ON BOUNDARY MAPS OF ANOSOV REPRESENTATIONS OF A SURFACE GROUP TO $\mathrm{SL}(3,\mathbb{R})$

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Abstract. We prove that Anosov representations from a closed surface group to $\mathrm{SL}(3,\mathbb{R})$ are uniquely determined by their boundary maps $S^1 \to \mathrm{Flag}(\mathbb{R}P^2)$ if and only if they do not factor over a completely reducible representation $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(3,\mathbb{R})$.

On the other hand, there are families of completely reducible representations which can not be distinguished neither by their boundary maps nor by the topological conjugacy class of the action on their domain of discontinuity.

We also prove that the quotient of the space of Anosov representations by the action of the mapping class group has at least $g+2$ components where g is the genus of the surface.

1. Introduction

Let Σ_g be an oriented, closed surface of genus $g \geq 2$. Then its fundamental group is presented as

$$\pi_1\Sigma_g = \left\langle a_1, b_1, \ldots, a_g, b_g \,\middle|\, \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle$$

Here $[a,b]$ denotes the commutator of a and b. Given a Lie group G, the representation variety $\mathrm{Hom}(\pi_1\Sigma_g, G)$ has a natural topology as a subset of G^{2g}. When $G = \mathrm{SL}(3,\mathbb{R})$, Hitchin ([7]) proved that the representation variety $\mathrm{Hom}(\pi_1\Sigma_g, \mathrm{SL}(3,\mathbb{R}))$ has three connected components, two of which correspond to topologically trivial representations with vanishing Stiefel-Whitney classes.

One of these two components is the so-called Hitchin component whose elements, the Hitchin representations, can be characterised by various equivalent properties. They are hyperconvex representations in the sense of [9] and correspond to convex projective structures on Σ_g by [3]. In particular they have Hölder-embedded circles Λ in the flag variety $\mathrm{Flag}(\mathbb{R}P^2)$ as limit curves. They are also characterized as representations with positive X-coordinates in the sense of [5]. This implies that all $\gamma \in \pi_1\Sigma_g$ are mapped to matrices with three distinct, positive real eigenvalues.

One important theme is that Hitchin representations are determined by and can be studied via their boundary map $\xi\colon S^1_\infty \to \mathrm{Flag}(\mathbb{R}P^2)$. In particular, the parametrisation of Hitchin representations by Bonahon-Dreyer ([2]) uses Fock-Goncharov's X-coordinates which are determined by the boundary map alone.

We are interested in the other component of representations with trivial Stiefel-Whitney class. This component contains the trivial representation and also all representations arising from the composition of the (lift of the) monodromy of a hyperbolic structure

$$\pi_1\Sigma_g \to \mathrm{SL}(2,\mathbb{R})$$

with the natural reducible representation by block matrices

$$\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(3,\mathbb{R}).$$

The latter examples and its deformations have been studied by Barbot [1], who proved that for so-called radial deformations, i.e., deformations arising by multiplication with $\begin{pmatrix} e^{u(\gamma)} & 0 & 0 \\ 0 & e^{-2u(\gamma)} & 0 \\ 0 & 0 & e^{u(\gamma)} \end{pmatrix}$ for some homomorphism $u\colon \pi_1\Sigma_g \to \mathbb{R}$, one always has the same boundary map $\xi\colon \partial_\infty \pi_1(\Sigma_g) \to \mathrm{Flag}(\mathbb{R}P^2)$, thus the same limit curve Λ_0 and a domain of discontinuity in the flag variety with quotient a 2-fold covering space of the unit tangent bundle $T^1\Sigma_g$.

In [1] Barbot asked whether the space of non-hyperconvex (i.e., non-Hitchin) representations is connected. Regarding this question, Thierry Barbot and Jaejeong Lee, in Daejeon 2014, observed that there are at least $2^{2g}+1$ connected components in the space of Anosov representations from a genus g surface group to $\mathrm{SL}(3,\mathbb{R})$, which gives a counterexample to the question. The reason behind is that eigenvalues, varying continuously, cannot change signs.

The mapping class group acts naturally on the space of Anosov representations $\mathcal{A}$ and for the action on components we have the following result.

Proposition 1.1. *There are at least $g+2$ orbits for the action of the mapping class group $\mathrm{MCG}(\Sigma_g)$ on $\pi_0\mathcal{A}$.*

The disconnectedness of non-Hitchin Anosov representations is due to completely reducible Anosov representations coming in tuples of 2^{2g} representations which all have to be in distinct components of $\mathcal{A}$. These 2^{2g} Anosov representations all share the same boundary map $\xi\colon S^1 \to \mathrm{Flag}(\mathbb{R}P^2)$. In particular they can not be distinguished in terms of invariants defined via boundary maps and in fact their Fock-Goncharov invariants are not well-defined.[1] Moreover we see in Section 3.2 that these representations can not be distinguished by the topological actions on their domains of discontinuity.

We show however in Section 5 that this is an exceptional behaviour, i.e., that these examples (and their radial deformations as considered by Barbot) are the only Anosov representations which are not determined by their boundary maps.

Theorem 1.2. *If $\rho_1, \rho_2\colon \pi_1\Sigma_g \to \mathrm{SL}(3,\mathbb{R})$ are Anosov representations with the same boundary map $\xi\colon \partial_\infty\pi_1(\Sigma_g) \to \mathrm{Flag}(\mathbb{R}P^2)$, then both factor over some completely reducible representation $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(3,\mathbb{R})$ and ρ_2 is obtained from ρ_1 by left multiplication with some conjugate of*

$$\begin{pmatrix} \lambda(\gamma) & 0 & 0 \\ 0 & \frac{1}{\lambda(\gamma)^2} & 0 \\ 0 & 0 & \lambda(\gamma) \end{pmatrix}$$

for some homomorphism $\lambda\colon \pi_1\Sigma_g \to \mathbb{R}\setminus\{0\}$.

