Abstract

Hyperbolic Surface Subgroups of Right-Angled Artin Groups and Graph Products of Groups

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We consider groups defined by graphs. These include right-angled Artin groups, right-angled Coxeter groups, and more generally, graph products of groups. We define an operation on finite graphs, called co-contraction. By showing that co-contraction of a graph induces an injective map between graph products of groups, we exhibit a family of graphs, without any induced cycle of length at least 5, such that the graph products of any non-trivial groups on those graphs contain hyperbolic surface groups. By applying this to the special case of right-angled Artin groups, we answer a question raised by Gordon, Long and Reid negatively.

We also give a family of right-angled Artin groups that do not contain hyperbolic surface groups. Let $A(\Gamma)$ denote the right-angled Artin group defined by a graph Γ. Using transversality, any π_1-injective map from a compact surface S to the standard Eilenberg-MacLane space X_Γ of $A(\Gamma)$ can be realized as a cubical map for some cubical structure on S. We examine the transversely oriented simple closed curves and the properly embedded arcs dual to this cubical structure. As a result, we prove that $A(\Gamma)$ does not contain a hyperbolic surface group for each Γ in an inductively defined family $\mathcal{F}$ of graphs. $\mathcal{F}$ is shown to contain each chordal graph, as well as each bipartite graph without any induced cycle of length at least 5.

Hyperbolic Surface Subgroups of Right-Angled Artin Groups and Graph Products of Groups

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To my wife,

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Chapter 1

Introduction

1.1 Background

By a *graph* we mean a finite graph without loops and without multi-edges.

A *right-angled Artin group* is a group defined by a presentation with a finite generating set, where the relators are certain commutators between the generators. If one assumes that each generator is an involution as well, we obtain a presentation for a *right-angled Coxeter group*. Such a presentation naturally determines the *underlying graph*, where the vertices correspond to the generators and the edges to the pairs of commuting generators. More generally, if a group G has a presentation such that the generating set consists of non-trivial elements of groups indexed by the vertex set of a given graph Γ, and the relators are the commutators of a pair of generators which lie in two different groups indexed by an adjacent pair of vertices in Γ, as well as the multiplication relators in each vertex group, then G is called the *graph product of groups over* Γ.

In this thesis, we study the properties of graph products of groups. The definition of graph products of groups is due to [Gre90], as well as the normal form theorem. In [HW99], linearity and residually finiteness properties of certain graph products are proved, using van Kampen diagrams as a geometric tool. Also, the uniqueness of graph product presentation for a given group is known for several cases ([Dro87b, Gre90, Rad03]).

As a special case of graph products of groups, we will mainly focus on right-angled Artin groups. It is known that the isomorphism type of a right-angled Artin group uniquely determines the isomorphism type of the underlying graph [Dro87b, KMLNR80]. Also, right-angled Artin groups possess various group theoretic properties. To name a few, right-angled Artin groups are linear [Hum94, HW99, DJ00], biorderable [DT92], biautomatic [Wyk94] and moreover, admitting free and cocompact actions on finite-dimensional CAT(0) cube complexes [CD95, MW95, NR98].

On the other hand, it is interesting to ask what we can say about the isomorphism type of the underlying graph, if a right-angled Artin group satisfies a given group theoretic property. Let Γ be a graph. We denote the vertex set and the edge set of Γ by $V(\Gamma)$ and $E(\Gamma)$, respectively. The *complement graph* of Γ is the graph $\overline{\Gamma}$ defined by $V(\overline{\Gamma}) = V(\Gamma)$ and $E(\overline{\Gamma}) = \{\{p, q\} : p, q \in V(\Gamma) \text{ and } \{p, q\} \notin E(\Gamma)\}$. For a subset S of $V(\Gamma)$ the *induced subgraph* on S, denoted by Γ_S, is defined to be the maximal subgraph of Γ with the vertex set S. This implies that $V(\Gamma_S) = S$ and $E(\Gamma_S) = \{\{p, q\} : p, q \in S \text{ and } \{p, q\} \in E(\Gamma)\}$. If Λ is another graph, an *induced* Λ in Γ means an induced subgraph isomorphic to Λ in Γ. C_n denotes the cycle of length n. That is, $V(C_n)$ is a set of n vertices, say $\{q_1, q_2, \ldots, q_n\}$, and $E(C_n)$ consists of the edges $\{q_i, q_j\}$ where $|i-j| \equiv 1 \pmod n$. Let $A(\Gamma)$ be the right-angled Artin group with its underlying graph Γ. Then, the following are true.

- $A(\Gamma)$ is coherent, if and only if Γ is *chordal*, i.e. Γ does not contain an induced C_n for any $n \geq 4$ [Dro87a]. This happens if and only if $[A(\Gamma), A(\Gamma)]$ is free [SDS89].

- $A(\Gamma)$ is a virtually 3-manifold group, if and only if each connected component of Γ is a tree or a triangle [Dro87a, Gor04]

- $A(\Gamma)$ is subgroup separable, if and only if no induced subgraph of Γ is a square or a path of length 3 [MR]. This happens if and only if every subgroup of $A(\Gamma)$ is also a right-angled Artin group [Dro87c].

In particular, we note

Theorem 1.1 ([SDS89]). *$A(\Gamma)$ contains a* hyperbolic surface group, *i.e. the fundamental group of a closed, hyperbolic surface, if there exists an induced C_n for some $n \geq 5$ in Γ.*

The question of whether a given group contains a hyperbolic surface group turns out to be important in several contexts. A famous question of this type is the *surface subgroup conjecture*, which asks whether a closed hyperbolic 3-mainfold group always contains a hyperbolic surface group. If a hyperbolic 3-manifold group is *LERF* in the sense of [Sco78], then an affirmative to this conjecture would imply that such a hyperbolic 3-manifold is virtually Haken.

Also, Gromov raised the question of whether any 1-ended word-hyperbolic linear group contains a hyperbolic surface group. In [GLR04], Gordon, Long and Reid proved that a word-hyperbolic (not necessarily right-angled) Coxeter group either is virtually-free or contains a hyperbolic surface group, settling the conjecture of Gromov for the case of Coxeter groups. They also showed that certain (again, not necessarily right-angled) Artin groups do not contain a hyperbolic surface group, raising the following question.

Question 1.2. *Does* $A(\Gamma)$ *contain a hyperbolic surface group if and only if* Γ *contains an induced* C_n *for some* $n \geq 5$*?*

An attempt to classify all the right-angled Artin groups that contain hyperbolic surface groups will be the theme of this thesis. As a result, we answer Question 1.2 negatively.

1.2 Hyperbolic surface groups in graph products of groups

Let Γ be a graph and B be a set of vertices of Γ such that Γ_B is connected. The *contraction* of Γ relative to B is the graph $\mathrm{CO}(\Gamma, B)$ obtained from Γ by collapsing Γ_B to a vertex, and deleting loops or multi-edges. We define the *co-contraction* $\overline{\mathrm{CO}}(\Gamma, B)$ of Γ relative to B by $\overline{\mathrm{CO}}(\Gamma, B) = \overline{\mathrm{CO}(\overline{\Gamma}, B)}$.

Suppose $\{G_q : q \in V(\Gamma)\}$ is a collection of groups indexed by the vertex set of a graph Γ. The *graph product of* $\{G_q : q \in V(\Gamma)\}$ *with the underlying graph* Γ is the group defined by $G = \langle S|R\rangle$, where

$$\begin{aligned} S &= \cup_q (G_q \setminus \{1\}) \\ R &= \{gh : g \text{ and } h \text{ belong to } G_q \setminus \{1\} \text{ for some } q, \text{ and } gh = 1\} \\ &\cup \{ghk : g, h \text{ and } k \text{ belong to } G_q \setminus \{1\} \text{ for some } q, \text{ and } ghk = 1\} \\ &\cup \{[g,h] : g \in G_p \setminus \{1\}, h \in G_q \setminus \{1\} \text{ for some } \{p,q\} \in E(\Gamma)\} \end{aligned}$$

For a group G, let $o_G(g)$ denotes the order of $g \in G$. The following is the main theorem of Chapter 4.

Theorem 1.3 (embedding induced by co-contraction)**.** *Let Γ be a graph and B be a set of vertices in Γ, such that $\overline{\Gamma_B}$ is connected. Write $\hat{\Gamma} = \overline{CO}(\Gamma, B)$, and let $\hat{v}$ be the vertex of $\hat{\Gamma}$ corresponding to B. Let G be the graph product of groups $\{G_q : q \in V(\Gamma)\}$ with the underlying graph Γ. Choose any $m \in \{o_G(g) : g \in (\cup_{q\in B} G_q) \setminus \{1\}\}$. For $q \in V(\hat{\Gamma}) = (V(\Gamma) \setminus B) \cup \{\hat{v}\}$, define*

$$\hat{G}_q = \begin{cases} \mathbb{Z}_m & q = \hat{v} \\ G_q & \textit{otherwise} \end{cases}$$

Let $\hat{G}$ be the graph product of $\{\hat{G}_q : q \in V(\hat{\Gamma})\}$ with the underlying graph $\hat{\Gamma}$. Then G contains a subgroup isomorphic to $\hat{G}$.

Let $C(\Gamma)$ denote the right-angled Coxeter group with the underlying graph Γ.

Corollary 1.4. *Let Γ be a graph and Γ_1 be a graph obtained from Γ by co-contraction. Fix $0 < m \leq \infty$, and let G and G_1 be the graph products of the cyclic groups of order m with the underlying graphs Γ and Γ_1, respectively. Then G_1 embeds into G. In particular, $A(\Gamma_1)$ and $C(\Gamma_1)$ embed into $A(\Gamma)$ and $C(\Gamma)$, respectively.*

From the above corollary, we see that $A(\overline{C_n})$ contains $A(\overline{C_5}) = A(C_5)$ (see Figure 4.1). Note that $A(C_5)$ contains a hyperbolic surface group (Theorem 1.1).

Corollary 1.5. *$A(\overline{C_n})$ contains a hyperbolic surface group, for any $n \geq 5$.*

An easy combinatorial argument shows that $\overline{C_n}$ does not contain an induced cycle of length at least 5, for $n > 5$ (Proposition 2.4), answering Question 1.2 negatively.

Theorem 1.3 is proved in the following steps.

In Chapter 2, we recall well-known results on graph products of groups and right-angled Artin groups. We also describe a *dual van Kampen diagram*, which is essentially the dual structure to a van Kampen diagram. We owe the notations to [CW04] where a closely related concept, a *dissection*, was defined and used with great clarity.

In Chapter 3, we prove an embedding result of hyperbolic surface groups into certain graph products of groups, extending Theorem 1.1.

Theorem 1.6 (a graph product on a long cycle contains a hyperbolic surface group)**.** *The graph product of any non-trivial groups on a cycle of length at least 5 contains a hyperbolic surface group.*

In Chapter 4, we define co-contraction of a graph, and show that co-contraction induces an embedding between graph products of groups (Theorem 1.3). The main tool used in this chapter is a dual of a van Kampen diagram. Also, we compute intersections of certain subgroups of right-angled Artin groups. From this, we describe some other choices of embeddings of hyperbolic surface groups into right-angled Artin groups.

1.3 $A(\Gamma)$ without hyperbolic surface subgroups

Knowing that $A(C_n)$ and $A(\overline{C_n})$ contain hyperbolic surface groups for any $n \geq 5$, it is natural to ask about other sufficient or necessary conditions for a right-angled Artin group to contain a hyperbolic surface group. We exhibit a set of conditions for a graph Γ, such that each of the conditions would imply that $A(\Gamma)$ does not contain a hyperbolic surface group. Most of the conditions we describe will be *recursive*, in the sense that they impose restrictions on strictly smaller induced subgraphs of Γ. For this, we use a geometric technique of studying a given map from a hyperbolic surface group into a right-angled Artin group [CW04].

For a graph Γ, let X_Γ denote the *standard Eilenberg-Maclane space of* $A(\Gamma)$. This means that X_Γ is a combinatorial (*cubed*) complex defined inductively as follows.

(i) $X_\Gamma^{(0)}$ is a single vertex.

(ii) For each complete subgraph of Γ with k-vertices, one attaches a k-cube to $X_\Gamma^{(k-1)}$ so that the image of the k-cube is a standard k-torus and the image of the boundary of the k-cube is the k copies of the $(k-1)$-tori in $X_\Gamma^{(k-1)}$.

We let K_n denote a complete graph on n vertices. K_0 is defined to be the empty set. For a graph Γ, let $\mathcal{K}(\Gamma)$ denote the set of the maximal complete subgraphs of Γ.

We define the graph classes $\mathcal{N}$ and $\mathcal{N}_\infty$, a “relative” version of $\mathcal{N}$. A map $f : X \to Y$ is π_1-*injective*, if the induced map $f_* : \pi_1(X) \to \pi_1(Y)$ is injective.

Definition 1.7 ($\mathcal{N}$ and $\mathcal{N}_\infty$)**.** Let Γ be a finite graph.

(1) $\mathcal{N}$ denotes the class of graphs Γ such that there does not exist an embedding of a hyperbolic surface group into $A(\Gamma)$.

(2) $\mathcal{N}_\infty$ is the class of the graphs Γ such that there does not exist a π_1-injective map $f : S \to X_\Gamma$ from a compact hyperbolic surface S satisfying the following.

> For each boundary component $\partial_0 S$ of S, there exists $K \in \mathcal{K}(\Gamma)$, such that $f(\partial_0 S) \subseteq X_K$.

An edge $\{a, b\}$ of a graph is called *bisimplicial*, if any vertex adjacent to a is either equal or adjacent to any vertex that is adjacent to b. Define $\mathcal{F}$ as the smallest family of finite graphs satisfying the following conditions.

(i) $K_1 = \bullet \in \mathcal{F}$.

(ii) If $\Gamma_1, \Gamma_2 \in \mathcal{F}$ and $\Gamma_1 \cap \Gamma_2 = K_n$ for some $n \geq 0$, then $\Gamma_1 \cup \Gamma_2 \in \mathcal{F}$.

(iii) $\Gamma_1, \Gamma_2 \in \mathcal{F}$, then $\mathrm{Join}(\Gamma_1, \Gamma_2) = \overline{\overline{\Gamma_1} \sqcup \overline{\Gamma_2}} \in \mathcal{F}$.

(iv) Suppose e is a bisimplicial edge of a graph Γ. If $\Gamma \setminus \mathring{e} \in \mathcal{F}$, then $\Gamma \in \mathcal{F}$.

(v) $\Gamma \in \mathcal{F}$ and B is an anticonnected subset of $V(\Gamma)$, then $\overline{\mathrm{CO}}(\Gamma, B) \in \mathcal{F}$.

Let $\mathcal{W} = \{\Gamma :$ there does not exist an induced C_n or $\bar{C}_n$ in $\Gamma\}$. By Corollary 1.5, $\mathcal{N} \subseteq \mathcal{W}$. Combined with this, the main result of Chapter 5 is summarized as the following theorem.

Theorem 1.8 (bounds for $\mathcal{N}$)**.** *$\mathcal{F} \subseteq \mathcal{N}_\infty \subseteq \mathcal{N} \subseteq \mathcal{W}$.*

This yields a large class of graphs as a lower bound for $\mathcal{N}$ and $\mathcal{N}_\infty$. A graph is *chordal bipartite*, if it is bipartite without any induced cycle of length at least 5.

Corollary 1.9 (chordal graphs and chordal bipartite graphs)**.** *If a graph* Γ *is chordal or chordal bipartite, then* $A(\Gamma)$ *does not contain a hyperbolic surface group.*

Question 1.10. *Is* $\mathcal{N}_\infty = \mathcal{W}$*?*

In [CW04], it was shown that any injective map ϕ from a hyperbolic surface group $\pi_1(S)$ into $A(\Gamma)$ gives rise to a cubical structure on S and a cubical map $f : S \to X_\Gamma$, inducing ϕ. The dual to this cubical structure is a set $\mathcal{H}$ of transversely oriented simple closed curves and properly embedded arcs, labeled by the vertices of Γ. This set $\mathcal{H}$, along with the transverse orientations and the labeling of the curves in $\mathcal{H}$, is called a *label-reading pair*. We examine label-reading pairs for the proof of Theorem 1.8.

In Chapter 2, we survey the technique of studying any map from a hyperbolic surface group into a right-angled Artin group using label-reading pairs.

In Chapter 3, we develop a method of simplifying (*normalizing*, Lemma 3.16 and 3.19) label-reading pairs, without changing the kernel of the map it is associated with.

In Chapter 5, we prove the main theorem (Theorem 1.8), delaying the proof of two crucial lemmas to the later part of the chapter. The first of those lemmas is that $\mathcal{N}_\infty$ is closed under complete graph amalgamations. This means that if $\Gamma = \Gamma_1 \cup \Gamma_2$ for some $\Gamma_1, \Gamma_2 \in \mathcal{N}_\infty$ and $\Gamma_1 \cap \Gamma_2$ is complete, then Γ is in $\mathcal{N}_\infty$ also (Lemma 5.9). A crucial step in the proof is use of a double $D(S)$ of a bounded surface S. Let $q : D(S) \to S$ denote the natural quotient map. We show that for a given $x \in \pi_1(D(S))$ if $T : D(S) \to D(S)$ is a composition of the Dehn twists along each of the boundary components for sufficiently many times, then $q_* \circ T_*(x)$ is non-trivial

(Lemma 5.25).

The second of the crucial lemmas state that if Γ is not in $\mathcal{N}_\infty$ and has a bisimplicial edge e, then the graph Γ' obtained by removing the interior of e from Γ is not in $\mathcal{N}_\infty$, either. For the proof, we use label-reading pairs and their simplification schemes (Lemma 3.16, 3.19).

Chapter 2

Preliminaries

2.1 Graphs

By a *graph*, we mean a finite 1-dimensional simplicial complex. In particular, we do not allow loops or multi-edges. For two graphs Γ_1 and Γ_2, we write $\Gamma_1 \cong \Gamma_2$, if there exists a combinatorial isomorphism between Γ_1 and Γ_2. For a graph Γ, let $V(\Gamma)$ and $E(\Gamma)$ be the set of its vertices and edges, respectively. Each edge is represented by a pair of vertices. A vertex p is *adjacent* to another vertex q if $\{p,q\} \in E(\Gamma)$. The *complement graph of* Γ is the graph $\overline{\Gamma}$ defined by the relations $V(\overline{\Gamma}) = V(\Gamma)$ and $E(\overline{\Gamma}) = \{\{p,q\} : p$ and q are non-adjacent vertices in $\Gamma\}$.

For each vertex q of Γ, the *link* of q is defined by $\mathrm{Link}_\Gamma(q) = \{p \in V(\Gamma) | \{p,q\} \in E(\Gamma)\}$. We simply write $\mathrm{Link}(q) = \mathrm{Link}_\Gamma(q)$, if the meaning is clear from the context. For an edge e, let $\mathring{e}$ denote the interior of e. The *open star of* q is the set $\mathring{\mathrm{Star}}_\Gamma(q) = \mathring{\mathrm{Star}}(q) = \cup\{\mathring{e} : e$ is an edge of Γ containing $q\} \cup \{q\}$.

The *degree* of q in Γ, is defined by $\mathrm{d}_\Gamma(q) = |\mathrm{Link}_\Gamma(q)|$. One simply writes $\mathrm{d}(q)$ instead of $\mathrm{d}_\Gamma(q)$, when there is no danger of confusion. A vertex q of Γ is called a *boundary vertex*, if $d(q) = 1$. Let $\partial\Gamma$ denote the set of the boundary vertices of Γ. A vertex which is not a boundary vertex is called an *interior vertex of* Γ. A *boundary edge* is an edge containing a boundary vertex.

Let Γ_1 and Γ_2 be graphs. Then the disjoint union of Γ_1 and Γ_2 is denoted by $\Gamma_1 \sqcup \Gamma_2$. We define $\mathrm{Join}(\Gamma_1, \Gamma_2)$ to be the graph obtained by taking the disjoint union of Γ_1 and Γ_2 and adding the edges in the set $\{\{q_1, q_2\} : q_1 \in V(\Gamma_1), q_2 \in V(\Gamma_2)\}$. This means that,

$$\mathrm{Join}(\Gamma_1, \Gamma_2) = \overline{\overline{\Gamma_1} \sqcup \overline{\Gamma_2}}$$

A graph is *discrete* if there exists no edge. We let D_n be a discrete graph on n vertices. A graph is *complete* if every pair of distinct vertices are adjacent. We let K_n be a complete graph on n vertices. For convention, we let K_0 be the empty set, also considered as a complete graph. A join of two discrete graphs is called a *complete bipartite graph*. We let $K_{m,n} = \mathrm{Join}(D_m, D_n)$. A *path on n vertices*, is the graph on n vertices, say, $q_1, q_2, \ldots, q_n$ such that the edge set consists of $\{q_1, q_2\}, \{q_2, q_3\}, \ldots, \{q_{n-1}, q_n\}$. Such a graph is also denoted by $(q_1, q_2, \ldots, q_n)$. P_n denotes a path on n vertices. A *cycle of length* n is a connected graph on n vertices such that each vertex has degree 2. We let C_n denote a cycle of length n. This means that one can write $V(C_n) = \{q_1, q_2, \ldots, q_n\}$ so that $E(C_n) = \{\{q_i, q_j\} : i - j \equiv 1 \pmod{n}\}$. A *triangle* is a graph isomorphic to C_3, and a *square* is a graph isomorphic to C_4. The complement of a cycle of length n is is called an *anti-cycle* of length n.

Notation 2.1. For a graph Γ, $\mathcal{K}(\Gamma)$ denotes the set of maximal complete subgraphs of Γ.

Definition 2.2 (induced subgraph, [Gol04])**.** Let Γ be a graph.

(1) Let $S \subseteq V(\Gamma)$. The *induced subgraph of* Γ *on* S, written as Γ_S, is the subgraph of Γ satisfying that $V(\Gamma_S) = S$ and $E(\Gamma_S) = \{\{p,q\} : p,q \in S \text{ and } \{p,q\} \in E(\Gamma)\}$. We also say that Γ_S is the *graph spanned by* S.

(2) A subgraph Γ_1 of Γ is called an *induced subgraph*, if $\Gamma_1 = \Gamma_S$ for some $S \subseteq V(\Gamma)$. One writes $\Gamma_1 \leq \Gamma$, in this case.

If $\Gamma_1 \leq \Gamma$, and $\Lambda \cong \Gamma_1$, then we say that Γ_1 *is an induced* Λ *in* Γ. If Γ does *not* contain an induced Λ, then we say that Γ is Λ-*free*. In particular, a graph is *triangle-free*, if it does not contain an induced C_3. The following proposition is immediate from the definition of induced subgraphs.

Proposition 2.3. *(1) Let* Γ *be a graph. An induced subgraph on* $S \subseteq V(\Gamma)$ *is the largest subgraph of* Γ*, having the vertex set* S*.*

(2) $\Gamma_1 \leq \Gamma_2 \leq \Gamma_3$ *implies* $\Gamma_1 \leq \Gamma_3$

(3) $\Gamma_1, \Gamma_2 \leq \Gamma_3$ *implies* $\Gamma_1 \cap \Gamma_2 \leq \Gamma_3$

(4) $\Gamma_1 \leq \Gamma_2$ *if and only if* $\overline{\Gamma_1} \leq \overline{\Gamma_2}$. □

The following properties of cycles and anti-cycles will be used later in Chapter 4.

Proposition 2.4 (no long cycle in $\overline{C_n}$)**.** *Let* $n \neq 5$*. Then* $\overline{C_n}$ *does not contain an induced* C_m*, for any* $m \geq 5$*.*

Proof) We have only to consider the case when $n > 5$ and $5 \leq m \leq n$. Any connected subgraph of C_n is either C_n or P_k for some k. Note that $\overline{C_n}$ and $\overline{P_k}$, when $k \geq 5$, are not cycles, since they have vertices of degree larger than 2. Hence C_n does not contain an induced $\overline{C_m}$. By Proposition 2.3 (4), $\overline{C_n}$ is C_m-free. □

2.2 Graph products of groups

A *graph product of groups* is first studied in [Gre90]. In [HW99], its fundamental properties are proved using a van Kampen diagram. In this section, we survey some of the basic facts on graph products of groups that will be used in this thesis. While doing so, we define and apply a *dual van Kampen diagram* as the main tool.

