \bigsqcup_{g \in G} g.V_i.$$

Let finally $\hat{\mathcal{U}} = \{gV_i\}_{i \in I, g \in G}$, which is a refinement of $\bar{\mathcal{U}}$ and well defined whenever the sets U_i are connected.

Proposition 2.4.8 *Let $\pi : \Sigma \rightarrow S$ be a Galois cover. Let $\mathcal{U}$ be a trivialising cover of S for π , so that the intersection of two open sets is connected. Then, we have a short exact sequence*

$$0 \rightarrow \pi_1^{\hat{\mathcal{U}}}(\Sigma) \rightarrow \pi_1^{\mathcal{U}}(S) \rightarrow G.$$

PROOF: The hypothesis on $\mathcal{U} = \{U_i\}_{i \in I}$ yields that we can lift uniquely every path $\mathcal{U}$ -path to a $\hat{\mathcal{U}}$ -path, which maps trivial loops to trivial loops. Hence we obtain the short exact sequence

$$0 \rightarrow \pi_1^{\hat{\mathcal{U}}}(S) \rightarrow \pi_1^{\mathcal{U}}(S) \rightarrow G.$$

Q.E.D.

2.5 Cohomology of Surfaces

This whole section is a warm up for the next chapter. We shall give – and prove – more general statements in this next chapter. However, we specialise in this section to the standard situation of – untwisted – cohomology groups so that the reader can parallel this construction to the more general one.

2.5.1 De Rham cohomology

In this section we shall introduce and compute the cohomology groups of a surface S . These can be defined in purely topological terms, but it is convenient to use the existence of a differentiable structure on S in order to give a simpler definition of these groups. This approach is due to de Rham and it uses differential forms. Thus let S be a closed surface, i.e. a compact connected smooth 2-manifold. At each point $m \in S$ we have a *tangent space* $T_m S$ to S at m . The dual space $T_m^* S$ is called the *cotangent space* of S at m . If M is a manifold, the space $T_m^* M$ may be described as follows. Let $C_0^0(M, m)$ be the vector space of functions that vanishes at a point m in M . Finally, we say two functions f and g of $C_0^0(M, m)$ have the *same derivative at m* if we can write

$$f - g = \sum_{i=1}^{i=n} \varphi_i \psi_i,$$

where ψ_i and φ_i belong to $C_0^0(M, m)$. We observe that this notion coincide with the classical one when $M = \mathbb{R}^n$ that is f and g have the same derivative at m if and only if $df(m) = dg(m)$.

Informally speaking, a differential form is a map which assigns to each point $m \in S$ a vector $\omega_m \in \bigwedge^k T_m^* S$ for some $k \geq 0$ which varies smoothly. We shall also call ω a k -form. By definition $\bigwedge^0 T_m^* S := \mathbb{R}$ and thus a (smooth) 0-form is just a (smooth) function. If f is a smooth function on S then its differential df is an endomorphism of the tangent space and thus can be regarded as a 1-form. If U is a contractible open coordinate chart with coordinates $x_1, \dots, x_n$ then

the differentials dx_j span the differential forms as an algebra. This means, that any k -form on U can be written as

$$\omega|_U = \sum_{I=(i_1, \dots, i_k)} \omega_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

with $\omega_I : U \rightarrow \mathbb{R}$. Now we can make precise what we mean by smoothly varying: A k -form ω is smooth if for every point $m \in S$ there exists a contractible open coordinate chart around m such that if $\omega|_U$ is written as above the functions ω_I are smooth. It is easy to see that this does not depend on the choice of chart. We will only consider smooth forms and thus we reserve both the terms *differential form* and *k-form* for such assignments which can be written in the above form with smooth coefficient functions. We shall use the notation $\Omega^k(S)$ to denote k -forms on S . Note that $\Omega^k(S) = \{0\}$ for $k > \dim S = 2$. It will be convenient for us to define $\Omega^k(S) = \{0\}$ also for $k < 0$. Then

$$\Omega^\bullet(S) := \bigoplus_{k \in \mathbb{Z}} \Omega^k(S)$$

becomes a $\mathbb{Z}$ -graded $C^\infty(S)$ -algebra, i.e. it is a vector space and module over $C^\infty(S)$ and the wedge product defines an algebra structure which maps $\Omega^k(S) \times \Omega^l(S)$ into $\Omega^{k+l}(S)$. All these definitions make sense for arbitrary manifolds.

If S_1 and S_2 are surfaces or, more generally, manifolds and $f : S_1 \rightarrow S_2$ is smooth then we can define a *pullback map*

$$f^* : \Omega^\bullet(S_2) \rightarrow \Omega^\bullet(S_1)$$

which for $\omega \in \Omega^k(S_2)$ and tangent vectors $X_1, \dots, X_k$ at some point m of S_1 is given by

$$f^* \omega_m(X_1, \dots, X_k) := \omega_{f(m)}(Tf(X_1), \dots, Tf(X_k)).$$

An important point about differential forms is that they can be integrated. If $U \subset \mathbb{R}^n$ is open and $\omega = f dx_1 \wedge \cdots \wedge dx_n \in \Omega^n(U)$ then we define

$$\int_U \omega := \int_U f dx_1 \wedge \cdots \wedge dx_n := \int_U f d\lambda^n(x_1, \dots, x_n),$$

where λ^n is the n -dimensional Lebesgue measure. This integral can of course be infinite in general. In order to make sure that this integral is finite we can restrict ourselves to the following subclass of differential forms: Define the

support of a differential form ω to be the closure of those points x for which ω can be expressed in a neighbourhood U of x in the standard form

$$\omega|_U = \sum_{I=(i_1, \dots, i_k)} \omega_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

in such a way, that at least one of the coefficient functions ω_I does not vanish at x . Then ω is called compactly supported, if its support is a compact subset of S . We denote the compactly supported k -forms on S by $\Omega_c^k(S)$. Of course $\Omega^k(S) = \Omega_c^k(S)$ if and only if S is compact. For a compactly supported n -form ω on $U \subset \mathbb{R}^n$ the integral $\int_U \omega$ will always be finite. If φ is any diffeomorphism then

$$\int_U \varphi^* \omega = \pm \int_{\varphi(U)} \omega,$$

where the sign is positive if φ preserves orientation and negative if φ reverses the orientations. As a consequence of this sign irritation we can only define integration (at least in our naive way) for oriented manifolds:

Proposition 2.5.1 *If M is an oriented n -dimensional manifold then there exists a unique linear map*

$$\Omega_c^n(M) \rightarrow \mathbb{R}, \quad \omega \rightarrow \int_M \omega$$

such that if ω is supported in a chart (U, x) with x orientation-preserving then

$$\int_M \omega = \int_{x(U)} (x^{-1})^* \omega.$$

Convention 2.5.2 Suppose $N \subset M$ is a k -dimensional oriented submanifold and $\iota : N \rightarrow M$ denotes the inclusion. Then we can integrate any k -form $\omega \in \Omega^k(M)$ over N . We will use a slight abuse of notation and denote this by

$$\int_N \omega := \int_N \iota^* \omega.$$

Integration of differential forms can also be defined for manifolds with boundary. Using integration one can evaluate k -forms on k -cycles of the manifold. This allows one to describe the cohomology of a manifold in terms of differential forms. The key lemma in this direction is the following existence result:

Proposition 2.5.3 *There exists a unique family of linear operators $d^k : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ with the following properties:*

(i) d^0 coincides with the usual differential $C^\infty(M) \rightarrow \Omega^1(M)$.

(ii) $d^k(f d^{k-1}\alpha) = d^0 f \wedge d^{k-1}\alpha$ for $f \in C^\infty(M), \alpha \in \Omega^{k-1}(M)$.

The proof is standard and will not be reproduced here. Uniqueness follows from the fact that locally any k form is a sum of expressions of the form $f d^{k-1}\alpha$ and existence is proved using a smooth partition of unity. Note that (ii) immediately implies

$$d^k d^{k-1} \equiv 0.$$

In particular, $\text{Im } d^{k-1} \subset \ker d^k$.

Definition 2.5.4 A k -form $\omega \in \Omega^k(M)$ will be called closed if $d^k\omega = 0$ and exact if there exists $\alpha \in \Omega^{k-1}$ with $\omega = d^{k-1}\alpha$. Denote by $Z^k(M)$ the space of closed k -forms on M and by $B^k(M)$ the space of exact ones. As we have just observed that every exact form is closed, we may then define the k -th de Rham cohomology group of M to be

$$H_{dR}^k(M) := Z^k(M)/B^k(M).$$

The dimension

$$\beta_k := \dim H_{dR}^k(M)$$

is called the k -th Betti number of M .

For the benefit of the reader familiar with singular cohomology we mention that the latter is naturally isomorphic to de Rham cohomology in the case of manifolds. This identification yields the following version of the Hurewicz theorem:

Theorem 2.5.5 [HUREWICZ THEOREM] Let M be a connected manifold. Then there is an isomorphism $h : H^1(M, \mathbb{R}) \rightarrow \text{Hom}(\pi_1(M), \mathbb{R})$ given by

$$h([\omega])([\gamma]) := \int_{\mathbb{T}^1} \gamma^* \omega \quad (\omega \in Z^1(M, \mathbb{R}), \gamma : \mathbb{T}^1 \rightarrow M).$$

Note that (as $\mathbb{R}$ is abelian and thus the conjugation action is trivial) we can identify

$$H_{dR}^1(M, \mathbb{R}) \cong \text{Hom}(\pi_1(M), \mathbb{R}) \cong \text{Rep}(\pi_1(M), \mathbb{R}).$$

Thus the first de Rham cohomology group of a manifold is our first example of a representation variety. Moreover, and this is more important for us in the sequel, we can identify $H_{dR}^1(M, \mathbb{R})$ with the *tangent space of the representation variety* at some fixed basepoint. We will see later on, that in the case of a general representation variety the tangent spaces (at the smooth points) can still be identified with a twisted version of de Rham cohomology. This is the motivation for us to study de Rham cohomology at this place. Arguably the most important result on de Rham cohomology is the following lemma:

Lemma 2.5.6 [POINCARÉ LEMMA] *If U is a contractible manifold, then every closed form is exact. Thus $H_{dR}^k(U) = 0$ for all $k > 0$.*

If one has a covering of a space by contractible submanifolds (which always exists for manifolds) then one can use Poincaré's Lemma in order to reduce the computation of $H_{dR}^k(M)$ to a combinatorial counting process. This is called the Mayer-Vietoris argument and will be discussed and applied below.

The relation between the de Rham differential of a manifold and integration is described by the following generalisation of the fundamental theorem of calculus:

Theorem 2.5.7 [STOKES FORMULA] *If M is an oriented manifold with boundary ∂M then*

$$\int_{\partial M} \omega = \int_M d\omega.$$

In particular, if $\partial M = \emptyset$ then the integral of a differential form only depends on its cohomology class. We close this subsection with an auxiliary result on extendability of differential forms:

Lemma 2.5.8 [EXTENSION LEMMA] *Let M be a manifold, V, U open subsets of M with $\bar{V} \subset U$. If $\omega = d\alpha$ on U for some $\alpha \in \Omega^k(U)$ then there exists $\tilde{\alpha} \in \Omega^k(M)$ such that $\omega|_V = d\tilde{\alpha}|_V$.*

The proof uses the existence of a smooth bump function with support in U which is 1 on $\bar{V}$ and 0 outside U and is left to the reader.

2.5.2 Ribbon graphs and cohomology

In the case where S is a compact orientable surface there is another approach to cohomology which is based on ribbon graphs rather than differential forms and which is more combinatorial. To explain this approach, let S be a surface and let Γ be a filling ribbon graph for S . As S is uniquely determined by Γ , it is obvious that the de Rham cohomology of S is also uniquely determined by Γ . It is less obvious, however, that there is an explicit way to compute the cohomology of S using Γ . We will describe such a method presently. We will consider a generalisation of this method in great detail later on when we discuss twisted cohomology. Therefore we only explain the basic constructions here and do not provide any proofs. The reader is invited to specialise the more general proofs given in Section 4.1.7 below to the present case.

Denote by V, E, F the sets the vertices, edges and faces of the filling ribbon graph Γ . We define the sets of combinatorial 0-cochains, 1-cochains and 2-cochains of Γ by

$$\begin{aligned} C_\Gamma^0(S) &:= \{f : V \rightarrow \mathbb{R}\}, \\ C_\Gamma^1(S) &:= \{f : E \rightarrow \mathbb{R} \mid f(\bar{e}) = -f(e)\}, \\ C_\Gamma^2(S) &:= \{f : F \rightarrow \mathbb{R}\}. \end{aligned}$$

We then define differentials

$$\begin{aligned} d^0 : C_\Gamma^0(S) &\rightarrow C_\Gamma^1(S), (d^0 f)(e) := f(e^+) - f(e^-), \\ d^1 : C_\Gamma^1(S) &\rightarrow C_\Gamma^2(S), (d^1 \alpha)(f) := \sum_{e \in f} \alpha(e). \end{aligned}$$

Then $d^1 \circ d^0 = 0$ because given a face f every vertex occurs as often as the origin of an edge of f as it occurs as a terminus of such an edge. It is convenient to define $C_\Gamma^k(S) := \{0\}$ for $k \notin \{0, 1, 2\}$ and $d^k : C_\Gamma^k(S) \rightarrow C_\Gamma^{k+1}(S)$ as the zero map for $k \notin \{0, 1\}$. Then $d^{k+1}d^k \equiv 0$ for all $k \in \mathbb{Z}$ and thus we may define:

Definition 2.5.9 *The group $H_\Gamma^k(S) := \frac{\ker d^k}{\text{Im } d^{k-1}}$ is called the k -th cohomology group of S with respect to Γ .*

Note that by definition, $\dim H_\Gamma^k(S) < \infty$.

We now relate the cohomology groups of Γ to the de Rham cohomology groups of M : We define maps $\iota_k : \Omega^k(S) \rightarrow C^k(\Gamma)$ by

$$\begin{aligned} \iota_0 : \Omega^0(S) &\rightarrow C_\Gamma^0(S), \quad \iota_0(f)(v) := f(v); \\ \iota_1 : \Omega^1(S) &\rightarrow C_\Gamma^1(S), \quad \iota_1(\alpha)(e) := \int_e \alpha; \\ \iota_2 : \Omega^2(S) &\rightarrow C_\Gamma^2(S), \quad \iota_2(\omega)(f) := \int_f \omega. \end{aligned}$$

We set $\iota_k \equiv 0$ for $k \notin \{0, 1, 2\}$. Here we have abused notation and identified a vertex, edge or face with its image in the geometric realisation of Γ , considered as a subset of S . For $\alpha \in \Omega^k(S)$ we will write $\hat{\alpha} := \iota_k(\alpha)$. Observe:

Proposition 2.5.10 *For any $\alpha \in \Omega^k(S)$,*

$$d\hat{\alpha} = \widehat{d\alpha}.$$

In particular, $\iota_k(Z^k(M)) \subset \ker d^k$, $\iota_k(B^k(M)) \subset \text{Im } d^k$ and thus ι_k induces a map

$$\iota_k : H_{dR}^k(M) \rightarrow H_\Gamma^k(S).$$

Exercise 2.5.11 Derive 2.5.10 from Stokes' formula.

In fact one has:

Theorem 2.5.12 *The maps $\iota_k : H_{dR}^k(S) \rightarrow H_\Gamma^k(S)$ defined by Proposition 2.5.10 are isomorphisms.*

Theorem 2.5.12 has a number of consequences. An obvious one is the following:

Corollary 2.5.13 (i) *The (de Rham) cohomology groups of a compact orientable surface are finite-dimensional.*

(ii) *The groups $H_\Gamma^k(S)$ do not depend on the choice of filling Ribbon graph which is used in order to define them.*

For a manifold with finite-dimensional cohomology spaces we define the *Euler characteristic* as the alternating sum over its Betti numbers, i.e.

$$\chi(M) := \sum_{i=0}^{\dim M} (-1)^i \beta_i.$$

Let us compute the Euler characteristic for compact orientable surfaces. For this we use the following easy algebraic fact:

Lemma 2.5.14 *Let $0 \rightarrow C^0 \xrightarrow{d^0} \dots \xrightarrow{d^{n-1}} C^n \rightarrow 0$ be a finite complex of finite-dimensional vector spaces C^k , i.e. $d^{k+1}d^k = 0$. Let*

$$H^k := \frac{\ker d^k}{\operatorname{Im} d^{k-1}}$$

be the associated cohomology groups. Then

$$\sum_{k=0}^n (-1)^k \dim H^k = \sum_{k=0}^n (-1)^k \dim C^k$$

We apply this in order to compute the Euler characteristic of a closed surface:

Corollary 2.5.15 *If $S \cong S_g$ is a compact orientable surface of genus g then its Euler characteristic is given by $\chi(S) = 2 - 2g$. In particular, a compact oriented surface is uniquely determined by its Euler characteristic.*

PROOF: If Γ is a filling ribbon graph with vertices V , edges E and faces F then

$$\dim C_{\Gamma}^0(S) = |V|, \quad \dim C_{\Gamma}^1(S) = \frac{1}{2}|E|, \quad \dim C_{\Gamma}^2(S) = |F|.$$

Thus Lemma 2.5.14 implies

$$\chi(S) = \dim H_{\Gamma}^0(S) - \dim H_{\Gamma}^1(S) + \dim H_{\Gamma}^2(S) = |V| - \frac{1}{2}|E| + |F|.$$

If we take for Γ the graph Γ_g then

$$|V| = |F| = 1, \quad \frac{1}{2}|E| = 2g$$

and the result follows. Q.E.D.

From this we can finally compute the cohomology of a compact oriented surface:

Corollary 2.5.16 *If $S \cong S_g$ is a surface of genus g then*

$$H_{dR}^0(S) \cong \mathbb{R}, \quad H_{dR}^1(S) \cong \mathbb{R}^{2g}, \quad H_{dR}^2(S) \cong \mathbb{R}.$$

PROOF: A function with trivial derivative on a connected manifold is constant and therefore

$$H_{dR}^0(S) = Z_{dR}^0(S) = \{f \in C^{\infty}(S) \mid d^0 f = 0\} = \{f \in C^{\infty}(S) \mid f \equiv c, c \in \mathbb{R}\} \cong \mathbb{R}.$$

The result for $H_{dR}^1(S)$ follows from the Hurewicz theorem and our computation of the surface group. Then $\chi(S) = 2 - 2g$ forces $H_{dR}^2(S) \cong \mathbb{R}$. Q.E.D.

Application: gluing ribbon graphs

We had to perform some intellectual gymnastics in order to identify the surface filled by a ribbon graph. It is much more easy by now: we just have to compute its Euler characteristic. This can be used to solve in a simple way Exercises 2.2.8. Here is an exercise as an application

Exercise 2.5.17 Let Γ_1 be a ribbon graph filling a surface of genus g_1 . Let Γ_2 be a ribbon graph filling a surface of genus g_2 . Let Γ_3 be the ribbon graph obtained in the following way : draw a circle such that Γ_1 and Γ_2 are in the two distinct connected component of the complementary, then join this circle to Γ_1 and Γ_2 as in Picture 2.19. Prove that Γ_3 fills a surface of genus $g_1 + g_2$.

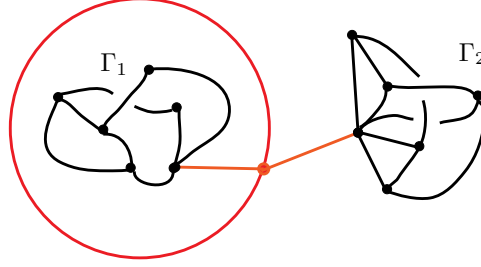


Figure 2.19: Gluing Graphs

2.5.3 The intersection form

In the next two subsections we will deal with symplectic structures on vector spaces. Recall the following basic definition:

Definition 2.5.18 *Let V be a vector space. A bilinear map $V \times V \rightarrow \mathbb{R}$ is called a linear symplectic form if it is alternating and non-degenerate.*

In this subsection we construct a symplectic form on $H_{dR}^1(S)$. As explained before, we view this cohomology group as the tangent space of the representation variety $\text{Rep}(\pi_1(S), \mathbb{R})$ at a base point. Therefore constructing a symplectic form on $H_{dR}^1(S)$ is the same as constructing a fibrewise symplectic form on the tangent space of $\text{Rep}(\pi_1(S), \mathbb{R})$. We will later on generalise this procedure to more general (and in particular, non-abelian) coefficient groups. Again the proofs will be given later on in the more general setting of Section 4.1. Here we will only explain the ingredients for the proof of the following theorem:

Theorem 2.5.19 *Let S be a surface. Then the map*

$$i : H_{dR}^1(S) \times H_{dR}^1(S) \rightarrow \mathbb{R}, \quad ([\alpha], [\beta]) \mapsto \int_S \alpha \wedge \beta$$

is a symplectic form.

The map i is called the *intersection form* of S . It is clear from the definition that i is alternating and bilinear. It is less clear, whether i is non-degenerate. To prove non-degeneracy we will use once again ribbon graphs and Theorem 2.5.12. Thus let S be a surface with filling ribbon graph Γ , and let Γ^* be its dual. The latter is again a ribbon graph and it is defined as follows: Denote by

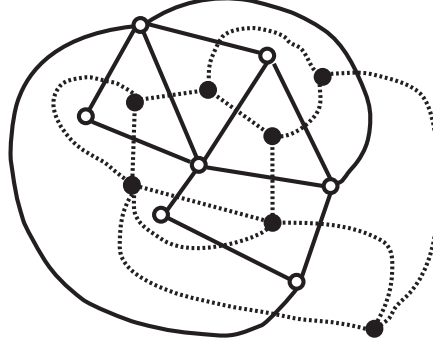


Figure 2.20: A dual graph

$V_\Gamma, E_\Gamma, F_\Gamma$ the vertices, edges and faces of Γ . We define $V_{\Gamma^*} := F_\Gamma$, $E_{\Gamma^*} := E_\Gamma$. If $(e, \bar{e})$ is a geometric edge in Γ then e is contained in a unique face $f \in F_\Gamma = V_{\Gamma^*}$ and $\bar{e}$ is contained in a unique face $\bar{f} \in V_{\Gamma^*}$. We then define

$$\varphi^* : E_{\Gamma^*} \rightarrow V_{\Gamma^*} \times V_{\Gamma^*}, \quad \varphi^*(e) := (f, \bar{f}).$$

Then $(V_{\Gamma^*}, E_{\Gamma^*}, \varphi^*)$ is a directed graph and the involution $e \leftrightarrow \bar{e}$ turn Γ^* into a graph. Geometrically Γ^* is obtained by drawing edges between adjoint faces as in Figure 2.20.

Exercise 2.5.20 Show that F_{Γ^*} and E_Γ are in canonical bijection.

Definition 2.5.21 Denote by $C_\Gamma^k, C_{\Gamma^*}^k$ the combinatorial cochains of Γ and Γ^* respectively. Then the dual pairing

$$\langle \cdot | \cdot \rangle : C_\Gamma^k \times C_{\Gamma^*}^{2-k} \rightarrow \mathbb{R}$$

is defined by

$$\langle \alpha | \beta \rangle := \sum_{v \in V_\Gamma = F_{\Gamma^*}} \alpha(v) \beta(v) \quad (\alpha \in C_\Gamma^0, \beta \in C_{\Gamma^*}^2);$$

$$\langle \alpha | \beta \rangle := \sum_{e \in E_\Gamma = E_{\Gamma^*}} \alpha(e) \beta(e) \quad (\alpha \in C_\Gamma^1, \beta \in C_{\Gamma^*}^1);$$

$$\langle \alpha | \beta \rangle := \sum_{f \in F_\Gamma = V_{\Gamma^*}} \alpha(f) \beta(f) \quad (\alpha \in C_\Gamma^2, \beta \in C_{\Gamma^*}^0)$$

and $\langle \cdot | \cdot \rangle \equiv 0$ for $k \notin \{0, 1, 2\}$.

Proposition 2.5.22 (i) *The dual pairing $\langle \cdot | \cdot \rangle : C_\Gamma^k \times C_{\Gamma^*}^{2-k} \rightarrow \mathbb{R}$ satisfies*

$$\langle d\alpha | \beta \rangle = \langle \alpha | d\beta \rangle \quad (\alpha \in C_\Gamma^k, \beta \in C_{\Gamma^*}^{1-k}).$$

(ii) *The dual pairing descends to a pairing*

$$\langle \cdot | \cdot \rangle : H_\Gamma^k \times H_{\Gamma^*}^{2-k} \rightarrow \mathbb{R}.$$

(iii) *The dual pairing is non-degenerate on $H_\Gamma^k \times H_{\Gamma^*}^{2-k}$ for $k \in \{0, 1, 2\}$.*

Recall the definition of the map $\iota_1 : \Omega^1(S) \rightarrow C_\Gamma^1$ from the last section. As $E_\Gamma = E_{\Gamma^*}$ and the involution on E_Γ and E_{Γ^*} agree we have $C_\Gamma^1 \cong C_{\Gamma^*}^1$ and thus we have also an isomorphism $\iota_1^* : \Omega^1(S) \rightarrow C_{\Gamma^*}^1$. We can use these maps to relate the intersection form on $\Omega^1(S)$ to the dual pairing. More precisely, we have the following result, which together with Proposition 2.5.22 implies Theorem 2.5.19:

Lemma 2.5.23 *Let $\alpha, \beta \in \Omega^1(S)$. Then*

$$\int_S \alpha \wedge \beta = \langle \iota_1(\alpha) | \iota_1^*(\beta) \rangle.$$

We prove later on a more generale that contains this lemma as a special case.

Chapter 3

Vector bundles and connections

3.1 Vector bundles

In order to proceed further in our study of surfaces we will need some basic facts and notions from differential geometry. This subsection is intended to provide a very brief crash course on the concepts which we shall need. More detailed expositions can be found in any good book on differential geometry. We start with the notion of a vector bundle. Informally, a vector bundle is a continuous family of vector spaces indexed by a topological space. The following definition makes this precise:

3.1.1 Definitions

A *vector bundle* is a triple $(\mathcal{L}, \pi, M)$, where $\mathcal{L}$ and M are topological spaces called the *total space* and the *base spaces* respectively and $\pi : \mathcal{L} \rightarrow M$ is a continuous map satisfying the following two conditions:

- (i) For all $m \in M$ the *fibre* $\mathcal{L}_m := \pi^{-1}(m)$ is a finite dimensional vector space isomorphic to a given vector space L .
- (ii) For all $m \in M$ there exists a neighbourhood U_m of m and a homeomorphism

$$\varphi : \pi^{-1}(U_m) \rightarrow U_m \times L,$$

which restricts to a linear isomorphism $\varphi : \mathcal{L}_{m'} \rightarrow \{m'\} \times L$ for any $m' \in U_m$.

The dimension $\dim L$ is called the *rank* of the vector bundle and denoted $\text{rk } \mathcal{L}$. If $\text{rk } \mathcal{L} = 1$ then $\mathcal{L}$ is called a *line bundle*. An open subset with the property in (ii) is called *trivialising*. By abuse of language we will simply call $\mathcal{L}$ a vector

bundle (over M). A *section* of $(\mathcal{L}, \pi, M)$ is a continuous map $\sigma : M \rightarrow \mathcal{L}$ such that $\sigma(m) \in \mathcal{L}_m$. We denote by $\Gamma(\mathcal{L})$ the space of sections of $(\mathcal{L}, \pi, M)$. If $\mathcal{L}$ and $\mathcal{L}'$ are vector bundles over the space base space M then a *morphism* $\psi : \mathcal{L} \rightarrow \mathcal{L}'$ is a fibre-preserving fibrewise linear map, i.e. the diagram

$$\begin{array}{ccc} \mathcal{L} & \rightarrow & \mathcal{L}' \\ \downarrow & & \downarrow \\ M & \xrightarrow{id} & M \end{array}$$

commutes and $\psi : \mathcal{L}_m \rightarrow \mathcal{L}'_m$ is linear. The group of automorphisms of $\mathcal{L}$ is called the *gauge group* and denoted $\mathcal{G}(\mathcal{L})$.

A more general definitions of vector bundles would allow the vector space L to jump from one connected component of B to the other. We will mostly be interested in connected base spaces and therefore we have no need for such a general definition here. An advantage of our definition is the fact that isomorphism classes of vector bundles in our sense can be described in a uniform combinatorial way. This description provided by the following definition is very convenient for performing constructions and computations with vector bundles.

3.1.2 A combinatorial interpretation through cocycles

Definition 3.1.1 *let M be a topological space. A cocycle over M with values in a vector space L is a pair $(\{U_i\}_{i \in I}, \{g_{ij}\}_{i,j \in I})$, where $\{U_i\}_{i \in I}$ is an open covering of M and $\{g_{ij}\}_{i,j \in I}$ is a collection of continuous maps $g_{ij} : U_i \cap U_j \rightarrow GL(L)$ subject to the cocycle condition*

$$g_{ij}g_{jk} = g_{ik}.$$

Two cocycles $(\{U_i\}_{i \in I}, \{g_{ij}\}_{i,j \in I})$, $(\{U_i\}_{i \in I}, \{g'_{ij}\}_{i,j \in I})$ with the same open covering are called directly equivalent if there exists a family of maps $h_i : U_i \rightarrow GL(L)$ such that

$$g'_{ij} = h_i g_{ij} h_j^{-1}.$$

A cocycle $(\{U'_j\}_{j \in J}, \{g_{ij}\}_{i,j \in J})$ is called a refinement of the cocycle $(\{U_i\}_{i \in I}, \{g_{ij}\}_{i,j \in I})$ if for each U'_j there exists $i(j) \in I$ with $U'_j \subset U_{i(j)}$ and

$$g_{j_1 j_2} = g_{i(j_1) i(j_2)}|_{U_{j_1} \cap U_{j_2}}.$$

Two cocycles will be called equivalent if they have directly equivalent refinements .

We claim that vector bundles over a topological space M up to isomorphism are in one-to-one correspondence with cocycles up to equivalence. Let us explain how the correspondence is obtained: Given a cocycle $(\{U_i\}_{i \in I}, \{g_{ij}\}_{i,j \in I})$, the cocycle condition ensures that we obtain an equivalence relation $\sim$ on the space

$$\coprod_{i \in I} \{i\} \times U_i \times L$$

by

$$(i, x, v) \sim (j, x, g_{ij}(x).v).$$

Then

$$\mathcal{L} := \left(\coprod_{i \in I} \{i\} \times U_i \times L \right) / \sim$$

is the total space of a vector bundle over M , where the projection is given by

$$\pi : \mathcal{L} \rightarrow M, \quad [(i, x, v)] \mapsto x.$$

Conversely, given a vector bundle $(\mathcal{L}, \pi, M)$ we may choose a trivialising covering $\{U_i\}_{i \in I}$ and trivialisations

$$\varphi_i : \pi^{-1}(U_i) \rightarrow U_i \times L.$$

Then we can trivialise the bundle over $U_i \cap U_j$ in two different ways, namely via

$$\varphi_{ij} : \pi^{-1}(U_i \cap U_j) \hookrightarrow \pi^{-1}(U_i) \xrightarrow{\varphi_i} U_i \times L$$

or

$$\varphi_{ji} : \pi^{-1}(U_i \cap U_j) \hookrightarrow \pi^{-1}(U_j) \xrightarrow{\varphi_j} U_j \times L.$$

Both φ_{ij} and φ_{ji} are fibre-preserving fibrewise linear homeomorphisms onto $(U_i \cap U_j) \times L$ and we can define the cocycle g_{ij} by

$$\varphi_{ji} \circ \varphi_{ij}^{-1} : (U_i \cap U_j) \times L \rightarrow (U_i \cap U_j) \times L, \quad (m, v) \mapsto (m, g_{ij}(m)v).$$

Exercise 3.1.2 Show that the constructions described above give indeed rise to a correspondence between isomorphism classes of vector bundles over M and equivalence classes of cocycles.

Example 3.1.3 1. For any topological space M there is the trivial bundle of rank n , $M \times \mathbb{R}^n$.