This was known from [5, Theorem 6.1] for positive representations, where it is just a consequence of the facts that a positive triple of flags determines a projective basis

[1]So this is different from the case of representations to $\mathrm{PSL}(2,\mathbb{R})$, where it is a consequence of Goldman's theorem that Anosov representations are uniquely determined by their boundary maps.

and that an element in $SL(3,\mathbb{R})$ is determined by its action on a projective basis. This argument also works for generic (not necessarily positive) representations and therefore the content of our proof is to analyze the situation in the non-generic case, when the boundary map is sending all triangles of an ideal triangulation to non-generic triples of flags which do not determine a projective basis.

In [5, Definition 1.9], Fock and Goncharov defined a universal higher Teichmüller space which in the case of $G = \mathrm{PGL}(3,\mathbb{R})$ consists of all positive maps $\xi\colon \mathbb{Q}P^1 \to \mathrm{Flag}(\mathbb{R}P^2)$ modulo the action of $\mathrm{PGL}(3,\mathbb{R})$, and they showed that a subset of it parametrises the Hitchin component. Our result shows that one can still parametrise the not completely reducible Anosov representations by a subset of the (not necessarily positive) maps $\xi\colon \mathbb{Q}P^1 \to \mathrm{Flag}(\mathbb{R}P^2)$ modulo the action of $\mathrm{PGL}(3,\mathbb{R})$

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2. Recollections

Throughout the paper Σ_g will be the closed, orientable surface of genus $g \geq 2$. We will freely use the identifications

$$H^1(\Sigma_g;\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(\pi_1\Sigma_g,\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(H_1(\Sigma_g;\mathbb{Z}),\mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{2g}.$$

We will always denote $G = \mathrm{PGL}(3,\mathbb{R}) = \mathrm{SL}(3,\mathbb{R})$ and $B \subset G$ will be the subgroup of upper triangular matrices.

2.1. **Anosov representations.**

2.1.1. *Definitions.* Recall that a flag in $\mathbb{R}P^2$ is a pair

$$([v]\,,[f]) \in P(\mathbb{R}^3) \times P(\mathbb{R}^{3*})$$

with $f(v) = 0$. Denote by

$$X := \mathrm{Flag}(\mathbb{R}P^2)$$

the flag variety of $\mathbb{R}P^2$ and by

$$Y := \mathrm{Frame}(\mathbb{R}P^2)$$

the frame variety, that is, the space of noncollinear triples of points in $\mathbb{R}P^2$.

There is a well-known open embedding $\iota\colon Y \to X \times X$ sending $([u]\,,[v]\,,[w])$ to

$$(([u]\,,[(uv)^*]),([w]\,,[(wv)^*]),$$

see [1, Section 2.3]. The image of this embedding is an open set and in particular T_yY is naturally identified with $T_{\iota(y)}(X\times X)$ for each $y \in Y$. The two direct summands of $T(X \times X) = TX \oplus TX$ are denoted by E^+ and E^-.

Two flags $([v_1]\,,[f_1])$ and $([v_2]\,,[f_2])$ are called transverse if $f_2(v_1) \neq 0$ and $f_1(v_2) \neq 0$.

Anosov representations were originally considered by Labourie in [9], the notion of P-Anosov representations for general parabolic subgroups $P \subset G$ was defined by Guichard-Wienhard in [6]. In this paper we will only consider the case that $P = B$ is the group of upper triangular matrices and henceforth abbreviate "B-Anosov representation" by "Anosov representation". Before giving a definition of Anosov representation, a closed surface Σ_g is assumed to be a hyperbolic surface.

Definition 2.1. *A representation $\rho\colon \pi_1\Sigma_g \to \mathrm{SL}(3,\mathbb{R})$ is an Anosov representation if there exist continuous, ρ-equivariant maps*

$$\xi^{\pm}\colon \partial_\infty \mathbb{H}^2 \to \mathrm{Flag}(\mathbb{R}P^2)$$

such that

(i) *$\xi^+(\eta)$ and $\xi^-(\eta)$ are transverse for each $\eta \in \partial_\infty \mathbb{H}^2$, so ξ^+ and ξ^- combine to a map*

$$\widetilde{\sigma}\colon T^1\mathbb{H}^2 \to \mathrm{Frame}(\mathbb{R}P^2).$$

(ii) *The lifted geodesic flow on $\widetilde{\sigma}^*E^+$ resp. $\widetilde{\sigma}^*E^-$ is dilating resp. contracting.*

In our case we can assume that $\xi^- = \partial_\infty s \circ \xi^+$, with $\partial_\infty s$ induced by the antidiagonal matrix $s \in \mathrm{SL}(3,\mathbb{R})$ permuting the basis vectors e_1 and e_3. Thus we will talk about "the" boundary map $\xi := \xi^+$, see [6, Section 4.5].

2.1.2. *Space of Anosov representations.* Let us denote

$$\mathcal{A} \subset \mathrm{Hom}(\pi_1\Sigma_g, \mathrm{SL}(3,\mathbb{R}))$$

the set of Anosov representations. By [9, Proposition 2.1] it is an open subset of the representation variety. The work of Labourie shows that $\mathcal{A}$ contains the Hitchin component and the work of Barbot exhibits Anosov representations in the other component of the topologically trivial representations. By [1, Corollary 6.6] only those two components can contain Anosov representations. Our aim is to distinguish components of $\mathcal{A}$ inside the non-hyperconvex component of topologically trivial representations in $\mathrm{Hom}(\pi_1\Sigma_g, \mathrm{SL}(3,\mathbb{R}))$.

Representations in this component have been studied in [1]. One of the results was that in all cases $\mathrm{Flag}(\mathbb{R}P^2)$ decomposes into the limit curve $\Lambda = \xi(\partial_\infty\mathbb{H}^2)$, a domain of discontinuity Ω homeomorphic to a solid torus, and two invariant Möbius bands with complicated dynamics. Moreover, the quotient of Ω by the $\rho(\pi_1(\Sigma_g))$-action is a circle bundle over Σ_g.

In [1, Section 8], Barbot asked the following questions for non-hyperconvex Anosov representations $\rho\colon \pi_1\Sigma_g \to \mathrm{SL}(3,\mathbb{R})$, noting that a positive answer to Question 2 would imply a positive answer to Question 1 (cf. [6, Theorem 9.12]).