We introduce general terms from group theory.

Definition 2.5 (word and letter)**.** (1) Let $G = \langle S|R\rangle$ be a group presentation. A *word in* G with respect to the given presentation is a sequence of elements in $S \cup S^{-1}$, and the *length of the word* is defined to be the length of the sequence. For $s_1, s_2, \ldots, s_m \in S \cup S^{-1}$, we denote the word $w = (s_1, s_2, s_3, \ldots, s_m)$ also by $w = s_1 s_2 s_3 \cdots s_m$. Each term s_i is called a *letter* of w. A *subword* is a subsequence consisting of consecutive terms. 1 denotes both the empty word and the trivial element in G, depending on the context.

(2) Each word corresponds to an element in G naturally. For two words w and w' on S, we write $w = w'$, if two words are identical (letter by letter). On the other hand, we write $w =_G w'$, if w and w' correspond to the same element in G. If $w =_G w'$, we say that w and w' are *equivalent*. We also write $w =_G g$, if the word w represents $g \in G$.

A *bigon* is a disk such that the boundary is considered as a loop with two vertices. Recall that a *combinatorial 2-complex* is a quotient space of a disjoint union of 0-, 1- or 2-cells, which are vertices, edges, bigons or polygons, by identifying certain pairs of their 0- or 1-dimensional faces [LS77, BH99]. An *edge-path* in a combinatorial 2-complex is a path, which can be decomposed as a sequence of edges. An edge-path is *closed*, if it is a loop.

Let $G = \langle S|R\rangle$ be a presentation. Consider a combinatorial 2-complex, such that each edge has a *label* in S, and an orientation. Let γ be an edge-path, that is, $\gamma = e_1^{p_1} \cdot e_2^{p_2} \cdots e_m^{p_m}$ written as a concatenation of oriented edges e_i and $p_i = \pm 1$. Let a_i be the label of e_i. Then the *word corresponding to* γ is $a_1^{p_1} \cdot a_2^{p_2} \cdots a_m^{p_m}$.

Now assume that each word in R has at least length 2, and w is a word representing the trivial element in G. Recall that a *van Kampen diagram for* w is a connnected, simply connected, planar combinatorial 2-complex $\tilde{\Delta}$, such that the following conditions are satisfied ([LS77, BH99]).

(i) Each edge is oriented and labeled by S.

(ii) The word corresponding to the boundary of each 2-cell, considered as a closed edge-path with a suitable choice of the basepoint and the orientation, is in R.

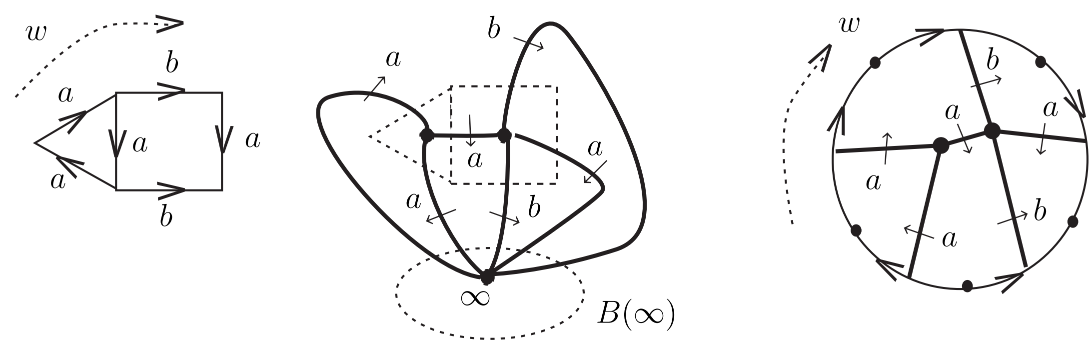
(iii) The word corresponding to a boundary cycle of $\tilde{\Delta}$, denoted by $\partial\tilde{\Delta}$, is w.

Proposition 2.6 ([vK33, LS77])**.** *Given a presentation for a group G, a van Kampen diagram exists for any word representing the trivial element in G.*

A van Kampen diagram, embedded in S^2, will define a 2-complex structure on S^2.

A dual to this structure often turns out to be useful.

Definition 2.7 (dual van Kampen diagram)**.** Let $G = \langle S|R \rangle$ be a group presentation. Suppose w is a word representing the trivial element in G. Let $\tilde{\Delta}$ be a van Kampen diagram for w, embedded in S^2. Let $(\tilde{\Delta})^*$ denote the dual structure to $\tilde{\Delta}$ on S^2 (Figure 2.1 (b)). Fix the vertex ∞ in $S^2 \setminus \tilde{\Delta}$, and let $\Delta = (\tilde{\Delta})^* \setminus B(\infty)$, where $B(\infty)$ denotes a sufficiently small ball around ∞, not touching $\tilde{\Delta}$. Then Δ is called a *dual van Kampen diagram for w*. The boundary of the remaining disk, written as $\partial\Delta$, is called the *boundary of* Δ.



(a) $\tilde{\Delta}$ (b) $(\tilde{\Delta})^*$ (c) $\Delta = (\tilde{\Delta})^* \setminus B(\infty)$

Figure 2.1: Construction of a dual van Kampen diagram for $w = abab^{-1}a$ in the group presented as $\langle a, b : [a, b] = 1, a^3 = 1 \rangle$.

Remark 2.8. (i) A dual van Kampen diagram can be considered as a graph embedded in a disk D, such that each edge is transversely oriented, and labeled by the generators.

(ii) From Proposition 2.6 and Definition 2.7, it is immediate to see that a dual van Kampen diagram exists for any word w representing the trivial element in the group.

Definition 2.9 (label-reading)**.** Let G be a group, X be a graph embedded in a surface S, and λ be a map from $E(X)$ to G. Suppose each edge of X is transversely oriented. Let γ be a closed curve or an arc on S, which is trasversely intersecting interiors of edges in $E(X)$. Decompose $\gamma = \gamma_1 \cdot \gamma_2 \cdots \gamma_m$ such that each γ_i intersects with exactly one edge of X (we let $m = 0$ if γ is disjoint from X). Assume γ_i intersects with an edge e_i. If the orientation of γ_i coincides with the transverse orientation of e_i, then we let $g_i = \lambda(e_i)$, and otherwise, $g_i = \lambda(e_i)^{-1}$. Then the word

$$w_\gamma = g_1 g_2 \cdots g_m$$

is called the *label-reading of* γ *with respect to* (X, λ). Here, $w_\gamma = 1$ if γ is disjoint from X.

Figure 2.2 shows an example of the label-reading of the curve ∂D with respect to a graph X in a disk D.

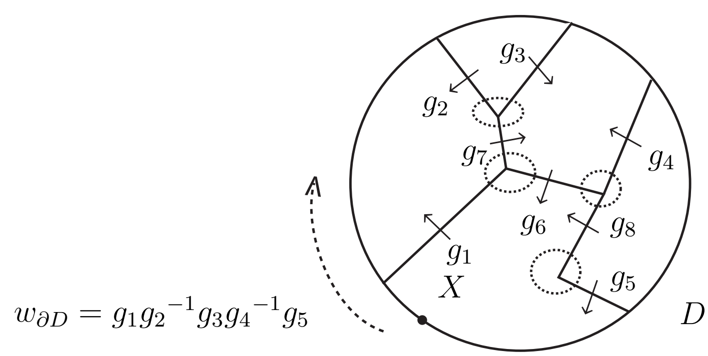


Figure 2.2: Label-reading of ∂D with respect to a graph X. Also, the smaller circles correspond to the words $g_1 g_7 g_6$, ${g_2}^{-1} g_3 {g_7}^{-1}$, ${g_6}^{-1} {g_4}^{-1} g_8$ and $g_5 {g_8}^{-1}$.

Remark 2.10. Let G be a group given by a presentation. Suppose w is a word representing the trivial element in G, and let Δ be a dual van Kampen diagram.

Then we always assume that $\partial\Delta$ is oriented, so that a label-reading of $\partial\Delta$ following its orientation is w.

Remark 2.11. Let $G = \langle S|R\rangle$ be a group presentation, and $w =_G 1$. A dual van Kampen diagram Δ for w, as defined in Definition 2.7, is actually a graph X embedded in a disk D, satisfying the following.

(i) There exists no isolated vertex of X.

(ii) $X \cap \partial D = \partial X$.

(iii) Each edge of X is transversely oriented and labeled by S. Moreover, the label-reading of a circle around (and sufficiently near from) any interior vertex represents the trivial element in G.

(iv) The label-reading of ∂D is w, with a suitable choice of the basepoint and the orientation of ∂D.

A dual van Kampen is a particularly useful tool to study *graph products of groups* defined as follows.

Definition 2.12 (graph product)**.** Let Γ be a graph.

(1) Suppose $\mathcal{G} = \{G_q : q \in V(\Gamma)\}$ is a collection of groups indexed by $V(\Gamma)$. The *graph product of* $\mathcal{G}$ *with the underlying graph* Γ, written as $\mathrm{GP}(\Gamma, \mathcal{G})$, is defined by $\mathrm{GP}(\Gamma, \mathcal{G}) = *\mathcal{G}/\langle\langle P\rangle\rangle$, where $*\mathcal{G}$ denotes the free product of the groups in $\mathcal{G}$, and $\langle\langle P\rangle\rangle$ denotes the normal closure of the set

$$P = \{[g, h] : g \in G_p, h \in G_q \text{ for an edge } \{p, q\} \text{ of } \Gamma\}$$

We call Γ as the *underlying graph* of $\mathcal{G}$, and each element of $\mathcal{G}$ as a *vertex group.*

(2) The *right-angled Artin group on* Γ, denoted by $A(\Gamma)$, is the graph product of infinite cyclic groups, with the underlying graph Γ.

(3) The *right-angled Coxeter group on* Γ, denoted by $C(\Gamma)$, is the graph product of cyclic groups of order 2, with the underlying graph Γ.

From this point on in this section, we let Γ be a graph, $\{G_q : q \in V(\Gamma)\}$ be a collection of groups indexed by $V(\Gamma)$ and $G = \mathrm{GP}(\Gamma, \{G_q\})$. The *graph product presentation for* G will mean the presentation of G where the generating set is $S = \sqcup\{G_q \setminus \{1\} : q \in V(\Gamma)\}$ and the relators are one of the following types.

(i) (length 2 relator) gh, where $g, h \in G_q \setminus \{1\}$ for some $q \in V(\Gamma)$, and $gh = 1$ in G_q,

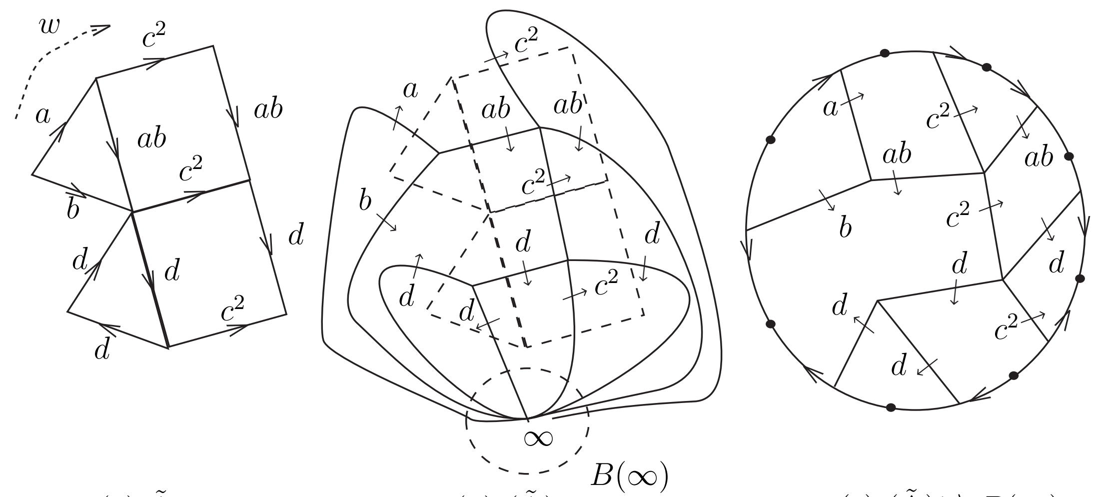
(ii) (multiplication table) ghk, where $g, h, k \in G_q \setminus \{1\}$ for some $q \in V(\Gamma)$, and $ghk = 1$,

(iii) (commuting relator) $[g, h]$ where $g \in G_p \setminus \{1\}, h \in G_q \setminus \{1\}$ for some $\{p, q\} \in E(\Gamma)$.

The terms *words, letters, lengths of words, equivalent words* and *dual van Kampen diagrams* of G shall make sense with respect to this presentation of G. (Figure 2.3).

Definition 2.13 (normal form, [Gre90])**.** Let Γ be a graph, and $G = \mathrm{GP}(\Gamma, \{G_q : q \in V(\Gamma)\})$.

(1) Let $w = \prod_{i=1}^{m} g_i$ be a word in G, where each letter g_i is in $G_{q_i} \setminus \{1\}$ for each i. An *elementary reduction of* w is a transformation from w to another word



(a) $\tilde{\Delta}$ (b) $(\tilde{\Delta})^*$ (c) $(\tilde{\Delta})^* \setminus B(\infty)$

Figure 2.3: Construction of a dual van Kampen diagram for $w = a{\cdot}c^2{\cdot}ab{\cdot}d{\cdot}c^{-2}{\cdot}d{\cdot}d{\cdot}b^{-1}$ in G. Here, we let Γ be the path (p, q, r), $G_p = \langle a, b | a^2 = 1\rangle, G_q = \langle c | - \rangle, G_r = \langle d | d^3 = 1\rangle$, and $G = \mathrm{GP}(\Gamma, \{G_p, G_q, G_r\})$.

$w' =_G w$, where w' is obtained from w by either

(i) (cancel) eliminating a subword of the form $g_i \cdot {g_i}^{-1}$,

(ii) (combine) combining $g_i \cdot g_{i+1} = (g_i g_{i+1})$, when $q_i = q_{i+1}$ and $g_{i+1} \neq {g_i}^{-1}$, or

(iii) (swap) changing $g_i \cdot g_{i+1}$ to $g_{i+1} \cdot g_i$, when q_i and q_{i+1} are adjacent

(2) A word w in G is in a *normal form*, if the length of w cannot be shortened by applying a sequence of elementary reductions.

Lemma 2.14 (reduction)**.** *Let $w = \prod_{i=1}^{m} g_i$ be a word in G ($m \geq 1$), where $g_i \in G_{q_i} \setminus \{1\}$ for each i. Then, w is in a normal form if and only if the following condition hold:*

if $i < j$ and $q_i = q_j$, then there exists $i < k < j$ such that q_k is not adjacent to q_i.

Proof)

($\Rightarrow$) Obvious, for otherwise canceling or combining elementary reduction will occur, after certain swaps.

($\Leftarrow$) Let w' be a shorter word for w, obtained by applying elemantary reductions to w. One can find a sequence of the words $w_1 = w, w_2, \ldots, w_k = w'$ such that w_i is obtained from w_{i-1} by applying elementary reduction once. Now consider the first elementary reduction that is not a swap. This means, find k_0 such that for $i < k_0$, w_i is obtained from w_{i-1} by applying the swap operation once, and w_{k_0} is obtained from w_{k_0-1} by applying a canceling or a combining operation. One can write $w_{k_0-1} = \cdots g \cdot g' \cdots$ where g and g' belong to the same vertex group. By considering the appearance of g and g' in $w = w_1$, one can can write $w = v_1 g v_2 g' v_3$ or $w = v_1 g' v_2 g v_3$, for some subwords v_1, v_2 and v_3 of w, such that each letter of v_2 commutes with g and g'. This violates the given condition. □

Theorem 2.15 (normal form theorem, [Gre90, HW99])**.** *If w is in a normal form, then w is not equivalent to any other word of a smaller length.*

For the rest of this section, we use a dual van Kampen diagram to prove Theorem 2.15. The idea of the proof will be revisited in Chapter 3 and 4.

Definition 2.16 (H-graph)**.** Let H be any group and D be a disk in $\mathbb{R}^2$. An *H-graph in D* is a pair (Y, λ) satisfying the following.

(i) Y is a connected planar graph contained in D, with transversely oriented edges.

(ii) λ, called a *labeling* of the edges, is a map from $E(Y)$ to $H \setminus \{1\}$.

(iii) For any $q \in V(Y)$, $0 < d(q) \leq 3$.

(iv) $Y \cap \partial D = \partial Y \neq \varnothing$.

(v) For each vertex q that is not on the boundary, let N_q be a sufficiently small disk containing q such that $\gamma = \partial N_q$ only intersects with the edges that are containing q. Then for any orientation and the choice of a basepoint of γ, $w_\gamma = 1$ in H.

Note that w_γ denotes the label-reading of γ as in Definition 2.9. The following is a crucial combinatorial lemma that is used in the proof of Theorem 2.15. One can view this as an equivalent statement to a result in [HW99, Lemma 4.3], by taking a dual.

Lemma 2.17 (label-reading of the boundary is trivial)**.** *(1) Let Z be a disjoint union of H-graphs in an oriented disk $D \subseteq \mathbb{R}^2$. Then the label-reading of ∂D with respect to Z is trivial in H.*

(2) Any H-graph has at least two boundary edges.

Proof) (1) We use an induction on $|E(Z)|$. Choose a boundary edge $e = \{p, q\}$ and suppose $d(p) = 1$. If $d(q) = 1$, then cut D along e and apply the inductive hypothesis to each of the pieces.

If $d(q) = 2$, then one can consider another disjoint unioin of H-graph Z', obtained by combining two edges containing q into one edge, removing the vertex q (Figure 2.4 (a)). Now let $d(q) = 3$, and $B(q)$ be a sufficiently small neighborhood of q in D. Cut

$D \setminus B(q)$ along $e \setminus B(q)$, to get D' (Figure 2.4 (b)). Then $Z' = Z \cap D'$ is another disjoint union of H-graphs. So the inductive hypothesis applies to Z'. Note that the label-reading of $\partial D'$ is equivalent in H to that of ∂D.

(2) Note that $w_{\partial D}$ is the multiplication of the labels of the boundary edges or their inverses in a certain order. So (2) follows from (1). □

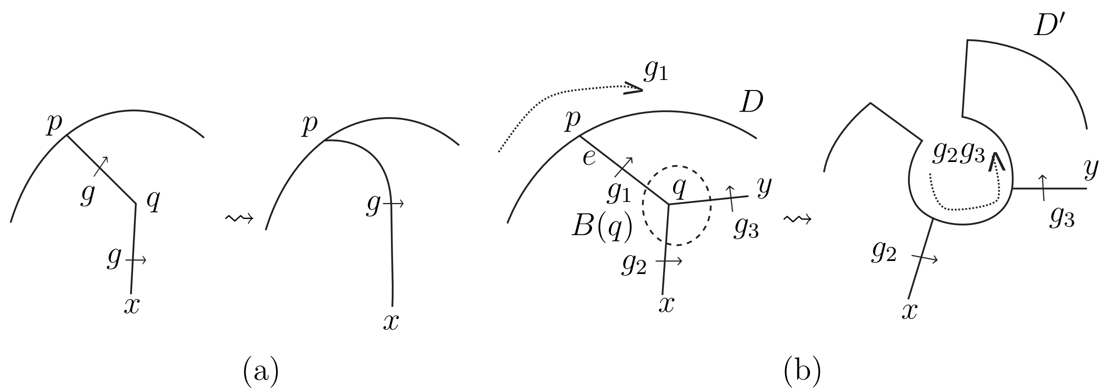


Figure 2.4: Reducing the number of edges for an H-graph. (a) "Smoothing out" a vertex of degree 2. (b) "Splitting" a boundary edge. Note that $g_1 = g_2 g_3$ in H.

Remark 2.18. Now let G be a graph product of groups $\{G_q : q \in V(\Gamma)\}$ with the underlying graph Γ. Suppose $w =_G 1$. Note that a dual van Kampen diagram Δ for w is a disk with an embedded graph X (Remark 2.8). For each $q \in V(\Gamma)$, let $X_q \subseteq X$ be the subgraph consisting of the edges labeled by elements in G_q. Then, X_p and X_q intersect *only if* p and q are adjacent in Γ. The intersection point will correspond to a commuting relator. It is obvious that each connected component of X_q is a G_q-graph (Definition 2.16). Moreover, by "smoothing out" vertices of degree 2 as described in the proof of Lemma 2.17, we may consider X as a union of transversely intersecting G_q-graphs for $q \in V(\Gamma)$.

So a dual van Kampen diagram determines a triple $(\mathcal{H}, \lambda, \{\mu_Y\}_{Y\in\mathcal{H}})$ where,

(i) $\mathcal{H}$ is a set of connected graphs in a disk D, $\lambda : \mathcal{H} \to V(\Gamma)$ and $\mu_Y : E(Y) \to G_{\lambda(Y)}$ for $Y \in \mathcal{H}$.

(ii) for each $Y \in \mathcal{H}$, (Y, μ_Y) is a $G_{\lambda(Y)}$-graph

(iii) Suppose Y and $Y' \in \mathcal{H}$ intersect and $Y \neq Y'$. Then $\lambda(Y)$ and $\lambda(Y')$ are adjacent in Γ. Moreover, Y and Y' transversely intersect, at interior points of the edges.

In this case, we write $\Delta = (\mathcal{H}, \lambda, \mu)$. Conversely, suppose a triple $\Delta = (\mathcal{H}, \lambda, \mu)$ satisfies the above conditions (i),(ii) and (iii). Let w be a label-reading of $\partial\Delta$. Then, by taking the dual, one obtains a van Kampen diagram for the word w. So one sees that Δ is a dual van Kampen diagram for w.

The boundary $\partial\Delta$ of a dual van Kampen diagram $\Delta = (\mathcal{H}, \lambda, \mu)$ can be divided into paths, called *segments*, so that each segment contains exactly one vertex of ∂Y for some $Y \in \mathcal{H}$. Such a segment has an orientation and a label, given by those of the edges of Y. A connected, contractible union of segments is called an *interval*. A *g-segment* (a *g-interval*, respectively) is a segment (an interval, respectively), the label-reading along which gives $g \in G$. Also, for a vertex q, a *G_q-segment* will mean a segment corresponding to an element in G_q. We will not hesitate to use these geometric terms, segments and intervals, to mean the corresponding algebraic terms also, namely letters and subwords.

A graph $Y \in \mathcal{H}$ is *innermost*, if all the segments of a certain component of $\partial\Delta \setminus Y$

intersect with distinct graphs in $\mathcal{H}$. If Y is innermost, the two letters bounding such a component is called an *innermost pair* (Figure 2.5).

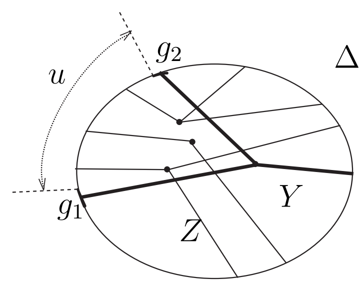


Figure 2.5: g_1 and g_2 are innermost pair of segments. Note that any two segments in u do not intersect with the same graph in $\mathcal{H}$.