2. These may be trivial or non-trivial. The Möbius strip (see Figure 2.1) is a non-trivial line bundle.

3. If M is a manifold, then the tangent bundle TM and the cotangent bundle T^*M are vector bundles. over the circle.

Constructions 3.1.4 1. All algebraic operations that can be performed with cocycles can be performed with vector bundles. As an example, we consider direct sums: Let $\mathcal{L}, \mathcal{F}$ be vector bundles over a manifold M . Then over some open covering $(U_i)_{i \in I}$ the bundles are represented by cocycles g_{ij} and h_{ij} respectively. Then we define a bundle $\mathcal{L} \oplus \mathcal{F}$ with fibre $(\mathcal{L} \oplus \mathcal{F})_m \cong \mathcal{L}_m \oplus \mathcal{F}_m$ via the block-matrix cocycle

$$g_{ij} \oplus h_{ij} := \begin{pmatrix} g_{ij} & 0 \\ 0 & h_{ij} \end{pmatrix}.$$

Similarly we may construct $\mathcal{L} \otimes \mathcal{F}$, $\bigwedge^k \mathcal{L}$, $\text{Hom}(\mathcal{L}, \mathcal{F})$, $\mathcal{L}^*$ etc.

2. The algebraic constructions described in (i) can be used to produce bundles over a fixed manifold M . There is also a construction which relates bundles over different manifolds. For this we consider a vector bundle $\pi : \mathcal{L} \rightarrow M$ with associated cocycle $(\{U_i\}_{i \in I}, \{g_{ij}\}_{i,j \in I})$ and a continuous map $f : N \rightarrow M$, where N is any topological space. Then the *pullback* of $\mathcal{L}$ via f is the bundle

$$f^*\mathcal{L} := \{(e, n) \in \mathcal{L} \times N \mid \pi(e) = f(n)\} \rightarrow N, \quad (e, n) \mapsto n.$$

This is a vector bundle with fibre $(f^*\mathcal{L})_n = \mathcal{L}_{f(n)}$ and cocycle $(\{f^{-1}(U_i)\}_{i \in I}, \{g_{ij} \circ f\}_{i,j \in I})$.

The following fact will be important for us, although we cannot prove it here, as the proof requires again some tools from Riemannian geometry:

Proposition 3.1.5 *Suppose that $f_0, f_1 : N \rightarrow M$ are homotopic maps and that $\mathcal{L}$ is a vector bundle over M . Then $f_0^*(\mathcal{L}) \cong f_1^*(\mathcal{L})$. In particular, any vector bundle over a contractible space is trivial.*

We will often need to perform operations like derivatives along a curve. A convenient way to express such operations is given by the following definition:

Definition 3.1.6 *Let $\gamma : [0, 1] \rightarrow M$ be a curve and $\mathcal{L} \rightarrow M$ be a vector bundle over M . A section along the curve γ is a map $v : [0, 1] \rightarrow \mathcal{L}$ with $\pi(v(t)) = \gamma(t)$. Equivalently, $v \in \Gamma(\gamma^*\mathcal{L})$.*

3.2 Vector bundles with structures

So far we have considered cocycles with values in $GL(L)$, where L is a vector space. These cocycles preserve the linear structure of the vector space L . However, we might also want to preserve some additional structure on L like an inner product or a symplectic form or some other tensor. This amounts to restricting the range of our cocycles. We formalise this as follows:

Let G be a topological group and $\rho : G \rightarrow GL(L)$ a continuous injective homomorphism – in other words a faithful linear *representation* of G .

Definition 3.2.1 [G -STRUCTURE] *A G -trivialisation of the vector bundle E over M is an open cover $\{U_i\}$ of M together with a trivialisation of $E|_{U_i}$ as $U_i \times L$ so that the changes of trivialisations are maps with values in G . Two G -trivialisations are equivalent if their union is still a G -trivialisation. Finally a G -structure is a class of G -trivialisations up to the equivalence above.*

In terms of cocycles, we obtain G -structures from G -valued cocycles which are cocycles $(\{U_i\}_{i \in I}, \{g_{ij}\}_{i,j \in I})$, so that g_{ij} are with values in G .

We also observe that a G -structure defines for every m in M a subgroup G_m of $GL(\mathcal{L}_m)$ isomorphic – although not canonically – to G . More generally, a G -structure defines for every m and n a subset $G_{m,n}$ of $\text{Hom}(\mathcal{L}_m, \mathcal{L}_n)$ equipped with a right transitive action of G_n and a left transitive action of G_m , so that $G_{m,p} = G_{m,n} \cdot G_{n,p}$ and $G_{m,m} = G_m$.

We need to extend the previous definitions

Definition 3.2.2 [G -MORPHISMS] *If we are given two vectors bundles $E \rightarrow M$ and $F \rightarrow N$ with a G -structure. A G -morphism is a map φ from E to F , which is an isomorphism on each fibre, such that the pull back of a G -trivialisation of F is a G -trivialisation of E .*

Definition 3.2.3 [GAUGE GROUP] *The gauge group $\mathcal{G}(E)$ of the G -structure is the set of G -automorphisms of the bundle. It can be identified with the group of sections $\sigma : M \rightarrow \text{End}(\mathcal{L})$ so that for every x $\sigma(x) \in G_x$.*

If $E = M \times L$ is a trivial bundle, then $\mathcal{G}(E) = C^0(M, G)$.

Example 3.2.4 Let B be a bilinear form on a vector space L . Denote by $\text{SO}(L, B)$ the group of linear automorphisms of L that preserve B . Then a $\text{SO}(L, B)$ -structure on a vector bundle $\mathcal{L}$ is the same as a continuously varying families of tensors of the same type as B in each fibre. Indeed, given $v, w \in \mathcal{L}_m$ we can compute $B(v, w)$ in a trivialisation, and as the coordinate changes are orthogonal with respect to B this quantity is independent of the trivialisation chosen.

Remark 3.2.5 Given a fibrewise tensor on a vector bundle as in the last example we can define a subgroup of $\text{SO}(\mathcal{L}_x) \subset \text{GL}(\mathcal{L}_x)$ for every $x \in M$ by requiring that the elements of $\text{SO}(\mathcal{L}_x)$ leave this tensor invariant. This group is isomorphic to $\text{SO}(L, B)$, but not canonically so. More general, given a G -structure on a vector bundle $\mathcal{L}_x$ we find a subgroup $G_x \subset \text{GL}(\mathcal{L}_x)$ for all $x \in M$ which is (non-canonically) isomorphic to G .

All we have said so far applies to topological spaces, topological groups and continuous maps. One can make the same theory work for smooth spaces, smooth groups and smooth maps. By smooth spaces we just mean manifolds and a smooth group is a group which is also a manifold in such a way that group multiplication and inversion are smooth maps. These groups are called *Lie groups*. Smooth maps are then defined by pretending charts to be smooth as we explained for surfaces. Then we call vector bundle smooth if the defining cocycles can be taken to be smooth and for such a bundle one can consider smooth sections. Pullbacks of smooth bundles via smooth maps are smooth and algebraic constructions do not lead out of the category of smooth bundles. In the following we will almost exclusively deal with smooth bundles. Therefore we will make the following conventions:

Convention 3.2.6 All vector bundles, cocycles, G -structures, ... are smooth unless otherwise mentioned and we denote by $\Gamma(\mathcal{L})$ the smooth sections of the smooth vector bundle $\mathcal{L}$.

For manifolds there are some bundles which are of particular importance. In the first place these are the tangent bundle TM and the cotangent bundle $T^*M := (TM)^*$. Via the algebraic constructions described in 3.1.4 we can construct new bundles out of these the bundles. We have encountered the bundles

$$\bigwedge^k T^*M \quad (k \geq 0)$$

before:

Exercise 3.2.7 Show that for $k \geq 0$ there is a canonical isomorphism $\Omega^k(M) \cong \Gamma(\bigwedge^k T^*M)$, i.e. a section of $\bigwedge^k T^*M$ is nothing but a k -form on M .

More generally, if $\mathcal{L}$ is a vector bundle over M , we denote by $\Omega^k(M, \mathcal{L})$ the vector space of *differential k -forms with values in $\mathcal{L}$* that is the space of sections of the bundle $\bigwedge^k(T^*M) \otimes \mathcal{L}$. By convention, $\bigwedge^0(T^*M) \otimes \mathcal{L} = \Gamma(\mathcal{L})$

3.3 Connections

Let us start with an informal explanation of the basic idea of connections: if $\pi : \mathcal{L} \rightarrow M$ is a vector bundle and m, m' two points in M then the fibres $\mathcal{L}_m$ and $\mathcal{L}_{m'}$ are isomorphic. However, there is no way to construct a natural isomorphism between different fibres. A connection is a procedure with associates to any path $\gamma : [0, 1] \rightarrow M$ from m to m' an isomorphism

$$\text{Hol}(\gamma) : \mathcal{L}_m \rightarrow \mathcal{L}_{m'},$$

called the holonomy of γ . This is done by lifting curves in M to curves $\mathcal{L}$. More precisely, given a curve $\gamma : [0, 1] \rightarrow M$ and a starting point $v_0 \in \mathcal{L}_{\gamma_0}$ a connection constructs for us a section $v(t)$ along γ with $v(0) = v_0$. Then the holonomy of γ is the map which maps the starting point v_0 to the endpoint $v(1)$ of v . If $M = \mathbb{R}^n$ and $\mathcal{L} = \mathbb{R}^n \times \mathbb{R}^n$ is the tangent bundle then there is an obvious connection: Given $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ and $(\gamma(0), v) \in \mathcal{L}$ we can define a constant lift $v(t) = (\gamma(t), v)$. This constant lift is uniquely determined by the property that its derivative in direction of the curve vanishes. Thus in order to define a connection we need to define a derivative along a curve. This is the idea of the following definition.

3.3.1 Koszul connection

A (Koszul) connection is a linear map

$$\nabla : \Gamma(\mathcal{L}) \rightarrow \Omega^1(M, \mathcal{L}), \quad \sigma \mapsto (X \mapsto \nabla_X \sigma)$$

such that $\nabla_X \sigma \in \mathcal{L}_m$ for $X \in T_m M$ and for all function f in $C^\infty(M)$ we have

$$\nabla_X(f\sigma) = df(X) \cdot \sigma + f\nabla_X \sigma.$$

Observe that by definition $\nabla_{fX}\sigma = f\nabla_X(\sigma)$.

Alternatively, we can think of a Koszul connection as a bilinear map from $\Gamma(TM) \times \Gamma(\mathcal{L})$ to $\Gamma(\mathcal{L})$ which associates vector field X on M and a section σ of $\mathcal{L}$, a new section $\nabla_X \sigma$ of $\mathcal{L}$ such that for all function f in $C^\infty(M)$, we have

$$\begin{aligned} \nabla_X f\sigma &= f\nabla_X(\sigma) + df(X) \cdot \sigma \\ \nabla_{fX}\sigma &= f\nabla_X(\sigma). \end{aligned}$$

Observe that by definition, for all function f on M , we have

$$\nabla_{fX}(f\sigma) = f\nabla_X \sigma$$

Convention 3.3.1 By definition we may plug a tangent vector $X \in T_m M$ and a section σ of $\mathcal{L}$ in a connection in order to obtain an element $\nabla_X \sigma$ of $\mathcal{L}_m$. But we might as well take a vector field $X \in \Gamma(TM)$ and a section σ of $\mathcal{L}$ in order to produce a section s of $\mathcal{L}$ according to the rule

$$s_m := \nabla_{X(m)} \sigma.$$

We will use the abbreviation

$$\nabla_X \sigma := s$$

for this construction. Thus, by definition,

$$(\nabla_X \sigma)(m) := \nabla_{X(m)} \sigma.$$

The motivating example for the notion of Koszul connection is the following one:

Example 3.3.2 Let $\mathcal{L} = M \times L$ be a trivial bundle. Then $\Gamma(M \times L) \cong C^\infty(M, L)$. In this case there is a preferred connection of $\mathcal{L}$ which we will always denote by D and which we call the trivial connection. It is given by directional derivative, i.e.

$$D_X(\sigma) = D\sigma(X).$$

In particular we see that every trivial bundle admits a connection. This can be used in order to derive:

Proposition 3.3.3 *Every vector bundle admits a connection.*

PROOF: If $f_1, \dots, f_n$ are functions on M such that $\sum f_i = 1$ and ∇_i are connections on M then

$$\nabla = \sum_i f_i \nabla_i$$

is also a connection. Thus we may take a trivialising cover for $\mathcal{L}$ over M , a corresponding partition of unity and combine trivial connections on the trivialising open sets using the partition of unity. Q.E.D.

Having proved existence of connections let us now turn to uniqueness. There will be many connections on a given vector bundle $\mathcal{L}$, but knowing one of them means already knowing all of them. To make this precise we introduce the following notion:

Definition 3.3.4 Let $\mathcal{L}, \mathcal{F}$ be vector bundles over the same manifold M . A map $A : \Gamma(\mathcal{L}) \rightarrow \Gamma(\mathcal{F})$ is called a tensor field if it is a morphism of $C^\infty(M)$ -algebras. With other words, $A \in \Gamma(\mathcal{L})^* \otimes \Gamma(\mathcal{F})$ is a tensor field if it is in the image of the map

$$\Gamma(\mathcal{L}^* \otimes \mathcal{F}) \rightarrow \Gamma(\mathcal{L})^* \otimes \Gamma(\mathcal{F}).$$

Then one has:

Proposition 3.3.5 If ∇^1, ∇^2 are connections on $\mathcal{L}$ then there exists a tensor field $A \in \Omega^1(M, \text{End}(\mathcal{L}))$ such that for all $m \in M$, $X \in T_m(M)$, $\sigma \in \Gamma(\mathcal{L})$

$$\nabla_X^1 \sigma - \nabla_X^2 \sigma = A_m(X)\sigma.$$

Conversely, given a connection ∇ and a tensor $A \in \Omega^1(M, \text{End}(\mathcal{L}))$ we obtain a new connection by $\nabla + A$. In particular, the connections on $\mathcal{L}$ form an affine space modelled on $\Omega^1(M, \text{End}(\mathcal{L}))$.

The algebraic constructions that we performed above for vector bundles generalise to bundles with connection. We explain some of these constructions here and leave the remaining constructions to the reader.

Constructions 3.3.6 (i) Suppose $\nabla^{\mathcal{L}}$ and $\nabla^{\mathcal{F}}$ are connections on bundles $\mathcal{L}$ and $\mathcal{F}$ over the same manifold M . Then there is a connection $\nabla = \nabla^{\mathcal{L}} \otimes \nabla^{\mathcal{F}}$ on $\mathcal{L} \otimes \mathcal{F}$ which for $X \in T_m M$ and a product section $\sigma = \sigma_{\mathcal{L}} \otimes \sigma_{\mathcal{F}}$ of $\mathcal{L} \otimes \mathcal{F}$ is given by

$$\nabla_X \sigma := \nabla_X^{\mathcal{L}} \sigma_{\mathcal{L}} \otimes \sigma_{\mathcal{F}}(m) + \sigma_{\mathcal{L}}(m) \otimes \nabla_X^{\mathcal{F}} \sigma_{\mathcal{F}}.$$

(ii) If ∇ is a connection on $\mathcal{L}$ then there is a unique connection ∇^* on the dual bundle $\mathcal{L}^*$ such that for all $X \in \Gamma(TM)$, $v \in \Gamma(\mathcal{L})$ and $w \in \Gamma(\mathcal{L}^*)$,

$$X \langle v | w \rangle = \langle \nabla_X v | w \rangle + \langle v | \nabla_X^* w \rangle,$$

where $\langle \cdot | \cdot \rangle : \Gamma(\mathcal{L}) \times \Gamma(\mathcal{L}^*) \rightarrow C^\infty(M)$ denotes the dual pairing and X acts as a differential operator on the smooth function $\langle v | w \rangle$.

(iii) If ∇ is a connection on a bundle $\mathcal{L}$ over M and $f : N \rightarrow M$ is a smooth map between manifolds then there is a unique connection $f^* \nabla$ on $f^* \mathcal{L}$ satisfying

$$(f^* \nabla)_X (f^* \sigma) = f^* (\nabla_{Tf(X)} \sigma)$$

for all $\sigma \in \Gamma(\mathcal{L})$ and $X \in \Gamma(TM)$.

3.3.2 Holonomy along a path

We have motivated the introduction of connections by considering the problem of lifting a curve. Now let us see that the formal machinery introduced in the last subsection solves the curve lifting problem:

Definition 3.3.7 [DERIVATION ALONG A CURVE] *Let $\mathcal{L}$ be a vector bundle over M and let ∇ be a connection on $\mathcal{L}$. Let $v(t)$ be section along a curve $\gamma : [0, 1] \rightarrow M$. Then the derivative of $v(t)$ along $\gamma(t)$ is the section along $\gamma(t)$ given by*

$$\nabla_{\dot{\gamma}} v := (\gamma^* \nabla)_{\partial_t} v,$$

where ∂_t is the fundamental vector field of the curve γ and v is regarded as a section of $\gamma^*(\mathcal{L})$.

Definition 3.3.8 [PARALLEL SECTION ALONG A CURVE] *Let $\mathcal{L}$ be a vector bundle over M and let ∇ be a connection on $\mathcal{L}$. A section $v(t)$ along a curve $\gamma : [0, 1] \rightarrow M$ is called ∇ -parallel along the curve γ if*

$$\nabla_{\dot{\gamma}} v = 0.$$

Now we can prove that a connection is the same as a way to lift curves:

Proposition 3.3.9 *Let $\pi : \mathcal{L} \rightarrow M$ be a vector bundle with connection ∇ . Let $\gamma : [0, 1] \rightarrow M$ a curve and $v_0 \in \mathcal{L}_{\gamma(0)}$. Then there exists a unique parallel section $v = v_{v_0}$ along γ such that $v(0) = v_0$. Thus ∇ defines a linear map*

$$\text{Hol}^\nabla(\gamma) : \begin{cases} \mathcal{L}_{\gamma(0)} & \rightarrow \mathcal{L}_{\gamma(1)} \\ v_0 & \mapsto v_{v_0}(1). \end{cases}$$

PROOF: As $[0, 1]$ is contractible, the bundle $\gamma^*\mathcal{L}$ is trivial. We fix a trivialisation and identify $\gamma^*\mathcal{L} \cong [0, 1] \times L$. By Proposition 3.3.5 the connection $\gamma^*\nabla$ takes the form $\gamma^*\nabla = D + A$ under this identification, where D denotes the trivial connection given by directional derivative and A is some tensor. Then $v \in \Gamma(\mathcal{L})$ is parallel if and only if

$$\dot{v} = B \cdot v,$$

where $B = -A(\partial_t)$. Thus lifting γ with initial point $v_0 = (\gamma(0), \tilde{v}_0)$ is the same as solving the initial value problem

$$\dot{v} = B \cdot v, \quad v(0) = \tilde{v}_0,$$

and the proposition follows from the existence and uniqueness theorem for ordinary differential equations. Q.E.D.

Definition 3.3.10 [HOLONOMY ALONG A PATH] *The map $\text{Hol}^\nabla(\gamma)$ from $\mathcal{L}_{\gamma(0)}$ to $\mathcal{L}_{\gamma(1)}$ is called the holonomy of ∇ along γ .*

Example 3.3.11 Let us consider once more the case of the trivial vector bundle $\mathcal{L} = M \times L$ over M with fibre L . As usual denote by D the trivial connection on $\mathcal{L}$. If $\gamma : [0, 1] \rightarrow M$ is any path then the constant section $\gamma_v \in \Gamma(\gamma^*\mathcal{L})$ given by $\gamma_v(t) := (\gamma(t), v)$ is parallel. Thus γ_v is the unique parallel lift of γ with initial value v and hence $\text{Hol}^D(\gamma)(v) = \gamma_v(1) = (\gamma(1), v)$. Thus identifying each fibre $\{x\} \times L$ with L via $(x, v) \rightarrow v$ we see that $\text{Hol}^D(\gamma)$ induces the identity between the two copies of L sitting over $\gamma(0)$ and $\gamma(1)$.

It is easy to see that the connection can be reconstructed from the associated holonomy maps, moreover parallel sections can be described using the holonomy. We leave this as an exercises:

Exercise 3.3.12 Let ∇, ∇' be connections on a vector bundle $\mathcal{L} \rightarrow M$. Suppose that for all $\gamma : [0, 1] \rightarrow M$ the maps $\text{Hol}^\nabla(\gamma)$ and $\text{Hol}^{\nabla'}(\gamma)$ coincide. Show that $\nabla = \nabla'$.

Exercise 3.3.13 Let σ be a section of $\mathcal{L}$. Then σ is parallel if and only if for every path γ from x to y , we have $\sigma(y) = \text{Hol}_\nabla(\gamma) \cdot \sigma(x)$.

3.3.3 Holonomy and linear differential equations

Parallel section and holonomies have interpretations using differential equations. We summarise this in the following Proposition whose proof is left to the reader.

Proposition 3.3.14 *Let $\gamma : [a, b] \rightarrow M$ be a path. We choose a trivialisation of $\mathcal{L}$ along γ as $[a, b] \times \mathcal{L}_0$ so that $\nabla = D + A$ where D is the trivial connection and $A \in \Omega^1([a, b]) \otimes \mathcal{L}_0$. Let $B = A(\partial_t)$. A section σ along γ is now interpreted as a map $X : [a, b] \rightarrow \mathcal{L}_0$. Then σ is parallel along γ if and only if X satisfies*

$$(*) \quad \frac{d}{dt}X(t) = B(t)X(t).$$

In particular the holonomy from $\gamma(u)$ to $\gamma(s)$ along γ is the resolvent of this equation.

3.3.4 Curvature

One should notice that a connection does *not* produce just one canonical isomorphism between different fibres, but rather one isomorphism for each curve γ in M joining the two base points. Different curves may lead to different holonomy maps. This may happen even if the two curves are homotopic relative their endpoints. We, however, will mainly be interested in connections whose holonomy depends only on the homotopy class (relative endpoints) of a curve. In order to characterise these connections we introduce the following notion:

Definition 3.3.15 *Let ∇ be a connection on a vector bundle $\pi : \mathcal{L} \rightarrow M$. Then the curvature of ∇ is*

$$R^\nabla : TM \times TM \rightarrow \text{End}(\mathcal{L})$$

which for vector fields $X, Y \in \Gamma(TM)$ and a section $\sigma \in \Gamma(\mathcal{L})$ is given by

$$R^\nabla(X, Y)\sigma := \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X, Y]}\sigma.$$

The connection is called flat if and only if R^∇ vanishes identically.

Exercise 3.3.16 Show that R^∇ defines a tensor field $R^\nabla \in \Omega^2(M, \text{End}(\mathcal{L}))$.

Example 3.3.17 (i) If D is the trivial connection of $\mathcal{L} = M \times L$ then $R^D = 0$ by Schwartz' Lemma and thus D is flat.

(ii) Suppose $\mathcal{L} = M \times R$ is the trivial line bundle over M . Then any connection on $\mathcal{L}$ is of the form $\nabla^\alpha = D + \alpha$, where α is a 1-form on M and

$$R^{\nabla^\alpha} = d\alpha.$$

Thus $D + \alpha$ is flat if and only if α is closed.

3.3.5 Flat connections

It turns out the curvature measures if the holonomy is invariant under homotopies:

Theorem 3.3.18 *Let ∇ be a connection on a vector bundle $\pi : \mathcal{L} \rightarrow M$. Then the following are equivalent:*

(i) ∇ is flat, i.e. $R^\nabla \equiv 0$.

(ii) If $\gamma, \gamma' : [0, 1] \rightarrow M$ with $\gamma(0) = \gamma'(0)$, $\gamma(1) = \gamma'(1)$ and $\gamma \simeq \gamma' \pmod{\{0, 1\}}$ then

$$\text{Hol}^\nabla(\gamma) = \text{Hol}^\nabla(\gamma').$$

(iii) The bundle $\mathcal{L}$ is trivial over every simply-connected open subset U of M and for any such U we may choose a trivialisation $i_U : \mathcal{L}|_U \xrightarrow{\cong} U \times L$ such that $\nabla|_U \cong i_U^* D_U$, where D_U is the trivial connection on $U \times L$.

PROOF: The obvious implication is (iii) $\Rightarrow$ (i), which follows from the fact that the trivial connection is flat. Let us show that (ii) implies (iii): Given a simply-connected open subset U of M fix a point $x_0 \in U$ and join any point $x \in U$ with x_0 via a path γ_x contained in U . Then the map $\text{Hol}(x) := \text{Hol}^\nabla(\gamma_x) : \mathcal{L}_x \rightarrow \mathcal{L}_{x_0}$ depends only on x but not on the choice of γ_x because any two paths in U joining x and x_0 are homotopic relative endpoints (even in U). Then the map

$$i : \mathcal{L}|_U \cong U \times \mathcal{L}_{x_0}, \quad v \mapsto (\pi(v), \text{Hol}(\pi(v))(v))$$

is a trivialisation with trivial holonomy and thus ∇ is trivial on U by Exercise 3.3.12. Finally, the hard part of the proof is to show that (i) $\Rightarrow$ (ii). For this we refer to the literature. Q.E.D.

Note that there is also an easy direct argument for the implication (iii) $\Rightarrow$ (ii). Namely, the holonomy of the trivial connection does not depend on a chosen path. A special case of the last theorem is given by the following corollary:

Corollary 3.3.19 *Let ∇ be a flat connection on a vector bundle $\pi : \mathcal{L} \rightarrow M$ and $x_0 \in M$. Then ∇ defines a representation*

$$\pi_1(M, x_0) \rightarrow GL(\mathcal{L}_{x_0}), \quad [\gamma] \mapsto \text{Hol}^\nabla(\gamma).$$

3.3.6 G -connections

When the bundle is equipped with a G -structure, it makes sense to speak of a G -connection. Let G be a subgroup of $GL(L)$. Let $\mathfrak{g}$ be the Lie algebra of G considered as a vector subspace of $\text{End}(L)$.

Definition 3.3.20 *Let M be a manifold and $\mathcal{L} \rightarrow M$ be a vector bundle equipped with a G -structure. A connection ∇ on $\mathcal{L}$ will be called a G -connection if, for any open set U , for any G trivialisation over U $\iota : \mathcal{L}|_U \rightarrow U \times L$, there exists a 1-form ω on U with values in $\mathfrak{g}$ such that*

$$\nabla|_U = \iota_i^* D + \omega,$$

where D is the trivial connection.

Example 3.3.21 Suppose $G = \text{SO}(B)$. Then a G -structure is the same as a fibrewise tensor of the same type as B , and a connection is a connection for which this family of tensors is parallel.

The following is a straight-forward verification:

Proposition 3.3.22 *Let ∇ be a G -connection on a vector bundle $\pi : \mathcal{L} \rightarrow M$ and $x_0 \in M$.*

(i) *There exists a 2-form $\Omega \in \Omega^2(M, \mathfrak{g})$ such that*

$$R^\nabla = \Omega.$$

(ii) *If ∇ is flat then it defines a homomorphism*

$$\pi_1(M, x_0) \rightarrow G_{x_0}, \quad [\gamma] \mapsto \text{Hol}^\nabla(\gamma).$$

(iii) *Moreover, if two connections are gauge equivalent, then the associated homomorphisms are conjugated.*

3.3.7 The holonomy map of a flat connection

The previous proposition associates to every flat connection a conjugacy class of homomorphism in G since G_{x_0} is isomorphic up to conjugation with G . This leads to the following definition.

Definition 3.3.23 [HOLONOMY REPRESENTATION OF A FLAT CONNECTION]
Let ∇ be a flat G -connection. Then the holonomy representation of ∇ is the conjugacy class of the representation of $\pi_1(M)$ in G described in Proposition 3.3.22.

The following proposition is crucial

Proposition 3.3.24 *Two gauge equivalent flat connections yields conjugate representations. Conversely, if two flat connections have the same holonomy they are conjugate.*

PROOF: The first part of the proposition is obvious. Let us concentrate on the second part. We produce effectively the Gauge equivalence using the following trick. Let ∇_1 and ∇_2 be two connections with the same holonomy. We fix a base point x_0 . Then we choose for every x in M a path c_x from x_0 to x . We finally define

$$G_x = (\text{Hol}^{\nabla_1}(c_x))^{-1} \text{Hol}^{\nabla_2}(c_x).$$

Since ∇_1 and ∇_2 are flat, G_x only depend on the homotopy class of c_x . Since moreover the two connections have the same holonomy, G_x is independent on the choice of the homotopy class. It follows that for every path joining γ two points x and y ,

$$G_y^{-1} \text{Hol}^{\nabla_1}(\gamma) G_x = \text{Hol}^{\nabla_2}(\gamma).$$

This implies that $x \rightarrow G_x$ realise a Gauge equivalence between ∇_1 and ∇_2 as in Exercise 3.3.12. Q.E.D.

3.4 Combinatorial versions of connections

We define in this section two combinatorial analogs of connections and flat connections together with the associated equivalence. In the next section we will show all these notions are equivalent.

3.4.1 Discrete connections on ribbon graphs

We now explain what is a discrete connection in the world of ribbon graphs. Let Γ be a ribbon graph.

Definition 3.4.1 [DISCRETE G - CONNECTIONS] *A discrete G -connection on Γ is a map*

$$\nabla : E \rightarrow G, \quad e \mapsto g_e,$$

satisfying $g_{\bar{e}} = g_e^{-1}$. We will write $\nabla = \{g_e\}_{e \in E}$. Denote by $(G^E)' \subset G^E$ the set of discrete connections. The group G^V is called the gauge group of Γ and it acts on the space of connections via

$$\{g_v\}_{v \in V} \cdot \{h_e\}_{e \in E} := \{g_{e_+} h_e g_{e_-}\}_{e \in E}.$$

Two discrete connections in the same G^V -orbit are called gauge equivalent.

Definition 3.4.2 [PATHS AND HOLONOMY] *Let $(e_1, \dots, e_n)$ be a path on Γ , that is a finite sequence of edges with $(e_j)_+ = (e_{j+1})_-$ for $j = 1, \dots, n-1$. Denote by $\mathcal{P}$ the space of discrete paths in Γ .*

If $\nabla = \{g_e\}_{e \in E}$ is a discrete connection then its holonomy is defined to be the map

$$\text{Hol}(\nabla) : \mathcal{P} \rightarrow G, \quad (e_1, \dots, e_n) \mapsto g_{e_1} \cdots g_{e_n}.$$

Definition 3.4.3 [FLAT CONNECTIONS] *As any face defines a path, we may define the curvature of ∇ to be*

$$R^\nabla := \{\text{Hol}(f)\}_{f \in F} \in G^F.$$

A discrete connection is called flat if $R^\nabla = \{e_G\}_{f \in F}$.

Definition 3.4.4 [HOLONOMY REPRESENTATION] *From the description of $\pi_1(S)$ using ribbon graphs given in Theorem 2.4.3, it follows the holonomy map described above of a flat connection give rises to the holonomy representation of the discrete G -connection which is the conjugacy class of homomorphisms of $\pi_1(S)$ in G . Observe again that two gauge equivalent discrete connections give rise to the same holonomy representation.*

3.4.2 Local systems

Definition 3.4.5 [G -LOCAL SYSTEMS] *Let M be a manifold and let $\mathcal{U} = \{U_i\}_{i \in I}$ be a covering. Given a Lie group G , a G -local system over M on $\mathcal{U}$ is a collection of elements $g_{ij} \in G$ for all $i, j \in I$ with $U_i \cap U_j \neq \emptyset$ such that*

$$g_{ij}g_{jk} = g_{ik}$$

whenever $U_{i,j,k} \neq \emptyset$. Observe that every local system on $\mathcal{U}$ gives rise to a local system on any refinement of $\mathcal{U}$.

Two G -local systems on the same covering $\mathcal{U}$ are called gauge equivalent as local systems if they lie in the same orbit of the G^I action on the space of local systems given by

$$(g_i)_{i \in I} \cdot (g_{ij}) := g_i g_{ij}^{-1} g_j.$$

Two G -local systems are equivalent as local systems if they are in they are gauge equivalent on some common refinement.

REMARK: If G is abelian, then in the language of Čech cohomology a G -local system is just a Čech-1-cocycle and an equivalence class of such systems is just the corresponding cohomology class. Then standard arguments in cohomology show that the set of equivalence classes of G -local systems are naturally identified with $H^1(M, G)$. Therefore one should think of the collection of equivalence classes of G -local systems as a non-abelian version of $H^1(M, G)$.