Question 1: Is the circle bundle $\rho(\Gamma)\backslash\Omega$ homeomorphic to the double covering of the unit tangent bundle of Σ_g?

Question 2: Is the space of non-hyperconvex Anosov representations connected?

As mentioned before, due to the observation of T. Barbot and J. Lee (see Section 4.1), it turns out that the answer for Question 2 is no. On the other had, Question 1 might still have a positive answer in view of the result of Section 3.2 below.

3. Completely reducible representations

In this section we consider the completely reducible representations ρ_ϕ which yield 2^{2g} different components of non-hyperconvex Anosov representations. The remainder of the section will not play a rôle for this paper, though it might be of independent interest: we show that the 2^{2g}-tuples of completely reducible representations with the same boundary map can also not be distinguished by the action on their domain of discontinuity, and we show that they are singular points of the character variety.

3.1. **Construction of a $(\mathbb{Z}/2\mathbb{Z})^{2g}$-action.** Assume a fixed representation

$$\rho_0 \colon \pi_1\Sigma_g \to \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(3,\mathbb{R}),$$

where we will as in [1] think of $\mathrm{SL}(2,\mathbb{R})$ embedded in $\mathrm{SL}(3,\mathbb{R})$ compatible with the embedding $(x,y) \to (x,0,y)$ of $\mathbb{R}^2$ in $\mathbb{R}^3$. Let us denote

$$J_{13} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

For each homomorphism

$$\phi \colon \pi_1\Sigma_g \to \mathbb{Z}/2\mathbb{Z} = \{0,1\}$$

we can consider the representation $\rho_\phi \colon \pi_1\Sigma_g \to \mathrm{SL}(3,\mathbb{R})$ defined by

$$\rho_\phi(\gamma) = \rho_0(\gamma) J_{13}^{\phi(\gamma)}$$

for all $\gamma \in \pi_1\Sigma_g$. This representation is well-defined because J_{13} commutes with all $\rho_0(\gamma)$ and hence the relation $\prod_{i=1}^g [\rho_0(a_i), \rho_0(b_i)] = 1$ for the standard generators $a_1, b_1, \ldots, a_g, b_g$ of $\pi_1\Sigma_g$ implies $\prod_{i=1}^g [\rho_\phi(a_i), \rho_\phi(b_i)] = 1$.

Observe that even though the images in $\mathrm{SL}(2,\mathbb{R})$ project to the same representations in $\mathrm{PSL}(2,\mathbb{R})$, this is not the case for the images in $\mathrm{SL}(3,\mathbb{R})$ in view of the equality $\mathrm{SL}(3,\mathbb{R}) = \mathrm{PGL}(3,\mathbb{R})$.

3.2. **Domains of discontinuity.** It is easy to check that all the ρ_ϕ are Anosov representations with the same boundary map as ρ_0, namely the embedding $\mathbb{R}P^1 \to \mathrm{Flag}(\mathbb{R}P^2)$ which is induced by the embedding $\mathbb{R}P^1 \to \mathbb{R}P^2$ given by $[x:y] \to [x:0:y]$. Let L be the image of the latter curve in $\mathbb{R}P^2$, and $L^* = \{[f] : f(e_2) = 0\}$, the image of the boundary map is $L \times L^*$ and one of the three components of its complement is

$$\Omega = \{([v],[f]) \colon v \notin L \text{ and } f \notin L^*\} \subset \mathrm{Flag}(\mathbb{R}P^2),$$

which can be interpreted as the projective tangent bundle of the disk $\mathbb{R}P^2 \setminus L$, and is thus equivariantly homeomorphic to $\mathrm{SL}(2,\mathbb{R})$. The action of $\rho_\phi(\pi_1\Sigma_g)$ on Ω is properly discontinuous as a special case of [1, Theorem 5.1]. We will argue that the actions of $\rho_\phi(\pi_1\Sigma_g)$ for different ϕ do all yield the same quotient manifold $\rho_\phi(\pi_1\Sigma_g)\backslash\Omega$.

The base space. A hyperbolic structure on Σ_g is given by its monodromy representation $\rho \colon \pi_1\Sigma_g \to \mathrm{PSL}(2,\mathbb{R})$. The quotient

$$\rho(\pi_1\Sigma_g)\backslash\mathrm{PSL}(2,\mathbb{R})$$

is the unit tangent bundle $T^1\Sigma_g$. Because this is a circle bundle we have an exact sequence

$$0 \to \mathbb{Z} \to \pi_1(T^1\Sigma_g) \to \pi_1\Sigma_g \to 1.$$

By Culler's theorem ρ can be lifted to a representation $\rho_0\colon \pi_1\Sigma_g \to \mathrm{SL}(2,\mathbb{R})$. We assume such a lift to be fixed.

The covering spaces. Since the Euler class of $T^1\Sigma_g$ is even, the spectral sequence for the homology with $\mathbb{Z}/2\mathbb{Z}$-coefficients degenerates at the E_2-term (in fact the only potentially nontrivial d_2-differential is multiplication by the Euler class) and thus we have

$$H_1(\pi_1(T^1\Sigma_g);\mathbb{Z}/2\mathbb{Z}) \simeq H_1(\mathbb{Z};\mathbb{Z}/2\mathbb{Z}) \oplus H_1(\pi_1\Sigma_g;\mathbb{Z}/2\mathbb{Z}).$$

As any homomorphism from $\pi_1(T\Sigma_g)$ to $\mathbb{Z}/2\mathbb{Z}$ has to factor through the abelianization $H_1(T^1\Sigma_g)$, this implies

$$\mathrm{Hom}(\pi_1(T^1\Sigma_g),\mathbb{Z}/2\mathbb{Z}) \simeq H^1(\mathbb{Z};\mathbb{Z}/2\mathbb{Z}) \oplus H^1(\pi_1\Sigma_g;\mathbb{Z}/2\mathbb{Z}) \simeq (\mathbb{Z}/2\mathbb{Z})^{2g+1}$$

In particular, for each homomorphism $\phi\colon \pi_1\Sigma_g \to \mathbb{Z}/2\mathbb{Z}$ we have a uniquely defined homomorphism

$$\Phi\colon \pi_1(T^1\Sigma_g) \to \mathbb{Z}/2\mathbb{Z} = \{0,1\}$$

which sends the generator of $H_1(S^1;\mathbb{Z}/2\mathbb{Z})$ to the nontrivial element $1 \in \mathbb{Z}/2\mathbb{Z}$ and agrees with ϕ on $H_1(\Sigma_g;\mathbb{Z}/2\mathbb{Z})$.