Lemma 2.19 (no pair in a normal form)**.** *Let $w_1 =_G 1$ and $\Delta = (\mathcal{H}, \lambda, \mu)$ be a dual van Kampen diagram for w_1. Suppose w_1 contains a subword w_2 which is in its normal form. Then for each graph Y in $\mathcal{H}$, Y does not intersect two distinct segments in w_2.*

Proof)

Suppose two segments g_1 and g_2 of w_2 are intersecting with a graph $Y \in \mathcal{H}$. Choose a nearest pair of such segments in the sense that the interval u, which is between the segments and is contained in w_2, does not contain any pair of segments intersecting with the same graph in $\mathcal{H}$ (Figure 2.5). Since Y is connected there exists a properly embedded (simple) arc γ from g_1 to g_2 in Y. Then for any other graph Z intersecting with a segment in u, $Z \cap \gamma \neq \varnothing$. Hence $\lambda(Z)$ is adjacent to $\lambda(Y)$ in Γ. This means that the letters between g_1 and g_2 in w_2 are commuting with both g_1 and g_2, which is a contradiction by Lemma 2.14. □

Proof of Theorem 2.15

We may assume the word is not empty. Let $w' =_G w$ be a shorter word for w. Draw a dual van Kampen diagram Δ for ww'^{-1}. First, I claim that there exist two segments of w on $\partial\Delta$ that intersect with the same $Y \in \mathcal{H}$. If not, each segment of w intersects with different $Y \in \mathcal{H}$. Since each Y has at least two boundary edges(Lemma 2.17), this would imply that the number of segments on $\partial\Delta$ is at least twice of the length of w, which is a contradiction to the assumption that w' is shorter than w.

Now two segments of w should intersect with the same graph $Y \in \mathcal{H}$. By Lemma 2.19, w is not in its normal form.□

Corollary 2.20. *Let Γ be a graph, and $\{G_q : q \in V(\Gamma)\}$ be a collection of non-trivial groups. Let $G = GP(\Gamma, \{G_q\})$.*

(1) Suppose $H_q \leq G_q$ for each $q \in V(\Gamma)$, and let let $H = GP(\Gamma, \{H_q\})$. Then $H \leq G$.

(2) If two normal forms $g_1 g_2 \cdots g_m$ and $h_1 h_2 \cdots h_{m'}$ are equivalent in G, then $m = m'$ and $\{h_1, h_2, \ldots, h_m\}$ is a permutation of $\{g_1, g_2, \ldots, g_m\}$.

Proof) (1) is immediate from Theorem 2.15, since a normal form in H is a normal for in G.

(2) $m = m'$ by Theorem 2.15. $g_1 g_2 \cdots g_m {h_m}^{-1} {h_{m-1}}^{-1} \cdots {h_1}^{-1} =_G 1$. By Lemma 2.14, there exists i, j such that the following hold.

(i) g_i and h_j belong to the same vertex group G_p.

(ii) For each $k > i$, $g_k \in G_q$ for some $q \in \text{Link}(p)$.

(iii) For each $k > j$, $h_k \in G_q$ for some $q \in \text{Link}(p)$.

After changing the order of the letters if necessary, we may assume that $i = j = m$, and both g_m and h_m belong to the same vertex group, say G_p. If $g_m = h_m$, the proof is complete by an induction on m.

Suppose $g_m \neq h_m$. Note that

$$g_1 g_2 \cdots g_{m-1} =_G h_1 h_2 \cdots h_{m-1}(h_m {g_m}^{-1})$$

The word on the right hand side is not in a normal form, although $h_1 h_2 \cdots h_{m-1}$ is. So there exists $i_0 < m$ such that h_{i_0} belongs to G_p, and for $i_0 < j < m$, h_j belongs to G_q for some $q \in \mathrm{Link}(p)$. Again, we may just assume that $i_0 = m - 1$. Then we have a contradiction, for h_{m-1} and h_m cannot belong to the same vertex group G_p. $\square$

2.3 Right-angled Artin groups

In this section, we describe group theoretic properties specific to right-angled Artin groups.

Let Γ be a graph. The right-angled Artin group on Γ has the group presentation

$$A(\Gamma) = \langle q \in V(\Gamma) \mid [a, b] = 1 \text{ if and only if } \{a, b\} \in E(\Gamma)\rangle$$

and whenever we talk about *words, letters, lengths of words, equivalent words* and *dual van Kampen diagrams* of $A(\Gamma)$, we will refer to that presenation. Note that the meanings of those terms when $A(\Gamma)$ is considered as a graph product of infinite cyclic groups, are slightly different. The key difference is, for a vertex a the word

a^n has length n in a right-angled Artin group, while the length of a^n is 1 in a graph product of infinite cyclic groups.

So, a *word of length k in $A(\Gamma)$* is an expression $w = \prod_{i=1}^{k} c_i^{e_i}$, where $c_i \in V(\Gamma)$ and $e_i = \pm 1$. Each $c_i^{e_i}$ is a *letter* of the word w.

We say the word w is *reduced*, if the length is minimal among the words representing the same element. Note that this is a weaker condition that w is in a normal form considering $A(\Gamma)$ as a graph product of infinite cyclic groups. For example, in the right-angled Artin group on an edge with vertices a and b, the word aba and a^2b are both reduced as elements of the right-angled Artin group, but the former is not in a normal form as an element of the graph product of two infinite cyclic groups $\langle a \rangle$ and $\langle b \rangle$.

For each $i_0 = 1, 2, \ldots, k$, the word $w_1 = \prod_{i=i_0}^{k} c_i^{e_i} \cdot \prod_{i=1}^{i_0-1} c_i^{e_i}$ is called a *cyclic conjugation* of $w = \prod_{i=1}^{k} c_i^{e_i}$. We say that a word w is *cyclically reduced*, if its cyclic conjugations are all reduced. By a *subword* of w, we mean a word $w' = \prod_{i=i_0}^{i_1} c_i^{e_i}$ for some $1 \leq i_0 < i_1 \leq k$. A letter or a subword w' of w is on the *left* of a letter or a subword w'' of w, if $w' = \prod_{i=i_0}^{i_1} c_i^{e_i}$ and $w'' = \prod_{i=j_0}^{j_1} c_i^{e_i}$ for some $i_1 < j_0$.

As in the previous section, The expression $w_1 = w_2$ means that w_1 and w_2 are equal as words (letter by letter), while $w_1 =_{A(\Gamma)} w_2$ means that the words w_1 and w_2 represent the same element in $A(\Gamma)$.

Let Δ be a dual van Kampen diagram for w in $A(\Gamma)$ (Figure 2.6). Recall that Δ is a graph X embedded in a disk, with edges labeled by $V(\Gamma)$ and transversely oriented (Remark 2.8). As in Remark 2.18, choose the subgraph X_a consisting of edges labeled by $a \in V(\Gamma)$. Since each 2-cell in the van Kampen diagram is a square corresponding to the commuting relator between two vertices, we see that each interior vertex of X_a has degree 2, and so, X_a is a disjoint union of properly embedded arcs and simple closed curves, after "smoothing out" degree 2 vertices (Remark 2.18). Let $\mathcal{H}$ denote the collection of such curves, for all $a \in V(\Gamma)$.

We always regard $\partial\Delta$ as divided into segments so that each segment intersects with exactly one arc in $\mathcal{H}$. We let the label and the orientation of each segment be induced from those of the arc that intersects with the segment.

We call each arc in $\mathcal{H}$ labeled by $q \in V(\Gamma)$ as a *q-arc*, and each segment in $\partial\Delta$ labeled by q as a *q-segment*. We will not hesitate to identify the letter $q^{\pm 1}$ of w with the corresponding q-segment of $\partial\Delta$. A connected contractible union of segments on $\partial\Delta$ is called an *interval*. For convention, a subword w_1 of w shall also denote the corresponding interval (called *w_1-interval*) on $\partial\Delta$.

Let Δ be a dual van Kampen diagram for w in $A(\Gamma)$. Similarly as in Remark 2.18 (see also Figure 2.6 (c)), Δ is determined by a pair $(\mathcal{H}, \lambda)$ where

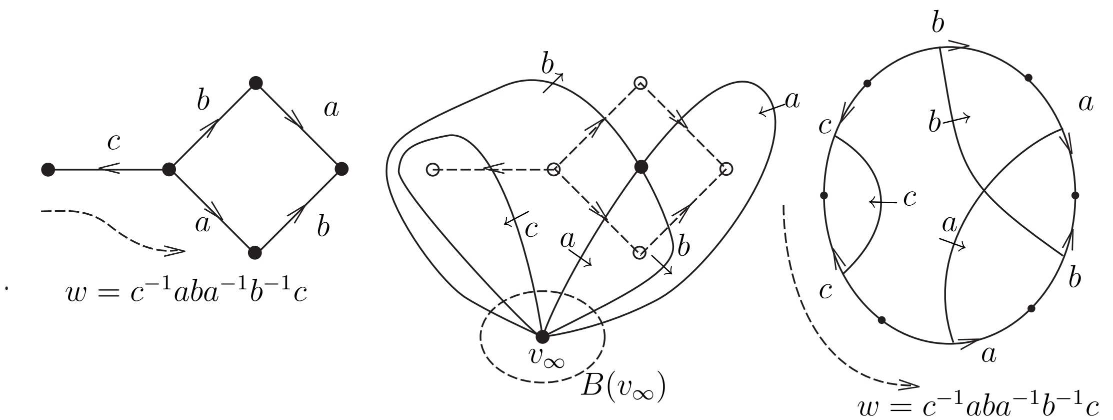
(i) $\mathcal{H}$ is a set of transversely oriented simple closed curves and properly embedded arcs in an oriented disk $D \subseteq \mathbb{R}^2$.

(ii) λ is a map from $\mathcal{H}$ to $V(\Gamma)$ such that γ and γ' in $\mathcal{H}$ are intersecting *only if* $\lambda(\gamma)$ and $\lambda(\gamma')$ are adjacent in Γ.

(iii) w is a label-reading of $\partial\Delta$.

As in Remark 2.8, we note that any pair $\Delta = (\mathcal{H}, \lambda)$ satisfying the above conditions (i), (ii), and (iii) is a dual van Kampen diagram for any label-reading of $\partial\Delta$.

Note that simple closed curves in a dual van Kampen diagram can always be assumed to be removed. Also, we may assume that two curves in Δ are minimally intersecting, in the sense that there does not exist any bigon formed by arcs in $\mathcal{H}$. See [CW04] for more details.



(a) $\tilde{\Delta}$ (b) $(\tilde{\Delta})^*$ (c) $(\tilde{\Delta})^* \setminus B(v_\infty)$

Figure 2.6: Constructing a dual van Kampen diagram from a van Kampen diagram $\tilde{\Delta}$, for $w = c^{-1}aba^{-1}b^{-1}$ in $\langle a, b, c \mid [a, b] = 1\rangle$.

Now let $\Delta = (\mathcal{H}, \lambda)$ be a dual van Kampen diagram on $D \subseteq \mathbb{R}^2$. Suppose γ is a properly embedded arc in D, which is either an element in $\mathcal{H}$ or in general position with $\mathcal{H}$. Then one can cut Δ along γ in the following sense. First, cut D along γ to get two disks D' and D''. Consider the intersections of the disks with the curves in $\mathcal{H}$. Then, let those curves in D' and D'' inherit the transverse orientations and the labelings from Δ (Figure 2.7). We obtain two dual van Kampen diagrams, one for each of D' and D''. An *innermost q-arc* γ of Δ is a q-arc such that the interior of D' or D'' does not intersect any q-arc. Conversely, we can glue two dual van Kampen diagrams along identical words.

Definition 2.21 (canceling pair)**.** Let Γ be a graph.

(1) Let w be a word representing the trivial element in $A(\Gamma)$, and Δ be a dual van Kampen diagram for w. Two segments on the boundary of Δ are called a *canceling q-pair* if there exists a q-arc joining the segments.

(2) For *any* word w_1, two letters of w_1 are called a *canceling q-pair* if there exists another word $w_1' =_{A(\Gamma)} w_1$ and a dual van Kampen diagram Δ for $w_1 w_1'^{-1}$, such that the two letters are a q-pair with respect to Δ.

(3) An *innermost canceling q-pair* is a canceling q-pair joined by an innnermost q-arc.

We let *canceling pair* mean a canceling q-pair for some $q \in V(\Gamma)$.

For a group G and its subset P, $\langle P \rangle$ denotes the subgroup generated by P. For a subgroup H of $A(\Gamma)$, $w \in H$ shall mean that w represents an element in H.

Lemma 2.22 (no pair if reduced)**.** *Let Γ be a graph and q be a vertex of Γ. If a word w in $A(\Gamma)$ has a canceling q-pair, then $w = w_1 q^{\pm 1} w_2 q^{\mp 1} w_3$ for some subwords w_1, w_2 and w_3 such that w_2 represents an element in $\langle Link(q) \rangle$. In this case, w is not reduced.*

Proof)

There exists a word $w' =_{A(\Gamma)} w$ and a dual van Kampen diagram Δ for ww'^{-1}, such that a q-arc joins two segments of w.

Write $w = w_1 q^{\pm 1} w_2 q^{\mp 1} w_3$, where the letters $q^{\pm 1}$ and $q^{\mp 1}$ (identified with the corresponding segments on $\partial\Delta$) are joined by a q-arc γ (Figure 2.7).

Cut Δ along γ, to get a dual van Kampen diagram Δ_0, which contains w_2 on its boundary. Give Δ_0 the orientation that coincides with the orientation of Δ on w_2. Let $\tilde{w}_2$ be the word, read off by following γ in the orientation of Δ_0. $\tilde{w}_2 \in \langle \mathrm{Link}(q) \rangle$, for the arcs intersecting with γ are labeld by vertices in $\mathrm{Link}(q)$. Since Δ_0 is a dual van Kampen diagram for the word $w_2\tilde{w}_2$, we have $w_2 =_{A(\Gamma)} \tilde{w}_2^{-1} \in \langle \mathrm{Link}(q) \rangle$. $\square$

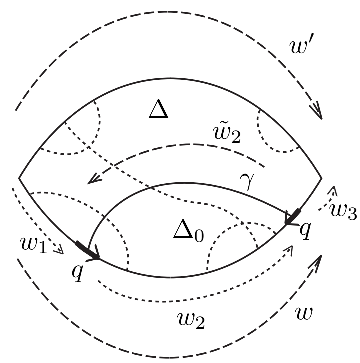


Figure 2.7: Cutting a dual van Kampen diagram Δ along a curve γ.

For $S \subseteq V(\Gamma)$, we let $S^{-1} = \{q^{-1} : q \in S\}$ and $S^{\pm 1} = S \cup S^{-1}$. The following lemma is standard, and we sketch the proof.

Lemma 2.23. *Let Γ be a graph and S be a subset of $V(\Gamma)$. Then the following are true.*

(1) $\langle S \rangle$ is isomorphic to $A(\Gamma_S)$.

(2) Each letter of any reduced word in $\langle S \rangle$ is in $S^{\pm 1}$.

Proof)

(1) The inclusion $V(\Gamma_S) \subseteq V(\Gamma)$ induces a map $f : A(\Gamma_S) \to A(\Gamma)$. Let w be a

word representing an element in $\ker f$. Since $w =_{A(\Gamma)} 1$, there exists a dual van Kampen diagram Δ for the word w in $A(\Gamma)$. Remove simple closed curves labeled by $V(\Gamma) \setminus V(\Gamma_S)$, if there is any. Since the boundary of Δ is labeled by vertices in $V(S)$, Δ can be considered as a dual van Kampen diagram for a word w in $A(\Gamma_S)$. So we get $w =_{A(\Gamma_S)} 1$.

(2) $w =_{A(\Gamma)} w'$ for some word w' such that the letters of w' are in S. Let Δ be a dual van Kampen diagram for ww'^{-1}. If w contains a q-segment for some $q \notin S$, then a q-arc joins two segments in Δ, and these segments must be in w. This is impossible by Lemma 2.22. $\square$

From this point on, $A(\Gamma_S)$ is considered as a subgroup of $A(\Gamma)$, whenever $S \subseteq V(\Gamma)$.

A right-angled Artin group can also be considered as a repeated HNN-extension as follows. Let H be a group and $\phi : C \to D$ be an isomorphism between subgroups of H. Then we define $H*_\phi = \langle H, t \mid t^{-1}ct = \phi(c),\ \text{for } c \in C\rangle$, which is the HNN extension of H with the amalgamating map ϕ and the stable letter t. Sometimes, we explicitly state what the stable letter is. If $C = D$ and ϕ is the identity map, then we let $H*_C = \langle H, t \mid t^{-1}ct = t \text{ for } c \in C\rangle$.

Lemma 2.24. *Let Γ be a graph. Suppose Γ' is an induced subgraph of Γ such that $V(\Gamma') = V(\Gamma) \setminus \{q\}$ for some $q \in V(\Gamma)$. Let C be the subgroup of $A(\Gamma')$ generated by $\mathrm{link}_\Gamma(q)$. Then the inclusion $A(\Gamma') \hookrightarrow A(\Gamma)$ extends to the isomorphism $f : A(\Gamma')*_C \to A(\Gamma)$ such that $f(t) = q$.*

Proof) Immediate from the definition of right-angled Artin groups.$\square$

We note the following general lemma, for its use in Section 4.3.

Lemma 2.25. *Let H be a group and $\phi : C \to D$ be an isomorphism between subgroups C and D. Suppose K is a subgroup of H and $J = \langle K, t\rangle \leq H*_\phi$. We let $\psi : J \cap C \to J \cap D$ be the restriction of ϕ. Then the inclusion $J \cap H \hookrightarrow J$ extends to the isomorphism $f : (J \cap H)*_\psi \to J$ such that $f(\hat{t}) = t$, where $\hat{t}$ and t denote the stable letters of $(J \cap H)*_\psi$ and $H*_\phi$, respectively.*

Proof) See Figure 2.8 for the inclusions between the given groups. $G = H*_\phi$ acts on a tree T, with a vertex v_0 and an edge $e_0 = \{v_0, t.v_0\}$ satisfying $\mathrm{Stab}(v_0) = H$ and $\mathrm{Stab}(e_0) = C$ [Ser03]. Let T_0 be the induced subgraph on $\{j.v_0 : j \in J\}$. For each vertex $j.v_0$ of T_0, write $j = k_1t^{\epsilon_1}k_2t^{\epsilon_2}\cdots k_mt^{\epsilon_m}$, where $k_i \in K$ and $\epsilon_i = \pm 1$ for each i. Then the following sequence in $V(T_0)$ gives rise to a path in T_0 from v_0 to $j.v_0$.

$$\begin{aligned} v_0 &= k_1.v_0, \\ k_1t^{\epsilon_1}.v_0 &= k_1t^{\epsilon_1}k_2.v_0, \\ k_1t^{\epsilon_1}k_2t^{\epsilon_2}.v_0 &= k_1t^{\epsilon_1}k_2t^{\epsilon_2}k_3.v_0, \\ &\cdots \\ k_1t^{\epsilon_1}k_2t^{\epsilon_2}k_3\cdots t^{\epsilon_m}.v_0 &= j.v_0 \end{aligned}$$

Hence T_0 is connected. Note that $\psi : J \cap C = \mathrm{Stab}_J(e_0) \to J \cap D = \mathrm{Stab}_J(e_0)^t$. Since J acts on a tree T_0, we have an isomorphism $J \cong \mathrm{Stab}_J(v_0)*_\psi = (J \cap H)*_\psi$.□

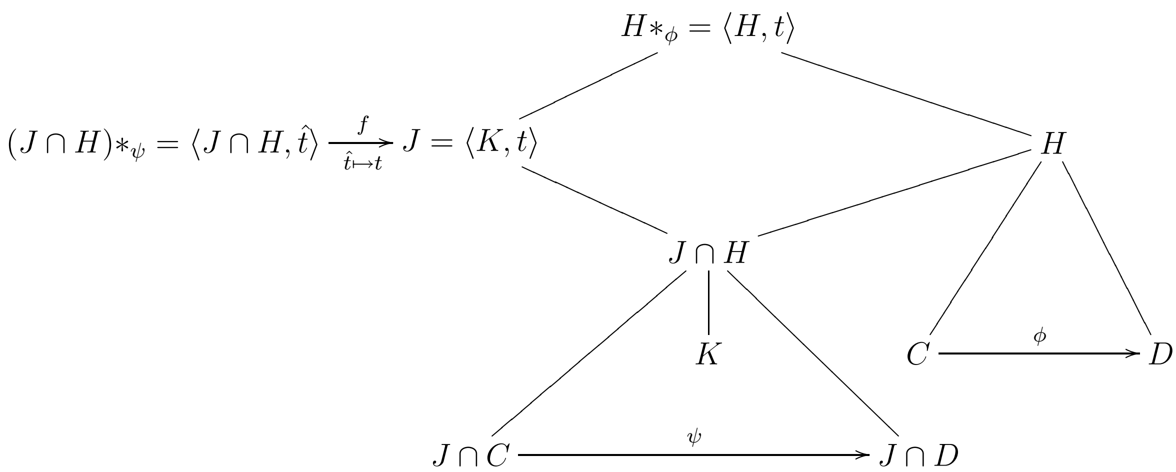


Figure 2.8: Lemma 2.25.

2.4 The space X_Γ

For each right-angled Artin group $A(\Gamma)$, there exists a "standard" $K(A(\Gamma), 1)$-space, called X_Γ. In this section, we define and describe this space X_Γ.

An *n-dimensional unit cube* is the set $I^n = [0, 1]^n \subseteq \mathbb{R}^n$. A *face* of I^n is recursively defined to be I^n itself, or a face of $(n-1)$-dimensional cube in I^n obtained by fixing one coordinate to be 0 or 1. A *cubed complex* X is a quotient space of a disjoint union of unit cubes $\{C_\alpha\}$ such that the interior of a face of each cube C_α injects into X and identifications occur only as isometry of faces of the unit cubes. A graph can be obviously considered as a cubed complex, for instance. In [BH99], the term a *cubical complex* is used to refer to a more restricted class, requiring that each *closed* unit cube to inject. For example, the n-torus, considered as a quotient space of I^n with each pair of opposite $(n-1)$ dimensional faces identified, is a cubed, but not cubical, complex. Any cubed complex can be considered as a cubical complex after a suitable subdivision.

The *k-skeleton of* X is the image of the k-dimensional cubes. The image of a cube (a face of a cube, respectively) in X is also called a *cube* (a *face* of a cube, respectively). A *vertex* and an *edge* are a 0- and 1-dimensional faces, respectively. Note that each edge has the length 1.

A map $f : X \to Y$ between cubed complexes is *cubical*, if the image of each face of a cube in X is mapped onto the image of a face of a cube in Y by an isometry on faces. An *edge-path (of length k)* of X is a cubical map from P_{k+1} into $X^{(1)}$ for some k. If the terminal point of an edge-path γ_1 coincides with the initial point of another edge-path γ_2, we denote the concatenation of the paths by $\gamma_1 \cdot \gamma_2$.

For a vertex v of a cubed complex X, the link of v is denoted by $\mathrm{Link}_X(v)$. $\mathrm{Link}_X(v)$ can be considered as a simplicial complex [BH99]. Each cubical map $f : X \to Y$ naturally induces a simplicial map $f_v : \mathrm{Link}_X(v) \to \mathrm{Link}_Y(f(v))$. A *subcomplex* of X is a union of faces in X. We say that $Y \subseteq X$ is a *1-full* subcomplex, and write $Y \leq X$, if Y is a maximal subcomplex of X with a fixed 1-skeleton.

Now let Y be a simplicial complex. $\mathrm{Link}_Y(v)$ denotes the link of a vertex v in Y. A subcomplex Z of Y is called a *full subcomplex*, and written as $Z \leq Y$, if Z is the maximal subcomplex with a fixed 0-skeleton. We say Y is a *flag complex*, or equivalently *determined by its 1-skeleton*, if every complete subgraph in $Y^{(1)}$ is contained in a simplex in Y. This condition is also called as *no-*Δ *condition* [Gro87].

Definition 2.26 (local isometry)**.** Let $f : X \to Y$ be a cubical map between cubed complexes. If for each vertex $v \in X^{(0)}$, f induces an injective map $f_v : \mathrm{Link}_X(v) \to \mathrm{Link}_Y(f(v))$ such that the image of f_v is a full subcomplex of $\mathrm{Link}_Y(f(v))$, then we say that f is a *local isometry*.

Remark 2.27. A finite dimensional cubed complex has a complete geodesic length metric by giving the Euclidean metric to the interior of each unit cube ([Gro87]). If this metric on Y is *non-positively curved* ([BH99]), then a local isometry (in the sense of Definition 2.26) $f : X \to Y$ between finite dimensional cubed complexes with the length metric is indeed locally an isometric embedding ([Gro87, Cha00]).