Definition 3.4.6 [HOLONOMY OF A G -LOCAL SYSTEM] *The holonomy of the path $(i_0, \dots, i_p)$ for the G -local system $\{g_{ij}\}$ is the element $g_{i_0, i_1} \dots g_{i_{p-1}, i_p}$ of G . By definition of a local system, the holonomy of a trivial loop is trivial. The holonomy representation is the map from $\pi_1^{\mathcal{U}}(M)$ to G which associates to each path its holonomy.*

The holonomy of a local system just depends on the equivalence class.

3.5 Four models of the representation variety

In this paragraph, we gather the information that we have collected on smooth and discrete connections, local systems and representations of the fundamental group of the surface. All these objects can be identified up to some equivalence. Let us be more precise: we have four collections of objects which are associated to "holonomy" representations, namely

- (1) Pairs $(\mathcal{L}, \nabla)$, where $\mathcal{L}$ is a G -vector bundle over the surface S and ∇ is a flat G -connection on $\mathcal{L}$, up to gauge equivalence.
- (2) Equivalence classes of discrete flat G -connections with respect to a filling ribbon graph Γ , again up to gauge equivalence.
- (3) Equivalence classes of G -local systems on S on a given covering.
- (4) Homomorphisms of $\pi_1(S)$ into G up to conjugation.

These four collections of objects give rise different models for the representation variety, each of which will turn out to be useful at some stage. The equivalence of the descriptions (1), (2) and (3) holds in fact for general manifolds and we shall prove them in this generality.

We shall denote by $\mathcal{M}_i$ with $i \in \{1, 2, 3, 4\}$ the corresponding set of equivalence classes in the four cases under consideration.

We shall also denote by $\text{Rep}(\pi_1(S), G) = \mathcal{M}_4$ the space $\text{Hom}(\pi_1(S), G)/G$ where G acts on $\text{Hom}(\pi_1(S), G)$ by conjugation.

3.5.1 Flat connections and local systems

We need a definition.

Definition 3.5.1 [GOOD COVERING] *An open covering $\{U_i\}_{i \in I}$ of a manifold M is called a good covering if it satisfies the following condition. For all $n \in \mathbb{N}$ and all $i_1, \dots, i_n \in I$ the subset*

$$U_{i_1, \dots, i_n} := U_{i_1} \cap \dots \cap U_{i_n}$$

is contractible.

Any manifold admits a good covering, but we will not prove this here. Anyhow, we are mainly interested in the case of surfaces and in this case one can construct good coverings associated to ribbon graphs.

Theorem 3.5.2 *Let M be a manifold. Then there exist a one-to-one correspondence preserving the holonomy between gauge equivalence classes of G -local systems on a good covering and flat G -connections.*

PROOF: Let (g_{ij}) be a G -local system for the good covering $\{U_i\}_{i \in I}$ of M . We view g_{ij} as a constant map $U_i \cap U_j \rightarrow G \subset \text{End}(L)$. Then g_{ij} defines a cocycle hence a vector bundle over M , which we call $\mathcal{L}$. By construction we have trivialisations $\mathcal{L}|_{U_i} \cong U_i \times L$ such that the transition maps are the constant functions g_{ij} . On $U_i \times V$ we have the trivial connection D_i and thus we obtain a connection ∇_i on $\mathcal{L}|_{U_i}$ via pullback. Since the g_{ij} are constant linear maps and the derivative of a constant map with respect to the trivial connection is the linear map itself we see that ∇_i and ∇_j coincide on $\mathcal{L}|_{U_{i,j}}$. Thus there exists a unique connection ∇ on $\mathcal{L}$ such that $\nabla|_{\mathcal{L}|_{U_i}} = \nabla_i$. Thus we have produced a bundle $\mathcal{L}$ with connection ∇ from a G -local system. Conversely, given a bundle with connection we conclude from Theorem 3.3.18 that it trivialises over each of the sets U_i and that the connection is the pullback of a trivial connection. Then the coordinate changes preserve the connection and thus are locally constant, hence constant. It is easy to check that this gives rise to the correspondence claimed in the proposition. Q.E.D.

3.5.2 Local systems and representations

Theorem 3.5.3 *Let M be a manifold and G be a Lie group. Then the holonomy map from the space of flat G -local systems on a good covering – up to gauge equivalence – to $\text{Rep}(\pi_1(M), G)$ is a bijection.*

Equivalently, using the previous correspondence

Theorem 3.5.4 *Let M be a manifold and G be a Lie group. Then the holonomy map from the space of flat G -connections up to gauge equivalence to $\text{Rep}(\pi_1(M), G)$ is a bijection.*

If G is abelian then this correspondence identifies $H^1(M, G)$ with $\text{Hom}(\pi_1(M), G)$. In this case, the correspondence is just the well-known Hurewicz isomorphism.

PROOF: Let us first remark that the universal covering $\tilde{M} \rightarrow M$ defines a $\pi_1(M)$ local system.

Indeed, let $\mathcal{U} = \{U_i\}_{i \in I}$ be a good covering. Observe that if $\pi : \tilde{M} \rightarrow M$ is the universal cover, then for every i in I , we have

$$\pi^{-1}(U_i) = \sqcup_{g \in \pi_1(M)} gV_i,$$

Then, every path in the covering $\{U_i\}_{i \in I}$ can be lifted as a path in the covering $\{g.V_i\}_{g \in \Lambda, i \in I}$. We now choose for every i in I a path p_i from i_0 to i in $\{U_i\}_{i \in I}$. Then if $U_i \cap U_j \neq \emptyset$ the loop p_i, p_j^{-1} lifts to an element $\lambda_{i,j}$ in Λ . Then, one checks that $\lambda_{i,j}$ is indeed a local system.

Therefore, if ψ is a representation of $\pi_1(M)$ into G , $\psi(\lambda_{i,j})$ defines a G -local system whose holonomy is precisely ψ . We have just proved that the holonomy map from G -local systems – on $\mathcal{U}$ – is surjective.

Using the previous correspondence and Proposition 3.3.24, we see that this map is also injective. Q.E.D.

3.5.3 Smooth and discrete connections

We have now established the correspondence (1) $\leftrightarrow$ (3) $\leftrightarrow$ (4) for general manifolds M . In the case of surfaces we can also give another equivalent description (2) using ribbon graphs. Thus let S be a compact oriented surface and let Γ be a filling ribbon graph for S .

Theorem 3.5.5 *Let S be a surface and Γ a filling ribbon graph for S . Then there exists a holonomy preserving one to one correspondence between flat connections up to gauge equivalence on S and flat discrete connections on Γ up to gauge equivalence.*

PROOF: Let E be a G -vector bundle on S . We assume Γ is embedded in S . We choose a G -trivialisation in the neighbourhood of every vertex of the graph. Then the holonomy of every edge is well defined as an element of G . Therefore we have associated to a flat connection a discrete connection on the ribbon graph with the same holonomy.

In order to complete the proof, by Theorem 3.5.4, it remains to prove that every discrete connection can be obtained this way. For that we first consider a subdivision Γ_0 of Γ so that all edges of Γ_0 connect different vertices and all faces are triangles.

This can be achieved by adding a vertex in the middle of every edge, and face and connect the created vertex in each face to all the vertex in the boundary of the face.

Then, we can construct a good covering $\{U_v\}_{v \in V}$ so that

- U_v is a contractible neighbourhood of the vertex v .
- U_v intersects U_w if and only if v and w are connected by an edge e . Moreover e intersects $U_v \cap U_w$.

- Finally $U_v \cap U_w \cap U_z \neq \emptyset$ if and only if the vertices v , w and w are the three vertices of a common face f .

We can associate to a discrete connection g on Γ_0 the G -local system such that $g_{v,w}$ is the value $g(e)$ of the connection on the edge from v to w . By construction, the smooth connection associated to this local system is such that the holonomy of e is $g(e)$. It follows that every discrete connection on Γ_0 can be realised as coming from a smooth connection.

We finally choose an injective map I from the set E of vertices of Γ to the set E_0 of vertices in Γ_0 , so that $I(e)$ is a subset of e . This defines a map I^* from the set of discrete flat connections on Γ to the set of discrete flat connection Γ_0 by the following rule: $I^*g(e) = g(e)$ if e belongs to $I(E)$ and the identity otherwise. By construction, if ∇ is a smooth connection such that its associated discrete connection on Γ_0 is $I^*(g)$, then its associated connection on Γ is g . Q.E.D.

Chapter 4

Twisted Cohomology

4.1 De Rham version of twisted cohomology

4.1.1 Motivation: variation of connection

Let us return to model (1) of the representation variety, the space of flat G -connections modulo gauge equivalence. Let $[\nabla]$ be an equivalence class in this model. The question that motivates for us the introduction of twisted cohomology is the following one: How should a candidate for the tangent space at $[\nabla]$ look like? The tangent space of the space of flat G -connections at ∇ should of course be something like the space of all possible variations of ∇ preserving the flatness condition. On the other hand, as we are dealing with equivalence classes, the tangent space at the class $[\nabla]$ should be a quotient of this space of variations by the subspace of those variations which come from a gauge transformation.

Let us try to make these ideas precise.

What does it mean to vary ∇ amongst flat connections ? Recall from Proposition 3.3.5 that any connection ∇' on the same bundle can be written as $\nabla' = \nabla + A$, where $A \in \Omega^1(M, \mathcal{L})$. Thus varying ∇ inside the space of all connections means to choose a smooth family of elements A_t in the vector space $\Omega^1(M, \mathcal{L})$ and to define $\nabla_t := \nabla + A_t$. An infinitesimal variation is then given by $\dot{\nabla} := \frac{d}{dt}\big|_{t=0} A_t$. However, we want to vary ∇ only in the subspace of flat connections. Thus we have to single out those infinitesimal variations coming from families ∇_t for which the curvature is zero. Let us call these infinitesimal variations *flat (infinitesimal) variations*. Let us define the infinitesimal change of curvature by $\dot{R} := \frac{d}{dt}\big|_{t=0} R^{\nabla_t}$. Then we want to express $\dot{R}$ in terms of $\dot{\nabla}$. This is indeed possible. We first introduce a notation. Let

$d_{\nabla}^1 : \Omega^1(M, \mathcal{L}) \rightarrow \Omega^2(M, \mathcal{L})$ be defined by

$$d_{\nabla}^1(\alpha)(X, Y) := \nabla_X(\alpha(Y)) - \nabla_Y(\alpha(X)) - \alpha([X, Y]),$$

where α belongs to $\Omega^1(M, \mathcal{L})$ and X, Y are tangent vector fields on M .

Proposition 4.1.1 *The infinitesimal variation of curvature is given by*

$$\dot{R} = d_{\nabla}^1 \dot{\nabla}.$$

PROOF: This follows from the fact that

$$R^{\nabla_t} = R^{\nabla} + d_{\nabla}^1 A_t + \frac{1}{2}[A, A],$$

where the commutator is defined by

$$[A, B](X, Y) := [A(X), B(Y)] - [A(Y), B(X)].$$

Q.E.D.

As a consequence, an infinitesimal variation $\dot{\nabla}$ is flat if and only if

$$d_{\nabla}^1 \dot{\nabla} = 0.$$

Now let us study those variations which come from a gauge transformation. These should be regarded as trivial and our space of cocycles should be divided by the space of these variations in order to obtain the tangent space of $[\nabla]$ (rather than that of ∇). To describe the trivial variations, let g_t be a smooth path with values in the gauge group with $g_0 = id$. Then a trivial variation is one of the form

$$\nabla_t = g_t^{-1} \nabla g_t.$$

We need to understand the corresponding infinitesimal variation. For this we need some more machinery. Let $\mathcal{L}$ denote the vector bundle on which ∇_0 lives. This is represented by a G -valued cocycle. If we replace the action of G on the typical fibre V of $\mathcal{L}$ by the adjoint action of G on its Lie algebra $\mathfrak{g}$, then this cocycle defines for us a bundle $\mathfrak{g}_{\mathcal{L}}$ with typical fibre $\mathfrak{g}$ over M . Moreover there is a canonical inclusion of $\mathfrak{g}_{\mathcal{L}}$ in $\text{End}(\mathcal{L})$. The above variation then defines a section $\dot{g}$ of $\mathfrak{g}_{\mathcal{L}}$ by

$$\dot{g}_m := \left. \frac{d}{dt} \right|_{t=0} g_t.$$

On the endomorphism bundle $\text{End}(\mathcal{L})$ there is a connection $\nabla^{\text{End}(\mathcal{L})}$ induced from ∇ and given by

$$(\nabla_X^{\text{End}(\mathcal{L})} A)(u) := \nabla_X(Au) - A(\nabla_X u)$$

where A is a section of $\text{End}(\mathcal{L})$ and u is a section of u . As ∇ is a G -connection the connection $\nabla^{\text{End}(\mathcal{L})}$ restricts to a connection on $\mathfrak{g}_{\mathcal{L}}$. Now we can finally describe the infinitesimal variation associated with ∇_t as above:

Proposition 4.1.2 *Let notation be as above. Then $\dot{\nabla} = \nabla^{\text{End}(\mathcal{L})} \dot{g}$.*

PROOF: As both sides of the equation are defined in local terms we can work in a trivialisation. Thus fix an open subset U of M such that $\mathcal{L}|_U \cong U \times L$. Under this trivialisation ∇ becomes identified with a connection of the form $D + A$, where D denotes as usual the trivial connection. Identifying $\text{End}(\mathcal{L})|_U \cong U \times \text{End}(V)$ the induced connection $\nabla^{\text{End}(\mathcal{L})}$ then becomes identified with $D + \text{ad}(A)$, where $\text{ad}(A) \in \Omega^1(U) \otimes \text{End}(\text{End}(\mathcal{L}))$ is given by

$$(\text{ad}(A)(X))(B) := [A(X), B].$$

Now,

$$\begin{aligned} (g_t^{-1} \nabla g_t)(u) &= (g_t^{-1} (D + A) g_t)(u) \\ &= g_t^{-1} D(g_t u) + (g_t^{-1} A g_t)(u) \\ &= g_t^{-1} g_t (Du) + g_t^{-1} (D g_t)(u) + (g_t^{-1} A g_t)(u) \\ &= (Du) + g_t^{-1} (D g_t)(u) + (g_t^{-1} A g_t)(u) \end{aligned}$$

and hence

$$\begin{aligned} \dot{\nabla}(u) &= (D \dot{g})(u) + \text{ad}(A) \dot{g}(u) \\ &= ((D + \text{ad}(A)) \dot{g})(u) \\ &= (\nabla^{\text{End}(\mathcal{L})} \dot{g})(u). \end{aligned}$$

Q.E.D.

We can restate the proposition as follows: Define

$$d_{\nabla}^0 : \Omega^0(M, \mathcal{L}) \rightarrow \Omega^1(M, \mathcal{L})$$

by the formula

$$d_{\nabla}^0(\alpha)(X) := \nabla_X \alpha,$$

where α is a section of $\mathcal{L}$, X a vector field on M and we identify $\Omega^0(M, \mathcal{L})$ with $\Gamma(\mathcal{L})$. Then an infinitesimal variation comes from a gauge change if and only if

$$d_{\nabla}^0(\dot{\nabla}) = 0.$$

This looks very much like a coboundary condition and thus our candidate for the tangent space of $\text{Rep}(\pi_1(S), G)$ at $[\nabla]$, the quotient

$$\frac{\ker d_{\nabla}^1}{\text{Im } d_{\nabla}^0},$$

looks very much like a cohomology space. Of course, for such an interpretation we need to know that $d_{\nabla}^1 \circ d_{\nabla}^0 = 0$. This is indeed the case:

Proposition 4.1.3 *Let ∇ be any (not-necessarily flat) connection and define d_{∇}^0 and d_{∇}^1 as above. Then $d_{\nabla}^1 \circ d_{\nabla}^0 = R^{\nabla}$ is precisely the curvature of ∇ . Thus $d_{\nabla}^1 \circ d_{\nabla}^0 = 0$ if and only if ∇ is flat.*

Now we may allow ourselves to write

$$H_{\nabla}^1(M) := \frac{\ker d_{\nabla}^1}{\text{Im } d_{\nabla}^0}$$

and to call it the first ∇ -twisted cohomology of M .

4.1.2 De Rham version of the twisted cohomology

We have seen in subsection 4.1.1 that a certain twisted cohomology group $H_{\nabla}^1(M)$ can be interpreted heuristically as some sort of tangent space to the representation variety at the point $[\nabla]$. This group is in fact only the first in a family of twisted cohomology groups, which we will now define. The study of these more general cohomology groups is not just for completeness: Recall that in order to prove non-degeneracy of the intersection form in the untwisted cohomology group $H^1(S)$ of a surface we also needed the groups $H^0(S)$ and $H^2(S)$. Similarly, we will need also their twisted analogs in the sequel. In order to define these one needs to prove the following proposition:

Proposition 4.1.4 *Let ∇ be a connection on a vector bundle $\pi : \mathcal{L} \rightarrow M$. Then there exists a unique operator*

$$d_{\nabla}^k : \Omega^k(M, \mathcal{L}) \rightarrow \Omega^{k+1}(M, \mathcal{L})$$

such that the following two properties are satisfied:

(i) *If $\omega \in \Omega^p(M)$ and $\alpha \in \Omega^q(M, \mathcal{L})$, then*

$$d_{\nabla}^{p+q}(\omega \wedge \alpha) = d^p \omega \wedge \alpha + (-1)^p \omega \wedge d_{\nabla}^q \alpha,$$

where d^p denote the untwisted de Rham differential.

(ii) If $\alpha \in \Omega^0(M, \mathcal{L}) \cong \Gamma(\mathcal{L})$ then

$$d_{\nabla}^0 \alpha = \nabla \alpha.$$

The operators d_{∇}^0 and d_{∇}^1 coincide with the operators defined above. Using Proposition 4.1.3 and property (i) it is easy to see that $d^{\nabla, k} \circ d^{\nabla, k-1} \equiv 0$ for all k if and only if ∇ is flat. Thus we may define:

Definition 4.1.5 Let $\pi : \mathcal{L} \rightarrow M$ be a vector bundle over M and let ∇ be a flat connection on $\mathcal{L}$. An $\mathcal{L}$ -valued k -form $\omega \in \Omega^k(M, \mathcal{L})$ will be called closed if $d_{\nabla}^k \omega = 0$ and exact if there exists $\alpha \in \Omega^{k-1} \otimes \mathcal{L}$ with $\omega = d_{\nabla}^{k-1} \alpha$. Denote by $Z_{\nabla}^k(M)$ the space of closed k -forms on M and by $B_{\nabla}^k(M)$ the space of exact ones. Then the ∇ -twisted k -th de Rham cohomology group of M is defined to be

$$H_{\nabla}^k(M) := Z_{\nabla}^k(M) / B_{\nabla}^k(M).$$

The dimension

$$\beta_k^{\nabla} := \dim H_{\nabla}^k(M)$$

is called the k -th ∇ -twisted Betti number of M and their alternating sum

$$\chi_{\nabla}(M) := \sum_{i=0}^{\dim M} (-1)^i \beta_i^{\nabla}$$

is called the ∇ -twisted Euler characteristic.

The notation is in the line of our convention that we drop the dependence on $\mathcal{L}$ from our notation and see ∇ as a short hand for the pair $(\mathcal{L}, \nabla)$. If we want to be very precise or want to stress the dependence on $\mathcal{L}$ we shall occasionally write $H_{\nabla}^k(M, \mathcal{L}) := H_{\nabla}^k(M)$.

Example 4.1.6 1. If $\mathcal{L} = M \times R$ is the trivial line bundle over M and $\nabla = D$ is the trivial connection then $H_D^k(M) = H_{dR}^k(M)$ is just the ordinary de Rham cohomology. Thus twisted cohomology generalises de Rham cohomology.

2. Let ∇ be flat connection on a vector bundle $\mathcal{L}$ and ρ the holonomy representation from $\pi_1(S, x_0)$ to $\text{End}(\mathcal{L}_{x_0})$. The space $H_{\nabla}^0(M)$ is the space of ∇ -parallel section of $\mathcal{L}$. Moreover, the map $\sigma \rightarrow \sigma(x_0)$ is an isomorphism from $H_{\nabla}^0(M)$ to

$$\mathcal{L}_{x_0}^{\rho(\pi_1(S, x_0))} := \{u \in \mathcal{L}_{x_0} \mid \forall \gamma \in \pi_1(S, x_0), \rho(\gamma)(u) = u\}.$$

In particular, $\beta_0^{\nabla}(M) = \text{rank}(\mathcal{L})$ if and only if the representation is trivial.

So far our considerations work for arbitrary manifolds. Now let us return to the special case of compact oriented surfaces: Most of the results stated above for ordinary de Rham cohomology can be generalised to the twisted case. The following theorem summarises the main results. Again the proofs will use a combinatorial model of twisted cohomology (via ribbon graphs) and are deferred to the next subsection.

Theorem 4.1.7 *Let $S \cong S_g$ be a compact oriented surface of genus g , let $\pi : \mathcal{L} \rightarrow S$ be a vector bundle and let ∇ be a flat connection on $\mathcal{L}$.*

(i) *The spaces $H_{\nabla}^k(S)$ are finite-dimensional.*

(ii) *The ∇ -twisted Euler characteristic of S is given by*

$$\chi_{\nabla}(S) = \text{rk}(\mathcal{L})(2 - 2g).$$

In particular it depends only on the genus of S and the rank of $\mathcal{L}$, but not on the choice of connection.

(iii) *If $\mathcal{L}^*$ is the dual bundle of $\mathcal{L}$ and ∇^* is the dual connection of ∇ , then the pairing*

$$i_{\nabla} : H_{\nabla}^1(S) \times H_{\nabla^*}^1(S) \rightarrow \mathbb{R}, \quad (\alpha, \beta) \mapsto \int_S \omega_{\alpha, \beta},$$

where

$$\omega_{\alpha, \beta}(X, Y) := \langle \alpha \wedge \beta \rangle(X, Y) := \langle \alpha(X) | \beta(Y) \rangle - \langle \alpha(Y) | \beta(X) \rangle,$$

and $\langle \cdot | \cdot \rangle$ denote dual pairing, is non-degenerate.

Before we turn to the proof, let us consider some applications of Theorem 4.1.7.

Example 4.1.8 Suppose $(\mathcal{L}, \nabla)$ is a vector bundle with flat connection over S which is equipped with a parallel inner product $\langle \cdot, \cdot \rangle$. Here, parallel means that

$$X.\langle v, w \rangle = \langle \nabla_X v, w \rangle + \langle v, \nabla_X w \rangle.$$

Then we can use $\langle \cdot, \cdot \rangle$ in order to identify $(\mathcal{L}, \nabla)$ and $(\mathcal{L}^*, \nabla^*)$. Thus we obtain a symplectic form

$$i_{\nabla} : H_{\nabla}^1(\mathcal{L}) \times H_{\nabla}^1(\mathcal{L}) \rightarrow \mathbb{R}, \quad (\alpha, \beta) \mapsto \int_S \langle \alpha \wedge \beta \rangle,$$

where

$$\langle \alpha \wedge \beta \rangle(X, Y) := \langle \alpha(X), \beta(Y) \rangle - \langle \alpha(Y), \beta(X) \rangle.$$

We will call i_∇ is called the ∇ -twisted intersection form on $H_\nabla^1(\mathcal{L})$. In the case of the trivial line bundle with the trivial connection we recover the usual intersection form.

The following example will be central for our treatment of the representation variety:

Example 4.1.9 Let S be a surface and G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. Let $\rho : \pi_1(S) \rightarrow G$ be a homomorphism which corresponds to a G -local system $(\{U_i\}_{i \in I}, \{g_{ij}\}_{i,j \in I})$ via the non-abelian Hurewicz theorem. Then $(\{U_i\}_{i \in I}, \{\text{ad}(g_{ij})\}_{i,j \in I})$ is a local system which defines a flat G -connection ∇_ρ on a vector bundle $\mathcal{L}_\mathfrak{g}$ over S with fibre $\mathfrak{g}$.

This example is so important for us, that we introduce a special notation for it:

Definition 4.1.10 Let $\rho : \pi_1(S) \rightarrow G$ be a representation and define $\mathcal{L}_\mathfrak{g}$ and ∇_ρ as in example 4.1.9. Then the twisted cohomology groups associated to the bundle with connection $(\mathcal{L}_\mathfrak{g}, \nabla_\rho)$ are denoted by

$$H_\rho^i(\mathfrak{g}) := H_{\nabla_\rho}^i(S).$$

Given a group G we write $Z(G)$ for its centre, and given a subgroup H of G we write

$$Z_G(H) := \{g \in G \mid gh = hg \text{ for all } h \in H\}$$

for its centraliser in G .

Proposition 4.1.11 Suppose that in the situation of Example 4.1.9 G has discrete centre and that we have $Z_G(\rho(\pi_1(S))) = Z(G)$. Then

$$H_\rho^0(\mathfrak{g}) = H_\rho^2(\mathfrak{g}) = \{0\}, \dim H_\rho^1(\mathfrak{g}) = \chi(S) \dim \mathfrak{g}.$$

PROOF: Suppose $H_\rho^0(\mathfrak{g}) \neq \{0\}$. Then there exists a nonzero map $\alpha : V \rightarrow \mathfrak{g}$ satisfying $\alpha(e^+) = g_e \alpha(e^-)$, where e is any edge and $\{g_e\}_{e \in \mathcal{L}}$ is the discrete connection associated with ρ . This means that for any vertex v the element $\alpha(v) \in \mathfrak{g}$ is invariant under the holonomy map

$$\pi_1(S, v) \rightarrow \text{Aut}(\mathfrak{g}).$$

As the holonomy map factors as

$$\pi_1(S, v) \xrightarrow{\rho} G \xrightarrow{\text{Ad}} \text{Aut}(\mathfrak{g}),$$

this implies that the one-parameter subgroup $\exp(t\alpha(v))$ commutes with the image $\rho(\pi_1(S))$ of $\pi_1(S)$. Thus $Z(G) = Z_G(\rho(\pi_1(S)))$ is non-discrete, which is a contradiction. Therefore $H_\rho^0(\mathfrak{g}) = \{0\}$ and also $H_\rho^2(\mathfrak{g}) = \{0\}$ by a duality similar to (iii) of Theorem 4.1.7. Finally, $\dim H_\rho^1(\mathfrak{g}) = \chi(S) \dim \mathfrak{g}$ follows from part (ii) of Theorem 4.1.7. Q.E.D.

4.2 A combinatorial version

In order to prove the final part of Theorem 4.1.7 we need to produce a combinatorial version of the twisted cohomology. We choose a ribbon graph Γ filling Σ . We associate to each edge e the oriented one dimensional manifold which is the interior of the geometric realisation and which we shall denote $\hat{e}$. In the same way we associate to every face f , the interior of each face which we denote by $\hat{f}$. By convention $v = \hat{v}$

4.2.1 The combinatorial complex

In order to smooth up our exposition, we have to consider the set of Γ_0 of oriented vertices, as well as the set Γ_2 of oriented faces, even though in our context vertices and faces have a canonical orientation. We denote as usual $\bar{\alpha}$ the element α of Γ_i with the opposite orientation. The boundary $\partial\alpha$ of an oriented element α of Γ_i is a tuple of elements of Γ_{i-1} , possibly with repetition. For instance, if e is an edge then

$$\partial e = (e_+, \bar{e}_-).$$

A covering

We now consider for every $\alpha \in \Gamma_i$ a contractible open set U_α which is a neighbourhood of the interior $\hat{\alpha}$, that is a vertex, the interior of the edge and the face. We denote by W_i the union of all the open sets U_α so that α is in Γ_i . Finally we choose for every pair (α, β) so that $\alpha \in \partial\beta$, open sets $U_{\alpha, \beta}$ homeomorphic to disks.

- $U_{\bar{\alpha}} = U_\alpha$ and $U_{\bar{e}, \bar{f}} = U_{e, f}$.
- $\forall i, \forall \alpha, \beta \in \Gamma_i, U_\alpha \cap U_\beta = \emptyset$.

- For every edge e , $U_e \cap W_0 = U_{e^+,e} \sqcup U_{\bar{e}^-,e}$,
- For every edge e , $U_e \cap W_1 = U_{e,f(e)} \sqcup U_{\bar{e},f(\bar{e})}$, where $f(a)$ is the unique face containing the edge a and g .
- For every face f , that $W_1 \cap U_f$ is an annulus

Vector spaces and homomorphisms

We now are given a vector bundle $\mathcal{L}$ equipped with a flat connexion ∇ .

We consider the vector space L_α , which consists of section parallel of $\mathcal{L}|_{U_\alpha}$. Observe that we have a canonical trivialisation of $\mathcal{L}|_{U_\alpha}$ by $L_\alpha \times U_\alpha$, and that $L_\alpha = L_{\bar{\alpha}}$.

Moreover, observe that for any pair (α, β) so that $\alpha \in \partial\beta$, there is a natural isomorphism $i(\alpha, \beta)$ from L_α to L_β : if u is parallel section along U_α , $i(\alpha, \beta)u$ is the unique parallel section along U_β which coincides with u on $U_{\alpha,\beta}$.

Exercise 4.2.1 Describe $i_{\alpha,\beta}$ using a trivialisation of the bundle at every vertex and the combinatorial connexion associated to ∇ .

A combinatorial complex

We consider the complex defined by

$$C_\Gamma^i = \{c^i : \Gamma_i \rightarrow \sqcup_{\alpha \in \Gamma_i} L_\alpha \mid c^i(\alpha) \in L_\alpha \text{ and } c^i(\bar{\alpha}) = -c^i(\alpha)\},$$

and the coboundary operators d by

$$d_i : C_\Gamma^i \rightarrow C_\Gamma^{i+1}, \quad d_i c^i(\beta_{i+1}) = \sum_{\alpha_i \in \partial\beta_{i+1}} i(\alpha_i, \beta_{i+1}) c^i(\alpha_i).$$

One checks that $d \circ d = 0$. We define

$$H_\Gamma^i(L) = \text{Ker}(d_i) / \text{Im}(d_{i-1}).$$

4.2.2 The isomorphism Theorem

First we need to build a map between complexes. We associate to an $\omega \in \Omega^i(S; L)$ the element $\hat{\omega}$ in C^i defined by

$$\omega(\alpha^i) = \int_{\alpha^i} \omega.$$

The integration is understood in the canonical trivialisation of $\mathcal{L}|_{U_{\alpha_i}}$ by L_{α_i} , since we have an identification $\Omega^i(U_{\alpha_i}; \mathcal{L}) = \Omega^i(U_{\alpha_i}) \otimes L_{\alpha_i}$. We now claim

Proposition 4.2.2

$$\widehat{d\omega} = d\hat{\omega}.$$

PROOF: This is an easy consequence of Stokes formula and we shall only check it when $i = 1$. We do it with technical details that we shall omit in the sequel. Let f be an element of Γ_2 . Let $\partial f = \{e_1, \dots, e_n\}$. We consider f as a map from the closed disk $\mathbb{D}$ to S . We observe that we can write $\partial\mathbb{D}$ as a reunion of closed intervals I_i so that $f|_{I_i}$ is a parametrisation of the edge e_i . By construction, the induced bundle $f^*\mathcal{L}$ is trivialised as $L_f \times \mathcal{D}$. As a consequence, if $\omega \in \Omega^1(S, L)$, then $f^*\omega \in \Omega(D) \otimes L_f$. Now

$$\begin{aligned} \widehat{d\omega}(f) &= \int_f \omega = \int_{\mathbb{D}} df^*\omega \\ &= \int_{\partial\mathbb{D}} f^*\omega \\ &= \sum_{i=1}^n \int_{I_i} f^*\omega. \end{aligned}$$

Finally we have to remark that

$$\int_{I_i} f^*\omega = i(e_i, f) \int_{e_i} \omega.$$

Hence

$$\widehat{d\omega}(f) = d\hat{\omega}(f).$$

Q.E.D.