Inspection shows that

$$\rho_\phi(\pi_1\Sigma_g)\backslash\mathrm{SL}(2,\mathbb{R})$$

is the 2-fold covering space of $\rho(\pi_1\Sigma_g)\backslash\mathrm{PSL}(2,\mathbb{R})$ which corresponds to the homomorphism $\Phi\colon \pi_1(T^1\Sigma_g) \to \mathbb{Z}/2\mathbb{Z}$.

Euler class. Circle bundles over Σ_g are classified by their Euler class

$$e \in H^2(\pi_1\Sigma_g;\mathbb{Z}) \cong \mathbb{Z}.$$

It is well-known that $T^1\Sigma_g$ is a circle bundle of Euler class $2-2g$. Our quotients

$$\rho_\phi(\pi_1\Sigma_g)\backslash\Omega \cong \rho_\phi(\pi_1\Sigma_g)\backslash\mathrm{SL}(2,\mathbb{R})$$

are fibre-wise double covers of $T^1\Sigma_g$ and therefore are circle bundles of Euler class $1-g$. So they are all isomorphic as circle bundles and in particular their total spaces are all homeomorphic to each other.

The homeomorphisms lift to equivariant homeomorphisms of the domains of discontinuity. So we see that the actions of the different $\rho_\phi(\pi_1\Sigma_g)$ on Ω are all topologically conjugate to each other.

3.3. **Deformations.** Although this will not be used in the remainder of the paper, we consider it worthwhile mentioning that the completely reducible representations are singular points of the character variety. Namely, it is well-known that the character variety $\mathrm{Hom}(\pi_1\Sigma_g,\mathrm{SL}(3,\mathbb{R}))//\mathrm{SL}(3,\mathbb{R})$ has dimension $16g-16$ and we will show that at completely reducible representations the dimension of the Zariski tangent space will be $16g-14$.

We use that the dimension of the Zariski tangent space at semisimple representations is $H^1(\Gamma, Ad\circ\rho)$, see [10]. To compute the latter we decompose the Lie algebra $\mathrm{SL}(3,\mathbb{R})$ as

$$\mathrm{SL}(3,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R}) \oplus \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R},$$

where one $\mathbb{R}^2$-summand is spanned by the elementary matrices e_{12}, e_{23}, the other $\mathbb{R}^2$-summand is spanned by e_{21}, e_{32} and the $\mathbb{R}$-summand is spanned by the diagonal matrix

$diag(-1,2,-1)$. The summands of this decomposition are orthogonal with respect to the Killing form and are preserved under the adjoint action Ad. We note that the action of Ad on the $\mathbb{R}$-summand is trivial. An explicit computation shows that the action of Ad on the $\mathbb{R}^2$-summands comes from the standard linear action ρ_{st} of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^2$. In particular, the group cohomology decomposes as a direct sum

$$\begin{aligned} & H^1(\pi_1\Sigma_g, Ad\circ\iota\circ\rho_0) \\ &= H^1(\pi_1\Sigma_g, Ad\circ\rho_0)\oplus H^1(\pi_1\Sigma_g,\mathbb{R})\oplus H^1(\pi_1\Sigma_g,\rho_{st}\circ\rho_0)\oplus H^1(\pi_1\Sigma_g,\rho_{st}\circ\rho_0) \\ &= T_{\rho_0}T(\Sigma_g)\oplus\mathbb{R}^{2g}\oplus H^1(\pi_1\Sigma_g,\rho_{st}\circ\rho_0)\oplus H^1(\pi_1\Sigma_g,\rho_{st}\circ\rho_0), \end{aligned}$$

where $T(\Sigma_g)$ means Teichmüller space and we use that $\pi_1\Sigma_g$ acts trivially on $\mathbb{R}$.

The Teichmüller space of Σ_g has dimension $6g-6$ and $H^1(\Sigma_g,\mathbb{R})$ has dimension $2g$. A variant of the Hopf trace formula argument shows

$$\begin{aligned} \dim\ H^0(\pi_1\Sigma_g,\mathbb{R}^2) - dim\ H^1(\pi_1\Sigma_g,\mathbb{R}^2) + dim\ H^2(\pi_1\Sigma_g,\mathbb{R}^2) \\ = \chi(\pi_1\Sigma_g)dim(\mathbb{R}^2) = 4-4g. \end{aligned}$$

The action of the cocompact lattice $\pi_1\Sigma_g\subset\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^2$ has no nonzero fixed vector, hence $H^0(\pi_1\Sigma_g,\mathbb{R}^2)=(\mathbb{R}^2)^{\pi_1\Sigma_g}=0$ and by Poincaré duality $H^2(\pi_1\Sigma_g,\mathbb{R}^2)=0$, thus

$$dim\ H^1(\pi_1\Sigma_g,\mathbb{R}^2)=4g-4.$$

Altogether

$$dim\ H^1(\pi_1\Sigma_g, Ad\circ\iota\circ\rho_0)=16g-14.$$

4. Disconnectedness: counting components modulo the mapping class group action

4.1. **Disconnectedness.** Barbot and Lee observed that there are at least 2^{2g} components in the space of non-hyperconvex Anosov representations. Their proof proceeds by showing that given a completely irreducible representation ρ_0 as in Section 3.1 all ρ_ϕ with $\phi\in H^1(\Sigma_g;\mathbb{Z}/2\mathbb{Z})$ belong to distinct path components of $\mathcal{A}$. For reader's convenience, we here sketch their proof.

The basic reason is that by [9, Proposition 3.2] Anosovness of ρ implies that for all $\gamma\in\pi_1\Sigma_g$ the matrix

$$\rho_\phi(\gamma)\in\mathrm{SL}(3,\mathbb{R})$$

has three distinct real eigenvalues.