The following is a widely known criteria for a cubical map to be π_1-injective. For a detailed proof, see [CW04].

Theorem 2.28 (local isometry is π_1-injective, [Gro87, CW04])**.** *Let X and Y be finite dimensional cubed complexes, such that the link of each vertex in Y is a flag complex. Then any local isometry $f : X \to Y$ is π_1-injective.*□

An *oriented and labeled* cubed complex is a cubed complex with extra data that are an orientation of each edge and a map, called a *labeling*, from the set of the edges into a given set.

Recall that $\mathcal{K}(\Gamma)$ denotes the set of the maximal complete subgraphs of a graph Γ.

Definition 2.29 (standard Eilenberg-Maclane space of $A(\Gamma)$, [CD95])**.** Let Γ be a graph. Then the *standard Eilenberg-Maclane space of* $A(\Gamma)$ is the cubed complex X_Γ satisfying the following.

(i) The 0-skeleton of X_Γ is a single vertex.

(ii) For each vertex q of Γ, there exists a unique oriented circle C_q in $X_\Gamma^{(1)}$.

(iii) For each K in $\mathcal{K}(\Gamma)$, a $|V(K)|$-dimensional torus is glued along the oriented circles $\{C_q : q \in V(K)\}$.

Then X_Γ is an oriented and labeled cubed complex, since each circle C_q is oriented and can be labeled by q.

Each vertex of $V(\Gamma)$ corresponds to a circle in X_Γ. For $\Gamma' \leq \Gamma$, $X_{\Gamma'}$ denotes the 1-full subcomplex of X_Γ determined by the circles corresponding the vertices of Γ'. For convention, we let $X_\varnothing$ be the set of the unique vertex of X_Γ.

$X^{(2)}$ is the Caley complex for $A(\Gamma)$ (see [LS77]), and so $\pi_1(X_\Gamma) = \pi_1(X_\Gamma^{(2)}) = A(\Gamma)$. Contractibility of the universal cover of X_Γ comes from the fact that X_Γ has a nonpositively curved metric ([BH99]). Hence, X_Γ is a $K(A(\Gamma), 1)$-space.

Example 2.30. (1) If Γ is discrete, then X_Γ is a bouquet of circles. Note that $A(\Gamma)$ is free in this case.

(2) If Γ is complete, then X_Γ is an n-torus. In this case, $A(\Gamma)$ is free abelian.

(3) Γ is C_5, then X_Γ is a union of five 2-dimensional tori such that the longitude of the $(i-1)$-th torus is identified with the meridian of the i-th torus, for $i = 1, 2, \ldots, 5 \pmod{n}$.

For a graph Γ, the *double of* Γ is the graph $D(\Gamma)$ satisfying $V(D(\Gamma)) = V(\Gamma) \times \{-1, 1\}$ and $E(D(\Gamma)) = \{\{(q, \epsilon), (q', \epsilon')\} : \epsilon, \epsilon' \in \{-1, 1\} \text{ and } \{q, q'\} \in E(\Gamma)\}$.

Let Γ be a graph and x_0 be the unique vertex of X_Γ. Let L_Γ be the link of x_0 in X_Γ. We can write $V(L_\Gamma) = V(\Gamma) \times \{-1, 1\}$, where $(q, 1)$ corresponds to the outgoing direction of the circle C_q for each vertex $q \in V(\Gamma)$, and $(q, -1)$ to the incoming direction (Figure 2.9). For $q, q' \in V(\Gamma)$ and $e, e' = \pm 1$, (q, e) and (q', e') span an edge if and only if C_q and $C_{q'}$ span a 2-torus, i.e. $\{q, q'\} \in E(\Gamma)$. Hence, $L_\Gamma^{(1)}$ is the double of Γ. For each complete subgraph of $L_\Gamma^{(1)}$, there exists a corresponding complete subgraph in Γ, and hence, a torus in X_Γ. It follows that L_Γ is a flag complex.

Proposition 2.31 (link of X_Γ)**.** *Let Γ be a graph, and v be the unique vertex of X_Γ. Then $Link^{(1)}_{X_\Gamma}(v) = D(\Gamma)$. Moreover, $Link_{X_\Gamma}(v)$ is a flag comlplex.*

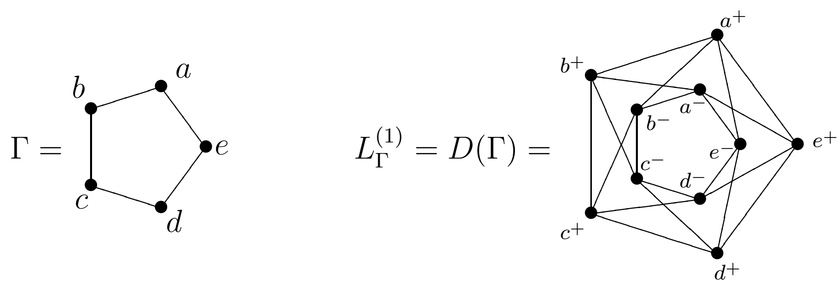


Figure 2.9: The link of the vertex in X_Γ, when $\Gamma = C_5$.

2.5 Maps from a surface into X_Γ

By a *surface*, we always mean a compact orientable surface unless specified otherwise. For a non-closed surface S, we also assume that each boundary component $\partial_1 S, \partial_2 S, \ldots, \partial_m S$ is oriented so that $\sum_i [\partial_i S] = 0$ in $H_1(S)$. We make a note of the following well-known property of an orientable surface, which will be used in Chapter 5.

Lemma 2.32 (not a proper power)**.** *(1) Let S be a surface, $c \in \pi_1(S)$ and $r \in \mathbb{Z}$.*

Suppose $x = c^r \in \pi_1(S) \setminus \{1\}$ *is represented by a simple closed curve. Then* $r = \pm 1$.

(2) *Let* S *be a hyperbolic surface. Suppose* x, y *are non-trivial elements of* $\pi_1(S)$, *represented by simple closed curves. If* $[x, y] = 1$, *then* $x = y$ *or* $x = y^{-1}$. *In particular, any two homotopy classes corresponding to two distinct boundary components do not commute in* $\pi_1(S)$.

Proof) (1) This is a standard result, from the orientability of S ([FHS82]).

(2) Since $\pi_1(S)$ is a one-relator group on at least 4 generators or a free group, $\langle x, y\rangle$ is a free group. Since $[x, y] = 1$, $\langle x, y\rangle$ is cyclic. Write $x = c^r$ and $y = c^s$ for some c. By (1), $|r|, |s| = 1$, and we conclude that $x = y^{\pm 1}$. □

A *curve* on a surface means a simple closed curve or a properly embedded arc.

Notation 2.33 (geometric intersection number)**.** Let S be a surface.

(1) Let α and β be curves on a surface S. Assume either

   (i) α and β are closed, and freely homotopic to each other, or

   (ii) α and β are arcs, and there exists a homotopy from α to β, so that during the homotopy the endpoints of the arcs lie on ∂S.

   Then we write $\alpha \sim \beta$.

(2) For two curves α, β, we let $i(\alpha, \beta)$ denote the geometric intersection number of

α and β, that is,

$$i(\alpha,\beta) = \min_{\substack{\alpha' \sim \alpha \\ \beta' \sim \beta}} |\alpha' \cap \beta'|$$

(3) Let α be a curve, and $A \subseteq S$. We write $\alpha \rightsquigarrow A$, if α can be homotoped into A (requiring that the endpoints to lie on ∂S during the homotopy, if α is an arc).

(4) Let A and B be subsurfaces of S. Then $A \sim B$ shall mean that there exists a homotopy from A to B.

For an example, see Figure 2.10. We say that α and β are *essentially intersecting* if $i(\alpha,\beta) \neq 0$.

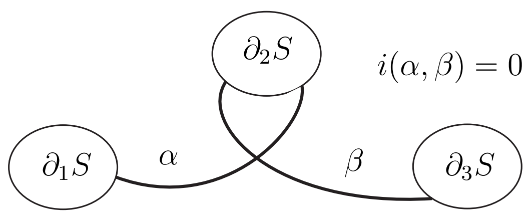


Figure 2.10: The case when $i(\alpha,\beta) = 0$.

A curve is *boundary-parallel*, if $\gamma \rightsquigarrow \partial_1 S$ for some boundary component $\partial_1 S$ of S. A curve is *essential*, if it is neither null-homotopic nor boundary-parallel. A set of essential curves are minimally intersecting, if the curves are transversely intersecting, and there exists no bigon formed by the curves. A set of essential curves $\mathcal{H}$ *fills* the surface, if they are minimally intersecting, and $S \setminus (\cup \mathcal{H})$ is a union of disks.

In Definition 2.9, we defined a *label-reading* of a curve with respect to a labeled graph with transversely oriented edges on a surface. We extend this notion to study maps from the fundamental groups of surfaces into right-angled Artin groups. This approach was taken in [CW04], to which we owe most of the definitions in this section.

Definition 2.34 (label-reading pair)**.** Let S be a surface, and Γ be a graph.

(1) Let $\mathcal{H}$ be a set of transversely oriented curves in general positions on S, and λ be a map from $\mathcal{H}$ into $V(\Gamma)$. Suppose for any $\alpha, \beta \in \mathcal{H}$, α and β intersect only if $\lambda(\alpha)$ and $\lambda(\beta)$ are equal or adjacent. Then $(\mathcal{H}, \lambda)$ is called a *label-reading pair on* S *with the underlying graph* Γ, and λ is called a *labeling*. Also, for $a \in V(\Gamma)$, a curve in $\lambda^{-1}(a)$ is called an *a-curve*.

(2) A label-reading pair $(\mathcal{H}, \lambda)$ on S is *cellular*, if $\mathcal{H}$ fills the surface and any two curves of the same label do not intersect.

(3) Let $(\mathcal{H}, \lambda : \mathcal{H} \to V(\Gamma))$ be a cellular label-reading pair. $\mathcal{H}$ induces a CW-complex structure on S as follows.

   (i) 0-cells correspond to the intersection points of the curves in $\mathcal{H}$ or in ∂S.

   (ii) 1-cells are the intervals on the curves in $\mathcal{H}$ or on ∂S, bounded by the 0-cells.

   (iii) 2-cells are the disks on S bounded by 1-cells.

   The dual to this CW-complex is a cubical complex, written as $X(S, \mathcal{H})$. Each edge of $X(S, \mathcal{H})$ is assumed to have an orientation and a label, consistent with the transverse orientation and the label of a curve in $\mathcal{H}$. $X(S, \mathcal{H})$, with the orientations and the labels of its edges, is called the *cubed structure on* S *induced by* $(\mathcal{H}, \lambda)$.

Definition 2.35 (label-reading map)**.** Let S be a surface and Γ be a graph.

(1) Fix a basepoint x_0 on S. Suppose $(\mathcal{H}, \lambda)$ be a label-reading pair with the underlying graph Γ. Then one can define an *label-reading map* $\phi : \pi_1(S) \to A(\Gamma)$ *associated with* $(\mathcal{H}, \lambda)$ by $\phi[\gamma] = w_\gamma$, where γ is a closed curve based at x_0, transversely intersecting with the curves in $\mathcal{H}$, and w_γ denotes the label-reading of γ with respect to $(\mathcal{H}, \lambda)$.

(2) Suppose $(\mathcal{H}, \lambda)$ is a cellular label-reading pair. The unique cubical map $f : X(S, \mathcal{H}) \to X_\Gamma$ respecting the orientation and the label of each edge, is called the *cubical map associated with* $(\mathcal{H}, \lambda)$.

Remark 2.36. From the conditions of the label-reading pair, an associated label-reading map $\phi : \pi_1(S) \to A(\Gamma)$ is a group homomorphism, well-defined by the relation $\phi[\gamma] = w_\gamma$ ([CW04]). While an associated label-reading map is determined up to the choice of the basepoint, the associated label-reading map is unique.

Definition 2.37 (equivalence of label-reading pairs)**.** (1) Two group homomorphisms $\phi_1, \phi_2 : G \to H$ are *equivalent*, if there exists an inner automorphism $\alpha : H \to H$ such that $\phi_1 = \alpha \circ \phi_2$.

(2) Let Γ be a graph, and S be a surface. Two label-reading pairs $(\mathcal{H}, \lambda)$ and $(\mathcal{H}', \lambda')$ are *equivalent*, if their associated label-reading maps are equivalent.

A change of the basepoint does not change the equivalence class of the associated label-reading map. So, to verify whether two label-reading maps are equivalent, the basepoints can be chosen arbitrarily.

If a label-reading pair $(\mathcal{H}', \lambda')$ is obtained from $(\mathcal{H}, \lambda)$ by removing non-essential curves, then the associated label-readings maps are equivalent ([CW04]). One can also remove bigons formed by two curves in $\mathcal{H}$, without changing the equivalence class of the label-reading pair. See Lemma 2.38 for more precise description of certain homotopies that do not change the equivalence class of the label-reading pairs (Figure 2.11).

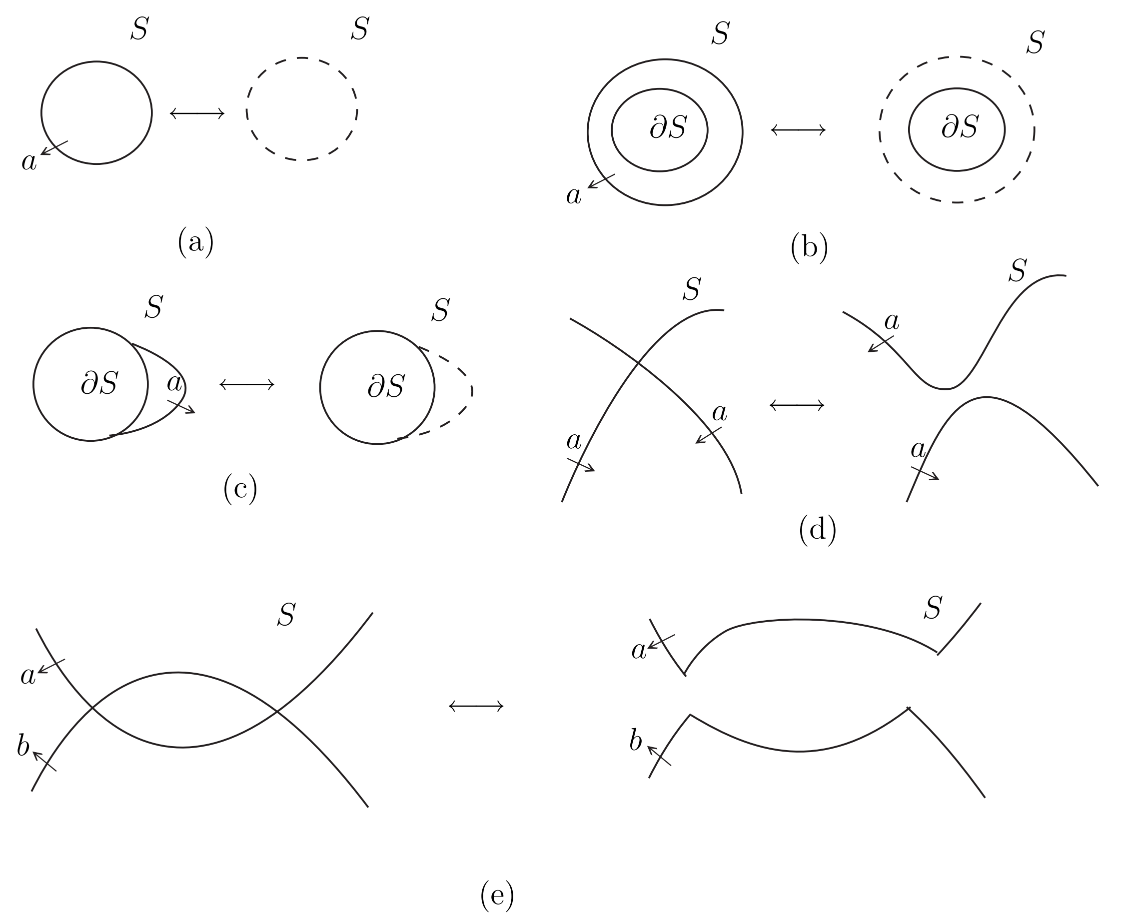


Figure 2.11: Homotopies on a label-reading pair, that do not change the equivalence class. (a) Removing a null-homotopic curves. (b), (c) Removing boundary-parallel curves. (d) Removing intersections of curves of the same label. (e) Removing bigons. Note that (e) is allowed only when a and b are equal or adjacent in Γ.

Suppose Γ be a graph, S be a surface, and $(\mathcal{H}, \lambda)$ be a label-reading pair. Let S' be a subsurface of S such that $\partial S'$ is in general position of the curves in $\mathcal{H}$. Then there exists an *induced label-reading pair* $(\mathcal{H}', \lambda')$ *on* S', where $\mathcal{H}'$ is the set of the components of $(\cup\mathcal{H}) \cap S'$, inheriting the transverse orientation and labeling (λ') from $(\mathcal{H}, \lambda)$.

Lemma 2.38 (disk swap)**.** *Let Γ be a graph and S be a surface. Let A be either*

(i) a disk in the interior of S,

(ii) a disk bounded by a properly embedded arc and an interval in ∂S, or

(iii) the annulus bounded by a boundary-parallel simple closed curve and a boundary component.

Consider two label-reading pairs $(\mathcal{H}_1, \lambda_1)$ and $(\mathcal{H}_2, \lambda_2)$, and their associated label-reading maps ϕ_1 and ϕ_2, respectively. Suppose that for each $i = 1, 2$, the curves in $\mathcal{H}_i$ are in general position with ∂A, and their induced label-reading pairs on $S \setminus A$ are equal. Then $(\mathcal{H}_1, \lambda_1)$ and $(\mathcal{H}_2, \lambda_2)$ are equivalent.

Proof) Up to equivalence, we may choose the basepoint outside A. Then for any $[\alpha] \in \pi_1(S)$, α can be homotoped into $S \setminus A$, so that $\phi_1([\alpha]) = \phi_2([\alpha])$.□

We note the following classical lemmas (See [SW79]).

Lemma 2.39. *Let X be a 2-dimensional CW-complex, and Y be any space. Then for any map $\phi : \pi_1(X) \to \pi_1(Y)$, there exists a map $f : X \to Y$ such that $f_* = \phi$.*□

Lemma 2.40 (transversality)**.** *Let X_0, X_1 and X_2 be finite CW-complexes. Suppose there exist π_1-injective embeddings $\phi_0 : X_2 \times \{0\} \to X_0$ and $\phi_1 : X_2 \times \{1\} \to X_1$.*

Define X to be the graph of spaces, equal to the quotient space $X_0 \cup_{\phi_0} (X_2 \times I) \cup_{\phi_1} X_1$, where ϕ_0 and ϕ_1 are amalgamating maps. Orient the interval $I = [0,1]$. Suppose $f : S \to X$ is a map from a surface S. Then there exists a map $g : S \to X$, homotopic to f, such that $g^{-1}(X_2 \times \{\frac{1}{2}\})$ is a set of disjoint, simple closed curves or properly embedded arcs, which carry transverse orientations induced by the orientation of the interval $[0,1]$. Henceforth, there exists an open regular neighborhood N of $g^{-1}(X_2 \times \{\frac{1}{2}\})$, such that each component of $S \setminus N$ is mapped into either X_0 or X_1 via the map g. The same conclusion holds, if we assume that X_0 and X_1 are identical. □

Now let S be a surface, Γ be a graph and $\phi : \pi_1(S) \to A(\Gamma)$ be any map. By Lemma 2.39, there exists $f : S \to X_\Gamma$ such that $f_* = \phi$. Choose a point p_q for each circle C_q corresponding to a vertex $q \in V(\Gamma)$, such that $p_q \notin X_\Gamma^{(0)}$. By transversality (Lemma 2.40), we may assume that $f^{-1}(p_q)$ is a collection of simple closed curves and properly embedded arcs, labeled by q and transversely oriented by the orientation of C_q. Let $\mathcal{H} = \cup_{q \in V(\Gamma)} f^{-1}(q)$ and define $\lambda : \mathcal{H} \to V(\Gamma)$ by $\lambda(\gamma) = q$ if $\gamma \in f^{-1}(q)$. Then ϕ is an associated label-reading map with respect to $(\mathcal{H}, \lambda)$ (Proposition 2.41).

This can be also seen as follows [CW04]. Let

$$\pi_1(S) =< x_1, x_2, \ldots, x_g, y_1, y_2, \ldots, y_g, d_1, d_2, \ldots, d_m | \prod_{i=1}^{g} [x_i, y_i] \prod_{i=1}^{m} d_i >$$

Here $d_1, d_2, \ldots, d_m$ correspond to the boundary components $\partial_1 S, \partial_2 S, \ldots, \partial_m S$ of S. Draw a dual van Kampen diagram Δ for the following word.

$$\prod_{i=1}^{g} [\phi(x_i), \phi(y_i)] \prod_{i=1}^{m} \phi(d_i)$$

For each i, glue $\partial\Delta$ along the pairs of words $\{\phi(x_i), \phi(x_i)^{-1}\}$ and also, $\{\phi(y_i), \phi(y_i)^{-1}\}$. For each i, "push" the interval $\phi(d_i)$ into the interior of Δ so that it becomes the boundary component $\partial_i S$ after gluing. Then one gets a set of transversely oriented and labeled curves $(\mathcal{H}, \lambda)$ on S, such that ϕ is an associated label-reading map. Moreover, if one has $\phi(d_i) = w_i'^{-1} w_i w_i'$ for some words w_i, w_i', then $(\mathcal{H}, \lambda)$ can be chosen so that any curve in $\mathcal{H}$ intersecting with the boundary component $\partial_i S$ is labeled by a letter (or its inverse) of w_i, by gluing the words w_i' and $w_i'^{-1}$ in our construction.

If ϕ is injective, $S \setminus \mathcal{H}$ is a union of disks after removing non-essential intersections and non-essential curves, and so, $(\mathcal{H}, \lambda)$ is cellular.

Proposition 2.41 (existence of a label-reading pair)**.** *Let S be a surface and Γ be a graph. Let $d_1, d_2, \ldots, d_m$ be the elements of $\pi_1(S)$ corresponding to the boundary components $\partial_1 S, \partial_2 S, \ldots, \partial_m S$ of S. Suppose $\phi : \pi_1(S) \to A(\Gamma)$ is a map, such that for each i, $\phi(d_i) = w_i'^{-1} w_i w_i'$, for some words w_i, w_i'.*

(1) There exists a label-reading pair $(\mathcal{H}, \lambda)$ on S with the underlying graph Γ such that ϕ is a label-reading map associated with $(\mathcal{H}, \lambda)$. Moreover, ϕ can be chosen so that, for each i, any curve in $\mathcal{H}$ intersecting with the boundary component $\partial_i S$ is labeled by a letter (or its inverse) of w_i.

(2) Suppose a label-reading map ϕ associated with $(\mathcal{H}, \lambda)$ is injective. Then there exists a label-reading map ϕ' associated with a cellular label-reading pair $(\mathcal{H}', \lambda')$ such that ϕ and ϕ' are equivalent.

By a *hyperbolic surface group*, we mean the fundamental group of a closed hyperbolic surface. The following is a well-known result ([SDS89, CW04]).

Theorem 2.42 ($A(C_n)$ contains a hyperbolic surface group, [SDS89])**.** *For any $n \geq$*

5, $A(C_n)$ contains a hyperbolic surface group.