It follows from this identification that we have a natural map $u \mapsto \hat{u}$ from $H_{\nabla}^i(S, \mathcal{L})$ to $H_{\Gamma}^i(S, \mathcal{L})$ so that

$$[\hat{\omega}] = \widehat{[\omega]}$$

We now prove

Theorem 4.2.3 [ISOMORPHISM THEOREM] *The map $u \mapsto \hat{u}$ from $H_{\nabla}^i(S, \mathcal{L})$ to $H_{\Gamma}^i(S, \mathcal{L})$ is an isomorphism*

Again, to shorten our exposition we only prove this result for $i = 1$. We prove this in two steps: injectivity and surjectivity of this map

Proposition 4.2.4 *The map $u \mapsto \hat{u}$ from $H_{\nabla}^1(S, \mathcal{L})$ to $H_{\Gamma}^1(S, \mathcal{L})$ is surjective.*

PROOF: We first prove that given $c^1 \in C^1$, there exists a neighbourhood U_1 of Γ , and a 1-form $\omega \in \Omega^1(S, L)$ so that

$$\begin{aligned} \hat{\omega} &= c^1 \\ d^\nabla \omega|_{W_2} &= 0. \end{aligned} \tag{4.1}$$

It suffices to show this for c^1 such that there exists an edge e so that $c^1(\alpha) = 0$ if $\alpha \neq e$. Let indeed φ be a real valued function defined on U_e so that $\varphi = 1$ on a neighbourhood of U_{e+} and $\varphi = 1$ on a neighbourhood of U_{e-} . We now consider

$$\sigma = \varphi.c^1(e) \in \Omega^0(U_e; \mathcal{L}).$$

Observe that $d^\nabla \sigma = 0$ on $U_{e\pm}$. It follows that $\beta = d\sigma$ can be extended smoothly to W_1 by zero outside U_e . If ψ is a function with support in W_1 which is equal to 1 on a neighbourhood of Γ , then

$$\omega = \psi\beta,$$

extended by 0 outside W^1 , fulfils our conditions.

Finally let's suppose that $dc^1 = 0$. Let γ_f be a circle in $W_f = f \cap W_2$ homotopic to the annulus $U_f \cap W_1$. We then have

$$\int_{\gamma_f} \omega = dc^1(f) = 0.$$

It follows that $\omega|_{W_f} = d\beta_f$. We extend β_f to U_f and therefore we have extended ω to a closed form on U_f . Performing this operation for every face f , we end up with a closed form ω so that $\hat{\omega} = c^1$. Hence $u \mapsto \hat{u}$ is indeed surjective. Q.E.D.

Proposition 4.2.5 *The map $u \mapsto \hat{u}$ from $H_\nabla^i(S, \mathcal{L})$ to $H_\Gamma^i(S, \mathcal{L})$ is injective.*

PROOF: Let's assume that ω is such that $\hat{\omega} = dc^0$. We wish to prove that ω is exact.

We proceed by steps again.

We first notice that we can as well assume that $\omega = 0$ on W_0 . Indeed ω is exact on a neighbourhood U_0 of W_0 :

$$\omega|_{U_0} = d\alpha.$$

Hence using a function φ with support in U_0 and equal to 1 on W_0 we reduce to this case by considering

$$\omega - d^\nabla(\varphi\alpha).$$

Now we show that we can as well assume that $\hat{\omega} = 0$. Indeed, we choose a parallel section σ on W_0 so that for every vertex v , we have $\sigma(v) = c^0(v)$. It follows that $\omega - d^\nabla \varphi v$ satisfies the required condition.

Next we show that we can reduce to the case that $\omega = 0$ on a neighbourhood of Γ . Indeed, for every edge e , $\omega|_{U_e} = d\alpha_e$, where α is parallel on $W_0 \cap U_e$. We can choose α_e so that $\alpha_e = 0$ on one of the connected component, hence also on the other since $\int_e \omega = \hat{\omega}(e) = 0$. Therefore, smoothing again by a function ψ with support in W_1 and equal to 1 on a neighbourhood of Γ , we get that

$$\omega - d\left(\sum_{e \in E} \alpha_e\right),$$

satisfies the condition.

Finally, we observe that for every face f , $\omega|_f = d\beta_f$. By our condition β_f is zero on a neighbourhood of Γ . It follows that we can choose β_f to be zero on a (connected) neighbourhood of Γ . Finally we get, that

$$\omega = \sum_{f \in F} d\beta_f.$$

4.2.3 Duality

Symplectic complexes

Definition 4.2.6 *We say a complex $C^\bullet : 0 \rightarrow C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2$ of degree 2 is symplectic, if we have a*

- *a symplectic form ω on C^1 ,*
- *a non degenerate pairing ω on $C^0 \times C^2$ such that*

$$\omega(d\alpha_0, \alpha_1) = \omega(\alpha_0, d\alpha_1).$$

Proposition 4.2.7 *The first cohomology is a symplectic vector space with symplectic form $[\omega]$ such that*

$$[\omega]([[\alpha], [\beta]]) = \omega(\alpha, \beta).$$

PROOF: this follows at once from the fact that

$$\ker(d_1)^\circ = \text{im}(d_0), \tag{4.2}$$

where V° denote the orthogonal with respect to ω of $V \subset C^1$. Q.E.D.

The dual graph

We realise geometrically the dual graph Γ^* in S . We denote by $\alpha \mapsto \alpha^*$ the map from Γ_i to γ_{2-i} . As far as the boundary is concerned, we observe that

$$v \in \partial e \implies \bar{e}^* \in \partial v^*, \quad (4.3)$$

$$e \in \partial f \implies f^* \in \partial e^*. \quad (4.4)$$

If $\alpha \in \Gamma_0^*$, we choose a neighbourhood U_α^* as above, requiring furthermore that for $v \in \Gamma^0$, $U_v \subset U_{v^*}^*$, for $e \in \Gamma^1$, $U_e \cap U_{e^*}^*$ is contractible, and for $f \in \Gamma^2$, $U_f^* \subset U_f$. We consider the dual bundle $\mathcal{L}^*$ to $\mathcal{L}$ together with its dual connection. We now observe that we have the canonical identification

$$(L_\alpha)^* = L_{\alpha^*}^*.$$

Let $C_\Gamma^\bullet$ be the complex associated to Γ^* and Let $C_{\Gamma^*}^\bullet$ be the complex associated to Γ^* . We consider that pairing between C_Γ^{n-i} and $C_{\Gamma^*}^i$ by

$$\langle \alpha, \beta \rangle = \frac{1}{2} \sum_{\gamma \in \Gamma^{n-i} = \Gamma_*^i} \langle \alpha(\gamma) | \beta(\gamma^*) \rangle.$$

Let now $C^\bullet = C_\Gamma^\bullet \oplus C_{\Gamma^*}^\bullet$. We consider the bilinear form Ω defined

- on $(C_\Gamma^0 \oplus C_{\Gamma^*}^0) \times (C_\Gamma^2 \oplus C_{\Gamma^*}^2)$ by

$$\Omega((c^0, c_*^0), (c^2, c_*^2)) = \langle c^0, c_*^2 \rangle + \langle c^2, c_*^0 \rangle. \quad (4.5)$$

- on $(C_\Gamma^1 \oplus C_{\Gamma^*}^1) \times (C_\Gamma^1 \oplus C_{\Gamma^*}^1)$ by

$$\Omega((c^1, c_*^1), (a^1, a_*^1)) = \langle c^1, a_*^1 \rangle - \langle a^1, c_*^1 \rangle. \quad (4.6)$$

We then have

Theorem 4.2.8 [DUALITY THEOREM] *The complex $C^\bullet$ equipped with Ω is a symplectic complex. Moreover, the corresponding symplectic structure on*

$$H^1(C^\bullet) = H_\Gamma^1(\mathcal{L}) \oplus H_{\Gamma^*}^1(\mathcal{L}^*),$$

is given in the de Ram interpretation by

$$\Omega([\alpha], [\alpha_*], ([\beta], [\beta_*])) = \int_S \langle \alpha \wedge \beta_* \rangle - \int_S \langle \beta \wedge \alpha_* \rangle,$$

where $\langle \alpha \wedge \beta_* \rangle \in \Omega^2(S)$ is given by

$$\forall X, Y \in T_s S, \quad \langle \alpha \wedge \beta_* \rangle(X, Y) = \langle \alpha(X) | \beta_*(Y) \rangle - \langle \alpha(Y) | \beta_*(X) \rangle.$$

Finally the bilinear form Ω defines a duality between the complex $C_{\Gamma^*}^\bullet$ and the complex to $C_\Gamma^\bullet$.

We split the proof in two parts :

The complex is a symplectic complex

We want to prove the first part of the theorem which amounts to the following proposition

Proposition 4.2.9 *We have*

$$\begin{aligned}\langle dc^0, c_*^1 \rangle &= -\langle c^0, dc_*^1 \rangle. \\ \langle c^1, dc_*^0 \rangle &= \langle dc^1, c_*^0 \rangle.\end{aligned}$$

PROOF: by definition

$$\begin{aligned}\langle 2dc^0, c_*^1 \rangle &= \sum_{e \in \Gamma_1} \langle dc^0(e), c_*^1(e) \rangle \\ &= \sum_{e \in \Gamma_1} \sum_{v \in \partial e} \langle c^0(v), c_*^1(e) \rangle\end{aligned}$$

We can now rearrange the sum and get

$$\begin{aligned}\langle 2dc^0, c_*^1 \rangle &= \sum_{v \in \Gamma_0} \langle c^0(v), \sum_{e|v \in \partial e} c_*^1(e^*) \rangle \\ &= - \sum_{v \in \Gamma_0 = \Gamma_*^2} \langle c^0(v), \sum_{e^* \in \partial v^*} c_*^1(e^*) \rangle \\ &= -2\langle c^0, dc_*^1 \rangle.\end{aligned}$$

The second equality have a symmetric proof. Q.E.D.

The de Rham interpretation

Again this amounts to the following

Proposition 4.2.10 *Using the mapping $\alpha \rightarrow \hat{\alpha}$ from $\Omega^\bullet(\Sigma, \mathcal{L})$ to $C_T^\bullet$. We have*

$$\int_S \langle \alpha \wedge \eta^* \rangle = \langle \hat{\alpha}, \hat{\eta}^* \rangle.$$

PROOF: Indeed, we decompose the integral along the oriented faces:

$$\int_S \langle \alpha \wedge \eta^* \rangle = \sum_f \int_{U_f} \langle \alpha \wedge \eta^* \rangle.$$

We now use the fact that U_f is contractible and write, on each U_f , $\eta^* = d\beta_f^*$. Using Stokes formula, we obtain

$$\int_S \langle \alpha \wedge \eta^* \rangle = - \sum_f \sum_{e \in \partial f} \int_e \langle \alpha | \beta_f^* \rangle.$$

We note that every oriented edge e appears exactly once on the boundary of a face $f(e) = ((e^*)_+)^*$. Rearranging, we get

$$\int_S \langle \alpha \wedge \eta^* \rangle = - \sum_{e \in \Gamma_1} \int_e \langle \alpha | \beta_{f(e)}^* \rangle \quad (4.7)$$

$$= \frac{1}{2} \sum_{e \in \Gamma_1} \int_e \langle \alpha | (\beta_{f(\bar{e})}^* - \beta_{f(e)}^*) \rangle \quad (4.8)$$

We now observe that, $\beta_{f(e)}^* - \beta_{f(\bar{e})}^*$ is a parallel section along the edge e and that

$$\beta_{f(\bar{e})}^* - \beta_{f(e)}^* = \int_{e^*} \eta_*.$$

Hence

$$\int_S \langle \alpha \wedge \eta^* \rangle = \frac{1}{2} \sum_e \langle \int_e \alpha | \int_{e^*} \eta_* \rangle \quad (4.9)$$

$$= \frac{1}{2} \sum_e \langle \alpha(e) | \eta_*(e_*) \rangle \quad (4.10)$$

$$= \langle \alpha | \eta_* \rangle \quad (4.11)$$

4.3 Torsion

4.3.1 Determinants

Definition 4.3.1 *If E is a finite dimensional vector space, we define $\det(E) = \Lambda^{\dim(E)} E$ and $\det(E)^{-1} = \Lambda^{\dim(E)} E^*$. Let $C^\bullet : 0 \rightarrow C^0 \rightarrow \dots \rightarrow C^n \rightarrow 0$ be a complex. We define the determinant of the complex*

$$\det(C^\bullet) = \bigotimes \det(C^i)^{(-1)^i}.$$

Observe that

$$\det(B^\bullet \oplus C^\bullet) = \det(B^\bullet) \otimes \det(C^\bullet).$$

4.3.2 An isomorphism between determinants

Proposition 4.3.2 *Let $C^\bullet$ a complex. Let $H^\bullet$ be its cohomology. There is a unique canonical isomorphism*

$$\tau_{C^\bullet} : \det(C^\bullet) \rightarrow \det(H^\bullet),$$

which satisfies

1. if $A^\bullet = B^\bullet \oplus C^\bullet$, then $\tau_{A^\bullet} = \tau_{B^\bullet} \otimes \tau_{C^\bullet}$,
2. let $C^\bullet$ be the exact sequence $0 \rightarrow \dots \rightarrow 0 \rightarrow C^k \xrightarrow{d} C^{k+1} \rightarrow 0 \dots \rightarrow 0$, whose cohomology is trivial, then

$$\tau_{C^\bullet} : \det(C^\bullet) = \det(C^k)^{(-1)^k} \otimes \det(C^{k+1})^{(-1)^{k+1}} \rightarrow \mathbb{R}$$

is the canonical pairing $\tau_{C^\bullet}(\alpha, \beta) = \langle d_* \alpha | \beta \rangle$.

PROOF: We indicate the sketch of the proof. Let us begin our construction. Any short exact sequence

$$\nu : 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0,$$

Give rise to an isomorphism τ_ν ,

$$\begin{cases} \det(A) \otimes \det(B)^* \otimes \det(C) & \rightarrow \mathbb{R} \\ a \otimes b \otimes c & \mapsto \langle b | f(a) \wedge \sigma(c) \rangle, \end{cases}$$

where $\sigma : C \rightarrow B$ is any map such that $g \circ \sigma = \text{Id}$. It follows that τ_ν – in the case $C = 0$ satisfies the second property of the proposition. This construction extends easily in two ways:

1. first, to any *acyclic* – i.e. with zero cohomology — complex

$$C^\bullet : 0 \rightarrow C^0 \xrightarrow{d} C^1 \xrightarrow{d} \dots \xrightarrow{d} C^n \rightarrow 0,$$

where it gives rise to a an isomorphism $\tau_{C^\bullet} : \det(C^\bullet) \rightarrow \mathbb{R}$.

2. to any short exact sequence of complexes $\xi : 0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$ where it gives rise to map $\tau_\xi : \det(A^\bullet) \otimes \det(B^\bullet)^{-1} \otimes \det(C^\bullet) \rightarrow \mathbb{R}$.

Finally, we can write any complex as $C^\bullet = H^\bullet \oplus A^\bullet$ where $A^\bullet$ is acyclic and $H^\bullet$ is the cohomology of $C^\bullet$. Then $\det(C^\bullet) = \det(H^\bullet) \otimes \det(A^\bullet)$ and, using the isomorphism $\tau_{A^\bullet}$, we indeed obtain an isomorphism of $\det(C^\bullet)$ with $\det(H^\bullet)$. This isomorphism is actually independent of the way to write the splitting of $C^\bullet$. From this construction, it also follows that $\tau_{C^\bullet \oplus B^\bullet} = \tau_{B^\bullet} \otimes \tau_{C^\bullet}$.

The reader is invited to check the uniqueness. Q.E.D.

4.3.3 The torsion of a metric complex

Definition 4.3.3 *We say a complex $C^\bullet$ is metric if we have a non degenerate metric on each C^i . In this context we define the torsion $\text{Tor}(C^\bullet) \in \det(H^\bullet)$ of the metric complex to be the image by τ of the volume forms on each of this C^i . Since we have not chosen any orientation, the torsion is defined only up to sign.*

It follows from the construction of the torsion that if we have a short exact sequence of metric vector spaces,

$$\xi : 0 \rightarrow D \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0,$$

such that f is an isometry then

$$\text{Tor}(\xi) = \det([g])^{-1},$$

where $[g]$ is the quotient map from $B/f(D)$ to C .

4.3.4 The torsion of a flat metric connexion

Let ∇ be a flat connection preserving a non degenerate metric on a vector bundle $\mathcal{L}$ over S . Let Γ be a ribbon graph. Observe first that the simplicial complex $C_\Gamma^\bullet$ is then a metric complex equipped with the metric

$$g(a_i, b_i) = \frac{1}{2} \sum_{\alpha_i \in \Gamma_i} g(a_i(\alpha_i), b_i(\alpha_i)).$$

We prove

Theorem 4.3.4 [REIDEMEISTER TORSION] *Let ∇ be a flat connection preserving a non degenerate metric g on a vector bundle $\mathcal{L}$ over S . Let Γ be a ribbon graph and $C_\Gamma^\bullet$ the associated metric complex. Then the torsion $\text{Tor}_g(C_\Gamma^\bullet)$ of this metric complex as an element of $\det(H_\nabla^\bullet(\mathcal{L}))$ does not depend of Γ . It is called the torsion of the connexion or the Reidemeister torsion and denoted $\text{Tor}_g(\mathcal{L})$. Moreover, if g and h are parallel metrics with the same volume form then $\text{Tor}_g(\mathcal{L}) = \text{Tor}_h(\mathcal{L})$*

PROOF: Since two ribbon graphs can be considered as subgraphs of the same graph, it suffices to show the theorem whenever Γ_a is a subgraph of Γ_b then $\tau_{\Gamma_a} = \tau_{\Gamma_b}$. This means that every edge, vertex, face of Γ_a can be written as union of edges, vertices, faces of Γ_b . This leads to a *restriction map* R from

$C_{\Gamma_b}^\bullet$ to $C_{\Gamma_a}^\bullet$. For instance if a face f of Γ_a splits as two faces (f_1, f_2) of Γ_b and if $c \in C_{\Gamma_b}^2$, we write

$$R(c)(f) = c(f_1) + c(f_2).$$

By Theorem 4.2.3 – or a direct check that we shall actually perform soon – this restriction map gives rise to an isomorphism at the cohomology level.

We want to understand the complex D given by the kernel of R . It suffices to understand the two following elementary cases which correspond to elementary subdivision.

1. Γ_b is obtained from Γ_a by adding a vertex v on an edge e , splitting it in two new edges e_1 and e_2 . In this case,

- $D^0 = \{c \mid \forall w \neq v, c(w) = 0\}$
- $D^1 = \{c \mid \forall e \notin \{e_1, e_2\}, c(e) = 0 \text{ and } c(e_1) + c(e_2) = 0\}$
- $D^2 = \{0\}$

In this case the metric restricted to $D^\bullet$ is non degenerate. Moreover, the determinant of d from D^0 to D^1 is $\sqrt{2}^{\text{rank}(\mathcal{L})}$. Hence $\text{Tor}(D^\bullet) = \sqrt{2}^{\text{rank}(\mathcal{L})}$.

2. Γ_b is obtained from Γ_a by adding an edge e between two - not necessarily - distinct vertices v_1 and v_2 , splitting a face f in two faces f_1 and f_2 .

- $D^0 = \{0\}$
- $D^1 = \{c \mid \forall \alpha \neq e, c(\alpha) = 0\}$
- $D^2 = \{c \mid \forall f \notin \{f_1, f_2\}, c(f) = 0 \text{ and } c(f_1) + c(f_2) = 0\}$

In this case the metric restricted to $D^\bullet$ is also non degenerate. Moreover, the determinant of d from D^1 to D^2 is $\sqrt{2}^{\text{rank}(\mathcal{L})}$. Hence $\text{Tor}(D^\bullet) = \sqrt{2}^{-\text{rank}(\mathcal{L})}$.

We can also observe that D is indeed acyclic and by the *snake lemma* [?] R is an isomorphism at the cohomology level. Now, let us compute the torsion of the short exact sequence of metric complexes

$$\xi : 0 \rightarrow D \rightarrow C_{\Gamma_b}^\bullet \xrightarrow{R} C_{\Gamma_a}^\bullet \rightarrow 0.$$

Let $[R]^\bullet$ be the quotient map from $C_{\Gamma_b}^\bullet/D^\bullet$ to $C_{\Gamma_a}^\bullet$. Then

1. in the first case,

- $\det([R^0]) = 1,$

- $\det([R^1]) = \sqrt{2}^{\text{rank}(\mathcal{L})}$,
- $\det([R^2]) = 1$.

It follows that $\text{Tor}(\xi = \sqrt{2}^{\text{rank}(\mathcal{L})}$. On the other hand,

$$\text{Tor}(C_{\Gamma_b}^\bullet) = \frac{\text{Tor}(D^\bullet)}{\text{Tor}(\xi)} \text{Tor}(C_{\Gamma_a}^\bullet).$$

Hence $\text{Tor}(C_{\Gamma_b}^\bullet) = \text{Tor}(C_{\Gamma_a}^\bullet)$.

2. Similarly, in the second case

- $\det([R^0]) = 1$,
- $\det([R^1]) = 1$,
- $\det([R^2]) = \sqrt{2}^{\text{rank}(\mathcal{L})}$.

It follows again that $\text{Tor}(C_{\Gamma_b}^\bullet) = \text{Tor}(C_{\Gamma_a}^\bullet)$.

We have shown that if Γ_a is a subgraph of Γ_b then

$$\text{Tor}(C_{\Gamma_b}^\bullet) = \text{Tor}(C_{\Gamma_a}^\bullet).$$

Q.E.D.

4.3.5 Torsion and symplectic complexes

Definition 4.3.5 *We say a symplectic complex $C^\bullet$ is compatible with the metric, if the corresponding identifications of C^{2-i} with $(C^i)^*$ given by the symplectic form is an isometry.*

We then have

Proposition 4.3.6 *Let $C^\bullet$ be a symplectic complex such that $H^0(C^\bullet) = H^2(C^\bullet) = 0$ then the torsion of this complex equals the volume form associated to the symplectic form on $H^1(C^\bullet)$:*

$$\text{Tor}(C^\bullet) = \pm \frac{[\omega]^d}{d!},$$

where $\dim(H^1(C^\bullet)) = 2d$.

PROOF: Let g_i be the metric on C^i . Let ω be the symplectic form on C^1 and the pairing between C^0 and C^2 given in the definition of a symplectic complex.

Observe first that by the definition of the compatibility of the metric and symplectic structure

$$\frac{\omega^{c_1}}{c_1!} = \pm \text{vol}(g_1).$$

where $\dim(C^1) = 2c_1$. Let now $\{c_a^i\}_{i \in \{1, \dots, \dim(C^0)\}}$ be an orthonormal basis of C^a , for $a \in \{0, 2\}$.

Let σ be a linear map from C^2 to C^1 so that $\sigma(C^2)$ is Lagrangian and $d \circ \sigma = \text{Id}$. Let W be a symplectic vector subspace of C^1 so that

$$\ker(d_1) = \text{im}(d_0) \oplus W.$$

Note that W is isomorphic to $H^1(C^\bullet)$ as a symplectic vector space. The existence of σ and W follows from the definition of the symplectic complex. Let us now consider the acyclic complex

$$D^\bullet : 0 \rightarrow C^0 \rightarrow \text{im}(d_0) \oplus \sigma(C^2) \rightarrow C^2 \rightarrow 0.$$

Observe that from the definition of a symplectic complex, the set

$$\{d_0(c_0^i), \sigma(c_2^j)\}_{i,j \in \{1, \dots, \dim(C^0)\}},$$

forms a symplectic basis of $\text{im}(d_0) \oplus \sigma(C^2)$. Hence

$$\tau_{D^\bullet} \left(\bigwedge_i c_0^i \wedge \frac{\omega^{\dim C_0}}{\dim(C_0)!} \wedge \bigwedge c_2^i \right) = 1.$$

We finally write

$$C^\bullet = D^\bullet \oplus W^\bullet,$$

where

$$W^\bullet : 0 \rightarrow 0 \rightarrow W \rightarrow 0 \rightarrow 0.$$

Using this splitting, it follows that

$$\begin{aligned} \text{Tor}(C^\bullet) &= \tau_{C^\bullet} \left(\bigwedge_i c_0^i \wedge \frac{\omega^{c_1}}{c_1!} \wedge \bigwedge c_2^i \right) \\ &= \tau_{D^\bullet} \left(\bigwedge_i c_0^i \wedge \frac{\omega^{\dim C_0}}{\dim C_0!} \wedge \bigwedge c_2^i \right) \otimes \tau_{W^\bullet} \left(\frac{\omega^{\frac{1}{2} \dim(W)}}{\left(\frac{\dim(W)}{2}\right)!} \right). \end{aligned}$$

To conclude the proof of the proposition, it suffices to remark that W is isomorphic as a symplectic vector space to $H^1(C^\bullet) = H^1(W^\bullet)$. The result follows. Q.E.D.

We obtain as a corollary

Corollary 4.3.7 *Let $\mathcal{L}$ equipped with a connexion ∇ be a flat vector bundle. Assume that $H_{\nabla}^0(\mathcal{L}) = H_{\nabla}^2(\mathcal{L}) = \{0\}$. Note that $\mathcal{L} \oplus \mathcal{L}^*$ is equipped with a symmetric non degenerate metric h given by*

$$h((a, a^*), (b, b^*)) = \langle a | b^* \rangle + \langle b | a^* \rangle.$$

Then, we have

$$\text{Tor}(\mathcal{L} \oplus \mathcal{L}^*) = \pm \frac{\Omega^{\dim(H_{\nabla}^1(\mathcal{L}))}}{\dim(H_{\nabla}^1(\mathcal{L}))!},$$

where Ω is the symplectic form on $H_{\nabla}^1(\mathcal{L}) \oplus H_{\nabla}^1(\mathcal{L}^*)$ described in Theorem 4.2.8.

PROOF: it follows at once from the definition of the symplectic structure on $C^{\bullet} = C_{\Gamma}^{\bullet} \oplus C_{\Gamma^*}^{\bullet}$ by Formulae (4.5) and (4.6) that the metric is compatible with the symplectic structure.

4.3.6 Parallel metric on bundle

We now come to the main result of this section

Theorem 4.3.8 [TORSION AND SYMPLECTIC VOLUME] *Let $\mathcal{L}$ be a vector bundle over S equipped with a flat connexion and a parallel metric g . Assume that $H_{\nabla}^0(\mathcal{L}) = H_{\nabla}^2(\mathcal{L}) = 0$. Let ω be the symplectic structure on $H_{\nabla}^1(\mathcal{L})$ given by*

$$\omega([\alpha], [\beta]) = \int_S g(\alpha \wedge \beta).$$

Then

$$\text{Tor}(\mathcal{L}) = \frac{\omega^{\frac{1}{2} \dim(H_{\nabla}^1(\mathcal{L}))}}{\frac{1}{2} \dim(H_{\nabla}^1(\mathcal{L}))!}.$$

REMARKS:

- This theorem was proved in [?] for the case of positive metric.
- Observe that Corollary 4.3.7 is a special case of this theorem. However, it will be the starting point of the proof.
- The proof of this theorem does not actually need the full strength of Theorem 4.3.4 but the immediate observation that the torsion of $C_{\Gamma}^{\bullet}$ is equal to the torsion of $C_{\Gamma^*}^{\bullet}$.

PROOF: Let $p = \frac{1}{2} \dim(H_{\nabla}^1(\mathcal{L}))$. We use the corollary 4.3.7 and its notations. In order to make our notations precise, we shall indicate by a subscript the metric with respect to which we compute the torsion. We first remark that the volume of h and $g \oplus g^*$ coincide on $\mathcal{L} \oplus \mathcal{L}^*$. It follows that

$$\mathrm{Tor}_h(\mathcal{L} \oplus \mathcal{L}^*) = \mathrm{Tor}_{g \oplus g^*}(\mathcal{L} \oplus \mathcal{L}^*) = \mathrm{Tor}_g(\mathcal{L}) \otimes \mathrm{Tor}_{g^*}(\mathcal{L}^*). \quad (4.12)$$

Let us use the metric to identify $\mathcal{L}$ and $\mathcal{L}^*$ and hence $H_{\nabla}^1(\mathcal{L})$ with $H_{\nabla}^1(\mathcal{L}^*)$. It follows from the duality Theorem 4.2.8 that

$$\mathrm{Tor}_g(\mathcal{L}) = \mathrm{Tor}_{g^*}(\mathcal{L}^*). \quad (4.13)$$

Let now λ be the real number such that

$$\mathrm{Tor}_g(\mathcal{L}) = \mathrm{Tor}_{g^*}(\mathcal{L}^*) = \lambda \frac{\omega^p}{p!2^p}.$$

By Assertions (4.12), (4.13) and the previous corollary, we obtain that

$$\frac{1}{(2p)!} \Omega^{2p} = \lambda^2 \left(\frac{1}{p!} i_*^0 \omega^p \wedge \frac{1}{p!} i_*^1 \omega^p \right), \quad (4.14)$$

where $i_0 : u \rightarrow (u, 0)$ and $i_1 : u \rightarrow (0, u)$ are the two embeddings of $H^1(\mathcal{L})$ as factors in $H_{\nabla}^1(\mathcal{L}) \oplus H_{\nabla}^1(\mathcal{L}^*)$.

Finally, note that $j^- : u \rightarrow \frac{1}{\sqrt{2}}(u, u)$ and $j^+ : u \rightarrow \frac{1}{\sqrt{2}}(u, -u)$ are symplectic embeddings of $H_{\nabla}^1(\mathcal{L})$ equipped with ω into $H_{\nabla}^1(\mathcal{L}) \oplus H_{\nabla}^1(\mathcal{L}^*)$ equipped with Ω . Therefore

$$\frac{1}{(2p)!} \Omega^{2p} = \lambda^2 \left(\frac{1}{p!} i_*^0 \omega^p \wedge \frac{1}{p!} i_*^1 \omega^p \right), \quad (4.15)$$

But $\det \left((j^+, j^-)^{-1} \circ (i_0, i_1) \right) = 1$. This and Formulae (4.15) and (4.15) implies that $\lambda^2 = 1$. This completes the proof of the following formula

$$\mathrm{Tor}_g(\mathcal{L}) = \pm \frac{\omega^p}{p!}.$$

Q.E.D.

Chapter 5

Moduli spaces

The purpose of this chapter is the study of representation varieties $\text{Rep}(\pi_1(S), G)$ for a compact oriented surface S and a Lie group G . We will abbreviate

$$\mathcal{M}(S, G) := \text{Rep}(\pi_1(S), G).$$

Here the letter $\mathcal{M}$ stands for *moduli space*. We shall use this term as a synonym for representation variety. By the results of the last chapter we may think of $\mathcal{M}(S, G)$ as a moduli space of flat connections or a moduli space of local systems or a moduli space of representations. As the term representation variety suggests that we are thinking of the latter model, we prefer to use the more general term *moduli space* unless we are really thinking of representations.

5.1 Topology and smooth structure

In the study of the moduli space $\mathcal{M}(S, G)$ we will need some elementary notions on (real) algebraic varieties which we summarise in the next definitions :

Definition 5.1.1 [AFFINE ALGEBRAIC VARIETIES] *An affine algebraic variety (over $\mathbb{R}$) is a subset Σ of $\mathbb{R}^n$ for which there exists a family of real polynomials $\{P_i\}_{i \in I}$ in n variables such that*

$$\Sigma = \bigcap_{i \in I} P_i^{-1}(\{0\}).$$

From the smooth point of view, a real algebraic manifold is a finite union of submanifolds V_i each of them of dimension i for $i \in \{0, \dots, n\}$. The submanifold V_n is the set of smooth points.

We now specialise these definitions to groups.

Definition 5.1.2 [ALGEBRAIC GROUPS AND ALMOST ZARISKI DENSE SUBGROUPS] *A subgroup G of $GL(n, \mathbb{R})$ is a (real) algebraic group if it is an affine algebraic variety as a subset of $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. The Zariski closure $\bar{A}^Z$ of a subset $A \subset \mathbb{R}^n$ is*

$$\bar{A}^Z = \bigcap \Sigma,$$

where the intersection is taken over all affine algebraic varieties Σ containing A . The set A is Zariski closed if $A = \bar{A}^Z$. A subgroup Γ of a real algebraic group G is almost Zariski-dense in G if the Zariski closure $\bar{\Gamma}^Z$ contains the identity component of G .

The definition is of course very much ad hoc, but it is good enough for our purposes.