So, decomposing the set of 3×3-matrices with real eigenvalues into the two sets

$$A_0=\{A\in\mathrm{SL}(3,\mathbb{R})\colon A \text{ has 3 positive eigenvalues}\},$$

$$A_1=\{A\in\mathrm{SL}(3,\mathbb{R})\colon A \text{ has 1 positive and 2 negative eigenvalues}\},$$

then each $\rho(\gamma)$ belongs either to A_0 or A_1 and there is an assignment

$$F\colon\mathcal{A}\to\mathrm{Map}(\{a_1,b_1,\ldots,a_g,b_g\},\mathbb{Z}/2\mathbb{Z})\simeq(\mathbb{Z}/2\mathbb{Z})^{2g}$$

by assigning for each Anosov representation $\rho\in\mathcal{A}$, each $k\in\{0,1\}$ and each of the standard generators $a_1,b_1,\ldots,a_g,b_g$ of $\pi_1\Sigma_g$

$$F(\rho)(\gamma)=k\iff\rho(\gamma)\in A_k.$$

(It seems unlikely that $F(\rho)$ is a homomorphism for arbitrary $\rho\in\mathcal{A}$, although this is true for the representations ρ_ϕ from Section 3.1.)

F is surjective because each $\phi \in \operatorname{Map}(\{a_1, b_1, \ldots, a_g, b_g\}, \mathbb{Z}/2\mathbb{Z})$ is realised by the representation ρ_ϕ from Section 3.1. On the other hand, F is constant on components of $\mathcal{A}$ because for a continuous path ρ_t of representations to $\mathrm{SL}(3, \mathbb{R})$, the value of $\rho_t(\gamma)$ for some fixed $\gamma \in \pi_1\Sigma_g$ can not switch from A_0 to A_1 while t is changing continuously. This proves that $\mathcal{A}$ has at least 2^{2g} components besides the Hitchin component.

4.2. **Action of the mapping class group.** The mapping class group $\mathrm{MCG}(\Sigma_g)$ (i.e., the group of diffeomorphisms modulo isotopy) of Σ_g acts canonically on $\operatorname{Hom}(\pi_1\Sigma_g, \mathrm{SL}(3, \mathbb{R}))$. It obviously maps Anosov representations to Anosov representations, so we can consider the quotient $\mathrm{MCG}(\Sigma_g)\backslash\mathcal{A}$ and we are going to show that $\mathrm{MCG}(\Sigma_g)\backslash\mathcal{A}$ has at least $g+2$ connected components. This subsection is devoted to the proof of the following proposition.

Proposition 4.1. *There are at least $g+2$ orbits for the action of the mapping class group $\mathrm{MCG}(\Sigma_g)$ on $\pi_0\mathcal{A}$.*

As in Section 3.1 we fix a representation $\rho_0\colon \pi_1\Sigma_g \to \mathrm{SL}(2, \mathbb{R}) \subset \mathrm{SL}(3, \mathbb{R})$ and will consider the finite subset $\mathcal{A}_0 \subset \mathcal{A}$ which consists of the representations ρ_ϕ for the 2^{2g} different choices of $\phi \in \operatorname{Hom}(\pi_1\Sigma_g, \mathbb{Z}/2\mathbb{Z})$. We want to show that the elements of $\mathcal{A}_0$ belong to $g+1$ different orbits of the maping class group. This implies the claim of Proposition 4.1 because the argument in Section 4.1 shows that all elements of $\mathcal{A}_0$ belong to pairwise distinct components of $\mathcal{A}$ (and of course the Hitchin component is preserved by the maping class group).

First we note that the action of the mapping class group on $\mathcal{A}_0$ and on $\operatorname{Hom}(\pi_1\Sigma_g, \mathbb{Z}/2\mathbb{Z})$ are compatible. Indeed, for $f \in \mathrm{MCG}(\Sigma_g)$ and $\rho = \rho_\phi \in \mathcal{A}_0$ with $\phi \in \operatorname{Hom}(\pi_1\Sigma_g, \mathbb{Z}/2\mathbb{Z})$ and for $F \in \operatorname{Hom}(\pi_1\Sigma_g, \mathbb{Z}/2\mathbb{Z})$ we have for any $\gamma \in \pi_1\Sigma_g$

$$F(f^*\rho)(\gamma) = F(\rho)(f_*\gamma)$$

because $F(f^*\rho)(\gamma) = 0$ is equivalent to $(f^*\rho)(\gamma)$ having three positive eigenvalues, which is of course equivalent to $\rho(f_*\gamma)$ having three positive eigenvalues, hence to $F(\rho)(f_*\gamma) = 0$.

So the orbits of the mapping class group on $\mathcal{A}_0$ are mapped to its orbits on

$$\operatorname{Hom}(\pi_1\Sigma_g, \mathbb{Z}/2\mathbb{Z}) = H^1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}).$$

As noted in Section 4.1 every $\phi \in \operatorname{Hom}(\pi_1\Sigma_g, \mathbb{Z}/2\mathbb{Z})$ is realised by the representation $\rho_\phi \in \mathcal{A}_0$ and thus Proposition 4.1 is a consequence of the following lemma.

Lemma 4.2. *There are $g+1$ orbits for the action of $\mathrm{MCG}(\Sigma_g)$ on $\operatorname{Hom}(\pi_1\Sigma_g, \mathbb{Z}/2\mathbb{Z})$.*

Proof: It is well-known (and easy to prove) that the *intersection form modulo 2*

$$i\colon H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}) \times H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$$

defines a symplectic form on the $\mathbb{Z}/2\mathbb{Z}$-vector space $H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})$ and that the action of the mapping class group preserves this symplectic form.

For each

$$\phi \in \operatorname{Hom}(\pi_1\Sigma_g, \mathbb{Z}/2\mathbb{Z}) = \operatorname{Hom}(H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z})$$

we let d_ϕ be the dimension of the maximal symplectic subspace on which ϕ is constant 0. This number is invariant under the action of the mapping class group, thus

$$d_{\phi_1} \neq d_{\phi_2}$$

implies that ϕ_1 and ϕ_2 are not in the same $\mathrm{MCG}(\Sigma_g)$-orbit.