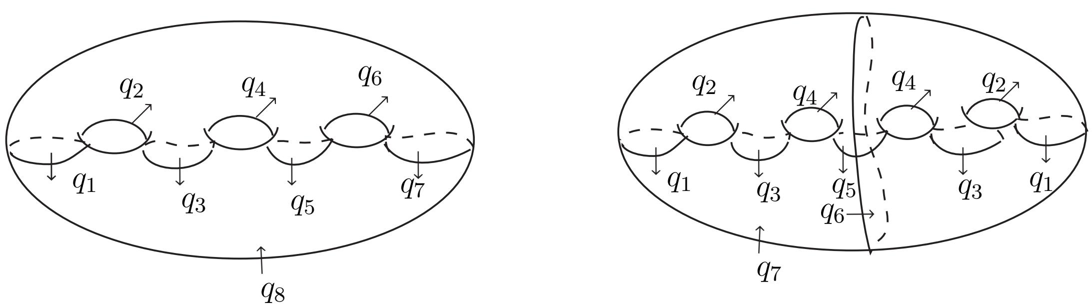
Proof)

Let the vertices of C_n be cyclically labeled by $q_1, q_2, \ldots, q_n$, and put $X = X_{C_n}$. Consider the case when n is even. A closed surface of the genus $\frac{n}{2} - 1$ contains a set $\mathcal{H}$ of n transversely oriented non-separating curves $\gamma_1, \gamma_2, \ldots, \gamma_n$ such that

$$i(\gamma_i, \gamma_j) = \begin{cases} 1, & |i - j| = 1 (\text{mod } n) \\ 0, & \text{otherwise} \end{cases}$$

as shown in Figure 2.12 (a). Let $\lambda(\gamma_i) = q_i$. Let $f_n : X(S, \mathcal{H}) \to X_\Gamma$ be the associated cubical map. Then $X(S, \mathcal{H})$ has four vertices, and the link at each vertex will be an n-cycle, which is mapped by f_n onto an induced n-cycle in $D(C_n)$. Hence f_n is a local isometry, and by Theorem 2.28, the label-reading map $\phi_n = (f_n)_*$ is injective.

For the case when n is odd, one has a similar construction with a closed surface of genus $n - 3$ (Figure 2.12 (b)).□



(a) $n = 8, g = 3$ (b) $n = 7, g = 4$

Figure 2.12: Examples of the standard embedding of a hyperbolic surface group into $A(C_n)$. (a) n is even. (b) n is odd.

The map $\phi_n : \pi_1(S) \to X_{C_n}$, used in the proof of Theorem 2.42 is called the *standard embedding of a hyperbolic surface group into $A(C_n)$.*

Remark 2.43. We briefly summarize two other proofs of Theorem 2.42.

(1) In [SDS89], Theorem 2.42 was proved by considering the following commutative diagram. Here, $\Gamma = C_n$.

$$\begin{array}{ccccccc}
p^{-1}(X_\Gamma^{(2)}) \cap I^n & \xrightarrow{f} & p^{-1}(X_\Gamma^{(2)}) & \longrightarrow & \tilde{X}_{K_n}^{(2)} & \longrightarrow & \tilde{X}_{K_n} \approx \mathbb{R}^n \\
 & & \downarrow & & \downarrow & & \downarrow p \\
 & & X_\Gamma^{(2)} & \longrightarrow & X_{K_n}^{(2)} & \longrightarrow & X_{K_n} \approx (S^1)^n
\end{array}$$

where $p : \tilde{X}_{K_n} \to X_{K_n}$ is a universal covering map so that $\tilde{X}_{K_n}$ is a cubical structure on $\mathbb{R}^n$. All the horizontal arrows denote embeddings and the vertical arrows denote covering maps. $f : p^{-1}(X_\Gamma^{(2)}) \cap I^n \to p^{-1}(X_\Gamma^{(2)})$ can be shown to be π_1-injective by a combinatorial argument. Theorem 2.42 follows from the observation that $p^{-1}(X_\Gamma^{(2)}) \cap I^n$ is a hyperbolic surface, for $n \geq 5$.

(2) Let $n \geq 5$ and P be the polygon on n vertices in the hyperbolic space such that each edge is a geodesic segment and all the dihedral angles are $\frac{\pi}{2}$. Then $C(C_n)$ is the reflection group with respect to P, which is discrete. So it has a finite index torsion-free subgroup (Selberg's lemma), which will be a hyperbolic surface group. In [DJ00], it is shown that $A(\Gamma)$ is commensurable with $C(D(\Gamma))$ for any Γ. Since $C(C_n)$ embeds into $C(D(C_n))$, it also follows that $A(C_n)$ contains a hyperbolic surface group [GLR04].

Chapter 3

On Label-Reading Maps

We have seen that the right-angled Artin group on a cycle of length at least 5 contains a hyperbolic surface group. In Section 3.1, we extend this result to show that any graph product of non-trivial groups on a cycle of length at least 5 contains a hyperbolic surface group (Theorem 3.6). For this, we realize an embedding of a hyperbolic surface group into a right-angled Artin group as a label-reading map, and examine the properties of the label-reading pairs. Section 3.2 will contain a technique of simplifying (*normalizing*) a given label-reading pair. This will play a crucial role in Chapter 5.

3.1 Surface subgroups of graph products of groups

We first prove the homotopy lifting property of a local isometry between cubed complexes. This is the key idea for the proof of the results in this section. For more details on edge-homotopy in a cubed complex, see [BH99].

Definition 3.1 (edge-homotopy)**.** Let X be a cubed complex, and γ be an edge-path in X.

(1) An *elementary homotopy* is a transformation sending γ to another edge-path γ' such that one of the following holds.

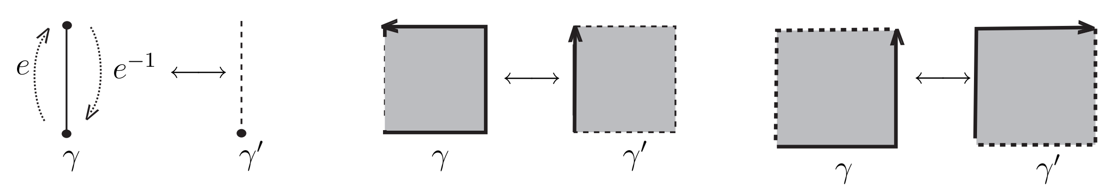
  (i) γ' is obtained by inserting or deleting a subpath of the form $e \cdot e^{-1}$ where e is an oriented edge in γ.

  (ii) One can write $\gamma = abc$ and $\gamma' = ab'c$ for some subpaths a, b, c, b' such that $b^{-1}b'$ is the boundary of a square.

An elementary homotopy is *non-increasing*, if the length of γ' is equal or smaller than that of γ (Figure 3.1).

(2) An *edge-homotopy* is a finite sequence of elementary homotopies. If each elementary homotopy is non-increasing, then we say that the edge-homotopy is *non-increasing*.

In a cubed complex if two edge-paths are homotopic relative to their boundaries, then there exists an edge-path homotopy from one to the other.



(a) (b) (c)

Figure 3.1: Elementary homotopies in a cubical complex. For each of (a),(b) and (c), the homotopy from the left to the right is non-increasing.

Lemma 3.2 (homotopy lifting property)**.** *Let $f : X \to Y$ be a local isometry between cubed complexes, and γ_0 and γ_1 be two edge-paths in Y. Assume that there exists a non-increasing edge-homotopy from γ_0 to γ_1. Suppose there exists an edge-path $\tilde{\gamma}_0$ in X such that $f(\tilde{\gamma}_0) = \gamma_0$. Then there exists another edge-path $\tilde{\gamma}_1$ in X and a non-increasing edge-homotopy from $\tilde{\gamma}_0$ to $\tilde{\gamma}_1$.*

Proof)

First, consider the case when γ_1 is obtained by a non-increasing elementary homotopy from γ_0.

Suppose γ_1 is obtained by deleting a subpath of the form ee^{-1} in γ_0, and let $\tilde{e}\tilde{e}'$ be the lifting of that subpath in X. Let $\tilde{x}$ be the terminal point of $\tilde{e}$. Since $f_{\tilde{x}} : \mathrm{Link}_X(\tilde{x}) \to \mathrm{Link}_Y(f(\tilde{x}))$ is injective, and $f(\tilde{e}) = e = f(\tilde{e}')^{-1}$, we have $\tilde{e}' = \tilde{e}^{-1}$. So $\tilde{\gamma}_0$ has a subpath $\tilde{e}\tilde{e}^{-1}$, and $\tilde{\gamma}_1$ is obtained by deleting this subpath.

Suppose γ_1 is obtained from γ_0 by replacing a concatenation of edges $a \cdot b \cdot c$ by an edge d so that $a \cdot b \cdot c \cdot d^{-1}$ is a boundary of a square C in Y. Let $\tilde{a} \cdot \tilde{b} \cdot \tilde{c}$ be the lifting of $a \cdot b \cdot c$ contained in $\tilde{\gamma}_0$. Since f is a local isometry and a and b belong to C, $\tilde{a}$ and $\tilde{b}$ belong to a square $\tilde{C} \subseteq X$. Since $f(\tilde{C}) = C$ and f is injective on links, $\tilde{c} \subseteq \tilde{C}$. Now let $\tilde{d}$ be the edge of $\tilde{C}$ such that $\tilde{a} \cdot \tilde{b} \cdot \tilde{c}$ is homotopic to $\tilde{d}$. Then we obtain the desired $\tilde{\gamma}_1$ by replacing $\tilde{a} \cdot \tilde{b} \cdot \tilde{c}$ in $\tilde{\gamma}_0$ by $\tilde{d}$.

The case when the elementary homotopy is replacing a subpath of length 2 in a square by another subpath of length 2 in the same square is similar.

Now the general case follows by an induction on the length of the sequence of elementary homotopies applied to γ_0.□

Let G be a group, and $g \in G$. We let $o_G(g)$ (or, $o(g)$ if the meaning is clear from the context) denote the order of g. $\mathbb{Z}_m$ denotes the finite cyclic group of order m. Let $\mathbb{Z}_\infty = \mathbb{Z}$, by convention. Recall that a word in a graph product of groups is in a normal form if the word cannot be shortened by applying one of the elementary reductions given in Definition 2.13.

Lemma 3.3. *Let Γ be a graph. To each $q \in V(\Gamma)$, we assign a (finite or infinite) cyclic group, the generator of which is denoted by $\bar{q}$. Let $p : A(\Gamma) \to G = GP(\Gamma, \{\langle \bar{q} \rangle\}_{q \in V(\Gamma)})$ be the natural quotient map so that $p(q) = \bar{q}$ for each $q \in V(\Gamma)$. If $w \in \ker p \setminus \{1\}$ is in a normal form, then w contains a subword of the form $q^{k \cdot o(\bar{q})}$ for some $k \neq 0$ and $q \in V(\Gamma)$.*

Proof) By Lemma 2.14, one can write $w = \prod_{i=1}^m a_i^{p_i}$, such that

(i) for each i, $a_i \in V(\Gamma)$ and $p_i \in \mathbb{Z} \setminus \{0\}$

(ii) if $i < j$ and $a_i = a_j$, then there exists $i < k < j$ such that a_k is not adjacent to a_i.

Since $w \in \ker p$, $p(w) =_G \prod_{i=1}^m \overline{a_i}^{p_i} =_G 1$. By Lemma 2.14 again, $\prod_{i=1}^m \overline{a_i}^{p_i}$ is in a normal form unless $\overline{a_i}^{p_i} =_G 1$ for some i. This implies that $o(a_i)$ divides p_i for some i. □

Lemma 3.4. *Suppose $(\mathcal{H}, \lambda)$ is a cellular label-reading pair on a surface S with the underlying graph Γ such that the associated cubical map $f : X(S, \mathcal{H}) \to X_\Gamma$ is a local isometry. Let γ be a closed curve transversely intersecting with $\mathcal{H}$, and w be a reduced word for a cyclic conjugation of w_γ. Then there exists a closed curve $\hat{\gamma}$ such* *that $w_{\hat{\gamma}} = w$. If $w =_{A(\Gamma)} w_\gamma$, then the basepoint of $\hat{\gamma}$ can be chosen to be same as γ.*

Proof) Note that any cyclic conjugation of w_γ is a label-reading of another curve γ' which is same as γ except for the basepoint. Now let w be a reduced word for $w_{\gamma'}$. Then w and $w_{\gamma'}$ correspond to edge-paths in X_Γ, which are edge-homotopic to each other. Since w can be obtained from $w_{\gamma'}$ by elementary reductions (Theorem 2.15), the edge-homotopy can be chosen to be non-increasing. By Lemma 3.2, γ' is homotopic to another curve $\hat{\gamma}$ such that $w_{\hat{\gamma}} = w$. □

Lemma 3.5. *For any $n \geq 5$, there exists an injective map ϕ from a hyperbolic surface group into the right-angled Artin group on C_n such that any reduced word w in the image of ϕ does not contain a subword in $\{q^{\pm 2} : q \in V(C_n)\}$.*

Proof) Let ϕ be the standard embedding of a hyperbolic surface group into $A(C_n)$ as in the proof of Theorem 2.42. Realize ϕ as a label-reading map as in Figure 2.12.

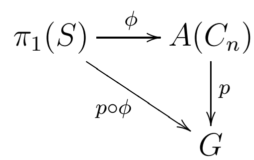
Suppose γ is any closed curve and w is a reduced word for w_γ. By Lemma 3.4, there exists another curve $\hat{\gamma}$ with the same basepoint as γ, such that $w_{\hat{\gamma}} = w$. On the other hand, one can see that the label-reading of any closed curve on S cannot contain a subword of the form $q^{\pm 2}$ for $q \in V(\Gamma)$ from the construction (Figure 2.12). □

For a group homomorphism f, we let $\ker f$ and $\mathrm{im}\ f$ denote the kernel and the image of f, respectively. The following is the main theorem of this section.

Theorem 3.6 (the graph product on a long cycle)**.** *The graph product of any non-trivial groups on a cycle of length at least 5 contains a hyperbolic surface group.*

Proof) Let G be the graph product of non-trivial groups on a cycle of length at least 5. Note that any non-trivial group contains a non-trivial cyclic subgroup. So by

Corollary 2.20 (1), we may assume that each vertex group is cyclic with its generator denoted by $\bar{q}$, for each $q \in V(C_n)$. Let $p : A(C_n) \to G$ be the natural quotient map, so that $p(q) = \bar{q}$ for each $q \in V(C_n)$. Let ϕ_n be the standard embedding of $\pi_1(S)$ into $A(C_n)$ for a closed hyperbolic surface S as in the proof of Lemma 3.5.



Suppose $w \in \ker p \setminus \{1\}$ is in a normal form. Then w contains a subword $q^{\pm 2}$ for some $q \in V(\Gamma)$ (Lemma 3.3). Then by Lemma 3.4, $w \notin \text{im } \phi$. So $\text{im } \phi \cap \ker p = \{1\}$ and $p \circ \phi : \pi_1(S) \to G$ is injective.□

3.2 Normalized label-reading pairs

Given a label-reading pair on a surface, we will consider a simplification (called, *normalization*) of the pair, without changing the equivalence class of the associated label-reading map. In Chapter 5, we will use this method to prove results on non-embeddability of closed hyperbolic surface groups into certain right-angled Artin groups.

Definition 3.7 (induced simple closed curve)**.** Let S be a surface, and $\mathcal{B}_0$ be a set of disjoint properly embedded arcs on S.

(1) A *set of arc representatives* is a maximal set $\bar{\mathcal{B}}_0 \subseteq \mathcal{B}_0$ such that two distinct arcs in $\bar{\mathcal{B}}_0$ are not homotopic. For each $\alpha \in \mathcal{B}_0$, the *arc representative for* α is the unique $\bar{\alpha} \in \bar{\mathcal{B}}_0$ such that $\alpha \sim \bar{\alpha}$.

(2) For each arc $\bar{\alpha} \in \bar{\mathcal{B}}_0$, we choose an embedding $\eta_{\bar{\alpha}} : I \times [-1, 1] \to S$, which is called a *strip*, such that the following conditions hold.

(i) $\eta_{\bar{\alpha}}(I \times \{0\}) = \bar{\alpha}$.

(ii) $\eta_{\bar{\alpha}}(I \times \{s\})$ is a properly embedded arc for each $s \in [-1, 1]$.

(iii) Suppose $\alpha \in \mathcal{B}_0$ is homotopic to $\bar{\alpha} \in \bar{\mathcal{B}}_0$. Then $\alpha \subseteq \eta_{\bar{\alpha}}(I \times (-1, 1))$.

(iv) For two distinct $\bar{\alpha}, \bar{\beta} \in \bar{\mathcal{B}}_0$, im $\eta_{\bar{\alpha}}$ and im $\eta_{\bar{\beta}}$ are disjoint.

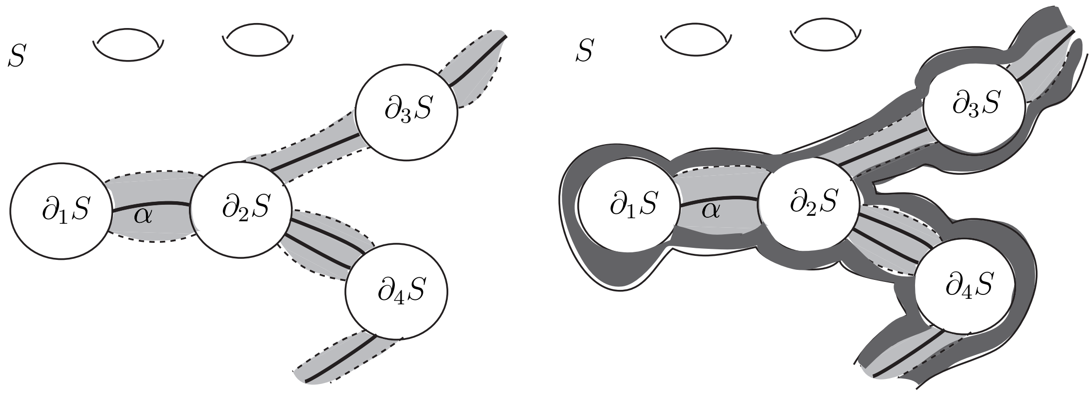
For any $\alpha \in \mathcal{B}_0$, we let $\eta_\alpha = \eta_{\bar{\alpha}}$ where $\bar{\alpha}$ is the arc representative for α.

(3) A *channel* is a connected component of $\partial S \cup (\cup_{\beta \in \mathcal{B}_0} \text{im } \eta_\beta)$. For $\alpha \in \mathcal{B}_0$, we denote the channel containing α by $\text{chan}(\alpha)$. A closed regular neighborhood of $\text{chan}(\alpha)$ is called a *channel surface of* α, and denoted by $\widetilde{\text{chan}}(\alpha)$.

(4) Let Y be a channel. A component of the frontier of Y is called an *induced simple closed curve*. (So, an induced simple closed curve is a component of $\partial(\overline{S \setminus Y}) \setminus \partial S$). If an induced simple closed curve $\hat{\alpha}$ intersects with the strip containing $\alpha \in \mathcal{B}_0$, then we say that $\hat{\alpha}$ *is an induced simple closed curve of* α, and also, that α *follows* $\hat{\alpha}$.

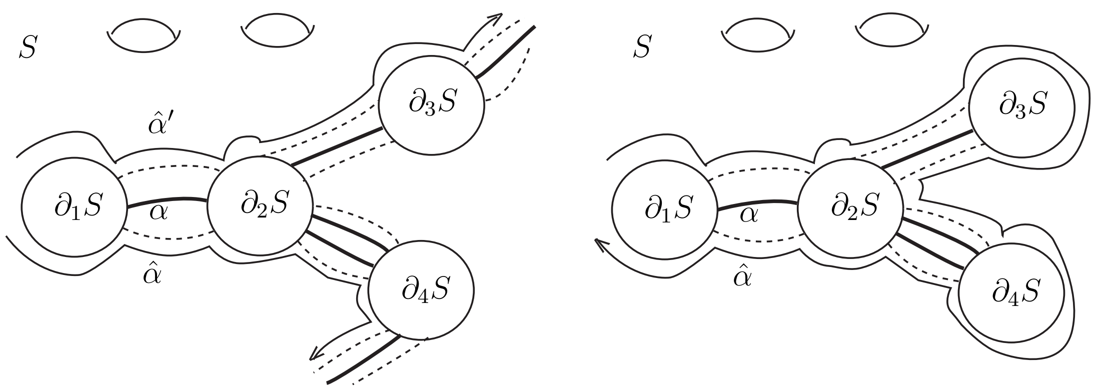
(5) An arc $\alpha \in \mathcal{B}_0$ is *one-sided*, if $\eta_\alpha(I \times \{-1\})$ and $\eta_\alpha(I \times \{1\})$ are contained in the same induced simple closed curve.

For examples, see Figure 3.2 and 3.3.



(a) (b)

Figure 3.2: Strips and channels. Here, the bold curves are properly embedded arcs in $\mathcal{B}_0$, and the dotted curves denote boundaries of strips $\{\eta_\gamma : \gamma \in \mathcal{B}_0\}$. In (a), the shaded region, along with the boundary components intersecting with the region, is chan(α). In (b), the darker region is $\widetilde{\text{chan}}(\alpha) \setminus \text{chan}(\alpha)$.



(a) (b)

Figure 3.3: Induced simple closed curves. The curves with arrows are showing homotopy classes of induced simple closed curves (an actual induced simple closed curve consists of intervals in ∂S and in the boundaries of strips). Note that in (a), α is not one-sided if $\hat{\alpha}$ and $\hat{\alpha}'$ are different. But in (b), α is one-sided.

Remark 3.8. (1) When we have a set of disjoint properly embedded arcs, we always assume that a set of arc representatives and strips are chosen *a priori.*

(2) Any induced simple closed curve can be written as a concatenation of paths

$$\hat{\alpha} = \alpha'_1 \cdot \delta_1 \cdot \alpha'_2 \cdot \delta_2 \cdot \alpha'_3 \cdots \alpha'_r$$

where for each i, $\delta_i \subseteq \partial S$ and α'_i is a properly embedded arc homotopic to some $\alpha_i \in \mathcal{B}_0$.

(3) Let $\hat{\alpha}$ be an induced simple closed curve of α. Consider the surface $S' = \overline{S \setminus \text{chan}(\alpha)}$. An inward unit normal vector field on $\hat{\alpha}$, considered as a boundary component of S', will give rise to an annulus in S'. So one can write $\widetilde{\text{chan}}(\alpha) = \text{chan}(\alpha) \cup A$, where $A = \sqcup_i A_i$ is a disjoint union of annuli $\{A_i\}$. The intersection of each A_i with $\text{chan}(\alpha)$ is an induced simple closed curve (each A_i corresponds to a darker region in Figure 3.2 (b)). Whenever needed, we implicitly assume that each annulus is sufficiently narrow for the purpose of the argument.

Lemma 3.9 (unique frontier of $\widetilde{\text{chan}}(\alpha)$)**.** *Let $\mathcal{B}_0$ be a set of disjoint properly embedded arcs on a surface S, such that each arc in $\mathcal{B}_0$ is one-sided. Let $\alpha \in \mathcal{B}_0$. Then $\widetilde{chan}(\alpha)$ has only one boundary component that is not in the boundary of S, and this boundary component separates S.*

Proof) We follow the notations in Definition 3.7. Fix $\alpha \in \bar{\mathcal{B}}_0$, and we let $\hat{\alpha}$ denote the (unique) induced simple closed curve of α.

We denote the boundary components of S by $\partial_1 S, \partial_2 S, \ldots, \partial_m S$. We say that a boundary component of S or a strip is *good*, if it intersects with $\hat{\alpha}$.

Claim 1. *If a strip is good, then so is any boundary component of S that the strip intersects.*

Suppose a strip $\eta_\beta : I \times [-1,1] \to S$ is good for some $\beta \in \mathcal{B}_0$. Then $\eta_\beta(I \times \{-1,1\}) \cap \hat{\alpha} \neq \varnothing$. Since β is one-sided, $\eta_\beta(I \times \{-1,1\}) \subseteq \hat{\alpha}$, and in particular, $\eta_\beta(\{0,1\} \times \{-1,1\}) \subseteq \hat{\alpha}$. Hence the boundary components of S that intersect with the strip η_β intersects with $\hat{\alpha}$.