Up to now we have treated the moduli spaces $\mathcal{M}(S, G)$ only as a set without topology. Now we are going to turn them into topological spaces. We make the following assumption:

Convention 5.1.3 From now on the letter S will always denote a compact connected surface (without boundary) of genus $g \geq 2$. Thus S is neither a sphere nor a torus. The letter G will denote an algebraic group whose only normal subgroups are either finite or with finite index.

In order to define a topology on $\mathcal{M}(S, G)$ we think of it as a space of discrete flat connections. Thus we fix a filling ribbon graph Γ of S with vertices V , edges E and faces F and think of $\mathcal{M}(S, G)$ as

$$\mathcal{M}(S, G) = \mathcal{D}_\Gamma(S, G)/G^V,$$

where

$$\mathcal{D}_\Gamma(S, G) = \{\nabla \in G^E \mid \nabla \text{ flat } G\text{-connection}\} \subset G^E.$$

denotes the set of discrete flat G -connections along Γ . If we equip $G^E = \prod_E G$ with the product topology then the inclusion $\mathcal{D}_\Gamma(S, G) \subset G^E$ induces a topology τ on $\mathcal{D}_\Gamma(S, G)$. The set $\mathcal{D}_\Gamma(S, G)$ is defined inside G^E by the vanishing of polynomials (namely $\nabla(e)\nabla(\bar{e}) - 1 = 0$ for edges e and $\text{Hol}_f(\nabla) = 0$ for faces f), and these polynomials and their derivatives have no common zeros. Moreover if $\bar{\Gamma}$ is a refinement of Γ , the restriction map sends $\mathcal{D}_\Gamma(S, G)$ to $\mathcal{D}_{\bar{\Gamma}}(S, G)$ in an equivariant way. Summarising, we get

Proposition 5.1.4 *The subspace $\mathcal{D}_\Gamma(S, G) \subset G^E$ is an affine subvariety. Moreover, the induced topology on $\mathcal{M}_\Gamma(S, G)$ is independent on the choice of Γ*

The topology on $\mathcal{D}_\Gamma(S, G)$ induces a topology on the quotient $\mathcal{M}(S, G)$. If the action of the Gauge group G^V on $\mathcal{D}_\Gamma(S, G)$ were proper and free then this quotient would be a nice manifold as well. Unfortunately, it is not so and the space $\mathcal{M}(S, G)$ with the quotient topology is not even Hausdorff in general. In order to complete our understanding of $\mathcal{M}(S, G)$ we need introduce the following subset of $\mathcal{D}_\Gamma(S, G)$.

Definition 5.1.5 [ZARISKI DENSE PART] *The Zariski dense part of the space of discrete flat connections is the subset*

$$\mathcal{D}_\Gamma^{zd}(S, G) = \{\nabla \in \mathcal{D}_\Gamma(S, G) \mid \rho_\nabla(\pi_1(S)) \text{ almost Zariski-dense in } G\},$$

We can now state our main result

Theorem 5.1.6 [SMOOTH STRUCTURE OF THE MODULI SPACE] *Let Γ be a ribbon graph.*

- (i) *The set $\mathcal{D}_\Gamma^{zd}(S, G)$ is a Zariski-open non-empty subset of $\mathcal{D}_\Gamma(S, G)$ which is a submanifold.*
- (ii) *The Gauge group G^V acts freely and properly on $\mathcal{D}_\Gamma^{zd}(S, G)$.*
- (iii) *The quotient*

$$\mathcal{M}^{zd}(S, G) := \mathcal{D}_\Gamma^{zd}(S, G)/G^V$$

is a manifold of dimension $-\chi(S) \dim G$ whose tangent space at the class of a representation $[\rho]$ is canonically identified with $H_\rho^1(\mathfrak{g})$.

- (iv) *The complement of $\mathcal{D}_\Gamma^{zd}(S, G)$ in $\mathcal{D}_\Gamma(S, G)$ is a finite union of G^V -orbits union of algebraic manifolds of the form $\mathcal{D}_\Gamma(S, H)$, where H varies over the algebraic subgroups of G . It is a finite union of submanifolds of dimension strictly less than the dimension of $\mathcal{D}_\Gamma^{zd}(S, G)$.*

In the next chapter, we shall explain in Theorem 6.1.4 how the smooth part of the moduli space describes smooth family of flat connections.

A way to visualise this theorem, is the following: although $\mathcal{M}(S, G)$ is not a manifold, it admits a stratification by manifolds and there is one big open stratum given by the representations with almost Zariski dense image.

We warn the reader that in general, the Zariski dense part is not dense as it follows from [?].

5.2 Proof of the main theorem

In the next preliminary paragraph, we describe almost Zariski dense subgroups in an algebraic group.

Then we prove Theorem 5.1.6 in several steps. We spend some time now to explain the structure of this proof. First we use the following

Convention 5.2.1 In what follows, we denote by L_0 the connected component of the identity of the group L . If H is a subgroup of L , we denote by $Z(H)$ the *centraliser* of H in L , that is the group of elements commuting with H .

Let us now introduce the following subsets of $\mathcal{D}_\Gamma(S, G)$:

Definition 5.2.2 [REGULAR, VERY REGULAR PARTS] *The regular part of $\mathcal{D}_\Gamma(S, G)$ is*

$$\mathcal{D}_\Gamma^r(S, G) = \{ \nabla \in \mathcal{D}_\Gamma(S, G) \mid Z(\rho_\nabla(\pi_1(S))) \text{ is finite} \}.$$

The very regular part of $\mathcal{D}_\Gamma(S, G)$ is

$$\mathcal{D}_\Gamma^{vr}(S, G) = \{ \nabla \in \mathcal{D}_\Gamma(S, G) \mid Z(\rho_\nabla(\pi_1(S))) = Z(G) \}.$$

We observe that we have the following inclusions when $Z(G) = Z(G_0)$.

$$\mathcal{D}_\Gamma^{zd}(S, G) \subset \mathcal{D}_\Gamma^{vr}(S, G) \subset \mathcal{D}_\Gamma^r(S, G).$$

Indeed if H is a group then $Z(Z(H))$ is an algebraic group that contains H . Thus, if H is almost Zariski dense, $G_0 \subset Z(Z(H))G$. It follows that $Z(H)$ is included in $Z(G_0) = Z(G)$, hence equal. This proves the first inclusion, the second is obvious.

We prove Theorem 5.1.6 using the following steps

1. In Paragraph 5.2.2, we study the holonomy map and prove in Proposition 5.2.9 that the very regular part is a smooth submanifold.
2. In Paragraph 5.2.3, we prove in Proposition 5.2.13 that the almost zariski dense part is non empty. This completes the proof of the first part of Theorem .
3. In Paragraph 5.2.4, we prove that the action of the Gauge group is proper and free on the regular part.

4. In Paragraph 5.2.5, we identify the tangent space to $\mathcal{M}^{zd}(S, G)$ with a cohomology group.
5. In Paragraph 5.2.6, we study the complementary set to $\mathcal{D}_\Gamma^{zd}(S, G)$.

We will not give complete proofs in full generality. This would require more knowledge about the structure of Lie groups than what we have assumed in this series of lectures. Nevertheless we will give the complete proof for $G = \mathrm{SL}(n, \mathbb{R})$ and give some hints about how to complete the proofs in the setting of a general semi-simple group.

5.2.1 The Zariski closure of a group

Here are some properties that we are going to use freely

Proposition 5.2.3 *1. An affine algebraic variety $\Sigma \subset \mathbb{R}^n$ has finitely many connected components when equipped with the topology induced from the metric topology on $\mathbb{R}^n$.*

- 2. The Zariski closure of a group is a group.*
- 3. An algebraic group is a submanifold in $\mathrm{SL}(n, \mathbb{R})$ and hence a Lie group.*
- 4. If G is algebraic, if H is a subgroup of G , then the normaliser of H is an algebraic subgroup of G .*

Almost Zariski-dense subgroups admit the following nice characterisation. In this proposition $\mathrm{Gr}_k(\mathfrak{g})$ denotes the Grassmannian of k -planes in $\mathfrak{g}$.

Proposition 5.2.4 *Let G be an algebraic group in $\mathrm{SL}(n, \mathbb{R})$. We assume that $\mathfrak{g}$ does not contain any non trivial ideal. Let Λ be an infinite group in G generated by $\{a_1, \dots, a_n\}$. Then the following are equivalent:*

- (i) Λ is almost Zariski-dense in G .*
- (ii) For any $0 < k < \dim \mathfrak{g}$ we have*

$$\bigcap_{i=1}^n \mathrm{Fix}_k(a_i) = \emptyset,$$

where for $g \in G$ we denote

$$\mathrm{Fix}_k(g) := \{W \in \mathrm{Gr}_k(\mathfrak{g}) \mid g.W = W\}.$$

We obtain as an immediate corollary

Corollary 5.2.5 *Let G be an algebraic group in $\mathrm{SL}(n, \mathbb{R})$. We assume that $\mathfrak{g}$ does not contain any non trivial ideal. Let Λ be an infinite group in G generated by a finite subset S . Then the set of homomorphisms of Λ in G_0 which are almost Zariski dense, seen as a subset of G^S , is an open Zariski subset.*

PROOF: Let G_0 be the identity component of G .

Assume that Λ is not Zariski-dense in G and let $H = \bar{\Lambda}^Z$ and $\mathfrak{h}$ be its Lie algebra.

We first claim that $0 < \dim \mathfrak{h} < \dim \mathfrak{g}$. Indeed, if $\dim \mathfrak{h} = \dim \mathfrak{g}$, then H contains G_0 and thus Λ is almost Zariski-dense. On the other hand, if $\dim \mathfrak{h} = 0$, then H is discrete. By Proposition 5.2.3, H is finite. Since H contains Λ , we obtain again a contradiction.

Let $k := \dim \mathfrak{h}$. As H contains Λ , the adjoint action of Λ fixes $\mathfrak{h}$. Thus

$$\mathfrak{h} \in \bigcap_{i=1}^n \mathrm{Fix}_k(a_i),$$

which violates (ii).

Conversely assume that Λ is almost Zariski-dense and that (ii) is violated. Then there exist k , with $0 < k < \dim \mathfrak{g}$, and

$$W \in \bigcap_{i=1}^n \mathrm{Fix}_k(a_i)$$

.

Let Σ be the stabiliser of W in G

$$\Sigma := \{g \in G \mid \mathrm{Ad}(g)W = W\}.$$

We observe that Σ is an algebraic group: indeed, we consider $\mathfrak{g}$, hence W , as a subspace of $M(n, \mathbb{R})$ and note that Σ is the intersection of G and the stabiliser of W in $\mathrm{SL}(n, \mathbb{R})$. Since Σ contains Λ , Σ also contains G_0 . Thus, G_0 stabilises W . Hence, by taking derivatives, for all $X \in \mathfrak{g}$,

$$\mathrm{ad}(X)W \subset W,$$

which means that W is an ideal. As $0 < k < \dim \mathfrak{g}$ this ideal is non-trivial which contradicts the simplicity of $\mathfrak{g}$. Q.E.D.

We have also the following criterion for products of almost Zariski-dense representations to be almost Zariski-dense:

Proposition 5.2.6 *Let G be an algebraic group with finite centre such that every normal subgroup is either finite or with finite index. Let G_0 be the connected component of the identity. Let Λ be a group and let $\rho_1 : \Lambda \rightarrow G_0$ and $\rho_2 : \Lambda \rightarrow G_0$ be two representations with almost Zariski-dense image. Then*

- *either there exists an automorphism h of G_0 such that ρ_1 and $h \circ \rho_2$ are equal,*
- *or $\rho_1 \times \rho_2 : \Lambda \rightarrow G \times G$ has also almost Zariski-dense image.*

PROOF: Let H be the Zariski closure of $(\rho_1 \times \rho_2)(\Lambda)$. Let π_1 and π_2 be the projections from $G \times G$ to the first factor and second factor.

We first prove that $\pi_i(H_0) = G_0$. First observe that the connected group $\pi_i(H_0) = G_0$ contains $\rho_i(\Lambda)$ and is thus non trivial. As a consequence, $\mathfrak{l}_i := T\pi_i(\mathfrak{h}_0)$, where $\mathfrak{h}_0$ is the Lie algebra of H_0 , is non trivial. Since $\mathfrak{l}_i$ is normalised by the adjoint action of $\rho_i(\Lambda)$ and the latter is almost Zariski dense, it follows that $\mathfrak{l}_i$ is the full Lie algebra of G_0 . Hence π_i is open, thus surjective.

We now prove that π_i is a covering. Let's consider $V = H_0 \cap G_0 \times \{e\}$.

Our first claim is that V is either finite or $V = G_0$. Observe that V is normalised by H , hence by $(\rho_1 \times \rho_2)(\Lambda)$, hence by $\rho_1(\Lambda)$ since the second factor acts trivially on V . It follows by Zariski density of $\rho_1(\Lambda)$ that V is a normal subgroup of G . Our claim follows by our hypothesis on G .

If $V = G_0$, since π_2 is surjective, it follows that $G_0 \times G_0 \subset H$ hence that $\rho_1 \times \rho_2(\Lambda)$ is almost Zariski dense.

Assume now that V is finite. This implies that π_1 and π_2 are finite coverings. They give rise to isomorphisms of $H_0/Z(H_0)$ with $G_0/Z(G_0)$. Therefore there exists an automorphism g of $G/Z(G)$ such that

$$H_0/Z(H_0) = \{(x, g(x)) \in G_0/Z(G_0) \times G_0/Z(G_0)\}.$$

We can lift g to an automorphism h of G_0 . It follows the graph of h and H_0 are two connected subgroups of finite index of $\pi^{-1}(H_0/Z(H_0))$ where π is the projection from $G \times G$ to $G/Z(G) \times G/Z(G)$. Hence these two groups are equal. This finishes the proof. Q.E.D.

We will only use the following special case of this :

Corollary 5.2.7 *Let G be as in the previous proposition. Let $\rho_1 : \Lambda \rightarrow G_0$ and $\rho_2 : \Lambda \rightarrow G_0$ be two representations with almost Zariski-dense image. If $\text{Tr } \rho_1(\gamma) = \text{Tr } \rho_2(\gamma)$ for all $\gamma \in \Lambda$ then ρ_1 and ρ_2 are conjugate by an automorphism in G_0 .*

PROOF: If the representations have the same traces then they are contained in a proper subvariety of $G \times G$ and thus not Zariski dense. Q.E.D.

5.2.2 More on the holonomy map

Let us denote by $G^{|E|}$ the set of discrete G -connections, i.e. maps $\alpha \in G^E$ with $\alpha(\bar{e}) = \alpha(e)^{-1}$ and let $\mu : G^{|E|} \rightarrow G^F$ be the holonomy map. Note that a discrete G -connection α is a flat if and only if $\mu(\alpha) = 1$ is the constant map. We claim that for $\alpha \in \mathcal{D}_\Gamma^{vr}(S, G)$ the tangent map

$$T_\alpha \mu : T_\alpha G^{|E|} \rightarrow T_1 G^F$$

is onto. Then (ii) will follow from some version of the implicit function theorem. Note that for any $g \in G$ we can identify $T_g G$ with $\mathfrak{g}$; if we view the elements of G and $\mathfrak{g}$ as matrices then this identification is given simply by $\dot{g} \mapsto \dot{g}g^{-1}$. Under this identification of $T_g G$ with $\mathfrak{g}$ the map $T_\alpha \mu$ becomes identified with a map

$$\dot{\mu} : \mathfrak{g}^{|E|} \rightarrow \mathfrak{g}^F.$$

Now one can identify the map $\dot{\mu}$ by checking the following proposition:

Proposition 5.2.8 *For $\alpha \in \mathcal{D}_\Gamma^{vr}(S, G)$ the diagram*

$$\begin{array}{ccc} T_\alpha G^{|E|} & \xrightarrow{T_\alpha \mu} & T_1 G^F \\ \downarrow & & \downarrow \\ \mathfrak{g}^{|E|} & \xrightarrow{d_\Gamma^1} & \mathfrak{g}^F \end{array}$$

commutes. Thus $\dot{\mu} = d_\Gamma^1$.

As a consequence, we have

Proposition 5.2.9 *The regular part $\mathcal{D}_\Gamma^r(S, G)$ is the set of regular points of μ in $\mathcal{D}_\Gamma(S, G)$. The regular part is an open Zariski set which is a submanifold.*

PROOF: The regular part is easily shown to be a Zariski open set. Therefore the full proposition is a consequence of the first statement. By Proposition 5.2.8, we have to understand when d_Γ^1 is surjective. By the definition of twisted cohomology d_Γ^1 is surjective exactly when $H_\Gamma^2(\mathfrak{g}) = \{0\}$. On the other hand by Poincaré duality $H_\Gamma^2(\mathfrak{g}) = \{0\}$ is dual to $H_\Gamma^0(\mathfrak{g}) = \{0\}$. Finally $H_\Gamma^0(\mathfrak{g}) = \{0\}$ is the Lie algebra of the centraliser of the holonomy. Hence the result follows. Q.E.D.

Regular elements

We need a refinement of the discussion in the last paragraph for $\Gamma = \Gamma_g$. We also need some properties of centralisers of elements. Let us begin with some definition

Definition 5.2.10 [SINGULAR AND REGULAR ELEMENTS] *Let G be an algebraic group. We say an element g of G is singular if there exists a subgroup K such that $\dim Z(K) > 0$ and such that $g \in [K, K]$, and regular otherwise.*

The reader is invited to check that for $SU(2)$ the set of singular elements is the centre. A real split element in $SL(n, \mathbb{R})$ is singular, if a subproduct of eigenvalues is equal to 1.

We now prove

Proposition 5.2.11 *Let μ be the map from G^{2g} to G given by*

$$(a_1, b_1, \dots, a_g, b_g) \mapsto \prod_{i=1}^{i=g} [a_i, b_i].$$

Then every regular element in G is a regular value of μ .

PROOF: As a first computation, let ν be the map from G^k to G ,

$$(c_1, \dots, c_k) \rightarrow c_1 \dots c_k.$$

Identifying $T_x G$ with $\mathfrak{g}$ by the map $u \rightarrow g^{-1}.u$, we obtain that $T\nu$ from $\mathfrak{g}^k$ to $\mathfrak{g}$ is given by

$$(u_1, \dots, u_k) \mapsto \sum_{i=1}^{i=k} (\text{ad}(c_{i+1} \dots c_k)^{-1} u_i).$$

A similar computation for μ yields that

$$\begin{aligned} & T\mu(u_1, v_1, \dots, u_g, v_g) \\ &= \sum_{i=1}^{i=g} \text{ad}\left(\prod_{j>i} [a_j, b_j]\right)^{-1} (\text{ad}(b_i a_i b_i^{-1}).u_i + \text{ad}(b_i a_i).v_i - \text{ad}(b_i a_i).u_i - \text{ad}(b_i).v_i) \\ &= \sum_{i=1}^{i=g} \text{ad}\left(\prod_{j>i} [a_j, b_j]\right)^{-1} (\text{ad}(b_i a_i b_i^{-1})(1 - \text{ad}(b_i))u_i + \text{ad}(b_i)(\text{ad}(a_i) - 1).v_i) \\ &= \sum_{i=1}^{i=g} (\text{ad}(A_i)(1 - \text{ad}(b_i))u_i + \text{ad}(B_i)(\text{ad}(a_i) - 1).v_i), \end{aligned}$$

where

$$A_i = \left(\prod_{j>i} [a_i, b_i] \right)^{-1} b_i \cdot a_i \cdot b_i^{-1}, \quad B_i = \left(\prod_{j>i} [a_i, b_i] \right)^{-1} b_i.$$

Observe now that the image of the endomorphism of $\mathfrak{g}$ given by $u \rightarrow (1 - \text{ad}(c))u$ is the orthogonal — with respect to the ad invariant metric — of the lie algebra of $Z(c)$. It follows that the orthogonal of the image of $T\mu$ is the Lie algebra of

$$H = \bigcap_{i=1}^{i=g} Z(A_i a_i A_i^{-1}) \bigcap_{i=1}^{i=g} Z(B_i b_i B_i^{-1}) = Z\langle a_1, b_1, \dots, a_g, b_g \rangle.$$

It follows that $(a_1, b_1, \dots, a_g, b_g)$ is a regular element for μ if and only if the centraliser of the group generated by these elements is finite.

Therefore, let $m = (a_1, \dots, a_g)$ and $\Lambda(m) = \langle a_1, \dots, a_g \rangle$, then m is a singular point for μ if $\dim(Z(\Lambda(m))) > 0$. Observe moreover that $\mu(m)$ belongs to $[\Lambda(m), \Lambda(m)]$. By definition, $\mu(m)$ is a singular point in G . Q.E.D.

5.2.3 The Zariski dense part is Zariski open and non-empty

It follows from Corollary 5.2.5 that $\mathcal{D}_F^{zd}(S, G)$ is Zariski open. We will now prove that it is non empty. We will give a complete proof only for $G = \text{SL}(n, \mathbb{R})$

Lemma 5.2.12 *Let G be a semi-simple algebraic group, then there exist two elements in G that generates a Zariski dense group.*

We will only prove this lemma for $G = \text{SL}(n, \mathbb{R})$ and admits the result for general semi-simple groups. We say that a matrix $a \in \text{SL}(n, \mathbb{R})$ is $\mathbb{R}$ -split if it is diagonalisable with distinct real eigenvalues.

PROOF: Let a be any real split matrix. Suppose that in the notation of Proposition 5.2.4 we have $W \in \text{Fix}_k(a)$. Then W is generated by eigenvectors of a (for the adjoint action): more precisely, if $\{e_i\}$ is the basis of eigenvectors of a in $\mathbb{R}^n$, and $\{e^i\}$ the dual basis, then W is generated by elements of the form $e^j \otimes e_i$. Thus by choosing the eigenvectors carefully it is possible to find another split matrix b such that $\text{Fix}_k(a) \cap \text{Fix}_k(b) = \emptyset$ for all $0 < k < \dim \mathfrak{g}$. Then the lemma follows from Proposition 5.2.4. Q.E.D.

Now we can prove

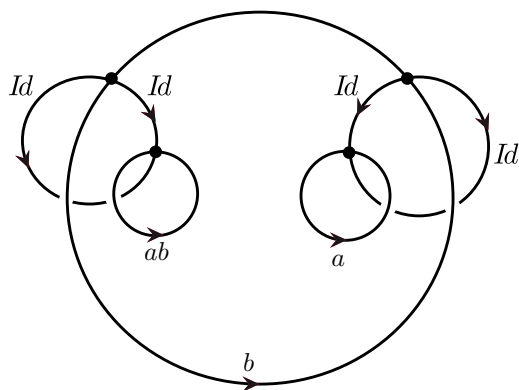


Figure 5.1: A flat discrete connection on a surface of genus 2

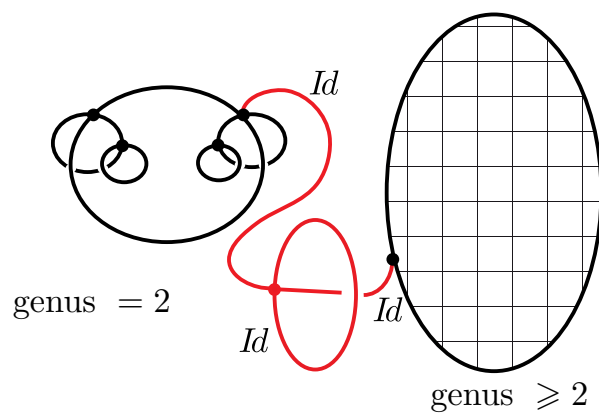


Figure 5.2: Very regular connections for higher genus

Proposition 5.2.13 *let G be an algebraic semi-simple group. Let S be a surface of genus greater than 1. Then $\mathcal{D}_\Gamma^{vr}(S, G)$ is non empty.*

PROOF: Choose a and b as in Lemma 5.2.12. For a pair of pants surface (i.e. a sphere with three holes) one can explicitly write down a flat discrete connection such that the image of the holonomy representation contains a and b and is therefore Zariski-dense. For example one can take the flat discrete connection depicted in Figure 5.1. Gluing two pairs of pants with this discrete connection as in Figure 5.2 yields a discrete connection on a surface of genus 2, whose holonomy representation still contains a and b and thus is still very regular. If S is a surface of genus > 2 then we can decompose it along a disc into a surface of genus $g - 1$ with one hole and a torus with one hole. Gluing connections as in Figure 5.2 we obtain a very regular connection on S by induction on g . Q.E.D.

5.2.4 The action of G^V on $\mathcal{M}^{zd}(S, G)$ is proper

We prove the second part of Theorem 5.1.6

Proposition 5.2.14 *The group G^V acts freely and properly on $\mathcal{D}_\Gamma^{zd}(S, G)$.*

The stabiliser of one element is identified with the centraliser of the holonomy. Hence the action is free on $\mathcal{D}_\Gamma^{vr}(S, G)$ which contains $\mathcal{D}_\Gamma^{zd}(S, G)$. The most difficult part is to verify that the G -action is proper. We cannot prove this in general here, so let us consider two special cases. The trivial case is the case, where G is compact. In this case, any action of G is proper and there is nothing to show. In the non-compact case one needs again some more results on algebraic groups. An elementary proof with linear algebra methods is possible if we make two simplifications. Firstly, let us assume $\Gamma = \Gamma_g$. Then we have

$$\mathcal{M}_{\Gamma_g}^{vr}(S, G) = \text{Hom}^{vr}(\pi_1(S), G)$$

and $G^V = G$. Thus the properness of the G^V -action is equivalent to the following statement:

(*) *Let $\{\rho_n\}_{n \in \mathbb{N}} \subset \text{Hom}^{vr}(A_g, G)$ be a convergent sequence. Then any sequence $\{g_n\}_{n \in \mathbb{N}} \in G$ for which $g_n \rho_n g_n^{-1}$ also converges, is bounded.*

Now let us prove the statement (*) and we shall only do it for $G = \text{SL}(p, \mathbb{C})$. For this we can as well assume that $\rho_n \rightarrow \bar{\rho}_0$ and that $\{g_n\}_{n \in \mathbb{N}} \in G$ satisfies $g_n \rho_n g_n^{-1} \rightarrow \bar{\rho}_1$. We need two lemmas

Lemma 5.2.15 *There exist $a, b \in \pi_1(S)$ such that $\bar{\rho}_0(a), \bar{\rho}_0(b), \bar{\rho}_1(a), \bar{\rho}_1(b)$ are all $\mathbb{C}$ -split and the axis' of $\bar{\rho}_0(a)$ (respectively $\bar{\rho}_1(a)$) are different from the axis' of $\bar{\rho}_0(b)$ (respectively $\bar{\rho}_1(b)$).*

PROOF: We distinguish two cases depending on whether $\bar{\rho}_0$ and $\bar{\rho}_1$ are conjugate or not. In the former case it suffices to choose a and b in such a way that $\bar{\rho}_0(a)$ and $\bar{\rho}_0(b)$ are $\mathbb{C}$ -split with distinct axis. Now the set of $\mathbb{C}$ -split elements of G is Zariski-open and non-empty and thus contains an element $\bar{\rho}_0(a)$ from the Zariski-dense subset $\bar{\rho}_0(A_g)$. Moreover, the set of $\mathbb{C}$ -split elements of G with axis' different from those of a is still Zariski open and this allows us to choose b . This settles the case where $\bar{\rho}_0$ and $\bar{\rho}_1$ are conjugate. In the other case it follows from Proposition 5.2.6 that $(\rho_0 \times \rho_1)(\pi_1(S))$ is Zariski dense in $G \times G$. Then the argument is reduced to the first case. Q.E.D.

Let us choose a and b as in the lemma. As $\rho_n(a) \rightarrow \bar{\rho}_0(a)$ and the set of $\mathbb{C}$ -split elements in G is open, we may assume that $\rho_n(a)$ is $\mathbb{C}$ -split for all $n \in \mathbb{N}$. Let $\alpha_n \in (\mathbb{P}(\mathbb{C}^p))^p / S^p$ denote the set of axis' of $\rho_n(a)$. Then $\alpha_n \rightarrow \bar{\alpha}_0$, where $\bar{\alpha}_0$ is the set of axis' of $\bar{\rho}_0(a)$. Similarly, $g_n \alpha_n \rightarrow \bar{\alpha}_1$, where $\bar{\alpha}_1$ is the set of axis' of $\bar{\rho}_1(a)$. The compact group $\mathrm{SU}(p) \subset \mathrm{SL}(p, \mathbb{C})$ acts transitively on the projective space and thus we can write

$$g_n = t_n u_n,$$

where $u_n \in \mathrm{SU}(p)$ and t_n is diagonal with respect to a basis of eigenvectors of $\bar{\alpha}_1$. As $\mathrm{SU}(p)$ is bounded we can reduce our discussion to the case where g_n are diagonal matrices with respect to a basis of eigenvectors of $\bar{\alpha}_1$.

For that we need the following lemma:

Lemma 5.2.16 *Let $\{g_n\}_{n \in \mathbb{N}}$ be an unbounded sequence of diagonal matrices in $\mathrm{SL}(p, \mathbb{C})$. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of elements of $\mathbb{P}(\mathbb{C}^p)$ which converges to an axis, which is not a coordinate axis. Then $\{g_n x_n\}_{n \in \mathbb{N}}$ converges to a coordinate axis.*

PROOF: We give the proof only for $p = 2$. We then have $g_n = \mathrm{diag}(\lambda_n, \lambda_n^{-1})$ and we may assume that $\lambda_n \rightarrow \infty$. If $x_n = [a_n : b_n]$ converges to $[a : b]$ with $a, b \neq 0$. Then

$$g_n x_n = [\lambda_n a_n : \lambda_n^{-1} b_n] \rightarrow [1 : 0].$$

Q.E.D.

Now we can finish the proof: Let $\beta_n, \bar{\beta}_0$ and $\bar{\beta}_1$ denote the set of axis' of $\rho_n(b), \bar{\rho}_0(b)$ and $\bar{\rho}_1(b)$ respectively. Then $g_n \beta_n \rightarrow \bar{\beta}_1$ and $\beta_n \rightarrow \bar{\beta}_0$. Write all

elements of G as matrices with respect to a basis of eigenvectors of $\bar{\alpha}_1$. Then the matrices $g_n \in \mathrm{SL}(p, \mathbb{C})$ are diagonal. Also as $\bar{\beta}_0 \neq \bar{\alpha}_1$, β_n converges to an axis', which is not a coordinate axis. Thus, by the lemma, if g_n was unbounded then $g_n \beta_n \rightarrow \bar{\alpha}_1$, which would imply $\bar{\alpha}_1 = \bar{\beta}_1$. But this contradicts the choice of a and b . Therefore $\{g_n\}_{n \in \mathbb{N}}$ must be bounded. This finishes our indication of the proof of (iii).

Hint of the proof in the general case. We just sketch here how the general proof works. First we need the following fact about actions of algebraic groups: to show that an algebraic action of an algebraic group on an algebraic variety is proper, it suffices to show that the orbits are closed. Indeed, by Rosenlicht theorem [?] the action is stratified in the following sense: there is a sequence of Zariski closed sets

$$V_n \subset V_{n-1} \subset V_{n-2} \subset \dots \subset V_1 \subset V_0 = V,$$

such that the action on $V_i \setminus V_{i+1}$ is proper.

In our case, the fact that the orbit are closed follows from Corollary 5.2.7: by this corollary, the G -orbit of a representation ρ , is

$$V_\rho = \{\bar{\rho} \mid \forall \gamma \in \pi_1(S) \quad \mathrm{Tr}(\rho(\gamma)) = \mathrm{Tr}(\bar{\rho}(\gamma))\}.$$

5.2.5 The moduli space $\mathcal{M}^{zd}(S, G)$ and its tangent space

We prove the third part of Theorem 5.1.6

Proposition 5.2.17 *The quotient*

$$\mathcal{M}^{zd}(S, G) := \mathcal{D}_\Gamma^{vr}(S, G)/G^V$$

is a manifold of dimension $-\chi(S) \dim G$ whose tangent space at the class of a representation $[\rho]$ is canonically identified with $H_\rho^1(\mathfrak{g})$.