The number d_ϕ can take integer even values in

$$\{0, 2, 4, \dots, 2g\}\,.$$

We claim that each of these $g+1$ values can indeed be realised for some ϕ. An explicit realisation for a given d is for example given as follows. Let

$$a_1, b_1, \dots, a_g, b_g$$

be the standard basis of $H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})$ with respect to which the intersection form is given by the standard symplectic form

$$i(a_k, b_l) = \delta_{kl}, i(a_k, a_l) = i(b_k, b_l) = 0$$

for $k, l = 1, \dots, g$. Then, for a given $d \in \{0, 2, 4, \dots, 2g\}$, the homomorphism

$$\phi_d(a_1) = \phi_d(b_1) = \ldots = \phi(a_d) = \phi(b_d) = 0,$$
$$\phi_d(a_{d+1}) = \phi_d(b_{d+1}) = \ldots = \phi_d(a_g) = \phi_d(b_g) = 1$$

obviously realises $d_\phi = d$.

This part of Lemma 4.2 actually suffices to prove Proposition 4.1 but for completeness we still show that there are *exactly* $g+1$ orbits for the action of the mapping class group on $\mathrm{Hom}(\pi_1\Sigma_g, \mathbb{Z}/2\mathbb{Z})$ (and thus that our completely reducible examples yield *exactly* g+1 distinct components of $\mathrm{MCG}(\Sigma_g)\backslash\mathcal{A}$). For this we have to show that any $\phi \in \mathrm{Hom}(\pi_1\Sigma_g, \mathbb{Z}/2\mathbb{Z})$ is in the $\mathrm{MCG}(\Sigma_g)$-orbit of ϕ_{d_ϕ}, where d_ϕ is the dimension of the maximal symplectic subspace on which ϕ is constant 0 and ϕ_{d_ϕ} is defined as in the previous paragraph.

By its definition, the symplectic structure i is standard, i.e. decomposes into the 2-dimensional subspaces generated by a_k, b_k for $k = 1, \dots, g$. If

$$\phi \neq \phi_{d_\phi},$$

then there is some set of 2-dimensional subspaces on which ϕ does not agree with ϕ_d but is not constant 0. Say these subspaces are generated by

$$a_{k_1}, b_{k_1}, \dots, a_{k_l}, b_{k_l}.$$

One can find some mapping class which sends a_{k_1} to a_1, b_{k_1} to b_1, ..., a_{k_l} to a_l, b_{k_l} to b_l, so we can w.l.o.g. assume $k_1 = 1, \dots, k_l = l$. (The existence of such a mapping class follows either from the Dehn-Nielsen Theorem [11], which says that any automorphism of a surface group is induced by some mapping class, or in this special case also from an explicit construction.)

For each $k \in \{1, \dots, l\}$ we can apply a Dehn twist to map the restriction of ϕ to the restriction of ϕ_{d_ϕ}. In formula: assume w.l.o.g.

$$\phi(a_k) = 0, \phi(b_k) = 1.$$

The Dehn twist $D_k := D_{b_k}$ at b_k sends a_k to $a_k + b_k$ and b_k to b_k. Thus

$$\phi(D_{b_k}(a_k)) = \phi(D_{b_k}(b_k)) = 1,$$

which means

$$\phi(D_{b_k}(.)) = \phi_{d_\phi}(.)$$

on the subspace generated by a_k, b_k. Composition of the Dehn twists $D_1, \dots, D_l$ then yields the wanted result:

$$\phi(D_l \dots D_1(h)) = \phi_{d_\phi}(h)$$

for all $h \in H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})$. ■

Remark: The same argument can be used to show that the lift of the Teichmüller space to $\mathrm{Hom}(\pi_1\Sigma_g, \mathrm{SL}(3,\mathbb{R}))$ falls into $g+1$ components modulo the action of the mapping class groups.

5. Boundary maps determine Anosov representations - almost always

In Section 3 we have seen distinct Anosov representations with the same boundary map. In this section we will see that these examples are essentially the only ones for which the boundary map does not determine the Anosov representation.

Theorem 5.1. *Let*

$$\rho_1, \rho_2 \colon \pi_1\Sigma_g \to \mathrm{SL}(3,\mathbb{R})$$

be Anosov representations such that there exists a map

$$\xi \colon \partial_\infty \mathbb{H}^2 \to \mathrm{Flag}(\mathbb{R}P^2)$$

satisfying the conditions of (the remark after) Definition 2.1 and equivariant for both, ρ_1 and ρ_2. Then either $\rho_1 = \rho_2$ or there exists a representation $r \colon \pi_1\Sigma_g \to \mathrm{SL}(2,\mathbb{R})$, a homomorphism $\lambda \colon \pi_1\Sigma_g \to \mathbb{R}\setminus\{0\}$ and some $A \in \mathrm{PGL}(3,\mathbb{R})$ such that

$$A\rho_1A^{-1} = \iota \circ r$$

$$A\rho_2A^{-1} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \frac{1}{\lambda^2} & 0 \\ 0 & 0 & \lambda \end{pmatrix} \circ \iota \circ r,$$

where $\iota \colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(3,\mathbb{R}) = \mathrm{PGL}(3,\mathbb{R})$ is the completely reducible representation from Section 3.

Proof: Consider an ideal triangulation Υ of Σ_g, let $\widetilde{\Upsilon}$ be the lifted ideal triangulation of $\mathbb{H}^2$ and (after identifying $\widetilde{\Sigma_g}$ to $\mathbb{H}^2$ via some hyperbolic metric) $\widetilde{\Upsilon}_0 \subset \partial_\infty\mathbb{H}^2$ its 0-skeleton.