Claim 2. *If a boundary component of S is good, then so is any strip intersecting with it.*

If $\partial_i S$ is good, then there exists a strip that contains an intersection point of $\hat{\alpha}$ and $\partial_i S$. So $\partial_i S$ is intersecting with at least one good strip. Suppose $\partial_i S$ intersects with a strip that is not good. By choosing a nearest pair, on $\partial_i S$, of a good strip X and a strip X' that is not good, one can find an induced simple closed curve $\hat{\beta}$ that intersects with X and X' and also with $\partial_i S$ (Figure 3.4). Since X is good, there exists a unique induced simple closed curve intersecting with X, namely $\hat{\alpha}$. So $\hat{\beta} = \hat{\alpha}$ and X' is also good, which is a contradiction.

By Claim 1 and 2, it immediately follows that

Claim 3. *If two boundary components of S are connected by an arc in $\mathcal{B}_0$, and one boundary component is good, then so is the other one.*

From Claim 3, we see that all the boundary components and the strips in $\mathrm{chan}(\alpha)$ are good, since $\mathrm{chan}(\alpha)$ is connected.

Now choose any component κ of $\partial\widetilde{\mathrm{chan}}(\alpha) \setminus \partial S$. There exists a unique induced simple closed curve $\hat{\beta} \subseteq \mathrm{chan}(\alpha)$ such that κ and $\hat{\beta}$ bound an annulus contained in $\overline{\widetilde{\mathrm{chan}}(\alpha) \setminus \mathrm{chan}(\alpha)}$ (Remark 3.8). Since $\hat{\beta}$ intersects with $\mathrm{chan}(\alpha)$, $\hat{\beta}$ intersects with a good strip, and since any arc in $\mathcal{B}_0$ is one-sided, $\hat{\beta} = \hat{\alpha}$. Hence κ is uniquely determined by $\hat{\alpha}$. This proves that $\partial\widetilde{\text{chan}}(\alpha)$ contains only one component that is not in ∂S.

From the construction $S \setminus \widetilde{\text{chan}}(\alpha)$ has an interior point, and so, $\widetilde{\text{chan}}(\alpha) \neq S \setminus \{\delta\}$. Any path from an interior point of $\widetilde{\text{chan}}(\alpha)$ to an interior point $S \setminus \widetilde{\text{chan}}(\alpha)$ will intersect δ, since $\partial\widetilde{\text{chan}}(\alpha) \setminus \partial S = \{\delta\}$. Hence δ separates S. □

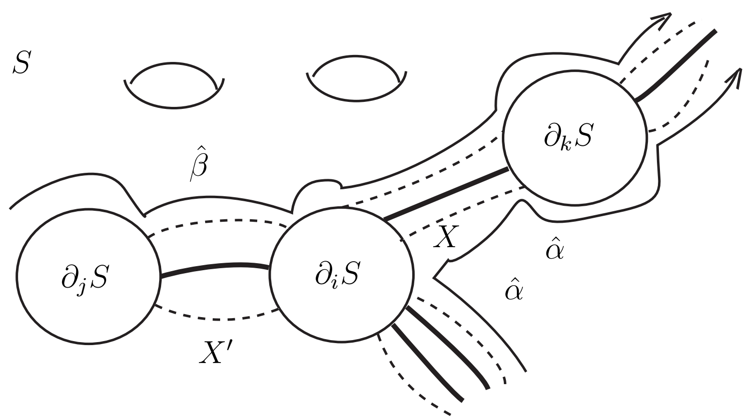


Figure 3.4: Proof of Claim 2 in Lemma 3.9.

We denote the interior of a surface S by $\text{Int}(S)$.

Lemma 3.10. *Let S be a surface. Suppose $\mathcal{B}_0$ is a set of disjoint properly embedded arcs on S such that each arc is one-sided. Let $\partial_0 S$ be the union of the boundary components of S that intersect with arcs in $\mathcal{B}_0$. Let $\hat{\alpha}$ be an induced simple closed curve of $\alpha \in \mathcal{B}_0$.*

(1) For any curve γ on S, either $\gamma \cap (\cup\mathcal{B}_0) \neq \varnothing$, or γ is a properly embedded arc intersecting $\partial_0 S$, or $\gamma \rightsquigarrow S \setminus \widetilde{chan}(\alpha)$.

(2) Suppose $\hat{\alpha}$ is null-homotopic. Then $\partial_0 S = \partial S$, and any essential simple closed curve on S intersects with an arc in $\mathcal{B}_0$.

Proof) Let $\bar{\mathcal{B}}_0$ denote a set of arc representatives.

(1) Suppose $\gamma \cap (\cup\mathcal{B}_0) = \varnothing$ and $\gamma \cap \partial_0 S = \varnothing$.

Case 1. γ is a simple closed curve.

We have

$$\gamma \subseteq S \setminus (\cup\mathcal{B}_0) \subseteq S \setminus (\cup\bar{\mathcal{B}}_0) \sim S \setminus (\cup_{\beta\in\mathcal{B}} \text{ im } \eta_\beta) \subseteq S \setminus \text{chan}(\alpha) \sim S \setminus \widetilde{\text{chan}}(\alpha)$$

The first homotopy is obtained by enlarging each arc in $\bar{\mathcal{B}}_0$ to a strip, and the second homotopy is retracting each annulus discussed in Remark 3.8 (3) onto a circle.

Case 2. γ is a properly embedded arc.

The argument for this case is almost the same as Case 1. One has only to show that that there exists a homotopy that maps γ into $S \setminus \widetilde{\text{chan}}(\alpha)$, leaving the endpoints on ∂S. For this, we choose $\widetilde{\text{chan}}(\alpha)$ as a sufficiently small regular neighborhood of chan(α) such that $\partial\gamma \cap \widetilde{\text{chan}}(\alpha) = \varnothing$. This is possible since γ does not intersect $\partial_0 S$. Then the homotopies in Case 1 do not move $\partial\gamma$. Note that in (1) we did not use the assumption that each arc is one-sided.

(2) Supose $\hat{\alpha} \sim 0$. From Lemma 3.9, there exists $\kappa \sim \hat{\alpha}$ such that $\partial\widetilde{\text{chan}}(\alpha) \subseteq \{\kappa\} \cup \partial_0 S$. Since κ separates, one can write $S = \widetilde{\text{chan}}(\alpha) \cup S'$ such that $\widetilde{\text{chan}}(\alpha) \cap S' = \kappa$. $\widetilde{\text{chan}}(\alpha) \not\sim 0$, since $\partial_i S \not\sim 0$. Hence S' is a disk, and $\partial S \subseteq \partial\widetilde{\text{chan}}(\alpha)$. So $\partial\widetilde{\text{chan}}(\alpha) = \{\kappa\} \cup \partial S$, and $\partial_0 S = \partial S$.

Let γ be any simple closed curve, not intersecting with any arc in $\mathcal{B}_0$. By (1), $\gamma \rightsquigarrow S \setminus \widetilde{\text{chan}}(\alpha) \subseteq S'$. This implies that γ is not essential. $\square$

If α and α' are homotopic properly embedded arcs, then a transverse orientation of α determines a unique transverse orientation of α', which is preserved by the homotopy. In this case, we say that the transverse orientation of α' *is induced by* that of α.

If α is a subarc of β which is a simple closed curve or a properly embedded arc, then a transverse orientation of α uniquely determines that of β so that the transverse orientation coincides on α. We say then, the transverse orientation of β *is induced by* (or, *inherits*) that of α.

Definition 3.11 (induced arc)**.** (1) Let $\mathcal{B}_0$ be a set of disjoint properly embedded arcs in S. Let $\hat{\alpha}$ be an induced simple closed curve $\alpha \in \mathcal{B}_0$. An *induced arc of* α is a path $\beta \subseteq \hat{\alpha}$, so that β can be written as a concatenation

$$\beta = \alpha'_1 \cdot \delta_1 \cdot \alpha'_2 \cdot \delta_2 \cdots \alpha'_r$$

where each α'_i is homotopic to a properly embedded arc α_i in $\mathcal{B}_0$, and $\delta_i \subseteq \partial S$. We say that α'_1 is the *initial arc* of β.

(2) Suppose each arc in $\mathcal{B}_0$ is transversely oriented. Then, in (1), this induces a transverse orientation of the initial arc α'_1, which β can inherit. This transverse orientation of β is said to be *according to its initial arc* α'_1. More generally, the *transverse orientation of* β *according to* α'_i is defined in a similar fashion.

Remark 3.12. An induced arc is not necessarily a properly embedded arc, but it is arbitrarily close to one (Figure 3.7 (a)).

Now let S be a surface and $(\mathcal{H}, \lambda)$ be a cellular label-reading pair with the underlying graph Γ. Write $\mathcal{H} = \mathcal{B} \sqcup \mathcal{C}$ where $\mathcal{B}$ is a set of properly embedded arcs and $\mathcal{C}$ is a set of simple closed curves. By *a strip, a channel, a channel surface, an induced simple* *closed curve and an induced arc* of $\alpha \in \mathcal{B}$ with respect to $(\mathcal{H}, \lambda)$, we mean those terms with respect to the set of properly embedded arcs $\mathcal{B} \cap \lambda^{-1}(\lambda(\alpha))$. Moreover, the boundary component $\hat{\alpha}'$ of $\widetilde{\text{chan}}(\alpha)$, which is homotopic to an induced simple closed curve $\hat{\alpha}$ of α, is always assumed to be transversely intersecting with $\mathcal{H}$, not intersecting with $\mathcal{C} \cap \lambda^{-1}(\lambda(\alpha))$.

Definition 3.13 (regular label-reading pair)**.** Let S be a hyperbolic surface with the boundary components $\partial_1 S, \partial_2 S, \ldots, \partial_m S$ (We put $m = 0$ if S is closed), and $(\mathcal{H}, \lambda)$ be a label-reading pair on S with the underlying graph Γ. We say that $(\mathcal{H}, \lambda)$ is *regular*, if the following conditions hold.

(i) Two curves of the same label do not intersect.

(ii) For any properly embedded arcs $\alpha, \beta \in \mathcal{H}$ intersecting with the same boundary component of S, $\lambda(\alpha)$ and $\lambda(\beta)$ are same or adjacent.

(iii) An associated label-reading map $\phi : \pi_1(S) \to A(\Gamma)$ is injective.

Remark 3.14. (1) Any label-reading pair with an injective associated label-reading map on a *closed* surface is regular, as long as two curves of the same label do not intersect.

(2) A regular label-reading pair is not necessarily cellular. Indeed, one can always generate an annulus component of $S \setminus (\cup\mathcal{H})$ by adding two homotopic simple closed curves with opposite transverse orientations and with the same label, inside $S \setminus (\cup\mathcal{H})$, without disturbing the regularity condition. The resulting label-reading pair will be equivalent to the original label-reading pair (Figure 3.5).

(3) In the case when a label-reading pair $(\mathcal{H}, \lambda)$ is cellular, then one can rephrase the condition (ii) in the Definition 3.13 as follows. For a proof, see Lemma 5.6.

> Let $f : X(S, \mathcal{H}) \to X_\Gamma$ be the associated cubical map. Then for any boundary component $\partial_i S$, there exists a complete graph $K \leq \Gamma$, such that $f(\partial_i S) \subseteq X_K$.

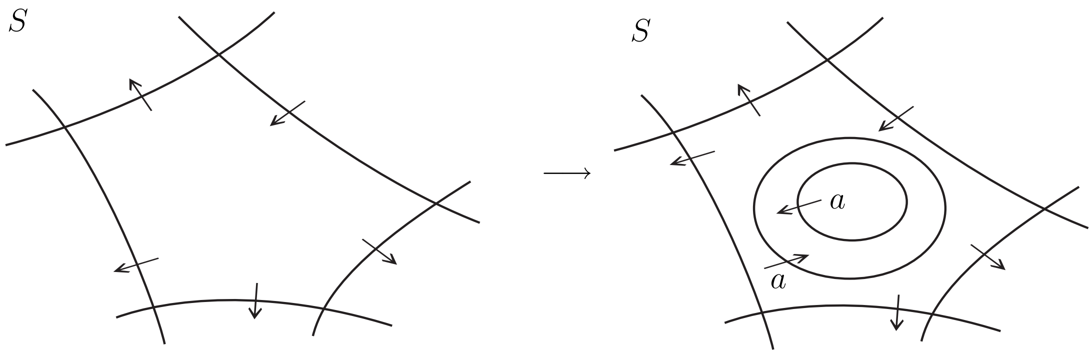


Figure 3.5: Generating an annulus component of $S \setminus \mathcal{H}$.

Definition 3.15 (normalized label-reading pair)**.** (1) Let $(\mathcal{H}, \lambda)$ be a regular label-reading pair on a hyperbolic surface S, and $\mathcal{B}$ be the set of properly embedded arcs in $\mathcal{H}$. Define the *complexity of* $\mathcal{H}$ to be the 4-tuple

$$c(\mathcal{H}) = (|(\cup\mathcal{B}) \cap \partial S|, \quad |\mathcal{H}/\sim|, \quad |\mathcal{B}/\sim|, \quad \sum_{\substack{(\alpha,\beta)\in\mathcal{H}\times\mathcal{H}\\ \alpha\neq\beta}} |\alpha \cap \beta|)$$

where $\sim$ denotes the homotopy equivalence relation on $\mathcal{H}$, and on $\mathcal{B}$. We denote the lexicographical ordering of the complexities by $\preceq$.

(2) We say that a regular label-reading pair $(\mathcal{H}, \lambda)$ is *normalized*, if for any other regular label-reading pair $(\mathcal{H}', \lambda')$ which is equivalent to $(\mathcal{H}, \lambda)$, $c(\mathcal{H}) \preceq c(\mathcal{H}')$.

It is obvious that any regular label-reading pair is equivalent to a normalized one.

Lemma 3.16 (normalization I)**.** *Let Γ be a graph, S be a hyperbolic surface, and $(\mathcal{H}, \lambda)$ be a normalized label-reading pair. $\mathcal{B}$ denotes the set of properly embedded arcs in $\mathcal{H}$.*

(1) If $\alpha, \beta \in \mathcal{B}$ have the same label, and intersect with the same boundary component $\partial_i S$, then the transverse orientation of α and that of β induce the same orientation on $\partial_i S$ at their intersections with $\partial_i S$. In particular, each properly embedded arc in $\mathcal{H}$ intersects with two distinct boundary components of S.

(2) Any two curves in $\mathcal{H}$ are minimally intersecting. This means that for any $\alpha \neq \beta \in \mathcal{H}$, $|\alpha \cap \beta| = i(\alpha, \beta)$.

(3) There does not exist any null-homotopic or boundary-parallel curves in $\mathcal{H}$.

Proof) (1) Let $a = \lambda(\alpha) = \lambda(\beta)$. Suppose the transverse orientations of α and β do not induce the same orientation on $\partial_i S$ at their intersection points $\{P_\alpha, P_\beta\}$. By choosing a nearest one among such a pair of intersection points on $\partial_i S$, We may assume a component of $\partial_i S \setminus \{P_\alpha, P_\beta\}$ does not intersect with any arc labeled by a. By Lemma 2.38, one can reduce $|(\cup\mathcal{B}) \cap \partial S|$ by 2 without changing the equivalence class of $(\mathcal{H}, \lambda)$ (Figure 3.6). Note that the new label-reading pair is still regular, for an intersection of two curves of the same label is not generated by this procedure. The second assertion follows from the orientability of S.

(2) and (3) immediately follow from Lemma 2.38. Note that one can always remove a bigon formed by two curves in $\mathcal{H}$. $\square$

Remark 3.17. By π_1-injectivity of an associated label-reading map, a simple closed curve $\gamma \subseteq S \setminus (\cup\mathcal{H})$ bounds a disk D. By (3), D does not contain a curve in $\mathcal{H}$. Since

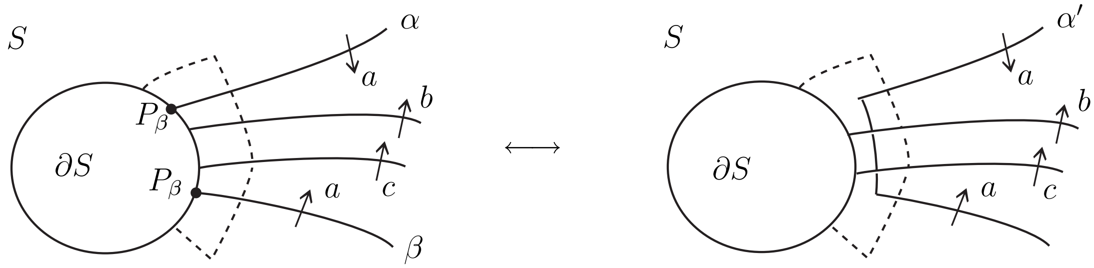


Figure 3.6: Reducing complexity. Here, the labels b, c of the curves between α and β are adjacent to a in Γ, by the regularity of $\mathcal{H}$. The change in the region bounded by the dotted curve is allowed by Lemma 2.38.

$\gamma \cap (\cup\mathcal{H}) = \varnothing$, it follows that $D \cap (\cup\mathcal{H}) = \varnothing$, and γ is null-homotopic in $S \setminus (\cup\mathcal{H})$. This proves that $\mathcal{H}$ fills the surface. Hence a normalized label-reading pair is always cellular ([CW04]).

For a given label-reading pair $(\mathcal{H}, \lambda)$ on S, we say that two curves α and β are *sufficiently close* if

(i) they are homotopic,

(ii) each of α and β either intersects $\cup\mathcal{H}$ transvsersely or is contained in $\mathcal{H}$, and

(iii) any curve in $\mathcal{H}$ transversely intersecting with one of α and β transversely intersects with the other.

Lemma 3.18 (eliminating an induced arc)**.** *Let S be a surface, $(\mathcal{H}, \lambda)$ be a normalized label-reading pair with the underlying graph Γ, and α be a properly embedded arc in $\mathcal{H}$. Suppose a properly embedded arc $\tilde{\alpha}$ is sufficiently close to an induced arc $\hat{\alpha}$ of α, with respect to $(\mathcal{H}, \lambda)$. Give a transverse orientation to $\hat{\alpha}$ according to its initial arc, and let $\tilde{\alpha}$ inherit this. Label $\tilde{\alpha}$ by $\lambda(\alpha)$. One gets another label-reading pair $(\mathcal{H}', \lambda')$ by adding the transversely oriented and labeled curve $\tilde{\alpha}$ to $(\mathcal{H}, \lambda)$. Then*

$(\mathcal{H}', \lambda')$ *is equivalent to another label-reading pair* $(\mathcal{H}'', \lambda'')$ *such that the set of simple closed curves of* $\mathcal{H}''$ *is same as that of* $\mathcal{H}$ *(and so, that of* $\mathcal{H}'$*), and the set of homotopy classes of properly embedded arcs in* $\mathcal{H}''$ *is contained in that of* $\mathcal{H}$*. Moreover,* $|\mathcal{H}'' \cap \partial S| \leq |\mathcal{H}' \cap \partial S|$*.*

Proof) Let $\mathcal{B}$ be the set of the properly embedded arcs in $\mathcal{H}$, and $a = \lambda(\alpha)$.

Suppose $\gamma \in \mathcal{H}$ intersects with $\tilde{\alpha}$. Since $\tilde{\alpha}$ is sufficiently close to $\hat{\alpha}$, either γ intersects with an a-arc, or a boundary component of S that meets a-arcs. Hence $\lambda(\gamma) \in \mathrm{Link}(a)$. Hence, $(\mathcal{H}', \lambda')$ is a label-reading pair, and two curves in $\mathcal{H}'$ of the same label do not intersect.

Write $\hat{\alpha}$ as a concatenation of paths (see Figure 3.7).

$$\hat{\alpha} = \alpha'_1 \cdot \delta_1 \cdot \alpha'_2 \cdot \delta_2 \cdot \alpha'_3 \cdots \alpha'_r$$

where for each i, $\delta_i \subseteq \partial S$ and α'_i is a properly embedded arc homotopic to $\alpha_i \in \mathcal{B} \cap \lambda^{-1}(a)$. Let $\partial_1 S$ be the boundary component of S that intersects both α_1 and α_2.

$\hat{\alpha}$ has the transverse orientation according to its initial arc α'_1. So, the transverse orientation of α'_2, as a subarc of $\hat{\alpha}$ will be opposite to the one induced by α_2 (Lemma 3.16 (1)). See Figure 3.7 (a). In this way, one sees that the transverse orientation of α'_i, as a subarc of $\hat{\alpha}$ will be opposite to the one induced by α_i if and only if i is even. Then one can obtain another label-reading pair $(\mathcal{H}'', \lambda'')$ by adding arcs homotopic to $\alpha_1, \alpha_3, \alpha_5, \ldots$ and removing arcs $\alpha_2, \alpha_4, \alpha_6, \ldots$ (Figure 3.7 (c)). By the construction, $|\mathcal{H}'' \cap \partial S| \leq |\mathcal{H}' \cap \partial S|$ is trivial. $\square$

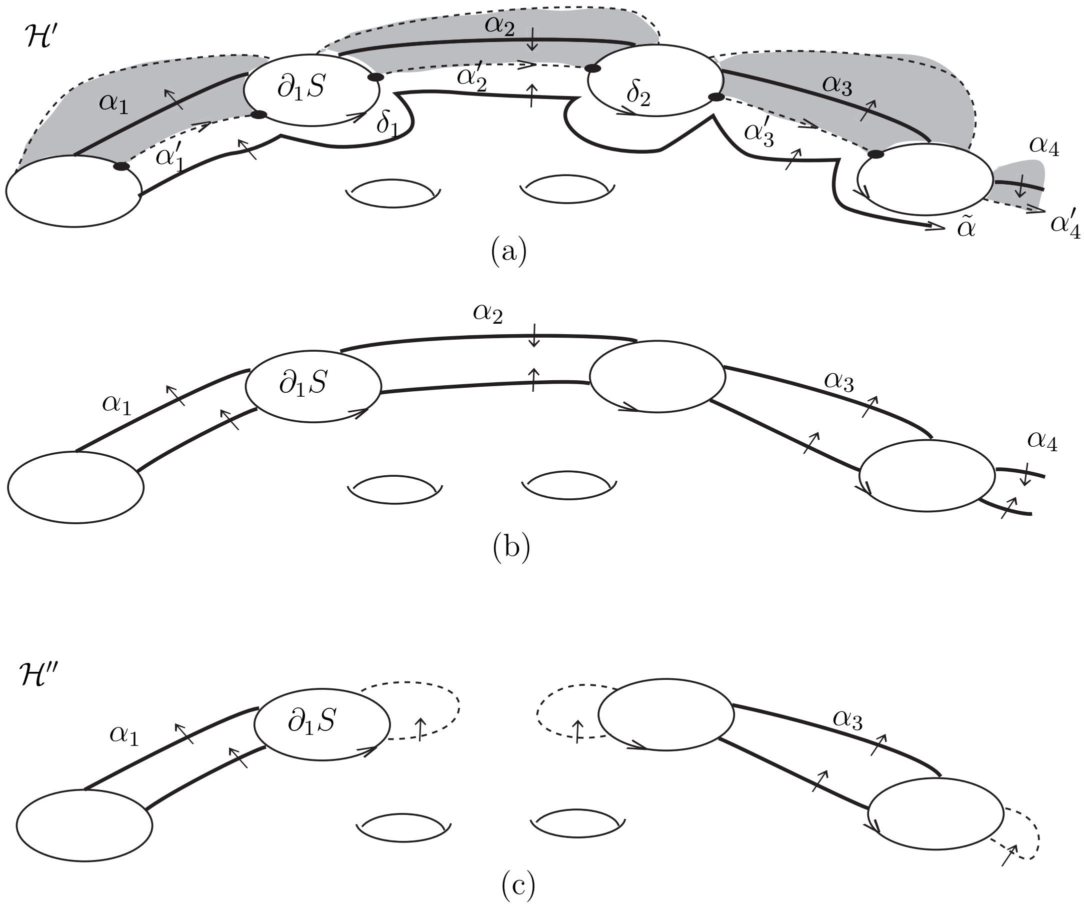


Figure 3.7: Proof of Lemma 3.18. Here, the shaded regions denote strips around a-arcs. (a) Bold curves are in $\mathcal{H}'$. $\tilde{\alpha}$ is a properly embedded arc homotopic to $\hat{\alpha} = \alpha_1' \cdot \delta_1' \cdot \alpha_2' \cdot \delta_2' \cdot \alpha_3' \cdots$. (b) An equivalent label-reading pair. (c) Bold curves are in $\mathcal{H}''$.