PROOF: We have already proved that $\mathcal{D}_\Gamma^{vr}(S, G)$ is a submanifold on which the group G^V acts freely and properly. Hence the quotient is a submanifold and it remains to understand its tangent space.

Let us denote by $G^{|E|}$ the set of discrete G -connections, i.e. maps $\alpha \in G^E$ with $\alpha(\bar{e}) = \alpha(e)^{-1}$ and let $\mu : G^{|E|} \rightarrow G^F$ be the holonomy map.

Similarly, the action of G gives rise to an orbit map $\mathfrak{g}^V \rightarrow T_\alpha G^{|E|}$ and the complex

$$\mathfrak{g}^V \rightarrow T_\alpha G^{|E|} \rightarrow T_{\mu(\alpha)} G^F$$

can be identified using Proposition 5.2.8 with the complex

$$\mathfrak{g}^V \xrightarrow{d_\Gamma^0} \mathfrak{g}^E \xrightarrow{d_\Gamma^1} \mathfrak{g}^F.$$

Thus the tangent space at $[\alpha]$ becomes identified with the quotient of $\ker d_\Gamma^1$ by the image of d_Γ^0 , which is precisely $H_\alpha^1(\mathfrak{g})$.

It remains to compute the dimension of $\mathcal{M}^{zd}(S, G)$. We know that

$$\dim \mathcal{M}^{zd}(S, G) = \dim H_\rho^1(\mathfrak{g}).$$

On the other hand, by using Proposition 4.1.11,

$$\dim(H_\Gamma^0(\mathfrak{g})) - \dim(H_\Gamma^2(\mathfrak{g})) + \dim(H_\Gamma^1(\mathfrak{g})) = -\chi(S) \dim(\mathfrak{g}).$$

Since for elements in $\mathcal{D}^{vr}(S, G)$, the cohomology groups $H_\Gamma^0(\mathfrak{g})$ $H_\Gamma^2(\mathfrak{g})$ vanish, the formula for the dimension of $\mathcal{M}^{zd}(S, G)$ follows. Q.E.D.

5.2.6 The complementary of $\mathcal{D}_\Gamma^{zd}(S, G)$

We prove the final part of Theorem 5.1.6

Proposition 5.2.18 *The complement of $\mathcal{D}_\Gamma^{zd}(S, G)$ in $\mathcal{D}_\Gamma^{zd}(S, G)$ is a finite union of G^V -orbits union of algebraic manifolds of the form $\mathcal{D}_\Gamma(S, H)$, where H varies over the algebraic subgroups of G . It has Hausdorff dimension strictly less than the dimension of $\mathcal{D}_\Gamma^{zd}(S, G)$.*

PROOF: We only sketch the proof here. By construction, for every homomorphism ρ not in $\mathcal{D}_\Gamma^{zd}$, there exists an algebraic group H which contains $\rho(\Gamma)$. Hence, the first part of the statement follows from the following fact [?] : there exists finitely many conjugacy classes of maximal algebraic subgroup of a semi-simple group. Q.E.D.

5.3 Connected components of the moduli space

5.3.1 Connected components I: the compact case

For general Lie groups G , the moduli space $\mathcal{M}(S, G)$ need not be connected. However, in some favourable cases this is indeed the case. For example one has:

Theorem 5.3.1 *If G is a compact, connected, simply-connected Lie group with finite centre, then $\mathcal{M}(S, G)$ is connected.*

We will only proof the theorem when $G = \mathrm{SU}(2)$. The general case requires a different proof than what we explain now :

To prove Theorem 5.3.1 we fix a genus g and consider once again the map $\mu : G^{2g} \rightarrow G$ given by $(a_i, b_i) \mapsto \prod [a_i, b_i]$. Then $\mathcal{M}(S, G) = \mu^{-1}(1)/G$ and thus $\mathcal{M}(S, G)$ will be connected if $\mu^{-1}(1)$ is connected.

Observe now that – in the case $G = \mathrm{SU}(2)$ – the set of singular elements is the centre of G .

It turns out that it is useful to study $\mu^{-1}(g)$ for $g \neq 1$ first. Let T be the maximal abelian subgroup of G – that is the group of diagonal matrices. Let C be the set of elements g such that $\dim Z(g) = \dim T$

Lemma 5.3.2 *The set C is dense, connected and is simply connected.*

PROOF: We can write

$$C = \{g \in G \mid i \operatorname{Tr}(g) \in]-2, 2[\}.$$

Let $\varphi : g \rightarrow i \operatorname{Tr}(g)$. Note that that φ is a fibration from C to its image and. Moreover the fibres of φ restricted to C are simply connected: they are homeomorphic to $\mathbb{CP}^1$. The lemma follows Q.E.D.

We also need the following lemma

Lemma 5.3.3 *The set $\mu^{-1}(C) \subset G^{2g}$ is connected.*

PROOF: Let us first consider the set R_0 of elements which are critical points of μ . A an element ρ is a critical point of μ , if its centraliser is not finite. It follows that R_0 is included in the finite union of G -orbits of sets of the form H^{2g} where H is a subgroup of G of positive codimension. It follows that R_0 is included in a finite union of submanifolds of codimension greater than 2. Hence $W = G^{2g} \setminus R_0$ is open and connected. By definition μ restricted to W is a submersion. Now $\mu^{-1}(C) = W \setminus \mu^{-1}\{\operatorname{Id}, -\operatorname{Id}\}$. Let

$$J = \mu^{-1}\{\operatorname{Id}, -\operatorname{Id}\} \cap W$$

has codimension equal to the dimension of $\mathrm{SU}(2)$ that is 3. It follows that $\mu^{-1}(C)$ is connected. Q.E.D.

For a regular element the connectedness of the fibre is easy to prove:

Lemma 5.3.4 *If $G = \mathrm{SU}(2)$ and $g \in C$, then $\mu^{-1}(g)$ is connected.*

PROOF: Let $E := \mu^{-1}(G \setminus C)$. Then the restriction $\mu : E \rightarrow G \setminus C$ is a fibration. By Lemma 5.3.2 the base $G \setminus C$ is simply connected and clearly E is connected. Now the long exact homotopy sequence of μ ends with

$$\dots \rightarrow \underbrace{\pi_1(G \setminus C)}_{=\{0\}} \rightarrow \pi_0(\mu^{-1}(g)) \rightarrow \underbrace{\pi_0(E)}_{=\{0\}} \rightarrow \pi_0(G \setminus C) \rightarrow 0.$$

Thus $\pi_0(\mu^{-1}(g)) = \{0\}$ as claimed. Q.E.D.

Of course 1 is a very singular element. However, using the fact that the regular elements are dense we can deduce Theorem 5.3.1 from the following approximation result:

Lemma 5.3.5 *Let K, P be topological spaces, K compact and let $\mu : K \rightarrow P$ be a continuous map. Suppose that $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in K such that $\mu^{-1}(x_n)$ is connected for all $n \in \mathbb{N}$. If $x_n \rightarrow x_0$ then $\mu^{-1}(x_0)$ is also connected.*

Exercise 5.3.6 Prove Lemma 5.3.5.

Now we may choose $K := G^{2g}$, $P = G$, μ as before and x_n a sequence of regular elements converging to the identity in order to deduce Theorem 5.3.1.

5.3.2 Connected Components II : an invariant

Notice that the simple connectedness of G entered crucially in the proof (via the long exact sequence of the fibration over the regular elements). Thus if we drop this assumption then it is not surprising that Theorem 5.3.1 fails. More precisely, let G be a connected semisimple compact Lie group, not necessarily simply connected. Denote by $\pi : \tilde{G} \rightarrow G$ its universal covering. We may identify $\pi_1(G)$ with the fibre over 1. Then $\pi_1(G)$ becomes a normal discrete subgroup of $\tilde{G}$ with $\tilde{G}/\pi_1(G) = G$ and π gets identified with the canonical quotient map.

Exercise 5.3.7 Show that a discrete normal subgroup Γ of a connected topological group H is central and deduce that $\pi_1(G) \subset Z(\tilde{G})$. (Hint: For $\gamma \in \Gamma$ consider the conjugation map $c_\gamma : H \rightarrow \Gamma$ given by $h \mapsto h\gamma h^{-1}$.)

Construction of the invariant

We construct a map

$$\chi : \mathcal{M}(S, G) \rightarrow \pi_1(G)$$

constant on connected components. The construction runs as follows and depends on some choices as follows

1. We fix a filling ribbon graph Γ in S .
2. We choose a discrete connection on Γ , that is a collection $\nabla = \{g_e\}_{e \in E}$ so that $g(\bar{e}- = (g(e))^{-1}$, where E denotes the edges of Γ , representing the element $[\nabla]$ of $\mathcal{M}(S, G)$.
3. For every $e \in E$ choose a lift $\tilde{g}_e \in \tilde{G}$ of g_e in such a way that $\tilde{g}_{\bar{e}} = \tilde{g}_e^{-1}$. Then $\tilde{\nabla} = \{\tilde{g}_e\}_{e \in E}$ defines a non-flat discrete $\tilde{G}$ -connection on Γ .
4. We choose a base point for every vertex in the face, so that we can define the holonomy of a face.

If F stands for the set of faces of Γ , we finally define

$$\chi([\nabla]) := \prod_{f \in F} \text{Hol}_{\tilde{\nabla}}(f).$$

One has to check that this is well-defined and independent of the choices made. We first prove

Proposition 5.3.8 *For every face f , the holonomy $\text{Hol}_{\tilde{\nabla}}(f)$ takes value in $\pi_1(G)$. Moreover*

1. $\text{Hol}_{\tilde{\nabla}}(f)$ is independent of the choice of the base point of the face f ,
2. $\prod_{f \in F} \text{Hol}_{\tilde{\nabla}}(f)$ is independent of the choice of the lift $\tilde{g}$,
3. $\prod_{f \in F} \text{Hol}_{\tilde{\nabla}}(f)$ is invariant under gauge transformations.

PROOF: First let us check that it takes values in $\pi_1(G)$: As ∇ is flat, the holonomy $\text{Hol}_{\nabla}(f)$ is trivial and thus $\pi(\text{Hol}_{\nabla}(f)) = 1$. Now let us check that it does not depend on the choice of lift: indeed, if we take another lift $\tilde{g}'_e$ with $\tilde{g}'_e = \tilde{g}_e \gamma$ for some $\gamma \in \pi_1(G)$, then we also have to take $\tilde{g}'_{\bar{e}} = \gamma^{-1} \tilde{g}_e^{-1}$. But by Exercise 5.3.7 the group $\pi_1(G)$ is abelian and thus γ and γ^{-1} cancel in the product over all faces. (Recall that every edge is contained in precisely one face.) Similarly, $\text{Hol}_{\nabla}(f)$ does not depend on the base point and is gauge-invariant. We leave this to the reader. Q.E.D.

One finally checks that $\prod_{f \in F} \text{Hol}_{\tilde{\nabla}}(f)$ is invariant under subdivision and thus that it is independent of the choice of Γ . From this it follows that $\chi[\nabla]$ is well defined.

Now let us discuss the fibres of the map χ . If we construct χ using the standard ribbon graph Γ_g , then it is easy to see that $\chi^{-1}(\alpha)$ is precisely the

subset of the representation variety of the fundamental group of the punctured surface $\pi_1(S \setminus \{*\})$ with values in $\tilde{G}$ given by the condition that the holonomy around the puncture is given by α . Using the fact that $\tilde{G}$ is simply-connected, a similar argument as in the proof of Theorem 5.3.1 will show that $\chi^{-1}(\alpha)$ is connected for all α . This proves:

Theorem 5.3.9 *If G is a connected semisimple compact Lie group, then the connected components of $\mathcal{M}(S, G)$ are in one-to-one correspondence with the elements of $\pi_1(G)$.*

Indeed, the correspondence is precisely given by the map χ above.

5.3.3 Connected components III: Milnor-Wood inequality

For non-compact Lie groups the statement of Theorem 5.3.9 fails. This can already be seen in the simple example $G = \mathrm{SL}(2, \mathbb{R})$. Indeed, $\pi_1(\mathrm{SL}(2, \mathbb{R})) \cong \mathbb{Z}$ is infinite, and thus Theorem 5.3.9 would predict an infinite number of connected components for $\mathcal{M}(S, G)$. However, this contradicts the fact that an algebraic variety – such as $\mathrm{Hom}^{zd}(\pi_1(S), G)$ – has only finitely many connected components. Therefore the problem to determine the number of connected components of $\mathcal{M}(S, G)$ cannot be solved by counting elements of the fundamental group unless G is compact.

It also turns out that in some cases – and we shall only see the case $G = \mathrm{SL}(2, \mathbb{R})$ – the Euler class of the representation, which is a priori bounded by the argument above, can be explicitly bounded

Theorem 5.3.10 [MILNOR-WOOD INEQUALITY] *Let $G = \mathrm{SL}(2, \mathbb{R})$. Then for any $[\nabla] \in \mathcal{M}(S, G)$,*

$$|\chi([\nabla])| \leq -\chi(S_g) = 2g - 2.$$

The universal cover of $\mathrm{SL}(2, \mathbb{R})$

We first observe that $\mathrm{SL}(2, \mathbb{R})$ admits a retraction by deformation on the subgroup of rotation $\mathrm{SO}(2, \mathbb{R})$. Thus $\pi_1(\mathrm{SL}(2, \mathbb{R}))$ is isomorphic to $\mathbb{Z}$.

Then, the universal cover of $\mathrm{SL}(2, \mathbb{R})$ can be constructed as follows. Observe that $\mathrm{SL}(2, \mathbb{R})$ acts on the circle $\mathbb{T}^1$ through the identification

$$\mathbb{T}^1 = (\mathbb{R}^2 \setminus \{0\})/\mathbb{R}^{>0}.$$

Let now $\pi : \mathbb{R} \rightarrow \mathbb{T}^1$ be the universal covering of the circle given by $t \rightarrow \exp(2it)$. Let $\tilde{G}$ be the subgroup of diffeomorphisms φ of $\mathbb{R}$ lifting the action of $G = \mathrm{SL}(2, \mathbb{R})$:

$$\tilde{G} := \{g \in \mathrm{Diff}(\mathbb{R}) \mid \exists h \in \mathrm{SL}(2, \mathbb{R}), \pi \circ g = h \circ \pi\}.$$

By construction, we have a group homomorphism π^* from $\tilde{G}$ to $G = \mathrm{SL}(2, \mathbb{R})$ and we have

Proposition 5.3.11 *The group $\tilde{G}$ is the universal cover of $G = \mathrm{SL}(2, \mathbb{R})$ and π^* is the covering map*

PROOF: We observe that the preimage of $\mathrm{SO}(2, \mathbb{R})$ is the group of translations. It follows from this observation that $\tilde{G}$ is connected and simply connected since it retracts on the group of translations. Q.E.D.

Proposition 5.3.12 *Let $G = \mathrm{SL}(2, \mathbb{R})$. There exists a – non continuous – section $\sigma : G \rightarrow \tilde{G}$ of the universal covering of G such that for all $x \in \mathbb{R}$ and $g \in G$,*

$$|\sigma(g).x - x| \leq \frac{1}{2}.$$

.

PROOF: Let us first consider an element $g \in G$ for which -1 is not contained in the spectrum $\mathrm{spec}(g)$ of g . In this case we find for every $x \in \mathbb{R}$ a unique $\tilde{g}_x$ such that $|\tilde{g}_x.x - x| < \frac{1}{2}$. Observe now that the function $y \mapsto |\tilde{g}_x.y - y|$ is continuous. Thus for y in a small neighbourhood of x we still have $|\tilde{g}_x.y - y| < \frac{1}{2}$. But then the uniqueness of $\tilde{g}_y$ implies that $\tilde{g}_x = \tilde{g}_y$. Therefore the map $x \mapsto \tilde{g}_x$ is locally constant and hence constant as $\mathbb{R}$ is connected. We may therefore define $\sigma(g) := \tilde{g}_x$ unambiguously.

Finally suppose $g \in G$ is arbitrary. Let $\{g_n\}_{n \in \mathbb{N}}$ be a sequence of elements in G converging to g so that for all n we have $-1 \notin \mathrm{spec}(g_n)$. We now observe that we can find a convergent subsequence which converges to an element h . We define $h = \sigma(g)$. Q.E.D.

A first inequality

In order to bound the number of connected components of the moduli space we consider, as in the compact case, the map $\chi : \mathcal{M}(S, G) \rightarrow \pi_1(G) \cong \mathbb{Z}$. Then we have:

Lemma 5.3.13 [FAKE MILNOR-WOOD INEQUALITY] *Let $G = \mathrm{SL}(2, \mathbb{R})$. Then for any $[\nabla] \in \mathcal{M}(S, G)$,*

$$-2g \leq \chi([\nabla]) \leq 2g.$$

PROOF: We compute χ via the standard ribbon graph Γ_g . Suppose $[\nabla]$ is represented by labellings $a_1, b_1, \dots, a_g, b_g \in G$ at the $2g$ edges. Let σ be the section of the universal covering $\tilde{G} \rightarrow G$ constructed in Proposition 5.3.12 and put $\tilde{a}_i := \sigma(a_i)$, $\tilde{b}_i := \sigma(b_i)$. Then

$$|\chi([\nabla])| = \prod_{i=1}^g |[\tilde{a}_i, \tilde{b}_i]|.$$

In general we have for $g, h \in G$ the estimate

$$|\sigma(g)\sigma(h).x - x| \leq |\sigma(g)(\sigma(h)x) - \sigma(h)x| + |\sigma(h)x - x|.$$

Thus we obtain inductively,

$$|\chi([\nabla])x - x| \leq |\tilde{a}_1(b_1 a_1^{-1} \cdots b_g x) - (b_1 a_1^{-1} \cdots b_g x)| + \cdots + |\tilde{b}_g^{-1}.x - x|.$$

Now by Proposition 5.3.12, each of the $4g$ summands is bounded by $\frac{1}{2}$ and thus

$$|\chi([\nabla])x - x| \leq 4g \cdot \frac{1}{2} = 2g,$$

from which the lemma follows. Q.E.D.

Proof of the Milnor-Wood inequality

Actually, we can improve the bound on $|\chi([\nabla])|$ by "stabilising it" and get the Milnor-Wood inequality.

PROOF: For any surface S we have $|\chi(\nabla)| \leq \chi(S) + 2$. Now consider an n -sheeted covering $S_n \rightarrow S$. View the elements of $\mathcal{M}(S, G)$ as equivalence classes of representations. Any representation $\rho : \pi_1(S) \rightarrow G$ lifts to a representation $\rho_n : \pi_1(S_n) \rightarrow G$. Now $\chi([\rho_n]) = n\chi([\rho])$ since a preimage of a ribbon graph as n times as many faces as the original ribbon graph and $\chi(S_n) = n\chi(S)$. Thus

$$|\chi([\rho])| = \frac{1}{n} \chi([\rho_n]) \leq \frac{1}{n} (-\chi(S_n) + 2) = -\chi(S) + \frac{2}{n}.$$

As n was arbitrary, the theorem follows. Q.E.D.

5.3.4 Connected components IV: Goldman's work for $\mathrm{SL}(2, \mathbb{R})$

For $\mathrm{SL}(2, \mathbb{R})$ the situation of connected components is pretty well understood thanks to W. Goldman's work. More precisely, fact that the Milnor-Wood inequality is sharp follows from the following theorem of Goldman:

Theorem 5.3.14 [GOLDMAN] *Let $G = \mathrm{SL}(2, \mathbb{R})$. Then for all p with $|p| \leq -\chi(S)$ the fibre $\chi^{-1}(p)$ is connected and non-empty. Thus $\mathcal{M}(S, G)$ has precisely $4g - 3$ connected components.*

The *maximal component* $\chi^{-1}(2g - 2)$ consists of discrete faithful representations. It can be identified with Teichmüller space, the space of oriented hyperbolisations of the surface S . (A discrete faithful representation yields a covering action of $\pi_1(S_g)$ on the upper half plane $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2, \mathbb{R})$ and thus a hyperbolisation.) The *minimal component* $\chi^{-1}(-2g + 2)$ corresponds to the Teichmüller space of S with opposite orientation. On these two components the action of the mapping class group, i.e. the group of outer automorphisms $\mathrm{Mod}_g := \mathrm{Out}(\pi_1(S_g))$, is proper. On the other hand one has the following conjecture of Goldman:

Conjecture 5.3.15 [GOLDMAN] *For every $p \notin \{\pm\chi(S_g)\}$ the action of Mod_g on $\chi^{-1}(p)$ is ergodic.*

Thus there is a stark contrast between the spaces of maximal/minimal representations and the other components of the moduli space.

Finally, let us state there exists fibres of the map χ which are not connected. The most striking case is the group $Sp(2g)$. Observe however that χ takes only finitely many values when G is an algebraic group. Indeed it is constant on connected components of $\mathrm{Hom}(\pi_1(S), G)$ which are in finite numbers since it is an affine variety.

Chapter 6

Symplectic structure of the moduli spaces

6.1 Universal connections and smooth structures

In this preliminary section, we relate the smooth part of the moduli space with smooth family of connections.

6.1.1 Lifting lemmas

Lifting small neighbourhoods

For a bundle E over S denote by $\mathcal{A}_E$ the affine space of connections and by $A_E := \Omega^1(S) \otimes \text{End}(E)$ the underlying vector space.

Lemma 6.1.1 [LIFTING LEMMA] *Let $[\rho_0]$ be an element of $\mathcal{M}^{zd}(S, G)$. Then there exists a neighbourhood U of $[\rho_0]$ in $\mathcal{M}^{zd}(S, G)$, a bundle E over S and a smooth map $f : U \rightarrow \mathcal{A}_E$ whose image consists of flat connections so that the holonomy representation of $f([\rho])$ is given by $[\rho]$.*

PROOF: Fix a good covering of S . If $\mathcal{X}^{zd}$ denotes the space of G -local systems with respect to this covering, which represent Zariski dense representations, then the projection $\mathcal{X}^{zd} \rightarrow \mathcal{M}^{zd}(S, G)$ is a locally trivial principal bundle. In particular, it admits local sections. Thus we find a neighbourhood U of $[\rho_0]$ and a section $\sigma : U \rightarrow \mathcal{X}^{zd}$.

We furthermore assume that U is diffeomorphic to a disc..

For every point in U we have a locally-constant G -valued cocycle on S . In other words, we have a G -valued cocycle on $S \times U$ constant along S . Observe that a cocycle gives rise to a concrete bundle E on $S \times U$ together with a flat G -connection along S . (Note that a local system really gives rise to a concrete bundle with connection, not just an equivalence class, and that it does so in a smooth way).

Now it suffices to show that this bundle is "constant" along U , or more precisely that it is equivalent to the pullback of a bundle over S via the projection $\pi : S \times U \rightarrow S$. This follows from the contractibility of U .

Let us explain this fact in some more details: Let $E_0 := E|_{S \times \{[\rho_0]\}}$, which is a bundle over $S \times \{[\rho_0]\} \cong S$. Let $\tilde{E} := \pi_0^* E_0$, where $\pi_0 : S \times U \rightarrow S \times \{[\rho_0]\}$ is the canonical projection. Our goal is to construct an isomorphism $F : E \rightarrow \tilde{E}$ which induces the identity on the base space $S \times U$. For this, we take any connection on E - without any relation whatsoever to any of the connections occurring before.

As U is a disc with centre $[\rho_0]$ we can join every point $u \in U$ to $[\rho_0]$ by a path $c_u : [0, 1] \rightarrow U$ so that the map $U \times [0, 1] \rightarrow U$ given by $(u, t) \mapsto c_u(t)$ is smooth. The bundle isomorphism F is defined by assigning to $v \in E_{(s,u)}$ its image by the holonomy along c_u :

$$F(v) := \text{Hol}_{\{s\} \times c_u}(v).$$

Q.E.D.

As a byproduct we obtain the following result:

Corollary 6.1.2 *If $[\nabla_0], [\nabla_1]$ are in the same connected component of $\mathcal{M}^{zd}(S, G)$, then they can be represented by connections on the same bundle E .*

6.1.2 Universal connections

We now elaborate on the Lifting lemma and introduce the following device.

Definition 6.1.3 *A universal connection is a connection ∇ on a bundle $\mathcal{E}$ over $S \times \mathcal{M}^{zd}(S, G)$ such that for every representation ρ belonging to $\mathcal{M}^{zd}(S, G)$, $\nabla|_{\{[\rho]\} \times S}$ is a flat connection with holonomy conjugate to ρ .*

The following result is a generalisation of the Lifting lemma.

Theorem 6.1.4 [UNIVERSAL CONNECTION] *For every surface S and group G there exists a universal connection $\nabla^\#$ over a bundle over $\mathcal{M}^{zd}(S, G)$.*

PROOF: we use the notation and ideas of the proof of the Lifting lemma.

The initial remark of this proof is that a smooth map from an open set U to $\mathcal{A}_E$ is the same object as a smooth connection on $E \rightarrow S \times U$ which is trivial along $\{s\} \times U$.

Therefore, assume that we have two open sets U and V in $\mathcal{M}^{zd}(S, G)$. Let f and g be two lifts from respectively U and V to $\mathcal{A}_E$. Let ∇_U and ∇_V be the associated connections on $S \times U$ and $S \times V$ respectively. Then by hypothesis, for any m in $U \cap V$, the connections $\nabla_U|_{\{m\} \times S}$ and $\nabla_V|_{\{m\} \times S}$ are gauge equivalent. Since the holonomies have Zariski dense image, the gauge equivalence is unique. In other words, there exists a unique $\sigma_m = \sigma_m^{U,V}$ in the Gauge group of $E \rightarrow S$ so that

$$\nabla_U|_{\{m\} \times S} = \sigma_m^* \nabla_V|_{\{m\} \times S}. \quad (6.1)$$

From the uniqueness it follows that $m \rightarrow \sigma_m$ is continuous. Moreover, the construction given in the proof of the lifting Lemma yields that $m \rightarrow \sigma$ is smooth.

Let now $\{U_i\}_{i \in I}$ be a covering of $\mathcal{M}^{zd}(S, G)$ by open sets such as in the Lifting lemma. From the uniqueness again, the collection of maps $\{\sigma_m^{U_i, V}\}$ defines a G -valued cocycle on E and a vector bundle $\mathcal{E}$ over $S \times \mathcal{M}^{zd}(S, G)$.

Moreover each connection $\nabla_i := \nabla_{U_i}$ gives rise to a connection on $\mathcal{E}|_{U_i}$ and by construction on $U_i \cap U_j$ we have by Equation 6.1

$$\nabla_i|_{\{m\} \times S} = \nabla_j|_{\{m\} \times S}. \quad (6.2)$$

We now produce a universal connection by setting

$$\nabla = \sum_i \varphi_i \nabla_i,$$

where φ_i is a partition of unity subordinated to the covering.

Q.E.D.

The final statement of this section is an elaboration on these ideas and follows from the proof

Proposition 6.1.5 *Let $\nabla^\#$ be a universal connection. Let M be a manifold, E be a vector bundle over M over $M \times S$, ∇ be a connection on E such that for any point m in M , $\nabla|_{\{m\} \times S}$ is a flat connection whose holonomy representation $[\rho]_m$ has almost Zariski dense image. Let Φ be the map from $M \times S$ to $\mathcal{M}^{zd}(S, G) \times S$ given by $(m, s) \rightarrow ([\rho]_m, s)$. Then*

1. *The map Φ is smooth.*

2. There exist an element σ of the Gauge group of E , so that for any m in M ,

$$\Phi^* \nabla^\#|_{\{m\} \times S} = \sigma^* \nabla|_{\{m\} \times S}.$$

6.2 The symplectic form

The aim of this subsection is to equip the Zariski dense part $\mathcal{M}^{zd}(S, G)$ of the moduli space with a symplectic structure in the sense of the following definition:

Definition 6.2.1 *A symplectic structure on a manifold M of dimension $2n$ is a 2-form $\omega \in \Omega^2(M)$ which satisfies*

- (i) $d\omega = 0$, i.e. ω is closed;
- (ii) ω is fibrewise non-degenerate, i.e. $\omega^{\wedge n}$ is nowhere vanishing.

The pair (M, ω) is then called a symplectic manifold.

Recall from Theorem 5.1.6 that the tangent space $T_{[\rho]} \mathcal{M}^{zd}(S, G)$ can be identified with $H_\rho^1(\mathfrak{g})$. On $H_\rho^1(\mathfrak{g})$ the intersection form i_ρ defines a non-degenerate alternating bilinear form. These forms fit together nicely:

Theorem 6.2.2 *Let S be a surface of genus $g \geq 2$. Then the intersection forms i_ρ on $T_{[\rho]} \mathcal{M}^{zd}(S, G) \cong H_\rho^1(\mathfrak{g})$ fit together to a symplectic form ω on $\mathcal{M}^{zd}(S, G)$.*

Definition 6.2.3 *The above symplectic form is called the Atiyah-Bott-Goldman symplectic form on the representation variety.*

Now we prove our theorem:

PROOF: [of Theorem 6.2.2] As we have already seen in Theorem 4.1.7 that the intersection forms are non-degenerate, we have only to show that ω is closed and smooth.

It suffices to show that for any $[\rho_0] \in \mathcal{M}^{zd}(S, G)$ there exists a neighbourhood U of $[\rho_0]$ such that $\omega|_U$ is the pull-back of a smooth exact form. By Lemma 6.1.1 we may assume that there exists a bundle E and a smooth map $f : U \rightarrow \mathcal{A}_E$ such that the holonomy representation of $f([\rho])$ is given by $[\rho]$. On the underlying vector space A_E we have a pairing Ω given by

$$\Omega(\alpha, \beta) := \int_S \text{Tr}(\alpha \wedge \beta),$$

where

$$\mathrm{Tr}(\alpha \wedge \beta)(x, y) = \mathrm{Tr}_{\mathrm{End}(E)}(\alpha(x)\beta(y) - \beta(y)\alpha(x)).$$

As the tangent space of $\mathcal{A}_E$ at any point is identified with A_E , Ω induces a smooth 2-form on $\mathcal{A}_E$. If ∇_0 is any fixed connection on E and ξ is the one-form on A_E given by

$$\xi_\nabla(\alpha) = \int_S \mathrm{Tr}((\nabla - \nabla_0) \wedge \alpha),$$

then $\Omega = d\xi$. Thus Ω is a smooth exact 2-form on $\mathcal{A}_E$. From the definition of the intersection form we see that $\omega|_U = f^*\Omega$ and the theorem follows. Q.E.D.

6.3 Classical mechanics and observables

In classical mechanics symplectic manifolds (M, ω) model configuration spaces. In this context, elements of $C^\infty(M)$ are regarded as *observables*. The dynamics of the observables can be expressed in term of their Poisson brackets, which are defined as follows: Using the symplectic form we can associate to every $f \in C^\infty(M)$ a vector field H_f called the *Hamiltonian vector field* of f and given by

$$df(u) = \omega(u, H_f).$$

Then the *Poisson bracket* $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ is defined by

$$\{f, g\} := \omega(H_f, H_g) = dg(H_f).$$

The set of observables has the following algebraic structure:

Proposition 6.3.1 *If (M, ω) is a symplectic manifold the $(C^\infty(M), \{\cdot, \cdot\}, \cdot)$ is a Poisson algebra, i.e.*

- $(C^\infty(M), \{\cdot, \cdot\})$ is a Lie algebra,
- $(C^\infty(M), \cdot)$ is a commutative associative algebra,
- the two structures are compatible in the sense that

$$\{f, gh\} = h\{f, g\} + g\{f, h\}.$$

Of course, it is rather naive to call any arbitrary function on M an observable - at least from a physical point of view. Namely, one is usually only interested in observing some special quantities like energy, momentum etc. and not just arbitrary functions. Thus there should be a subset of $C^\infty(M)$ consisting of *interesting observables*. As the dynamics of the observable is governed by the

Poisson algebra structure, a collection of interesting observables of M should be a Poisson subalgebra of $C^\infty(M)$. In the case where $M = \mathcal{M}^{zd}(S, G)$ is the Zariski dense part of the moduli space there are various such subalgebras of geometric interest. In this subsection and the next we discuss two such subalgebras, which are related to the topology of the surface S and feature prominently in topological quantum field theory.