Assume $\rho_1 \neq \rho_2$, so there is some $\gamma_0 \in \pi_1\Sigma_g$ with $\rho_1(\gamma_0) \neq \rho_2(\gamma_0)$. For every vertex $v \in \widetilde{\Upsilon}_0$ we have $\rho_1(\gamma_0)\xi(v) = \xi(\gamma_0 v) = \rho_2(\gamma_0)\xi(v)$ and thus

$$\rho_1^{-1}(\gamma_0)\rho_2(\gamma_0) \in \mathrm{Stab}(\xi(v)).$$

In particular, for every ideal triangle $T = (v_0, v_1, v_2) \in \widetilde{\Upsilon}$, $\rho_1^{-1}(\gamma_0)\rho_2(\gamma_0)$ stabilizes the associated triple of flags $(\xi(v_0), \xi(v_1), \xi(v_2))$.

By part i) of Definition 2.1 we have that $\xi(v)$ and $\xi(w)$ are transverse for all $v, w \in \widetilde{\Upsilon}_0$. An elementary argument, given in [8, Section 2] shows that every triple of pairwise transverse flags in $\mathbb{R}P^2$ is in the $\mathrm{PGL}(3,\mathbb{R})$-orbit of one of the following triples:

$$\left\{ \left((e_1, e_3^\perp), (e_3, e_1^\perp), (e_1 - e_2 + e_3, (e_1 + (1+X)e_2 + Xe_3)^\perp\right), X \in \mathbb{R}\setminus\{0,-1\} \right\}$$

$$((e_1, e_3^\perp), (e_3, e_1^\perp), (e_1 + e_3, (e_1 + e_2 - e_3)^\perp))$$

$$((e_1, e_3^\perp), (e_3, e_1^\perp), (e_1 + e_2 + e_3, (e_1 - e_3)^\perp))$$

$$(e_1, e_3^\perp), (e_3, e_1^\perp), (e_1 + e_3, (e_1 - e_3)^\perp)$$

One easily checks that the last triple is the only one of these possibilities which has a nontrivial stabilizer in $\mathrm{PGL}(3,\mathbb{R})$. In fact, the stabilizer of a triple in the $\mathrm{PGL}(3,\mathbb{R})$-orbit of $(e_1, e_3^\perp), (e_3, e_1^\perp), (e_1+e_3, e_1^\perp - e_3^\perp))$ is conjugate to

$$\left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \frac{1}{\lambda^2} & 0 \\ 0 & 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{R} \setminus \{0\} \right\}.$$

So $\rho_1^{-1}(\gamma_0)\rho_2(\gamma_0) \in \mathrm{Stab}(\xi(v)) \setminus \{id\}$ for every $v \in \widetilde{\Upsilon}_0$ implies that for every ideal triangle $T = (v_0, v_1, v_2) \in \widetilde{\Upsilon}$ the associated triple of flags

$$(\xi(v_0), \xi(v_1), \xi(v_2))$$

must be in the $\mathrm{PGL}(3,\mathbb{R})$-orbit of

$$((e_1, e_3^\perp), (e_3, e_1^\perp), (e_1+e_3, (e_1-e_3)^\perp).$$

In other words, if we denote $\xi(v_i) = (p_i, l_i)$ for $i = 0, 1, 2$, then p_2 must be on the line through p_0 and p_1 and l_2 goes through the intersection point of l_0 and l_1.

We claim that this implies that all $\xi(v), v \in \widetilde{\Upsilon}_0$ are of the form $\xi(v) = (p_v, l_v)$ with all p_v lying on the same line l, and all l_v intersecting in the same point p. To see this. let $\lambda^{closed} \subset \Sigma_g$ be the union of closed leaves of Υ, let $\tilde{\lambda}^{closed} \subset \mathbb{H}^2$ its lift to $\widetilde{\Sigma}_g \simeq \mathbb{H}^2$ and let $\widetilde{U}$ be the component of $\mathbb{H}^2 \setminus \tilde{\lambda}^{closed}$ containing T. Looking at the dual tree of the ideal triangulation $\widetilde{\Upsilon}\mid_{\widetilde{U}}$ we can enumerate its triangles such that each T_k is adjacent to exactly one triangle T_j with $j < k$, see the proof of [2, Lemma 21]. Using this enumeration we see by induction that the claim is true for all vertices of triangles in $\widetilde{U}$. Further this also applies to the leaves of $\tilde{\lambda}^{closed}$ adjacent to $\widetilde{U}$, because one of their ideal vertices is actually a vertex of $\widetilde{\Upsilon}\mid_{\widetilde{U}}$, while the other ideal vertex is an accumulation point of vertices and so the claim holds by continuity of ξ. Next we can extend this argument to the components of $\mathbb{H}^2 \setminus \tilde{\lambda}^{closed}$ adjacent to $\widetilde{U}$. Namely, the same argument shows the claim (with a priori possibly different (p, l)) for all triangles in an adjacent component. But since the (p, l) agree on both ideal vertices of the leaf in $\tilde{\lambda}^{closed}$ along which they are adjacent, the (p, l) must actually be the same for both adjacent components, just because there is a unique line through two points and a unique point common to two lines. Finally we use the dual tree to the decomposition into components of $\mathbb{H}^2 \setminus \tilde{\lambda}^{closed}$ to enumerate these components such that each U_k is adjacent to exactly one component U_j with $j < k$, as in the proof of [2, Lemma 24], so that we can induct on the components and finally get the claim for all triangles in all components.

Now fix an ideal triangle $T = (v_0, v_1, v_2) \in \widetilde{\Upsilon}$ and a projective map $A \in \mathrm{PGL}(3,\mathbb{R})$ that sends

$$(\xi(v_0), \xi(v_1), \xi(v_2)) = ((p_0, l_0), (p_1, l_1), (p_2, l_2))$$

to

$$((e_1, e_3^\perp), (e_3, e_1^\perp), (e_1+e_3, e_1^\perp - e_3^\perp)).$$

The map

$$A\xi \colon \partial_\infty \mathbb{H}^2 \to \mathrm{Flag}(\mathbb{R}P^2)$$

is equivariant for $A\rho_1 A^{-1}$ and $A\rho_2 A^{-1}$, and still satisfies the conditions from Definition 2.1.