Lemma 3.19 (normalization II)**.** *Let S be a hyperbolic surface, and $(\mathcal{H}, \lambda)$ be a normalized label-reading pair on S with the underlying graph Γ. Then each properly embedded arc in $\mathcal{H}$ is one-sided with respect to $(\mathcal{H}, \lambda)$.*

Proof)

Suppose there exists a properly embedded arc α_1, which is not one-sided. Let $\hat{\alpha}$ be one of the two induced simple closed curves of α_1 with respect to $(\mathcal{H}, \lambda)$, transversely oriented according to α_1.

First consider the case when no other curve in $\mathcal{H}$ is homotopic to α_1.

Let $a = \lambda(\alpha_1)$. Write

$$\hat{\alpha} = \alpha_1' \cdot \delta_1 \cdot \alpha_2' \cdot \delta_2 \cdots \alpha_r' \cdot \delta_r$$

where for each i, α_i' is a properly embedded arc homotopic to an a-arc $\alpha_i \in \mathcal{B}$, and $\delta_i \subseteq \partial S$. Consider an embedding $g : S^1 \times I \to S$, such that $g(S^1 \times \{0\}) = \hat{\alpha}$ (Remark 3.8 (3)). Put $\gamma_{\frac{1}{2}} = g(S^1 \times \{\frac{1}{2}\})$ and $\gamma_1 = g(S^1 \times \{1\})$. We may assume γ_1 is sufficiently close to $\hat{\alpha}$. Let γ_1 have the transverse orientation induced from $\hat{\alpha}$, and $\gamma_{\frac{1}{2}}$ have the opposite orientation (Figure 3.8 (a)). By adding $\gamma_{\frac{1}{2}}$ and γ_1 to $\mathcal{H}$, we have a new label-reading pair $(\mathcal{H}_1, \lambda_1)$ which is equivalent to $(\mathcal{H}, \lambda)$ (Remark 3.14). Then $(\mathcal{H}_1, \lambda_1)$ is equivalent to another label-reading pair $(\mathcal{H}_2, \lambda_2)$ by removing α_1 and replacing $\gamma_{\frac{1}{2}}$ by a properly embedded arc γ' which is homotopic $\alpha_2' \cdot \delta_2 \cdots \alpha_r'$. The transverse orientation of α_2 induces that of $\alpha_2' \cdot \delta_2 \cdots \alpha_r'$, and in turn, that of γ' (Figure 3.8 (b)).

Note that $\mathcal{H}_2$ does not contain any properly embedded arc in the homotopy class of α_1. $\alpha_2' \cdot \delta_2 \cdots \alpha_r'$ is an induced arc with respect not only to $(\mathcal{H}, \lambda)$, but also to $(\mathcal{H}_2 \setminus \{\gamma'\}, \lambda_2)$, since for each $i > 1$, $\alpha_i' \not\simeq \alpha_1$ (recall that α_1 is not one-sided). By Lemma 3.18, $(\mathcal{H}_2, \lambda_2)$ is equivalent to another label-reading pair $(\mathcal{H}_3, \lambda_3)$, with $\mathcal{B}_3$ denoting the set of the properly embedded arcs in $\mathcal{H}_3$, such that $|\mathcal{H}_3/\simeq| \leq |\mathcal{H}/\simeq|$, and $|\mathcal{B}_3/\simeq| \leq |\mathcal{B}/\simeq| - 1$. This is because the homotopy class of α_1 does not exist in $\mathcal{B}_3$. We have a contradiction to that $(\mathcal{H}, \lambda)$ is normalized.

In the case when there exists $l > 1$ properly embedded arcs in $\mathcal{H}$ homotopic to α_1, we consider a set of disjoint, transversely oriented simple closed curves $\gamma_{\frac{1}{2l}}, \gamma_{\frac{2}{2l}}, \ldots, \gamma_{\frac{2l}{2l}}$ all parallel to $\hat{\alpha}$, such that $\gamma_{\frac{1}{2l}}, \gamma_{\frac{2}{2l}}, \ldots, \gamma_{\frac{l}{2l}}$ have the opposite transverse orientations to that of $\hat{\alpha}$. By letting $\mathcal{H}_1 = \mathcal{H} \cup \{\gamma_{\frac{1}{2l}}, \gamma_{\frac{2}{2l}}, \ldots, \gamma_{\frac{2l}{2l}}\}$, a similar argument applies. $\square$

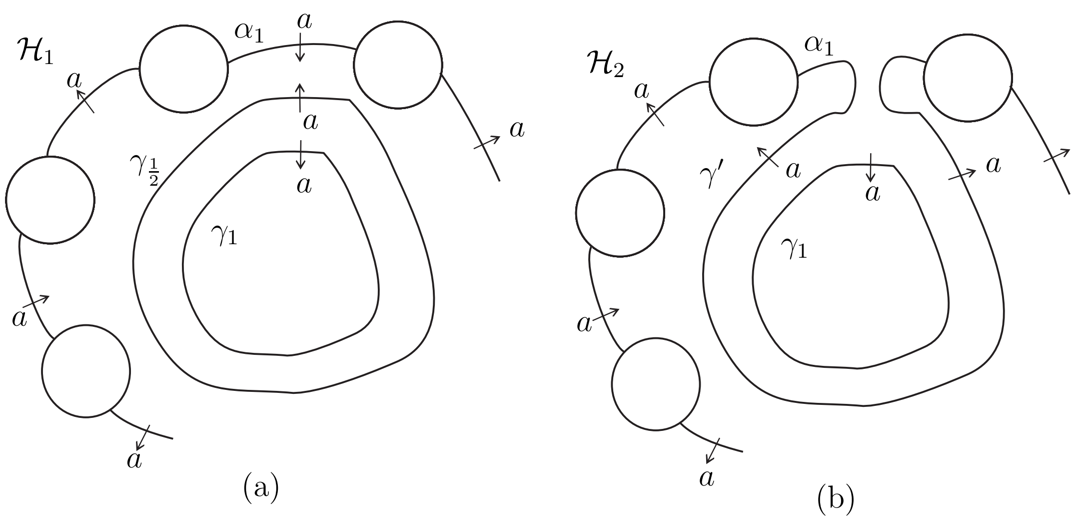


Figure 3.8: Reducing complexity, if α_1 is not one-sided.

Chapter 4

Co-contraction of Graphs

In Chapter 3, we have seen that the graph product of any non-trivial groups on a cycle of length at least 5 contains a hyperbolic surface group. In this Chapter, we will define an operation on graphs, called *co-contraction*, and show that co-contraction induces an embedding between graph products of groups. As a corollary, we prove that a graph product of non-trivial groups on any anti-cycle of length at least 5 contains a hyperbolic surface group for $n \geq 5$. We also exhibit some other choices of such embeddings in the case of right-angle Artin groups.

A dual van Kampen diagram, introduced in Chapter 2, will be used crucially for the proof of the main theorem (Theorem 4.10) in this chapter. Other than that, the proof will be largely independent of the material in Chapter 2 and 3.

4.1 Co-contraction of graphs

Let Γ be a graph and $B \subseteq V(\Gamma)$. We say B is *connected*, if Γ_B is connected. B is *anticonnected*, if $\overline{\Gamma_B}$ is connected.

Suppose $B \subseteq V(\Gamma)$. A vertex q in Γ is a *neighbor* of B, if q is adjacent to some vertex in B and $q \notin B$. q is called a *common neighbor* of B, if q is adjacent to each vertex in B and $q \notin B$.

Definition 4.1 ((co-)contraction)**.** Let Γ be a graph and $B \subseteq V(\Gamma)$.

(1) If B is connected, the *contraction of* Γ *relative to* B is the graph $\mathrm{CO}(\Gamma, B)$ defined by:

$$\begin{aligned} V(\mathrm{CO}(\Gamma, B)) &= (V(\Gamma) \setminus B) \cup \{v_B\} \\ E(\mathrm{CO}(\Gamma, B)) &= E(\Gamma_{V(\Gamma)\setminus B}) \cup \{\{v_B, q\} : q \text{ is a neighbor of } B\} \end{aligned}$$

(2) If B is anticonnected, the *co-contraction of* Γ *relative to* B is the graph $\overline{\mathrm{CO}}(\Gamma, B)$ defined by:

$$\begin{aligned} V(\overline{\mathrm{CO}}(\Gamma, B)) &= (V(\Gamma) \setminus B) \cup \{v_B\} \\ E(\overline{\mathrm{CO}}(\Gamma, B)) &= E(\Gamma_{V(\Gamma)\setminus B}) \cup \{\{v_B, q\} : q \text{ is a common neighbor of } B\} \end{aligned}$$

(3) More generally, if $B_1, B_2, \ldots, B_m$ are disjoint connected subsets of $V(\Gamma)$, then inductively define

$$\mathrm{CO}(\Gamma, (B_1, B_2, \ldots, B_m)) = \mathrm{CO}(\mathrm{CO}(\Gamma, (B_1, B_2, \ldots, B_{m-1})), B_m)$$

and if $B_1, B_2, \ldots, B_m$ are disjoint anticonnected subsets, then similarly,

$$\overline{\mathrm{CO}}(\Gamma, (B_1, B_2, \ldots, B_m)) = \overline{\mathrm{CO}}(\overline{\mathrm{CO}}(\Gamma, (B_1, B_2, \ldots, B_{m-1})), B_m)$$

A vertex of the (co-)contraction corresponding to some B_i is called a *contracted vertex*.

(4) In (3), let $\hat{\Gamma}$ denote the (co-)contraction. Then the *(co-)contraction map of* Γ *relative to* $B_1, B_2, \ldots, B_m$ is the map $p : V(\Gamma) \to V(\hat{\Gamma})$ defined by:

$$p(q) = \begin{cases} v_{B_i} & \text{if } q \in B_i \text{ for some } i \\ q & \text{otherwise} \end{cases}$$

For any subgraph $\hat{\Lambda}$ of $\hat{\Gamma}$, the *pre-image of* $\hat{\Lambda}$ *with respect to the (co-)contraction map* is the subgraph of Γ induced by $p^{-1}(V(\hat{\Lambda}))$.

Remark 4.2. (1) If B is connected, then $\mathrm{CO}(\Gamma, B)$ is obtained by (homotopically) collapsing Γ_B onto the contracted vertex v_B and removing any loops or multi-edges.

(2) In (1) and (2) of Definition 4.1, let $p : \Gamma \to \hat{\Gamma}$ be the (co-)contraction map, and $\hat{\Lambda} \leq \hat{\Gamma}$. If $\Lambda \leq \Gamma$ is the pre-image of $\hat{\Lambda}$ with respect to the (co-)contraction map, then $V(\Lambda)$ either contains or is disjoint from B, depending on whether the contracted vertex v_B is in $\hat{\Lambda}$ or not. Hence, $\hat{\Lambda}$ is a (co-)contraction of Λ, with respect to $V(\Lambda) \cap B$.

The following observations are immediate from the definition (see Figure 4.1).

Proposition 4.3. *If B is an anticonnected set of vertices in Γ, then*

$$\overline{CO}(\Gamma, B) = \overline{CO(\overline{\Gamma}, B)}. \qquad \square$$

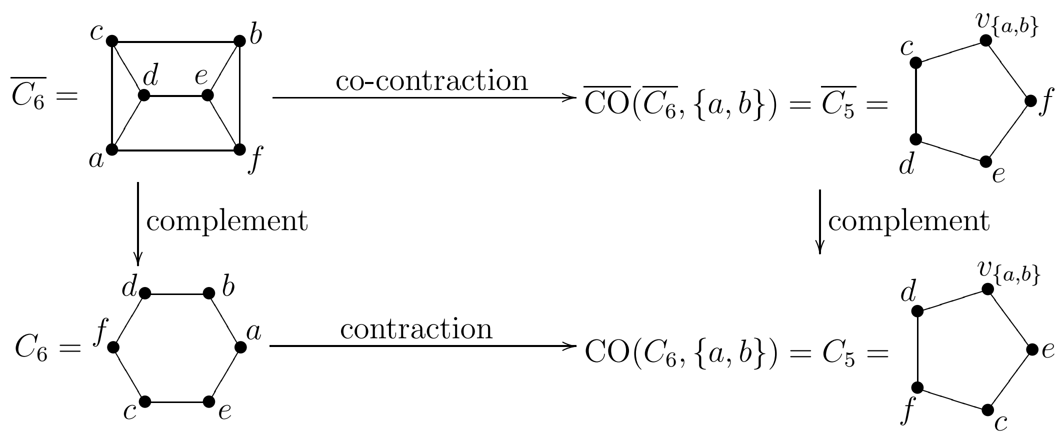


Figure 4.1: c and f are common neighbors of $\{a, b\}$ in $\overline{C_6}$. Hence $v_{\{a,b\}}$ is adjacent to c and f in $\overline{\mathrm{CO}}(\overline{C_6}, \{a, b\})$. This can be also viewed by looking at the complement graph of $\overline{C_6}$, namely C_6, and collapsing the edge $\{a, b\}$.

Lemma 4.4 (contracting two vertices at a time)**.** Γ *is a graph.*

(1) Let $B \subseteq V(\Gamma)$ be connected. Then there exists a sequence of graphs

$$\Gamma_0 = \Gamma, \Gamma_1, \Gamma_2, \ldots, \Gamma_p = CO(\Gamma, B)$$

such that for each $i = 0, 1, \ldots, p-1$, Γ_{i+1} is a contraction of Γ_i relative to a pair of adjacent vertices of Γ_i.

(2) Let $B \subseteq V(\Gamma)$ be anticonnected. Then there exists a sequence of graphs

$$\Gamma_0 = \Gamma, \Gamma_1, \Gamma_2, \ldots, \Gamma_p = \overline{CO}(\Gamma, B)$$

such that for each $i = 0, 1, \ldots, p-1$, Γ_{i+1} is a co-contraction of Γ_i relative to a pair of non-adjacent vertices of Γ_i.□

Lemma 4.5. *(1) If B is a connected subset of p vertices of C_n, then $CO(C_n, B) \cong C_{n-p+1}$.*

(2) If B is an anticonnected subset of p vertices of $\overline{C_n}$, then $\overline{CO}(\overline{C_n}, B) \cong \overline{C_{n-p+1}}$.

Proof) (1) is obvious from the definition. From (1), one gets (2) by taking complements. □

Definition 4.6 (weakly chordal graph[Hay85]). A graph Γ is called *weakly chordal* if Γ does not contain an induced C_n or $\overline{C_n}$ for any $n \geq 5$.

Note that a graph Γ is weakly chordal if and only if so is $\overline{\Gamma}$. We define

$$\mathcal{W} = \{\Gamma : \Gamma \text{ is a weakly chordal graph}\}$$

We will later show that $\mathcal{N}$ is closed under co-contraction (Corollary 4.11). The following theorem shows one of the similarities between $\mathcal{N}$ and $\mathcal{W}$ (see also Question 4.14). The proof is given in the appendix to this chapter.

Theorem 4.7. *$\mathcal{W}$ is closed under contraction and co-contraction.*

4.2 Embeddings between graph products of groups

We let Γ be a graph, $B \subseteq V(\Gamma)$ be anticonnected and $A = V(\Gamma) \setminus B$. Let $\hat{\Gamma} = \overline{\mathrm{CO}}(\Gamma, B)$ and $\hat{v}$ denote the contracted vertex v_B of $\hat{\Gamma}$. For each $q \in A$, $\hat{q}$ denotes the corresponding vertex in $V(\hat{\Gamma})$, and let $\hat{A} = \{\hat{q} : q \in A\} \subseteq V(\hat{\Gamma})$ (see Figure 4.3). Consider a collection of non-trivial groups $\{G_q : q \in V(\Gamma)\}$ indexed by $V(\Gamma)$, and let $G = \mathrm{GP}(\Gamma, \{G_q\})$. Fix an arbitrary word $w_0 \in \langle \{G_q : q \in B\} \rangle \leq G$, and define $\hat{G}_{\hat{v}}$ to be the cyclic group generated by one element $\hat{v}$, such that $o(\hat{v}) = o_G(w_0)$. Let $\hat{G}_{\hat{q}} = G_q$ for $q \in A$. This defines a graph product of groups $\hat{G} = \mathrm{GP}(\hat{\Gamma}, \{\hat{G}_x : x \in V(\hat{\Gamma})\})$.

Consider the subgroup $\hat{H} = \langle \{\hat{G}_{\hat{q}} : q \in A\} \rangle \cong \mathrm{GP}(\hat{\Gamma}_{\hat{A}}, \{\hat{G}_{\hat{q}}\}_{\hat{q} \in \hat{A}})$. Let H be the corresponding subgroup of G, i.e. $H = \langle \{G_q : q \in A\} \rangle \cong \mathrm{GP}(\Gamma_A, \{G_q\}_{q \in A})$. Since $\hat{\Gamma}_{\hat{A}} \cong \Gamma_A$ and $G_q = \hat{G}_{\hat{q}}$ for each $q \in A$, we have $\hat{H} \cong H$ (Figure 4.2).

If a vertex $\hat{q}$ of $\hat{\Gamma} = \overline{\mathrm{CO}}(\Gamma, B)$ is adjacent to $\hat{v} = v_B$, then q is a common neighbor of B in Γ, and so, for any $z \in \hat{G}_{\hat{q}} = G_q$, $[z, w_0] =_G 1$. So we have a map $\phi : \hat{G} \to G$ satisfying that

$$\phi(z) = \begin{cases} w_0 & \text{if } z = \hat{v} \\ z & \text{if } z \in \hat{G}_{\hat{q}} = G_q \text{ for some } q \in A = V(\Gamma) \setminus B \end{cases}$$

$$\begin{array}{ccc} \hat{\Gamma} & \xleftarrow{\overline{\mathrm{CO}}(\ ,B)} & \Gamma \\ | & & | \\ \hat{\Gamma}_{\hat{A}} & \xleftarrow[\cong]{} & \Gamma_A \end{array} \qquad \begin{array}{ccc} \hat{G} = \mathrm{GP}(\hat{\Gamma}, \{\hat{G}_{\hat{q}}\}_{\hat{q} \in \hat{A}} \cup \{\hat{G}_{\hat{v}} = \langle \hat{v} \rangle\}) & \xrightarrow{\phi} & G = \mathrm{GP}(\Gamma, \{G_q\}_{q \in A} \cup \{G_q\}_{q \in B}) \\ | & & | \\ \hat{H} = \mathrm{GP}(\hat{\Gamma}_{\hat{A}}, \{\hat{G}_{\hat{q}}\}_{\hat{q} \in \hat{A}}) & \xrightarrow[\cong]{} & H = \mathrm{GP}(\Gamma_A, \{G_q\}_{q \in A}) \end{array}$$

Figure 4.2: A map induced by co-contraction.

Example 4.8. Let the vertices of $\Gamma \cong \overline{C_6}$ be labeled as in Figure 4.3. Consider vertex groups $\{G_q : q \in V(\Gamma)\}$ defining $G = \mathrm{GP}(\Gamma, \{G_q : q \in V(\Gamma)\})$. Put $\hat{\Gamma} = \overline{\mathrm{CO}}(\Gamma, \{a, b\}) \cong C_5$ and label the vertices of $\hat{\Gamma}$ as above, so that the contracted vertex is denoted by $\hat{v}$, and c, d, e and f are mapped onto $\hat{c}, \hat{d}, \hat{e}$ and $\hat{f}$ by the co-contraction map $p : V(\Gamma) \to V(\hat{\Gamma})$. Choose $a_0 \in G_a$ and $b_0 \in G_b$. Define $\hat{G}_{\hat{q}} = G_q$ for $q \in V(\Gamma) \setminus \{a, b\}$ and $\hat{G}_{\hat{v}} = \langle \hat{v} \rangle$ where $o(\hat{v}) = o(a_0)$. We let $\hat{G} = \mathrm{GP}(\hat{\Gamma}, \{\hat{G}_x : x \in V(\hat{\Gamma})\})$. Then there exists a map $\phi : \hat{G} \to G$ satisfying $\phi(\hat{v}) = {b_0}^{-1} a_0 b_0$, such that ϕ maps the subgroup $\hat{H} = \langle \hat{G}_{\hat{c}}, \hat{G}_{\hat{d}}, \hat{G}_{\hat{e}}, \hat{G}_{\hat{f}} \rangle$ isomorphically onto $H = \langle G_c, G_d, G_e, G_f \rangle$.

The main theorem is that ϕ is injective for a suitable choice of w_0. The crucial case is when $|B| = 2$ (Lemma 4.9).

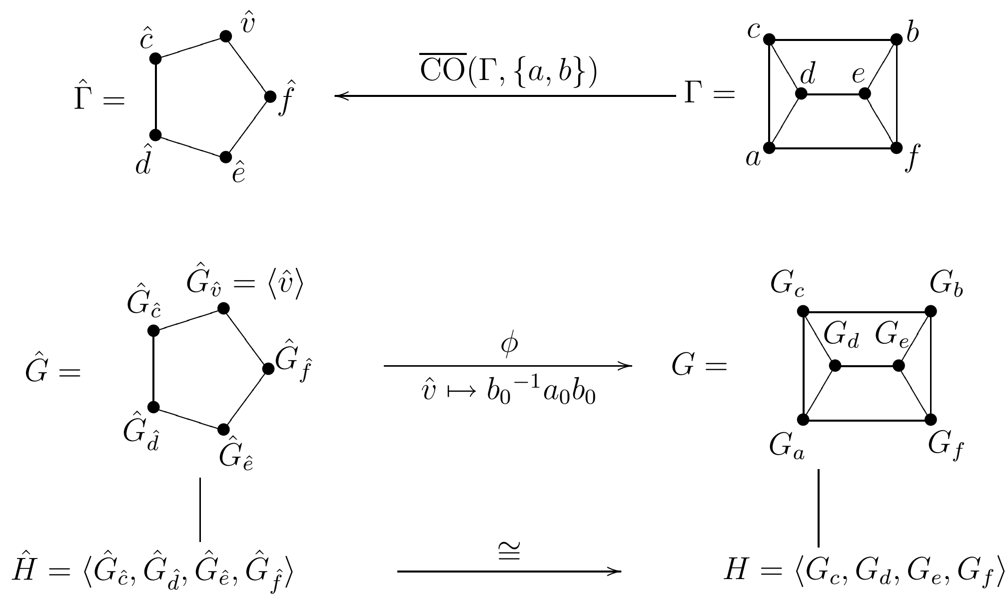


Figure 4.3: Example 4.8. A co-contraction induced map.

Lemma 4.9 (co-contraction of two vertices)**.** *Let* Γ *be a graph, and* $\hat{\Gamma} = \overline{CO}(\Gamma, \{a, b\})$*, where* $\{a, b\}$ *is a pair of non-adjacent vertices in* Γ*. Let* G *be the graph product of groups* $\{G_q : q \in V(\Gamma)\}$ *with the underlying graph* Γ*. Fix* $a_0 \in G_a \setminus \{1\}$ *and* $b_0 \in G_b \setminus \{1\}$*. Let* $\hat{q}$ *denote the vertex of* $\hat{\Gamma}$ *corresponding to* $q \in V(\Gamma) \setminus \{a, b\}$*, and let* $\hat{G}_{\hat{q}} = G_q$*. Also, let* $\hat{v}$ *denote the contracted vertex of* $\hat{\Gamma}$*, and define* $\hat{G}_{\hat{v}}$ *to be the cyclic group of order* $o(a_0)$*, the generator of which is denoted also by* $\hat{v}$*. Define* $\hat{G} = GP(\hat{\Gamma}, \{\hat{G}_x : x \in V(\hat{\Gamma})\})$*. Then the map* $\phi : \hat{G} \to G$ *uniquely determined by the following condition is injective.*

$$\phi(z) = \begin{cases} {b_0}^{-1} a_0 b_0 & \textit{if } z = \hat{v} \\ z & \textit{if } z \in \hat{G}_{\hat{q}} = G_q \textit{ for some } q \in V(\Gamma) \setminus \{a, b\} \end{cases}$$

Proof) Let $A = V(\Gamma) \setminus \{a, b\}$, $\hat{A} = \{\hat{q} : q \in A\}$, $H = \langle \{G_q : q \in A\} \rangle$ and $\hat{H} = \langle \{\hat{G}_{\hat{q}} : \hat{q} \in \hat{A}\} \rangle$. See Figure 4.3, for an illustration of the notations.