6.4 Special observables I: Wilson loops and the Goldman algebra

We first recall some elementary facts about loops and conjugacy classes in the fundamental group. The free homotopy class of a loop is the homotopy class of the loop as a map from the circle to the surface S .

Choosing a base point x in S and path α from x to a point in the loop γ , we obtain a loop $\eta = \alpha \cdot \gamma \alpha^{-1}$ based at x and thus an element in $\pi_1(S, x)$. Different choices of the connecting path lead to conjugate elements in $\pi_1(S, x)$. More generally, a free homotopy class of loop defines a conjugacy class in $\pi_1(S, s)$. Conversely every conjugacy class in $\pi_1(S, x)$ defines a free homotopy class of loop. We denote by $[\gamma]$ the conjugacy class of γ in $\pi_1(S, x)$, or equivalently the free homotopy class of the loop γ .

Note finally that if ρ is a representation of $\pi_1(S, s)$ in G , the conjugacy class $[\rho(\gamma)]$ of $\rho(\gamma)$ in G only depends on the conjugacy classes of γ and ρ . This remark allows us the following definition since traces only depends on conjugacy classes.

Definition 6.4.1 [WILSON LOOPS] *Let $[\gamma]$ be a conjugacy class in $\pi_1(S)$. Let λ be a linear representation of G . The following function*

$$\mathbf{w}_{[\gamma]} : \mathcal{M}^{zd}(S, G) \rightarrow \mathbb{R}, \quad [\rho] \mapsto \text{Tr}(\lambda[\rho]([\gamma])),$$

is called, following physic's literature, the Wilson loop associated to $[\gamma]$.

Exercise 6.4.2 Let ∇ be the flat connection associated to λ and ρ . Let $[\gamma]$ be a homotopy class of loop. Check that

$$\mathbf{w}_{[\gamma]}([\nabla]) = \text{Tr}(\text{Hol}_\nabla(\gamma)).$$

In order to understand the dynamics of the Wilson loops it is important to understand the structure of the Poisson algebra that they generate in $C^\infty(\mathcal{M}^{zd}(S, G))$. For this it is enough to understand the Poisson bracket of two Wilson loops, as the general case can be reduced to this using the axioms of a

Poisson algebra. In order to carry out this computation, we introduce another closely related algebra, which can be defined in purely geometric terms.

Let $\mathcal{G}_S$ be the real vector space spanned by the free homotopy classes of loops, where the trivial loop is identified with zero.

We begin with a proposition. First we recall that if two oriented curves c_1 and c_2 in S meet transversally at a point x , we define $\epsilon_x(\gamma_1, \gamma_2)$ to be 1 if $(\dot{c}_1, \dot{c}_2)$ is oriented and -1 otherwise. Then

Proposition 6.4.3 *Let γ_1, γ_2 be two loops in S with transversal intersections. Then, the following element of $\mathcal{G}$*

$$\{\gamma_1, \gamma_2\} := \sum_{x \in \gamma_1 \cap \gamma_2} \epsilon_x(\gamma_1, \gamma_2) [\gamma_1 \#_x \gamma_2].$$

only depends on the classes $[\gamma_1], [\gamma_2]$. Moreover, if we define $\{[\gamma_1], [\gamma_2]\} := [\{\gamma_1, \gamma_2\}]$, then $(\mathcal{G}, \{, \})$ is a Lie algebra.

This yields the following definition.

Definition 6.4.4 *The Lie algebra $(\mathcal{G}_S, \{, \cdot\})$ is called the Goldman algebra.*

The precise relation between the Goldman algebra and the Poisson algebra generated by the Wilson loops depends on the target group G . For example one has:

Theorem 6.4.5 [GOLDMAN] *Let S be a surface, G a linear Lie group and let $[\gamma_1], [\gamma_2]$ be two free homotopy classes of loops in S with associated Wilson loops $\mathbf{w}_{[\gamma_1]}, \mathbf{w}_{[\gamma_2]} \in C^\infty(\mathcal{M}^{zd}(S, G))$.*

(i) *If $G = GL(n, \mathbb{R})$ then*

$$\{\mathbf{w}_{[\gamma_1]}, \mathbf{w}_{[\gamma_2]}\} = \mathbf{w}_{\{[\gamma_1], [\gamma_2]\}}$$

(ii) *If $G = SL(n, \mathbb{R})$ then*

$$\{\mathbf{w}_{[\gamma_1]}, \mathbf{w}_{[\gamma_2]}\} = \mathbf{w}_{\{[\gamma_1], [\gamma_2]\}} - \frac{1}{n} i([\gamma_1, \gamma_2]) \mathbf{w}_{[\gamma_1]} \mathbf{w}_{[\gamma_2]},$$

where $i([\gamma_1, \gamma_2]) = \sum_{x \in \gamma_1, \gamma_2} \epsilon_x(\gamma_1, \gamma_2)$ denotes the intersection number of γ_1 and γ_2 .

This result was preceded and inspired by computations of S. Wolpert in Teichmüller theory [1]. It has generated lot of research. As a recent example, String Topology, as described by M. Chas and D. Sullivan [2], is a far reaching generalisation in higher dimension of the Goldman algebra.

Hamiltonian vector field of Wilson loops

Our first step in the proof Theorem 6.4.5 is to compute the Hamiltonian vector field of a Wilson loop. Thus let us fix a loop γ and denote by $H := H_{\mathbf{w}_{[\gamma]}}$ the Hamiltonian vector field associated to the corresponding Wilson loop $\mathbf{w}_{[\gamma]}$.

We identify $T_{[\rho]}\mathcal{M}^{zd}(S, G)$ for $[\rho]$ in $\mathcal{M}^{zd}(S, G)$ with $H_\rho^1(\mathfrak{g})$. Recall from Definition 4.1.10 that this $H_\rho^1(\mathfrak{g})$ is the twisted cohomology of a certain bundle with connection $(\mathcal{L}_{\mathfrak{g}}, \nabla_{\mathfrak{g}})$ whose typical fibre is the Lie algebra $\mathfrak{g}$ of G .

Thus H is represented by a class of 1-forms H_{∇} with values in the bundle $\mathcal{L}_{\mathfrak{g}}$. We are going to construct this 1-form explicitly.

We assume that $G \subset GL(n, \mathbb{R})$ is reductive – as are our classical examples $SU(n)$, $SL(n, \mathbb{R})$. On $M_n(\mathbb{R})$ we have a non-degenerate inner product given by $(A, B) \mapsto \text{Tr}(AB)$. The reductivity assumption translates in the fact that $M_n(\mathbb{R})$ splits as

$$M_n(\mathbb{R}) = \mathfrak{g} \oplus \mathfrak{g}^\perp.$$

From the point of view of bundle, we consider an associated G -bundle E of rank n equipped with a flat connection, so that $(\mathcal{L}_{\mathfrak{g}}, \nabla_{\mathfrak{g}})$ is a subbundle of the bundle $\text{End}(E)$. The above splitting translates in a parallel orthogonal splitting

$$\text{End}(E) = \mathcal{L}_{\mathfrak{g}} \oplus \mathcal{L}_{\mathfrak{g}}^\perp.$$

Denote by $X \mapsto \hat{X}$ the projection onto the first summand.

In order to simplify things, let us assume that γ does not have any self-intersection.

We describe the step of the construction and first make the following choices. Let U be a tubular neighbourhood of γ and β_γ a one-form with support in U so that for any closed 1-form α defined on U and for any transverse curve c to γ

$$\int_H \alpha \wedge \beta_\gamma = \int_\gamma \alpha, \tag{6.3}$$

$$\int_c \beta_\gamma = i(c, x). \tag{6.4}$$

$$\tag{6.5}$$

Such a form β_γ is easily constructed out of the derivative of a bump function.

For any point $x \in U$, the holonomy $\text{Hol}_\gamma(x)$ of a loop based at x and freely homotopic to γ in U with respect to the connection $\nabla = \nabla_{\mathfrak{g}}$ is an automorphism of the fibre E_x . Observe that $\text{Hol}_\gamma : U \rightarrow \text{End}(E)$ is a well defined section which is furthermore parallel (cf Exercise 3.3.13).

By a slight abuse of language we denote by $\widehat{\text{Hol}}_\gamma \cdot \beta_\gamma$ the 1-form, with values in $\mathcal{L}_\mathfrak{g}$ which vanishes outside U and is actually equal to $\widehat{\text{Hol}}_\gamma \cdot \beta_\gamma$, on U . One easily checks that this form is closed.

We claim that the cohomology class of this form is the symplectic gradient of the Wilson loop.

Proposition 6.4.6 *Let notation be as above. Then the class of $\widehat{\text{Hol}}_\gamma \cdot \beta_\gamma$ in $H_\rho^1(\mathfrak{g})$ represents H_∇*

PROOF: We choose a family of ∇_t a family of flat connections deforming $\nabla = \nabla_0$. We denote by $[\dot{\nabla}]$ the tangent vector $t = 0$ to the curve in $\mathcal{M}^{zd}(S, G)$ given by $t \rightarrow [\nabla_t]$. We have a representative of $[\dot{\nabla}]$ given by

$$\dot{\nabla} = \frac{d}{dt}\bigg|_{t=0}(\nabla_t - \nabla_0) \in \Omega^1(S, \mathcal{L}_\mathfrak{g}).$$

We want to compute

$$\frac{d}{dt}\bigg|_{t=0} \text{tr}(\text{Hol}_{\nabla_t}(\gamma)).$$

We consider γ as a $\mathbb{Z}$ periodic map from $\mathbb{R}$ to S . We choose a trivialisation of γ^*E so that $\gamma^*\nabla_0$ is the trivial connection D and write $\gamma^*\nabla_t = D + A_t$. Let $\frac{\partial}{\partial s}$ be the canonical vector field on $\mathbb{R}$ and

$$B_t(s) := A_t\left(\frac{\partial}{\partial s}\right).$$

Then by Proposition 3.3.14 the holonomy of ∇_t is obtained by solving the following linear ordinary differential equation

$$(*) \quad \frac{d}{ds}(X_t) = B_t X_t.$$

More precisely, recall that the resolvent $u, s \rightarrow {}^tR_u^s$ of this equation is by definition such that $X_t(s) = {}^tR_u^s X_t(u)$ is the solution of $(*)$ with initial condition $X_t(u)$ at time u . Then,

$$\text{Hol}_{\nabla_t}(\gamma) = {}^tR_0^1.$$

The method of the variation of constants yields

$$\frac{d}{dt}\bigg|_{t=0} \text{Tr}(\text{Hol}_{\nabla_t}(\gamma)) = \frac{d}{dt}\bigg|_{t=0} \text{Tr}({}^tR_0^1) = \int_0^1 \text{Tr}({}^tR_s^{1+s} \dot{B}(s)) ds,$$

where

$$\dot{B} = \frac{d}{dt}\bigg|_{t=0} B_t(s) = \dot{\nabla}\left(\frac{\partial f}{\partial s}\right).$$

We now interpret the right-hand side term and rewrite this integral as

$$\int_0^1 \text{Tr}({}^0R_s^{1+s} \dot{B}(s)) ds = \int_0^1 \text{Tr}(\text{Hol}_{\nabla_0}(\gamma)_{\gamma(s)}) \dot{\nabla} \left(\frac{\partial \gamma}{\partial s} \right) ds \quad (6.6)$$

$$= \int_{\gamma} \text{Tr}(\text{Hol}_{\gamma} \cdot \dot{\nabla}) \quad (6.7)$$

$$= \int_{\gamma} \text{Tr}(\widehat{\text{Hol}}_{\gamma} \cdot \dot{\nabla}) \quad (6.8)$$

In the last equality, we have used that $\text{Tr}(au) = \text{Tr}(\hat{a}u)$ if u belongs to $\mathfrak{g}$. Let α be the 1-form defined on U by

$$\alpha = \text{Tr}(\widehat{\text{Hol}}_{\gamma} \cdot \dot{\nabla}).$$

Note that α is closed. It follows by Equation (6.3), that

$$\int_U \alpha \wedge \beta_{\gamma} = \int_{\gamma} \alpha$$

Therefore combining the previous equations, we obtain that

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \text{Tr}(\text{Hol}_{\nabla_t}(\gamma)) &= \int_{\gamma} \alpha \\ &= \int_U \alpha \wedge \beta_{\gamma} \\ &= \int_U \text{Tr}(\widehat{\text{Hol}}_{\gamma} \dot{\nabla}) \wedge \beta_{\gamma} \\ &= - \int_S \text{Tr}(\widehat{\text{Hol}}_{\gamma} \cdot \beta_{\gamma} \wedge \dot{\nabla}). \end{aligned}$$

Using the symplectic form ω on $\mathcal{M}^{zd}(S, G)$ we finally obtain

$$d\mathbf{w}_{\gamma}(\rho) = - \int_S \text{Tr}(\widehat{\text{Hol}}_{\gamma} \cdot \beta_{\gamma} \wedge \dot{\nabla}) = -\omega(\widehat{\text{Hol}}_{\gamma} \cdot \beta_{\gamma}, \dot{\nabla}).$$

Thus by definition

$$H_{\nabla_0} = [\widehat{\text{Hol}}_{\gamma} \cdot \beta_{\gamma}].$$

Q.E.D.

Exercise 6.4.7 Deduce Theorem 6.4.5 from Proposition 6.4.6.

6.5 Special observables II: Spin networks

A larger class of observables on $\mathcal{M}^{zd}(S, G)$ which includes Wilson loops 6.4.1 as a special case can be constructed using embeddings of so-called spin networks. The starting definition is the following:

Definition 6.5.1 [SPIN NETWORKS] *A spin network $\mathcal{S}$ for G consists of the following data*

- an oriented graph $\Gamma = \Gamma^{\mathcal{S}} = (V, E)$ called the underlying graph of $\mathcal{S}$.
- for every $e \in E$, a vector space $I_e = I_e^{\mathcal{S}}$ and a representation $\lambda_e = \lambda_e^{\mathcal{S}} : G \rightarrow \text{Aut}(I_e)$.
- for any $v \in V$, a linear map

$$i_v = i_v^{\mathcal{S}} \in \text{Hom} \left(\bigotimes_{\{e \in E \mid e_+ = v\}} I_e, \bigotimes_{\{e \in E \mid e_- = v\}} I_e \right),$$

which intertwines the G -actions in the sense that for any $g \in G$,

$$i_v \circ \left(\bigotimes_{\{e \in E \mid e_+ = v\}} \lambda_e(g) \right) = \left(\bigotimes_{\{e \in E \mid e_- = v\}} \lambda_e(g) \right) \circ i_v.$$

We observe that the group G^E naturally acts on the space of spin networks with a common underlying graph. Namely, for every edge e we conjugate the representation by $g(e)$, and for every vertex we perform the coherent transformation on the intertwiner. This action is called the *gauge action on spin networks*.

Definition 6.5.2 [EMBEDDING OF A SPIN NETWORK] *Let S be a surface equipped with a bundle $\mathcal{L}$. An embedding of the spin network $\mathcal{S}$ into S is by definition*

- an embedding of $\Gamma^{\mathcal{S}}$ into S ,
- for every vertex v in the underlying graph, an identification of $\mathcal{L}_{\Psi(v)}$ with a fixed vector space L .

We say two embeddings of spin networks are isotopic under the gauge action if the corresponding embeddings of the underlying graph are isotopic. For instance, fixing the embedding of the graph, and changing the identification of the fibres at images of vertices with L , yield isotopic embeddings of spin networks. An isotopy class is called an embedded spin network.

6.5.1 An observable on the space of all flat connections

Now, we shall define a function on the space of flat connections on $\mathcal{L}$ associated to every embedding of a spin network in S .

Let $\mathcal{S}$ be a spin network and let $\Psi : \mathcal{S} \rightarrow S$ be an embedding into some surface S . Let ∇ be a flat connection on some bundle $\mathcal{L}$ with fibre L . Using the identification of $\mathcal{L}_{\Psi(v)}$ with L , for every edge e , holonomy $H(e) := \text{Hol}_{\nabla}(\Psi(e))$ becomes an element. Thus $\lambda_e(H(e)) \in \text{End}(I_e)$. We now define

$$\Lambda_{\Psi}(\nabla) := \bigotimes_{e \in E} \lambda_e(H(e)) \in \text{Aut} \left(\bigotimes_{e \in E} I_e \right).$$

On the other hand, we have

$$\begin{aligned} I_{\Psi} := \bigotimes_{v \in V} i_v^{\mathcal{S}} &\in \bigotimes_{v \in V} \left(\text{Hom} \left(\bigotimes_{\{e \in E \mid e_+ = v\}} I_e, \bigotimes_{\{e \in E \mid e_- = v\}} I_e \right) \right) \\ &= \text{Hom} \left(\bigotimes_{v \in V} \bigotimes_{\{e \in E \mid e_+ = v\}} I_e, \bigotimes_{v \in V} \bigotimes_{\{e \in E \mid e_- = v\}} I_e \right) \\ &= \text{End} \left(\bigotimes_{e \in E} I_e \right) \end{aligned}$$

Then we can multiply

$$\Lambda_{\Psi}(\nabla) \cdot I_{\Psi} \in \text{End} \left(\bigotimes_{e \in E} I_e \right),$$

Thus, we define an observable on the space of all flat connections by

$$f_{\Psi} : \nabla \mapsto \text{Tr}(\Lambda_{\Psi}(\nabla) \cdot I_{\Psi}).$$

Definition 6.5.3 [SPIN NETWORK OBSERVABLE] *The function f_{Ψ} is called the associated spin network observable to Ψ .*

REMARK:

1. Observe that example extends the Wilson loop as a loop is nothing but an embedding of the trivial spin network whose underlying graph has one vertex and one edge and whose only intertwiner i_v is the identity.

An observable on $\mathcal{M}^{zd}(S, G)$

The following proposition says that the spin network is a well defined observable on M

Proposition 6.5.4 [INVARIANCE OF THE SPIN NETWORK] *The value of f_Ψ at ∇ is invariant under gauge action on ∇ , and isotopies of the embedding of the spin network. Hence, it yields a function f_Ψ on $\mathcal{M}^{zd}(S, G)$*

Definition 6.5.5 [SPIN NETWORK OBSERVABLE (BIS)] *By a new abuse of language, we call again the corresponding function a spin network observable.*

6.5.2 Poisson bracket of spin network observables

We define the space $\mathfrak{S}$ of spin networks on S as the $\mathbb{R}$ -module generated by the equivalence classes of spin networks - actually it should be a quotient of that space By the above definition, the map $\Psi \mapsto f_\Psi$ extend to a linear map

$$\text{ev} : \mathfrak{S} \rightarrow C^\infty(\mathcal{M}^{zd}(S, G)).$$

We will now define a Lie bracket on $\mathfrak{S}$ which generalises the Poisson bracket on the Goldman algebra. For this let $\mathcal{S}_1, \mathcal{S}_2$ be two spin networks and let

$$\begin{aligned} \Psi_1 : \mathcal{S}_1 &\rightarrow S \\ \Psi_2 : \mathcal{S}_2 &\rightarrow S. \end{aligned}$$

be two embeddings. Assume that the images of the underlying oriented graphs Γ_1, Γ_2 intersect transversely. As in the case of the Goldman algebra we shall define for each intersection point $x \in \Gamma_1 \cap \Gamma_2$ a sign ϵ_x and new spin network $\mathcal{S}_1 \#_x \mathcal{S}_2$ as well as an embedding Ψ_x of this spin network. We finally define

$$[\Psi_1, \Psi_2] := \sum_{x \in \Gamma_1 \cap \Gamma_2} \epsilon_x \cdot \Psi_x.$$

Let us make this definition precise.

- The definition of ϵ_x is as in the case of the Goldman algebra: ϵ_x is 1 if the orientations of Γ_1 and Γ_2 are such that Γ_2 crosses Γ_1 at x from right to left and -1 otherwise.
- The underlying graph of $(\mathcal{S}_1 \#_x \mathcal{S}_2, \Psi_x)$ has as vertices the union of the vertices of Γ_1 and Γ_2 and all intersection points of Γ_1 and Γ_2 . These are connected by edges as in Γ_1 and Γ_2 respectively and the embedding is the one given by Ψ_1 and Ψ_2 .

- Any edge is a subedge of an edge of either Γ_1 or Γ_2 and it is labelled by the corresponding G -module.
- The vertices of Γ_1 and Γ_2 are labelled by the same intertwiners as before.
- Finally, suppose y is an intersection point of an edge e_1 of Γ_1 with an edge e_2 of Γ_2 . Let $(I_1, \lambda_1), (I_2, \lambda_2)$ be the corresponding G -modules. Then we have to define an intertwiner $i_y : I_1 \otimes I_2 \rightarrow I_2 \otimes I_1$. Such intertwiners can be constructed using the Lie algebra representations $d\lambda_j : \mathfrak{g} \rightarrow \text{End}(I_j)$, $j = 1, 2$: Assume that there exists a non-degenerate invariant bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. For a simple Lie algebra we can take the Killing form, but such a form may also exist for non-simple Lie algebras, for example $\mathfrak{gl}(n, \mathbb{R})$. Let $e_1, \dots, e_k$ be a basis of $\mathfrak{g}$ which is orthonormal with respect to $\langle \cdot, \cdot \rangle$. It is a well-known fact from the representation theory of Lie algebras that the *Casimir intertwiner*

$$\Delta := \sum_{i=1}^k d\lambda_1(e_i) \otimes d\lambda_2(e_i) \in \text{End}(I_1 \otimes I_2)$$

is independent of the choice of orthonormal basis and intertwines the tensor product action of G on $I_1 \otimes I_2$ with itself. Thus we define

$$i_y := \begin{cases} \Delta, & \text{if } y = x \\ \text{Id}, & \text{if } y \neq x \end{cases}$$

This finishes the construction of $(\mathcal{S}_1 \#_x \mathcal{S}_2, \Psi_x)$. As in the case of Wilson loops the Poisson bracket of spin network observables is closely related to the Lie bracket of embedded spin networks. In the case of $GL(n, \mathbb{R})$ one has:

Theorem 6.5.6 *Let $G = GL(n, \mathbb{R})$ and let Ψ_1, Ψ_2 be embedded spin networks in a surface S . Then the associated spin network observables $f_{\Psi_1}, f_{\Psi_2} \in C^\infty(\mathcal{M}^{zd}(S, G))$ satisfy*

$$\{f_{\Psi_1}, f_{\Psi_2}\} = f_{[\Psi_1, \Psi_2]}.$$

In particular one gets

Corollary 6.5.7 *The associative algebra generated by $\text{ev}(\mathfrak{S})$ is a Poisson subalgebra.*

6.6 Volumes of moduli spaces

6.6.1 Witten's formula

Recall that the very regular part carries a symplectic form.

Theorem 6.6.1 [WITTEN'S FORMULA] *Let S be a surface of genus g and let G be a simply-connected simple compact Lie group with finite centre $Z(G)$. Then the volume of the moduli space $\mathcal{M}(S, G)$ with respect to the symplectic form is given by*

$$\text{vol}(\mathcal{M}(S, G)) = |Z(G)| \sum_{\lambda \in \hat{G}} \left(\frac{\text{vol}(G)}{\dim V_\lambda} \right)^{2g-2},$$

where the volume of G is computed with respect to the Killing form, $\hat{G}$ is the set of finite dimensional representations λ of G and V_λ is the vector space associated to λ .

REMARKS:

1. [ON NORMALISATION] Given a symplectic manifold (M, ω) of dimension $2d$, the associated volume form is the *Liouville volume* $\frac{1}{d!} \omega^d$.
2. In this theorem, everything – the symplectic form and the volume of the group – is computed with respect to the Killing form k . We could have chosen another normalisation. Let just see what happens when we choose instead of k , the metric αk . Then

$$\text{vol}_\alpha(G) = \alpha^{\frac{\dim G}{2}} \text{vol}(G),$$

and

$$\text{vol}_\alpha(\mathcal{M}(S, G)) = \alpha^{(g-1) \dim G} \text{vol}(\mathcal{M}(S, G)).$$

Hence the formula is homogeneous.

For reason that are going to be clearer in the next section, it is also quite natural to choose $\frac{1}{4\pi^2} k$ as it is done in Witten's' original paper. This normalisation is motivated by the fact that $\frac{1}{4\pi^2} \omega$ is integral (as we shall see below) and is used e.g. by Witten. Therefore the original Witten formula reads as

$$\text{vol}^*(\mathcal{M}^{zd}(S, G)) = \left(\frac{1}{2\pi} \right)^{-\chi(S) \dim G} \sum_{\lambda \in \hat{G}} |Z(G)| \cdot \frac{(\text{vol}(G))^{2g-2}}{(\dim V_\lambda)^{2g-2}},$$

which differs from our formula precisely by the factor $(\frac{1}{2\pi})^{-\chi(S) \dim G}$.

3. For $G = \mathrm{SU}(2)$, we have $\widehat{G} \cong \mathbb{N}$ and $\dim V_n = n$. Moreover, $Z(\mathrm{SU}(2)) = \{\pm 1\}$. The volume of G with respect to the Killing form is $2^{\frac{5}{2}}\pi^2$. Thus we obtain

$$\mathrm{vol}(S, \mathcal{M}(\mathrm{SU}(2))) = 2(2^{(5g-5)}\pi^{4g-4}) \sum_{n=1}^{\infty} n^{-(2g-2)}.$$

We rather choose the metric $\frac{1}{4\pi^2}k$. Hence we get,

$$\mathrm{vol}_{\frac{1}{4\pi^2}}(\mathcal{M}(S, G)) = 2 \frac{(2^{(5g-5)}\pi^{4g-4})}{2^{6g-6}\pi^{6g-6}} \sum_{n=1}^{\infty} n^{-(2g-2)} = \frac{\zeta(2g-2)}{2^{g-2}\pi^{2g-2}}$$

where $\zeta(z)$ denotes the Riemann zeta function. In particular, the volume with respect to this normalisation is rational.

The proof goes in two steps. The first step is to identify this measure with a measure obtained by disintegration in Proposition 6.6.3. And the second step is to compute by induction in Proposition 6.6.6 the volume with respect to this measure.

6.6.2 Volume and the disintegrated measure

We first recall the definition of a disintegrated measure. In a presence of a map between two measured spaces, there is a general way to construct a family of measures on the fibres.

Proposition 6.6.2 *Let (X, ν_X) and (Y, ν_Y) be finite measure spaces and let $f : X \rightarrow Y$ be a measurable map, such that $f_*\nu_X$ is absolutely continuous with respect to ν_Y , then there exists a unique measurable family of measures $\{\nu_y\}_{y \in Y}$, where ν_y is a measure on $f^{-1}(y)$, such that for all measurable $g : X \rightarrow \mathbb{R}$ the identity*

$$\int_X g(x) d\nu_X(x) = \int_Y \left(\int_{f^{-1}(y)} g(x) d\nu_y(x) \right) d\nu_Y(y).$$

holds.

The family $\{\nu_y\}_{y \in Y}$ is said to be obtained by *disintegration* of (X, ν_X) and (Y, ν_Y) with respect to f .

In the next proof we will not use the full force of this result and just prove and use it in a geometric differential context.

We now interpretate the symplectic volume using a desintegrated measure. Let μ be the map from G^{2g} to G be given by

$$\mu(a_1, b_1, \dots, a_g, b_g) = \prod_{i=1}^{i=g} [a_i, b_i].$$

We equip G with the Killing form.

Theorem 6.6.3 *Let ν be the desintegrated measure on μ^{-1} . Then*

$$\int_{\mu^{-1}(\text{Id})} \nu = \frac{|Z(G)|}{\text{vol}(G)} \int_{\mathcal{M}(S,G)} \frac{\omega^n}{n!},$$

where $n = \frac{1}{2} \dim(\mathcal{M}(S, G))$ and ω is the symplectic form on $\mathcal{M}(S, G)$.

Preliminaries

We first need to understand in a geometric differential way the desintegration of measures. This is the purpose of the next lemma.

Lemma 6.6.4 *We use the same notation as in Proposition 6.6.2. When X and Y are smooth, f is a submersion, and ν_X and ν_Y come from volume forms Ω_X and Ω_Y , the measures ν_y also comes from volume forms Ω_y . This family of measures satisfy*

$$\Omega_X = \Omega_y \otimes \Omega_Y,$$

where we use the identification

$$\det(T_x X) = \det(\ker T_x f) \otimes \det(T_{f(x)} Y).$$

As a hint of the proof, let us just say that this is a local result and thus it suffices to consider the case where X is a product and f is a projection. The same proof gives the following proposition

Lemma 6.6.5 *Let K be a compact Lie group equipped with a biinvariant measure realised by a volume form Ω_K . Let M be a smooth manifold on which K acts freely, equipped with a K -invariant volume form Ω_M . Then there exists a unique volume form $\Omega_{M/K}$ on M/K characterised by*

$$\Omega_M = \Omega_{M/K} \otimes \Omega_K,$$

when we use the exact sequence

$$0 \rightarrow \mathfrak{K} \rightarrow TM \rightarrow T(M/K) \rightarrow 0,$$

where the first arrow is the orbital map and the second arrow is the projection. Moreover

$$\text{vol}(M/K) = \frac{\text{vol}(M)}{\text{vol}(K)}.$$

Proof of Theorem 6.6.3

We saw in Paragraph 5.2.5 that on the very regular part the sequence

$$0 \rightarrow TG \rightarrow TG^{2g} \xrightarrow{T\mu} TG \rightarrow 0,$$

is identified by the complex

$$0 \rightarrow C^0(\Gamma_g) \rightarrow C^1(\Gamma_g) \rightarrow C^2(\Gamma_g) \rightarrow 0.$$

From the construction of Lemmas 6.6.4 and 6.6.5 and the definition 4.3.4 of the torsion, it follows that,

$$\lambda = \text{Tor}_h(\mathcal{G})$$

where h is the Killing form. From Theorem 4.3.8, it follows that

$$\lambda = \frac{\omega^{\dim G(g-1)}}{\dim G(g-1)!},$$

since $\dim(H_{\nabla}^1(\mathcal{L})) = (2g-2)(\dim G)$.

Therefore, we have

$$\int_{\mu_{zd}^{-1}(\text{Id})} \nu = \frac{|Z(G)|}{\text{vol}(G)} \int_{\mathcal{M}(S,G)} \frac{\omega^n}{n!},$$

where $\mu_{zd}^{-1}(\text{Id})$ is the intersection of $\mu^{-1}(\text{Id})$ with the set of homomorphisms of Zariski dense image.

Finally, a closer inspection of the structure of the stratification of $\mu^{-1}(\text{Id})\mu^{-1}(\text{Id})$ is a reunion of smooth manifolds of dimension strictly less than that of $\mu^{-1}(\text{Id})$ – see Lemma 5.2.6 and have measure zero with respect to the desintegrated measure. It follows that

$$\int_{\mu_{zd}^{-1}(\text{Id})} \nu = \int_{\mu^{-1}(\text{Id})} \nu,$$

and the equality finishes the proof of Theorem 6.6.3

6.6.3 Characters and the disintegrated measure

We now move to the last step needed in order to prove Witten's formula

Proposition 6.6.6 *Let S be a surface of genus g and let G be a simply-connected simple compact Lie group with finite centre $Z(G)$. Then*

$$\int_{\mu^{-1}} \nu = \sum_{\lambda \in \widehat{G}} \frac{(\text{vol}(G))^{2g-1}}{(\dim V_{\lambda})^{2g-2}}$$

In order to prove this proposition, we need properties of characters of G .

characters

Let $\widehat{G}$ be the set of finite dimensional representations of G . Let $\lambda : G \rightarrow GL(V_\lambda)$ be an element of $\widehat{G}$. Such a representation is uniquely determined by its *character* $\chi_\lambda : G \rightarrow \mathbb{C}$, which is defined by $\chi_\lambda(h) := \text{Tr} \lambda(h)$. Note that $\chi_\lambda(1)$ is the dimension of the module V_λ .

The characters χ_λ , $\lambda \in \widehat{G}$ form an orthogonal basis of the subspace of $L^2(G)$ spanned by the conjugation invariant function. The following summarises the properties of characters as functions in L^2 that we shall need.