We note that

$$Ap = Al_0 \cap Al_1 \cap Al_2 = e_3^\perp \cap e_1^\perp \cap (e_1 - e_3)^\perp = e_2$$

and that Al is the line containing $Ap_0 = e_1, Ap_1 = e_3$ and $Ap_2 = e_1 - e_3$ and thus

$$Al = e_2^{\perp}.$$

For each $\gamma \in \pi_1\Sigma_g$, we have (because $\widetilde{\Upsilon}$ is defined by lifting Υ) that $\gamma T = (\gamma v_0, \gamma v_1, \gamma v_2)$ is one of the ideal triangles of $\widetilde{\Upsilon}$. Then $A\rho_1(\gamma)A^{-1}e_1$ is the point component of

$$A\rho_1(\gamma)A^{-1}(A\xi(v_0)) = A\xi(\gamma v_0)$$

and thus belongs to $e_2^{\perp}$. In the same way, $A\rho_1(\gamma)A^{-1}e_3$ is the point component of

$$A\rho_1(\gamma)A^{-1}(A\xi(v_1)) = A\xi(\gamma v_1)$$

and thus belongs to $e_2^{\perp}$. This shows that $A\rho_1(\gamma)A^{-1}$ sends $e_2^{\perp}$ to itself. Similarly, considering the line components, we can show that $A\rho_1(\gamma)A^{-1}$ sends e_2 to itself. Since this is true for any $\gamma \in \pi_1\Sigma_g$ we have that the image of $A\rho_1A^{-1}$ is in the image of the completely reducible representation

$$\iota\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(3,\mathbb{R}) = \mathrm{PGL}(3,\mathbb{R}).$$

The same argument applies to $A\rho_2A^{-1}$.

Let

$$r\colon \pi_1\Sigma_g \to \mathrm{SL}(2,\mathbb{R})$$

be the representation over which $A\rho_1A^{-1}$ factors. Then, for each $\gamma \in \pi_1\Sigma_g$, both $A\rho_1(\gamma)A^{-1}$ and $A\rho_2(\gamma)A^{-1}$ send $A\xi(T)$ to $A\xi(\gamma T)$ and thus $A\rho_1^{-1}(\gamma)\rho_2(\gamma)A^{-1}$ is in the stabilizer of

$$A\xi(T) = ((e_1, e_3^{\perp}), (e_3, e_1^{\perp}), (e_1 + e_3, (e_1 - e_3)^{\perp})).$$

So there is a unique $\lambda(\gamma) \in \mathbb{R} \setminus \{0\}$ with

$$A\rho_1^{-1}(\gamma)\rho_2(\gamma)A^{-1} = \begin{pmatrix} \lambda(\gamma) & 0 & 0 \\ 0 & \frac{1}{\lambda(\gamma)^2} & 0 \\ 0 & 0 & \lambda(\gamma) \end{pmatrix},$$

from which the claimed formula for $A\rho_2(\gamma)A^{-1}$ follows.

From the fact that the diagonal matrices of the form $\mathrm{Diag}(\lambda, \frac{1}{\lambda^2}, \lambda)$ commute with the image of the completely reducible representation ι, one easily concludes that λ is a homomorphism. Indeed, this follows from the computation

$$\begin{aligned}
&\begin{pmatrix} \lambda(\gamma_1\gamma_2) & 0 & 0 \\ 0 & \frac{1}{\lambda(\gamma_1\gamma_2)^2} & 0 \\ 0 & 0 & \lambda(\gamma_1\gamma_2) \end{pmatrix} (\iota \circ r(\gamma_1\gamma_2)) \\
= \;& A\rho_2(\gamma_1\gamma_2)A^{-1} \\
= \;& A\rho_2(\gamma_1)A^{-1}A\rho_2(\gamma_2)A^{-1} \\
= \;& \begin{pmatrix} \lambda(\gamma_1) & 0 & 0 \\ 0 & \frac{1}{\lambda(\gamma_1)^2} & 0 \\ 0 & 0 & \lambda(\gamma_1) \end{pmatrix} (\iota \circ r(\gamma_1)) \begin{pmatrix} \lambda(\gamma_2) & 0 & 0 \\ 0 & \frac{1}{\lambda(\gamma_2)^2} & 0 \\ 0 & 0 & \lambda(\gamma_2) \end{pmatrix} (\iota \circ r(\gamma_2)) \\
= \;& \begin{pmatrix} \lambda(\gamma_1) & 0 & 0 \\ 0 & \frac{1}{\lambda(\gamma_1)^2} & 0 \\ 0 & 0 & \lambda(\gamma_1) \end{pmatrix} \begin{pmatrix} \lambda(\gamma_2) & 0 & 0 \\ 0 & \frac{1}{\lambda(\gamma_2)^2} & 0 \\ 0 & 0 & \lambda(\gamma_2) \end{pmatrix} (\iota \circ r(\gamma_1\gamma_2)),
\end{aligned}$$

where the last equality uses the fact that $\mathrm{Diag}(\lambda(\gamma_2), \frac{1}{\lambda(\gamma_2)^2}, \lambda(\gamma_2))$ commutes with $\iota \circ r(\gamma_1)$ and that $\iota \circ r$ is a homomorphism. ∎

It is perhaps worth mentioning that, given a completely reducible Anosov representation ρ_1, not every homomorphism λ will yield an Anosov representation ρ_2 as in Theorem 5.1. Actually, [1, Theorem 4.2] gives a precise condition for the stable norm of $\log(\lambda)$ to guarantuee that ρ_2 also is Anosov.

Let us finally mention that the assumption of Theorem 5.1 can not be weakened to consider only the boundary map with image in $\mathbb{R}P^2$ or its dual, rather than in $\mathrm{Flag}(\mathbb{R}P^2)$. Barbot constructs in [1, Lemma 4.5] a family of (reducible, but not completely reducible) representations which all have the same boundary map to the dual space of $\mathbb{R}P^2$, however their boundary maps to $\mathbb{R}P^2$ do not agree. Similarly, one can construct a family of representations with the same boundary map to $\mathbb{R}P^2$, but their boundary maps to the dual space and hence to $\mathrm{Flag}(\mathbb{R}P^2)$ will not agree.

These examples are not Zariski-dense and so they show that the class of representations determined by their boundary map is strictly larger than the set of Zariski-dense representations.

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