Proposition 6.6.7 *Let $\lambda, \lambda' \in \widehat{G}$. Then*

$$\int_G \chi_\lambda(ag) \chi_{\lambda'}(g^{-1}b) = \frac{\text{vol}(G)}{\chi_\lambda(1)} \chi_\lambda(ab) \delta_{\lambda, \lambda'}. \quad (6.9)$$

$$\int_G \chi_\lambda(ahbh^{-1}) dh = \frac{\text{vol}(G)}{\chi_\lambda(1)} \chi_\lambda(a) \chi_\lambda(b). \quad (6.10)$$

Moreover, $f \in L^2(G)$ is invariant under conjugation then

$$f(g) = \sum_{\lambda \in \widehat{G}} \frac{\chi_\lambda(g)}{\text{vol}(G)} \int_G f(h) \chi_\lambda(h^{-1}) dh. \quad (6.11)$$

Using these properties of the characters we can now deduce the main result of this section:

Proof of Proposition 6.6.6

Let

$$F(g) = \int_{\mu^{-1}(h)} d\nu_h$$

where $\{\nu_h\}_{h \in G}$ the family of measures obtained by disintegration along μ . One checks that this is a conjugation invariant L^2 -function on G and thus Proposition 6.11 yields

$$F(1) = \sum_{\lambda \in \widehat{G}} \frac{\chi_\lambda(1)}{\text{vol}(G)} \int_G F(h) \chi_\lambda(h^{-1}) dh.$$

Thus by Proposition 6.6.2

$$\begin{aligned} \int_G F(h) \chi_\lambda(h^{-1}) dh &= \int_G \left(\int_{\mu^{-1}(h)} \chi_\lambda \circ \mu d\lambda_h d\lambda_G(h) \right) \\ &= \int_{G^{2g}} \chi_\lambda \left(\prod_{i=1}^{i=g} [a_i, b_i] \right) da_1 db_1 \dots da_g db_g. \end{aligned}$$

It remains to compute the integrals on the right hand side. Using Proposition 6.10 we obtain

$$\begin{aligned} \int_G \chi_\lambda \left(\prod_{i=1}^{i=g} [a_i, b_i] \right) da_1 &= \int_G \chi_\lambda(a_1 b_1 a_1^{-1} b_1^{-1} \cdot \prod_{i=2}^g [a_i, b_i]) da_1 \\ &= \chi_\lambda(b_1^{-1} \cdot \prod_{i=2}^g [a_i, b_i]) \chi_\lambda(b_1) \frac{\text{vol}(G)}{\chi_\lambda(1)}. \end{aligned}$$

Applying Proposition 6.9 yields

$$\begin{aligned} \int_{G^2} \chi_\lambda \left(\prod_{i=1}^{i=g} [a_i, b_i] \right) da_1 db_1 &= \frac{\text{vol}(G)}{\chi_\lambda(1)} \int_G \chi_\lambda(b_1) \chi_\lambda(b_1^{-1} \cdot \prod_{i=2}^g [a_i, b_i]) db_1 \\ &= \left(\frac{\text{vol}(G)}{\chi_\lambda(1)} \right)^2 \chi_\lambda \left(\prod_{i=2}^g [a_i, b_i] \right). \end{aligned}$$

Thus we see by induction that

$$\int_{G^{2g}} \chi_\lambda \left(\prod_{i=1}^{i=g} [a_i, b_i] \right) da_1 db_1 \dots da_g db_g = \left(\frac{\text{vol}(G)}{\chi_\lambda(1)} \right)^{2g} \chi_\lambda(1).$$

Thus using $\chi_\lambda(1) = \dim V_\lambda$, we obtain

$$\int_{\mu^{-1}(1)} \nu = \sum_{\lambda \in \hat{G}} \frac{\chi_\lambda(1)}{\text{vol}(G)} \left(\frac{\text{vol}(G)}{\chi_\lambda(1)} \right)^{2g} \chi_\lambda(1) = \sum_{\lambda \in \hat{G}} \frac{(\text{vol}(G))^{2g-1}}{(\dim V_\lambda)^{2g-2}}.$$

which is what we wanted to prove Q.E.D.

Chapter 7

Three manifolds and integrality questions

7.1 Integrality

We prove in this section

Proposition 7.1.1 [THE SYMPLECTIC FORM IS INTEGRAL] *Let ω be the symplectic form on $\mathcal{M}^{zd}(S, G)$. For any map from f to the 2-sphere S^2 to $\mathcal{M}^{zd}(S, G)$, we have*

$$\frac{1}{4\pi^2} \int_{S^2} f^* \omega \in \mathbb{Z}.$$

More generally, the symplectic form on $\mathcal{M}^{zd}(S, G)$ satisfies

$$\frac{1}{4\pi^2} \omega \in H^2(\mathcal{M}^{zd}(S, G), \mathbb{Z}).$$

REMARKS:

- We have not defined – and will not – what is $H^2(\mathcal{M}^{zd}(S, G), \mathbb{Z})$. Accordingly, we shall not prove the last statement in full details
- It follows from this proposition and a little more work that that the symplectic structure suitably normalised can be realised as the first Chern class of a line bundle over $\mathcal{M}^{zd}(S, G)$ [?] : more precisely, there exists a complex line bundle L over $\mathcal{M}^{zd}(S, G)$ and a connection ∇ on L with curvature R^∇ such that

$$\frac{i}{2\pi} \text{Tr}(R^\nabla)$$

is cohomologous to $\frac{1}{4\pi^2}\omega$. For a general symplectic manifold with integral symplectic form, this line bundle is in general not unique.

In our case, we could actually bypass this construction and define by analytic methods a line bundle L whose curvature is ω : the line bundle is the *holomorphic determinant bundle* $\det_{\mathbb{C}}(T\mathcal{M}^{zd}(S, G))$, where the complex structure on $\mathcal{M}^{zd}(S, G)$ is induced from any choice of complex structure on S , and for ∇ one can take the so-called *Quillen connection*. [?] This construction – which relies on deeper analytic methods – proves in particular that the symplectic form is integral without appealing to the argument that we present in the next sections.

7.1.1 The Cartan form

We continue to assume that G is a simply-connected simple compact Lie group and we fix an invariant non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$.

Definition 7.1.2 *We define the Cartan 3-form ω_C as the left-invariant 3-form on G given by*

$$\omega_C(A, B, C) := \langle [A, B], C \rangle$$

for A, B, C in $\mathfrak{g}$.

The form $\frac{1}{4\pi^2}\omega_C$ satisfies the following property: for any smooth map $F : M \rightarrow G$ from a 3-dimensional manifold M into G one has

$$\frac{1}{4\pi^2} \int_M F^* \omega_C \in \mathbb{Z}.$$

The reader is invited to check this property at least when $G = \mathrm{SU}(2)$. The general results are proved in [?]

7.1.2 Integrating the Cartan form

As a convention in this section, if q is a bilinear form on a vector space V with values in W , and if α and β are 1-forms with values in V , $q(\alpha \wedge \beta)$ is the 2-form with values in W given by

$$q(\alpha \wedge \beta) = q(\alpha(u), \beta(v)) - q(\alpha(v), \beta(u)).$$

Lemma 7.1.3 *Let $M = S \times \mathbb{T}^1$, where S is a surface. Let F be a map from M to G . For t in $[0, 1]$, we write $F_t(m) = F(m, t)$. We have*

$$\int_{S \times \mathbb{T}^1} F^* \omega_C = \int_{\mathbb{T}^1} \left(\int_S \langle F_s^{-1} dF_s \wedge \frac{d}{dt} \Big|_{t=s} (F_t^{-1} dF_t) \rangle \right) ds$$

PROOF: We first give a detailed pedestrian proof, where consider G as a group – via the adjoint representation – of $n \times n$ -matrices. Therefore, if s belongs to $[0, 1]$, we denote by $\dot{F}_s$ the map from S to $M_n(\mathbb{R})$ given by

$$\dot{F}_s : m \rightarrow \left. \frac{d}{dt} \right|_{t=s} F_t.$$

We start from the integrand in the right hand side of the equation. Applying the chain rule yields

$$\begin{aligned} & \int_S \text{Tr}(F_s^{-1} dF_s \wedge \left. \frac{d}{dt} \right|_{t=s} (F_t^{-1} dF_t)) \\ &= \int_S \text{Tr}(F_s^{-1} dF_s \wedge F_s^{-1} d\dot{F}_s) \\ &= \int_S \text{Tr}(F_s^{-1} dF_s \wedge (F_s^{-1} dF_s (\partial_t) F_s^{-1} dF_s)) \end{aligned} \quad (7.1)$$

Now, we observe that for u and v in $T_m S$, we have

$$\begin{aligned} (F_s^{-1} dF_s \wedge F_s^{-1} d\dot{F}_s)(u, v) &= F_s^{-1} dF_s(u) F_s^{-1} d\dot{F}_s(v) - F_s^{-1} dF_s(v) F_s^{-1} d\dot{F}_s(u) \\ &= -d(\text{Tr}(F^{-1} d\dot{F}))(u, v). \end{aligned}$$

Hence, by Stokes Formula

$$\int_S \text{Tr}(F_s^{-1} dF_s \wedge F_s^{-1} d\dot{F}_s) = - \int_S d(\text{Tr}(F^{-1} d\dot{F})) = 0. \quad (7.2)$$

Moreover, observe that the pullback of ω_C by F is

$$\begin{aligned} F^* \omega_C(u, v, w) &= \text{Tr}(F^{-1} dF(u) F^{-1} dF(v) F^{-1} dF(w)) \\ &\quad - \text{Tr}(F^{-1} dF(v) F^{-1} dF(u) F^{-1} dF(w)). \end{aligned}$$

It follows that if u and v belongs to $T_m S$, we have

$$\text{Tr}(F_s^{-1} dF_s \wedge (F_s^{-1} dF_s (\partial_t) F_s^{-1} dF_s))(u, v) = F^* \omega_C(u, v, \partial_t)$$

Finally, combining Equations (7.1), (7.2) and (7.3) we obtain the assertion.

We now give a short proof which is just a conceptual way to write down the above pedestrian computation. Let μ be the Maurer-Cartan 1-form on G with value in $\mathfrak{g}$ given by $\mu_g(u) = g^{-1}u$. The Maurer-Cartan form satisfies

$$d\mu + \frac{1}{2}[\mu \wedge \mu] = 0, \quad (7.3)$$

We now use freely the Lie-Cartan calculus. Then the integrand in the right-hand term can be written as

$$\int_S \langle F_s^{-1} dF_s \wedge \frac{d}{dt} \Big|_{t=s} (F_t^{-1} dF_t) \rangle = \int_S \langle F^* \mu \wedge L_{\partial_t} F^* \mu \rangle$$

Using the fundamental formula of the Lie-Cartan formalism

$$L_X = i_X d + di_X,$$

Stokes Formula and Equation 7.3, we obtain

$$\begin{aligned} \int_S \langle F^* \mu \wedge L_{\partial_t} F^* \mu \rangle &= \frac{1}{2} \int_S L_{\partial_t} F^* \langle \mu \wedge \mu \rangle \\ &= \frac{1}{2} \int_S i_{\partial_t} F^* \langle \mu \wedge [\mu \wedge \mu] \rangle. \end{aligned}$$

Since

$$\langle \mu \wedge [\mu \wedge \mu] \rangle = \omega_C,$$

this colludes the proof of our proposition Q.E.D.

Proof of Proposition 7.1.1

We first prove that for any map f from the sphere S^2 to $\mathcal{M}^{zd}(S, G)$ whose range contains the trivial connection we have

$$\int_{S^2} f^* \omega$$

is an integral multiple of $4\pi^2$. Then we shall explain the idea of the general proof.

Let m_0 be a point in S^2 which is mapped to the trivial connection realise S^2 as $S^2 = D/\partial D$, where the boundary of the disc is identified with m_0 . As D is contractible, the map $f : S^2 \rightarrow \mathcal{M}^{zd}(S, G)$ can be lifted to a map $F : D \rightarrow \mathcal{A}_E$, where the latter denotes the space of flat connection on E – as in the proof of Theorem 6.2.2. In this proof, we have seen that the symplectic form on $\mathcal{M}^{zd}(S, G)$ is obtained locally as the pullback of a certain form Ω via a smooth local section. This implies that

$$\int_{S^2} f^* \omega = \int_D F^* \Omega.$$

Recall from the proof of Theorem 6.2.2 that $\Omega = d\beta$, where

$$\beta_{\nabla}(A) = \int_S \text{tr}((\nabla - \nabla_0) \wedge A)$$

for any fixed choice of connection ∇_0 on E . Thus Stokes' theorem yields

$$\int_{S^2} f^* \omega = \int_{\partial D} F^* \beta.$$

As $S^2 = D/\partial D$ the boundary ∂D is mapped to the fibre over $f(m_0)$ of the infinite-dimensional bundle $\mathcal{A}_E \rightarrow \mathcal{M}^{zd}(S, G)$. This fibre is identified with the gauge group of E and as E was assumed to be trivial this gauge group is just the group of smooth maps from S to G . In particular, we have a smooth map

$$H : S \times \partial D \rightarrow G, \quad (x, t) \mapsto H_t(x)$$

such that

$$F(t) = D_t := H_t^{-1} D_0 H_t,$$

where D_0 denotes the trivial connection on E . More precisely our notation means that

$$(D_t)_x(u) = H_t^{-1}(D_0)_x(H_t(u)).$$

In particular we have

$$\int_{S^2} f^* \omega = \int_{\partial D} F^* \beta = \int_{S \times \partial D} \text{Tr}((D_0 - D_t) \dot{D}_t)$$

Now $D_0 - D_t = H_t^{-1} dH_t$ and $\dot{D}_t = \frac{\partial}{\partial t}(H_t^{-1} dH_t)$ and thus by Lemma 7.1.3,

$$\int_{S^2} f^* \omega = \int_{S \times \partial D} \text{Tr} \left(H_t^{-1} dH_t \frac{\partial}{\partial t} (H_t^{-1} dH_t) \right) = \int_{S \times \partial D} H^* \omega_C \in 4\pi^2 \mathbb{Z}.$$

This finishes the proof of the first part of the proposition.

We explain briefly how to obtain the second part using a similar scheme.

Let Σ be a triangulated closed surface. Using the previous argument, we can lift every simplex Δ to a simplex $\tilde{\Delta}$ in $\mathcal{A}_E$. Then by construction the integral of ω on Δ is the integral of the symplectic form Ω of $\mathcal{A}_E$ on $\tilde{\Delta}$. Let e be an edge of the triangulation, the lift of the simplex on each side of the triangulation do not coincide on the boundary and the defect is given by a map from e to the gauge group. Since the gauge group is connected, we can find an homotopy of this map to the identity. In other words, we can find a map from $e \times [0, 1]$ to $\mathcal{A}_E$ which connects the two lifts of the two simplexes bounded by e such that the image of $\{a\} \times [0, 1]$ lies in the orbit of the gauge group for each a in e . Observe now that the integral of Ω on the image of this map is zero. It follows that – after "fattening" the edge, we have constructed a map Φ from $\Sigma \setminus V$ – where V is the set of vertices of the triangulation – in $\mathcal{E}$ so that

$$\int_{\Sigma \setminus V} \Phi^* \Omega = \int_{\Sigma} f^* \omega.$$

We now use as before the fact that $\Omega = d\beta$ and obtain that

$$\int_{\Sigma \setminus V} \Phi^* \Omega = \sum_{v \in V} \int_{c_v} \beta,$$

where, for each vertex v , c_v is a curve in an orbit of the gauge group. Just as before, we obtain a map H from $S \times [0, 1]$ to G so that

$$\int_{c_v} \beta = \int_{M \times [0, 1]} H^* \omega_C \in 4\pi^2 \mathbb{Z}.$$

This shows that

$$\int_{\Sigma} f^* \omega \in 4\pi^2 \mathbb{Z},$$

and concludes this sketch of proof. Q.E.D.

7.2 Handlebodies and Lagrangians

Let M be a 3-manifold such that $\partial M = S$. The inclusion from S into M gives rise to a map i_* from $\pi_1(S)$ to $\pi_1(M)$, hence to a map I from $\text{Rep}(\pi_1(M), G)$ to $\text{Rep}(\pi_1(S), G)$.

Let $\mathcal{L}_M = I(\text{Rep}(\pi_1(M), G))$ and $\mathcal{L}_M^{zd} := \mathcal{L}_M \cap \mathcal{M}^{zd}(S, G)$. In other words

$$\mathcal{L}_M^{zd} := \{[\rho] \in \mathcal{M}^{zd}(S, G) \mid \exists \tilde{\rho} : \pi_1(M) \rightarrow G, \rho = \tilde{\rho}|_{\pi_1(S)}\}$$

We will consider the special case when M is a handlebody in which case i_* is an injection and $\pi_1(M)$ is generated by half the generators of $\pi_1(S)$.

Theorem 7.2.1 [LAGRANGIAN SUBMANIFOLD] *Let M be an handlebody with boundary S , then $\mathcal{L}_M^{zd}$ is a Lagrangian submanifold of $\mathcal{M}^{zd}(S, G)$.*

The particular importance of handlebodies in the theory of 3-manifolds stems from the following fact:

Proposition 7.2.2 *Heegaard If M is an arbitrary 3-manifold, then there exists handlebodies M_1 and M_2 and a homeomorphism $f : \partial M_1 \rightarrow \partial M_2$ such that*

$$M = M_1 \cup_f M_2$$

The surface appearing as the common boundary of the two handlebodies is constructed as the boundary of tubular neighbourhood of the 1-skeleton of a triangulation of the 3-manifold. The observation is then that this surface is

also the boundary of the tubular neighbourhood of the 1-skeleton of the dual graph.

A decomposition of M as in Proposition 7.2.2 is called a *Heegaard splitting*. Clearly a flat connection on M restricts to flat connections on M_1 on M_2 and vice-versa. Thus the gauge equivalence classes of flat connection on the 3-manifold M are parametrised by $\mathcal{L}_{M_1} \cap \mathcal{L}_{M_2} \subset \mathcal{M}(S, G)$. We will apply this fact in the next subsection in order to compute an important invariant for flat connections on 3-manifolds.

7.2.1 Submanifolds of the moduli space

We first prove

Proposition 7.2.3 *Let M be an handlebody with boundary S , then $\mathcal{L}_M^{zd}$ is a smooth submanifold of $\mathcal{M}^{zd}(S, G)$ with*

$$\dim \mathcal{L}_M = \frac{1}{2} \dim \mathcal{M}^{zd}(S, G).$$

PROOF: The fundamental group of the handlebody is a free group with g generators. Let U be the open set of G^g consisting of those g -tuples $a = \{a_1, \dots, a_g\}$ such that the group $\Gamma(a)$ generated by $\{a_1, \dots, a_g\}$ is Zariski dense. The group G acts by conjugation on U and the stabiliser of a is the centraliser of $\Gamma(a)$ hence of its Zariski closure. It follows that $G/Z(G)$ acts freely on U and we denote by $\text{Rep}^{zd}(\pi_1(M), G)$ the quotient $(G/Z(G)) \backslash U$. The space $\text{Rep}^{zd}(\pi_1(M), G)$ is a manifold of dimension

$$(g - 1) \dim G = \frac{1}{2} \dim \mathcal{M}^{zd}(S, G).$$

By construction, $\text{Rep}^{zd}(\pi_1(M), G)$ is embedded as a submanifold of $\mathcal{M}^{zd}(S, G)$ and its image is $\mathcal{L}_M^{zd}$. Q.E.D.

7.2.2 Isotropic submanifold

We prove the second part of Theorem 7.2.1.

Proposition 7.2.4 *The restriction of the symplectic form $\omega|_{\mathcal{L}_M^{zd}}$ vanishes identically.*

REMARK: Actually, this statement generalises when M is not a handlebody. The same result – and proof – holds when we replace $\omega|_{\mathcal{L}_M^{zd}}$ – which is not necessarily a submanifold – by any smooth submanifold embedded in $\omega|_{\mathcal{L}_M^{zd}}$.

PROOF: Suppose that $A, B \in T_{[\rho]}\mathcal{M}^{zd}(S, G) \cong H_\rho^1(S)$ are contained in $T_{[\rho]}\mathcal{L}_M^{zd}$. Then they can be extended to cohomology classes $\tilde{A}, \tilde{B}$ in $H_\rho^1(M)$ and

$$\omega_\rho(A, B) = \int_S \text{Tr}(A \wedge B) = \int_M d \text{Tr}(\tilde{A} \wedge \tilde{B}) = 0.$$

Q.E.D.

This means that $\mathcal{L}_M^{zd}$ is a Lagrangian submanifold of $\mathcal{M}^{zd}(S, G)$ in the sense of the following definition:

Definition 7.2.5 *If (M, ω) is a symplectic manifold and $N \subset M$ is a submanifold, then N is called Lagrangian if it maximal degenerate, i.e. $\dim N = \frac{1}{2} \dim M$ and $\omega|_N \equiv 0$.*

7.2.3 Chern-Simons invariants

In this final paragraph, we discuss some application of the moduli spaces $\mathcal{M}(S, G)$ for $G = \text{SU}(n)$ to 3-manifolds.

Chern-Simons invariants for connections and representations

Let E be a rank n complex vector bundle over a 3-manifold M . By choosing a hermitian structure on E we can view E as a G -bundle for $G = \text{SU}(n)$. Denote by $\mathcal{A}_E$ the affine space of flat G -connections on E and by A_E the underlying vector space. We view $\mathcal{A}_E$ as an infinite-dimensional manifold with tangent space A_E at each point.

Definition 7.2.6 [CHERN-SIMONS FORM] *We define the Chern-Simons form as the real valued 1-form ω^{CS} on the affine space $\mathcal{A}_E$ given by*

$$\omega_\nabla^{CS}(A) := \int_M \text{Tr}(A \wedge R^\nabla),$$

where ∇ is a G -connection, A is an element of $A_E \cong T_\nabla \mathcal{A}_E$, where R^∇ is the curvature of ∇ and we set

$$A \wedge R^\nabla(X_1, X_2, X_3) := \frac{1}{6} \sum_{\sigma \in \mathfrak{S}(3)} A(X_{\sigma(1)}) R^\nabla(X_{\sigma(2)}, X_{\sigma(3)}).$$

Proposition 7.2.7 *Assume that the 3-manifold M is closed. Suppose ∇_1 and ∇_2 are gauge-equivalent connections on E and $\gamma : [0, 1] \rightarrow \mathcal{A}_E$ is a path from ∇_1 to ∇_2 . Then*

$$\frac{1}{4\pi^2} \int_\gamma \omega^{CS} \in \mathbb{Z}$$

In particular, ω^{CS} is closed.

As a first consequence of this proposition, when the 3-manifold is closed, we can define the Chern-Simons action difference for connections.

Definition 7.2.8 [CHERN-SIMONS ACTION FOR CONNECTIONS] *Let ∇_0 and ∇_1 be two connections defined over the same vector bundle E . Then we define the Chern-Simons action difference to be the real number*

$$CS(\nabla_1, \nabla_0) = \frac{1}{4\pi^2} \int_{\gamma} \omega^{CS}(\dot{\nabla}_t) dt,$$

where γ is any path of connections joining ∇_0 to ∇_1 .

As a second consequence, we can define the Chern-Simons action difference for gauge equivalence classes of connections – and in particular for representations from $\pi(M)$ to $SU(n)$ – as an element of $\mathbb{R}/\mathbb{Z}$, in the following way

Definition 7.2.9 [CHERN-SIMONS ACTION FOR EQUIVALENCE CLASSES OF CONNECTIONS] *Let ρ_1 and ρ_2 be two Gauge equivalence classes of connections defined over the same vector bundle E . Let ∇_1 and ∇_2 be two connections on E representing respectively ρ_1 and ρ_2 , then we define the Chern-Simons action difference to be*

$$CS(\rho_1, \rho_2) = CS(\nabla_1, \nabla_2) \in \mathbb{R}/\mathbb{Z}.$$

Observe that, whenever it makes sense,

$$CS(\rho_1, \rho_2) = CS(\rho_1, \rho_3) + CS(\rho_3, \rho_2).$$

We now prove Proposition 7.2.7.

PROOF: Define a bundle $\mathcal{E}$ over $M \times [0, 1]$ by $\mathcal{E} := p_1^*(E)$, where p_1 denotes the projection onto the first factor. Then there exists a unique connection ∇ on $\mathcal{E}$ which satisfies the following two conditions

1. Let $i_t : M \rightarrow M \times \{t\}$ be the injection $m \rightarrow (m, t)$ then $i_t^* \nabla = \gamma(t)$.
2. Let $j_m : [0, 1] \rightarrow \{m\} \times [0, 1]$ be the injection $t \rightarrow (m, t)$. Observe that by construction $j_m^* \mathcal{E}$ is the trivial bundle $\mathcal{E}|_{m \times [0, 1]}$. We furthermore require that for each fixed m in M , $J_m^* \nabla$ is the trivial connection.

Moreover, the curvature of ∇ is given by

$$R^\nabla = dt \wedge \dot{\nabla} + R^{\gamma(t)}.$$

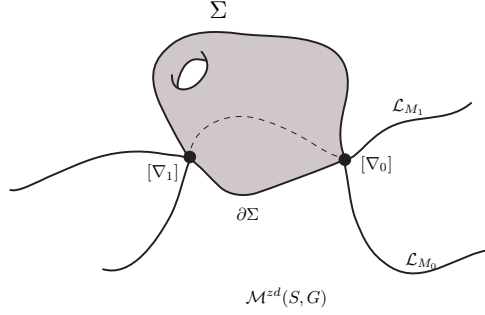


Figure 7.1: Connections in good position

Therefore,

$$\int_{\gamma} \omega^{CS} = \int_0^1 \int_M \text{Tr}(\dot{\nabla} \wedge R^{\gamma(t)}) dt = \frac{1}{2} \int_{M \times [0,1]} \text{Tr}(R^{\nabla} \times R^{\nabla}).$$

Now consider $P := M \times \mathbb{T}^1$, which we view as a quotient of $M \times [0, 1]$ by identifying the endpoints of the interval. Using the gauge equivalence between ∇_1 and ∇_2 we can identify $\mathcal{E}|_{M \times \{0\}}$ and $\mathcal{E}|_{M \times \{1\}}$ in order to obtain a bundle with connection $(\mathcal{E}', \nabla')$ over P . Then we have

$$\frac{1}{4\pi^2} \int_{\gamma} \omega^{CS} = \frac{1}{8\pi^2} \int_P \text{Tr}(R^{\nabla'} \wedge R^{\nabla'}) = p_1(\mathcal{E}'),$$

where p_1 denotes the first Pontryagin class. Then the proposition follows from the fact that the Pontryagin classes of the bundle $\mathcal{E}'$ are integral. Q.E.D.

7.2.4 Chern Simons action difference and the symplectic form

We explain how to compute the Chern-Simons action difference, using a Heegaard splitting and the symplectic geometry of the moduli space.

For this we fix a Heegaard splitting $M = M_1 \cap_f M_0$, where M_1, M_0 are handlebodies with boundary S . Then ∇_1, ∇_0 correspond to points in $\mathcal{L}_{M_1} \cap \mathcal{L}_{M_0} \subset \mathcal{M}(S, G)$. Let us make a number of simplifying assumptions

- The connections ∇_1, ∇_0 correspond to regular points in $\mathcal{M}^{zd}(S, G)$.

- There is a surface Σ with one connected boundary component with two marked points x_1, x_0 on the boundary and a smooth map $\iota : \Sigma \rightarrow \mathcal{M}^{zd}(S, G)$ such that $\iota(x_j)$ corresponds to ∇_j and that the two components of $\partial\Sigma \setminus \{x_1, x_0\}$ are mapped to $\mathcal{L}_{M_1}$ and $\mathcal{L}_{M_0}$ respectively.

Then one has

Theorem 7.2.10 *Let ∇_1, ∇_0 be flat connections on M satisfying the above regularity requirements and defining a surface Σ in $\mathcal{M}^{zd}(S, G)$ as above. Then*

$$\text{CS}([\nabla_1], [\nabla_0]) = \frac{1}{4\pi^2} \int_{\Sigma} \omega,$$

as elements of $\mathbb{R}/\mathbb{Z}$.

As an exercise, we leave the reader go through the proof below and check which orientation on S – either as a boundary of M_0 or as a boundary of M_1 and which orientation on Σ gives the correct sign in the formula above.

PROOF: We only sketch this proof. We triangulate Σ in such a way that x_1 and x_0 are the only vertices of this triangulation that belong to the boundary. Let b_1 and b_0 be the two edges of this triangulation in the boundary of Σ , where b_1 joins x_0 to x_1 .

Proceeding as alluded in the end of the proof of Proposition 7.1.1, we construct a map Φ from $\Sigma \setminus V$ – where V is the set of vertices of the triangulation – in $\mathcal{A}_E$ lifting ι so that

$$\int_{\Sigma \setminus V} \Phi^* \Omega = \int_{\Sigma} \iota^* \omega.$$

Moreover, blowing up x_1 and x_0 to arcs c_1 and c_0 and the interiors points of V to loops c_v which all lies in orbits on the gauge group, we obtain that

$$\int_{\Sigma \setminus V} \Phi^* \Omega = \sum_{v \in V \setminus \{x_1, x_0\}} \int_{c_v} \beta + \int_{c_1} \beta + \int_{c_0} \beta + \int_{b_1} \beta + \int_{b_0} \beta.$$

Working in $\mathbb{R}/\mathbb{Z}$, and observing again that $\int_{c_v} \beta$ belongs to $\mathbb{Z}$, we are left to understanding the integral of β along some arcs.

Since $\mathcal{A}_E$ is contractible, we can find a map C from $[0, 1] \times [0, 1]$ filling up the loop given by c_1, b_1, c_0 and b_0 . We choose this map so that $t \rightarrow C(0, t)$ is a parametrisation of c_1 with the respective obvious conventions for the other sides. We also denote by $\nabla_{s,t}$ the connection on the bundle E over S defined by $C(s, t)$.

Therefore it remains to compute

$$\int_C \Omega = \int_0^1 \left(\int_0^1 \Omega(\partial_s C, \partial_t C) ds \right) dt.$$

By definition,

$$\int \Omega(\partial_s C, \partial_t C) ds = \int_{[0,1]} \int_S \text{Tr}(\partial_s \nabla_{s,t}, \partial_t \nabla_{s,t}) ds$$

As usual, the map $t \rightarrow \nabla_{s,t}$ defines a connection ∇_s on $S \times [0, 1]$ which restricts to $\nabla_{s,t}$ on $S \times \{t\}$ and to a trivial connection along $\{m\} \times [0, 1]$. Thus

$$R^{\nabla_s} = dt \wedge \partial_t \nabla_{s,t} + R^{\nabla_{s,t}}.$$

It follows that

$$\int_{[0,1]} \int_S \text{Tr}(\partial_s \nabla_{s,t}, \partial_s \nabla_{s,t}) = \int_{S \times [0,1]} \text{Tr}(\partial_s \nabla_{s,t}, R^{\nabla_t}).$$

Since $\nabla_{s,0}$ and $\nabla_{s,1}$ extends respectively to flat connections on M_0 and M_1 . The connection ∇_s extends to a flat connection $\hat{\nabla}_s$ on $\hat{M} = M_0 \cup S \times [0, 1] \cup M_1$. We leave as an exercise that we can choose $\hat{\nabla}_s$ to be smooth (*Hint* : use local systems). Thus

$$\begin{aligned} \int_{[0,1]} \int_S \text{Tr}(\partial_s \nabla_{s,t}, \partial_s \nabla_{s,t}) &= 1 \int_{S \times [0,1]} \text{Tr}(\partial_s \nabla_s, R^{\nabla_s}) \\ &= \int_M \text{Tr}(\partial_s \hat{\nabla}_s, R^{\hat{\nabla}_s}) \\ &= \omega_{\hat{M}}^{CS}(\partial_s \hat{\nabla}_s). \end{aligned}$$

It follows that

$$\frac{1}{4\pi^2} \int_C \Omega = CS(\hat{\nabla}_1, \hat{\nabla}_0).$$

By construction, the connection $\hat{\nabla}_1$ is gauge equivalent to ∇_1 , and $\hat{\nabla}_0$ is gauge equivalent to ∇_0 . Therefore as elements of $\mathbb{R}/\mathbb{Z}$.

$$\frac{1}{4\pi^2} \int_C \Omega = \omega_{\hat{M}}^{CS}(\nabla_1, \nabla_0).$$

Q.E.D.