Suppose ϕ is not injective. Let $\hat{w} = \prod_{i=1}^{m} \hat{g}_i$ be a word of minimal length in $\ker \phi \setminus \{1\}$. Let $g_i = \phi(\hat{g}_i)$ for each i, and put $w = \prod_{i=1}^{m} g_i$. Let Δ be a dual van Kampen diagram for w. Since $\hat{w} \notin \hat{H}$, $\hat{g}_j = \hat{v}^k$ for some j and $k \neq 0$, and in this case, we note that $g_j = \phi(\hat{g}_j) = \phi(\hat{v}^k) = {b_0}^{-1} \cdot a_0^k \cdot b_0$, which is of length 3 in G.

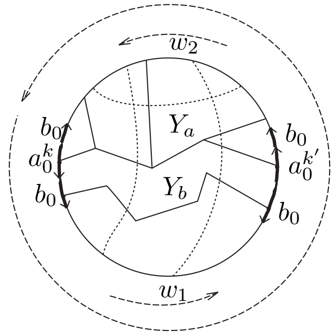
Choose an innermost G_a-graph Y_a in Δ. This means that a component of $\partial\Delta \setminus \partial Y_a$ does not intersect with any other G_a-graphs in Δ. Cyclically conjugating w if necessary, one may write

$$w = {b_0}^{-1} \cdot a_0^k \cdot b_0 \cdot w_1 \cdot {b_0}^{-1} a_0^{k'} \cdot b_0 \cdot w_2$$

for some subwords w_1 and w_2 such that Y_a intersects with a_0^k and $a_0^{k'}$, and w_1 does not contain any G_a-segment. In w, a G_b-segment is always next to a G_a-segment, since a G_b-segment appears only in the subword of the form ${b_0}^{-1} a_0^l b_0$, for some l. Hence, w_1 does not contain any G_b-segment, either. Since Y_a does not intersect with any G_b-graph, the b_0-segment, which is between a_0^k and w_1, is connected by a G_b-graph, say Y_b, to the ${b_0}^{-1}$-segment between w_1 and $a_0^{k'}$ (Figure 4.4).

One can write $\hat{w} = \hat{v}^k \hat{w}_1 \hat{v}^{k'} \hat{w}_2$, so that $\hat{w}_1 \in \hat{H}$ and $\phi(\hat{w}_i) = w_i$ for $i = 1, 2$. Note that any subword of a word in a normal form is again in a normal form. So $\hat{w}_1 \neq 1$ is in a normal form, and since $\hat{H} \cong H$ (Figure 4.2), w_1 is in a normal form also.

Suppose that G_c-graph Y_c intersects with a segment in w_1, for some $c \in A = V(\Gamma) \setminus \{a, b\}$. Then Y_c does not intersect with any other segment in w_1, since w_1 is in a normal form (Lemma 2.19). So Y_c intersects with segments in w_2 (Lemma 2.17). This implies that Y_c intersects Y_a and Y_b, and so, the vertex c is adjacent to both a and b in Γ. This is true for any segment in w_1. So $[\hat{w}_1, \hat{v}] = 1$, and $\hat{w}$ is equivalent to a shorter word $\hat{v}^{k+k'}\hat{w}_1\hat{w}_2$.□



$w = {b_0}^{-1} \cdot a_0^k \cdot b_0 \cdot w_1 \cdot {b_0}^{-1} \cdot a_0^{k'} \cdot b_0 \cdot w_2$

Figure 4.4: Δ in the proof of Lemma 4.9.

Note that in the proof of Lemma 4.9, $o({b_0}^{-1}a_0b_0) = o(a_0)$. The main theorem of this chapter follows, using routine induction argument.

Theorem 4.10 (co-contraction induced embedding)**.** *Let Γ be a graph, and $B_1, B_2, \ldots, B_m$ be disjoint, anticonnected sets of vertices of Γ. Consider $\{G_q : q \in V(\Gamma)\}$, which is a collection of non-trivial groups indexed by $V(\Gamma)$. Put $G = GP(\Gamma, \{G_q\})$.*

Put $\hat{\Gamma} = \overline{CO}(\Gamma, \{B_1, \ldots, B_m\})$. Let $\hat{q}$ denote the vertex of $\hat{\Gamma}$ corresponding to $q \in V(\Gamma)\backslash \cup_i B_i$, and $\hat{v}_i$ denote the contracted vertex v_{B_i} of $\hat{\Gamma}$ corresponding to $B_i \subseteq V(\Gamma)$.

For each $1 \le i \le m$, choose $m_i \in \{o(g) : g \in (\cup_{q \in B_i} G_q) \setminus \{1\}\}$. For $x \in V(\hat{\Gamma})$, define

$$\hat{G}_x = \begin{cases} \langle \hat{v}_i \rangle \cong \mathbb{Z}_{m_i} & \text{if } x = \hat{v}_i, \text{ for some } i \\ G_q & \text{if } x = \hat{q}, \text{ for some } q \in V(\Gamma) \setminus \cup_i B_i \end{cases}$$

Put $\hat{G} = GP(\hat{\Gamma}, \{\hat{G}_x\}_{x \in V(\hat{\Gamma})})$. Then there exists an embedding from $\hat{G}$ into G.

Proof) The case when $m = 1$ and $|B_1| = 2$ is precisely Lemma 4.9. The general case follows from induction (Lemma 4.4). □

Corollary 4.11 (co-contraction induced embedding in $A(\Gamma)$)**.** *Let Γ be a graph and Γ_1 be a graph obtained from Γ by co-contraction. Fix $0 < m \leq \infty$, and let G and G_1 be the graph products of cyclic groups of order m with the underlying graphs Γ and Γ_1, respectively. Then G_1 embeds into G. In particular, $A(\Gamma_1)$ and $C(\Gamma_1)$ embed into $A(\Gamma)$ and $C(\Gamma)$, respectively.*

Using the theorem, we can extend the result of Proposition 2.42.

Corollary 4.12 ($A(C_n)$ and $A(\overline{C_n})$ contain hyperbolic surface groups)**.** *Let Γ be a cycle or an anti-cycle, of length at least 5. Then the graph product of non-trivial groups on Γ contains a hyperbolic surface group. In particular, the right-angled Artin group and the right-angled Coxeter group on Γ contain hyperbolic surface groups.*

Proof) Let G be the graph product of non-trivial groups on Γ.

The case when Γ is a cycle of length at least 5 is in Theorem 3.6. Now suppose $\Gamma = \overline{C_n}$ for some $n \geq 6$. By Lemma 4.5, $\overline{C_5} = C_5$ is obtained from Γ by co-contraction. From Theorem 4.10, G contains the graph product of certain non-trivial cyclic groups on C_5. By Theorem 3.6 again, G contains a hyperbolic surface group.$\square$

Now we give the negative answer to Question 1.2.

Corollary 4.13. *There exists an infinite family $\mathcal{A}$ of graphs satisfying the following.*

(i) *Each element in $\mathcal{A}$ does not contain an induced C_n for $n \geq 5$.*

(ii) *Each element in $\mathcal{A}$ is not an induced subgraph of another element in $\mathcal{A}$.*

(iii) *for each $\Gamma \in \mathcal{A}$, the graph product of non-trivial groups with the underlying graph Γ contains a hyperbolic surface group. In particular, the right-angled*

Artin group and the right-angled Coxeter group on Γ *contain hyperbolic surface groups.*

Proof) Let $\mathcal{A}$ be the set of anti-cycles of length at least 6. (i) is in Proposition 2.4. (ii) is obvious by looking at the complements. (iii) is Corollary 4.12.□

Recall that $\mathcal{N}$ denotes the family of the graphs Γ such that the right-angled Artin group on Γ does not contain a hyperbolic surface group. From Corollary 4.12, it follows that any graph in $\mathcal{N}$ is weakly chordal (Definition 4.6). Moreover, $\mathcal{N}$ and $\mathcal{W}$ are both closed under co-contraction (Corollary 4.11, Theorem 4.7). We ask whether $\mathcal{N}$ and $\mathcal{W}$ are actually equal.

Question 4.14 ($\mathcal{N} = \mathcal{W}$?). *(1) Does* $A(\Gamma)$ *contain a hyperbolic surface group if and only if* Γ *is not weakly chordal?*

(2) More generally, does a graph product of non-trivial cyclic groups contain a hyperbolic surface group if and only if the underlying graph is not weakly chordal?

In the next chapter, we describe similarities between $\mathcal{N}$ and $\mathcal{W}$ in more detail.

4.3 Contraction words

The material in this section will not be referred to in this thesis, and thus, may be skipped at the first reading.

From this point on, we restrict our attention to the case when all the vertex groups are infinite cyclic, and so, the graph products are right-angled Artin groups. Let Γ be a graph and B be an anticonnected set of vertices of Γ. Recall that $V(\Gamma)$ can be considered as a set of generators for $A(\Gamma)$. Following the notations in Section 4.2, we let $\hat{\Gamma} = \overline{\mathrm{CO}}(\Gamma, B)$, $\hat{v} = v_B$, $A = V(\Gamma) \setminus B$, $\hat{A} = \{\hat{q} : q \in A\} \subseteq V(\hat{\Gamma})$, $H = \langle A \rangle$ and $\hat{H} = \langle \hat{A} \rangle$. Choose a word $w_0 \in \langle B \rangle$, to define a map $\phi : A(\hat{\Gamma}) \to A(\Gamma)$ by

$$\phi(x) = \begin{cases} w_0 & \text{if } x = \hat{v} \\ q & \text{if } x = \hat{q}, \text{ for some } q \in A = V(\Gamma) \setminus B \end{cases}$$

From the proof of Lemma 4.9, it follows that ϕ is injective in the case when $B = \{a, b\}$ and $w_0 = b^{-1}ab$. In this section, we will give other sufficient conditions for the word w_0, so that the map ϕ is injective.

Definition 4.15 (contraction word)**.** (1) Let Γ_0 be an anticonnected graph. A sequence $b_1, b_2, \ldots, b_p$ of vertices in Γ_0 is a *contraction sequence of* Γ_0, if the following holds: for any $(b, b') \in V(\Gamma_0) \times V(\Gamma_0)$, there exists $l \geq 1$ and $1 \leq k_1 < k_2 < \cdots < k_l \leq p$ such that, $b_{k_1}, b_{k_2}, \ldots, b_{k_l}$ is a path from b to b' in $\overline{\Gamma}$.

(2) Let Γ be a graph and B be an anticonnected set of vertices of Γ. A reduced word $w = \prod_{i=1}^{p} b_i^{e_i}$, where $b_i \in B$ and $e_i = \pm 1$ for each i, is called a *contraction word of* B if $b_1, b_2, \ldots, b_p$ is a contraction sequence of Γ_B. An element of $A(\Gamma)$ is called a *contraction element*, if it can be represented by a contraction word.

Example 4.16. (1) If a and b are non-adjacent vertices in Γ, then any reduced word in $\langle a, b \rangle \setminus \{a^m b^n : m, n \in \mathbb{Z}\}^{\pm 1}$ is a contraction word of $\{a, b\}$.

(2) Suppose $(a_1, a_2, \ldots, a_m)$ is a (possibly redundant) path in $\overline{\Gamma}$ such that two consecutive terms are different, and $V(\Gamma) = V(\overline{\Gamma}) = \{a_1, a_2, \ldots, a_m\}$. Then $(a_m, a_{m-1}, \ldots, a_2, a_1, a_2, a_3, \ldots, a_m)$ is a contraction sequence. Hence,

$$a_m{}^{-1} a_{m-1}{}^{-1} \cdots a_2{}^{-1} a_1 a_2 a_3 \cdots a_m$$

is a contraction word. From this, we deduce that for any anticonnected set $B \subseteq V(\Gamma)$, there exists a contraction word of B.

We first note the following general lemma.

Lemma 4.17 (reduced word for a power)**.** *Let Γ be a graph and $g \in A(\Gamma)$. Then $g =_{A(\Gamma)} u^{-1}vu$ for some words u, v such that $u^{-1}v^m u$ is reduced for each $m \neq 0$.*

Proof) Choose words u, v such that $u^{-1}vu$ is a reduced word representing g and the length of u is maximal. We will show that $u^{-1}v^m u$ is reduced for any $m \neq 0$.

Assume that $u^{-1}v^m u$ is not reduced for some $m \neq 0$. We may assume that $m > 0$. Let w be a reduced word for $u^{-1}v^m u$. Draw a dual van Kampen diagram Δ for $u^{-1}v^m uw^{-1}$. Let v_i denote the v-interval on $\partial\Delta$ corresponding to the i-th occurrence of v from the left in $u^{-1}v^m u$ (Figure 4.5 (a)).

By Lemma 2.22, there exists a q-arc γ joining two q-segments of $u^{-1}v^m u$ for some $q \in V(\Gamma)$. Let w_0 denote the interval between those two q-segments. We may choose q and γ so that the number of the segments in w_0 is minimal. Then any arc intersecting with a segment in w_0 must intersect γ. It follows that any letter in w_0 should commute with q. Moreover, w_0 does not contain any q-segment.

Case 1. The intervals u^{-1} and u do not intersect with γ.

Since w_0 does not contain any q-segment, γ joins v_i and v_{i+1} for some i (Figure 4.5 (b)). Then one can write $v = w_1 q^{\pm 1} w_2 q^{\mp 1} w_3$ for some subwords w_1, w_2, w_3 of v such that and $w_0 = w_3 w_1$. Since any letter in w_0 commutes with q, $[w_3, q] =_{A(\Gamma)} 1 =_{A(\Gamma)} [w_1, q]$. So $u^{-1}vu =_{A(\Gamma)} u^{-1}q^{\pm 1}w_1 w_2 w_3 q^{\mp 1}u$, which contradicts to the maximality of

u.

Case 2. γ intersects u- or u^{-1}-interval.

Suppose u^{-1} intersects γ. Since $u^{-1}v = u^{-1}v_1$ is reduced, γ cannot intersect v_1. So, w_0 contains v_1. Since w_0 does not contain any q-segment, v does not contain the letters q or q^{-1} and so, γ cannot intersect any v_i for $i = 1, \ldots, m$. γ should intersect with the u-interval of $u^{-1}v^m u$ (Figure 4.5 (c)). This implies that γ intersects with the rightmost q-segment of the u^{-1}-interval, and with the leftmost q-segment of the u-interval, in $u^{-1}v^m u$. One can write $u^{-1}v^m u = {u_2}^{-1}q^{\pm 1}{u_1}^{-1}v^m u_1 q^{\mp 1}u_2$ such that any letter in $w_0 = {u_1}^{-1}v^m u_1$ commutes with q, hence, $[q, u_1] =_{A(\Gamma)} 1 =_{A(\Gamma)} [q, v]$. But then $u^{-1}vu =_{A(\Gamma)} {u_2}^{-1}{u_1}^{-1}vu_1u_2$, which is a contradiction to the assumption that $u^{-1}vu$ is reduced.□

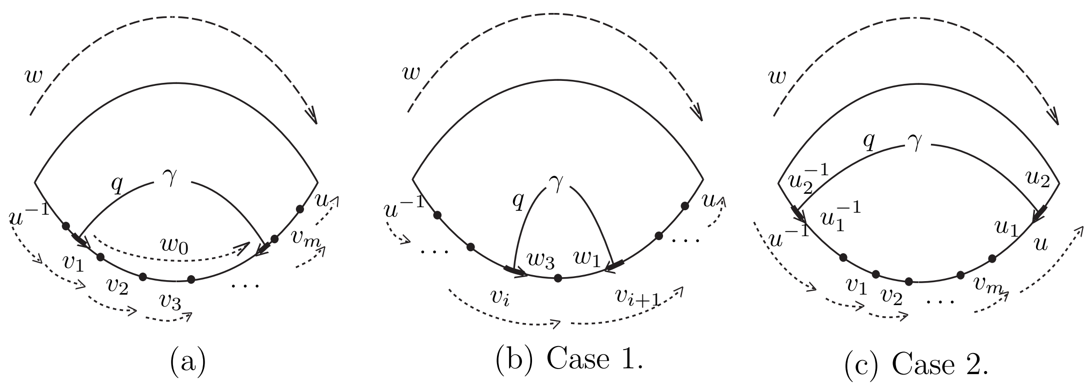


Figure 4.5: Proof of Lemma 4.17.

Lemma 4.18 (power of a contraction word). *(1) Any reduced word for a contraction element is a contraction word.*

(2) Any non-trivial power of a contraction element is a contraction element.

Proof) (1) Let $w = \prod_{i=1}^{p} b_i^{e_i}$ be a contraction word of an anticonnected set B in $V(\Gamma)$. Here, $b_i \in B$ and $e_i = \pm 1$ for each i. Suppose w' is a reduced word, such that $w' =_{A(\Gamma)} w$. There exists a dual van Kampen diagram Δ for ww'^{-1}. Note that any properly embedded arc of Δ meets both of the intervals w and w', since w and w' are reduced (Lemma 2.22). Now let $b, b' \in B$. w is a contraction word, so one can find $l \geq 1$ and $1 \leq k_1 < k_2 < \cdots < k_l \leq p$ such that, b_{k_i} and $b_{k_{i+1}}$ are non-adjacent for each $i = 1, \ldots, l-1$, and $b = b_{k_1}, b' = b_{k_l}$. Let γ_i be the arc that intersects with the segment b_{k_i} of w. Since $\gamma_1, \gamma_2, \ldots, \gamma_l$ are all disjoint, the boundary points of those arcs on w' will yield the desired subsequence of the letters of w'.

(2) Let $u^{-1}vu$ be a reduced word for a contraction element g as in Lemma 4.17. Note that a sequence, containing a contraction sequence as a monotonic subsequence, is again a contraction sequence. So the reduced word $u^{-1}v^m u$ is a contraction word of B, for each $m \neq 0$. □

Definition 4.19 (canonical expression)**.** Let Γ be a graph, and P and Q be disjoint subsets of $V(\Gamma)$. Suppose P_1 is a set of words in $\langle P \rangle \leq A(\Gamma)$. A *canonical expression for $g \in \langle P_1, Q \rangle$ with respect to $\{P_1, Q\}$* is a word $\prod_{i=1}^{k} c_i^{e_i}$, where

(i) for each i, $c_i \in P_1 \cup Q$ and $e_i = 1$ or -1,

(ii) $\prod_{i=1}^{k} c_i^{e_i} =_{A(\Gamma)} g$

such that k is minimal. k is called the *length* of the canonical expression.

Remark 4.20. In the case when $P_1 \subseteq P$, a word is a canonical expression with respect to $\{P_1, Q\}$, if and only if it is reduced in $A(\Gamma)$.

Now we compute intersections of certain subgroups of $A(\Gamma)$.

Lemma 4.21 (intersections)**.** *Let Γ be a graph, P, Q be disjoint subsets of $V(\Gamma)$ and P_1 be a set of words in $\langle P\rangle \leq A(\Gamma)$. Let R be any subset of $V(\Gamma)$.*

(1) If w is a canonical expression with respect to $\{P_1, Q\}$, then there does not exist a q-pair of w for any $q \in Q$.

(2) $\langle P_1, Q\rangle \cap \langle R\rangle \subseteq \langle P_1, Q \cap R\rangle$. Moreover, the equality holds if $P \subseteq R$.

(3) Suppose P is anticonnected. Let g be a contraction element of P, and $P_1 = \{g\}$. Assume $P \not\subseteq R$. Then $\langle P_1, Q\rangle \cap \langle R\rangle = \langle Q \cap R\rangle$.

Proof)

(1) Let w be a canonical expression, Suppose there exists a q-pair of w for some $q \in Q$. Then by Lemma 2.22, one can write $w = w_1 q^{\pm 1} w_2 q^{\mp 1} w_3$ for some subwords w_1, w_2 and w_3 such that $w_2 \in \langle \mathrm{Link}_\Gamma(q)\rangle$. It follows that $w =_{A(\Gamma)} w'' = w_1 w_2 w_3$. Since $P \cap Q = \varnothing$, w_1, w_2 and w_3 are also canonical expressions with respect to $\{P_1, Q\}$. This contradicts to the minimality of k.

(2) Let w be a canonical expression for an element in $\langle P_1, Q\rangle \cap \langle R\rangle$, and $w' =_{A(\Gamma)} w$ be a reduced word. Consider a dual van Kampen diagram Δ for ww'^{-1}.

Suppose that there exists a q-segment in w, for some $q \in Q$. Then by (1), the q-segment should be joined, by a q-arc, to another q-segment of w'. Since w' is a reduced word representing an element in $\langle P_1, Q\rangle \cap \langle R\rangle \subseteq \langle R\rangle$, each segment of w' is labeled by $R^{\pm 1}$ (Lemma 2.23 (2)). Therefore, $q \in Q \cap R$. This shows $\langle P_1, Q\rangle \cap \langle R\rangle \subseteq \langle P_1, Q \cap R\rangle$.

If $P \subseteq R$, then $\langle P_1, Q \cap R\rangle \subseteq \langle P_1, Q\rangle \cap \langle R\rangle$ is obvious.

(3) $\langle Q \cap R \rangle \subseteq \langle P_1, Q \rangle \cap \langle R \rangle$ is trivial.

To prove the converse, suppose $w \in (\langle P_1, Q \rangle \cap \langle R \rangle) \setminus \langle Q \cap R \rangle$. w is chosen so that w is a canonical expression with respect to $\{P_1, Q\}$, and the length (as a canonical expression) is minimal.

Let $\tilde{w}$ be a reduced word for g. One can write $\tilde{w} = u^{-1}vu$ such that $u^{-1}v^m u$ is reduced for any $m \neq 0$ (Lemma 4.17).

Write $w = \prod_{i=1}^{k} c_i^{e_i}$ ($c_i \in \{P_1, Q\}$, $e_i = \pm 1$). From the proof of (2), $c_i \in P_1 \cup (Q \cap R) = \{\tilde{w}\} \cup (Q \cap R)$ for each i. Any shorter canonical expression than w, for an element in $\langle P_1, Q \rangle \cap \langle R \rangle$ is in $\langle Q \cap R \rangle$. This implies that $c_1, c_k \notin Q \cap R$, and hence, $c_1 = \tilde{w} = c_k$. So we can write $w = \tilde{w}^m w_1 \tilde{w}^e w_2$ for some subwords w_1, w_2 of w, $m \in \mathbb{Z} \setminus \{0\}$ and $e \in \{1, -1\}$. Here, w_1 is chosen so that the letters of w_1 are in $(Q \cap R)^{\pm 1}$. We will show that $[w_1, \tilde{w}] =_{A(\Gamma)} 1$.

Consider another word $w' = u^{-1}v^m u w_1 \tilde{w}^e w_2 =_{A(\Gamma)} w$. Let w'' be a reduced word for w', and Δ be a dual van Kampen diagram for $w' w''^{-1}$.

Fix $b \in P \setminus R$. Let $\beta \subseteq \Delta$ be the b-arc that intersects with the leftmost b-segment of w', considered as a subset of $\partial\Delta$. This leftmost b-segment will be contained in the contraction word $u^{-1}v^m u$ (Figure 4.6). Since $u^{-1}v^m u$ is reduced and $b \notin Q \cap R$, β intersects with w'' or $\tilde{w}^e w_2$ on $\partial\Delta$. Since $b \notin R$, β intersects with $\tilde{w}^e w_2$.

Let b' be any element in P. Using the assumption that $u^{-1}v^m u$ is a contraction word, one can find a sequence of arcs $\beta_1, \beta_2, \ldots, \beta_l \in \mathcal{H}$ such that

(i) $\lambda(\beta_1) = b, \lambda(\beta_l) = b'$,

(ii) $\lambda(\beta_i)$ and $\lambda(\beta_{i+1})$ are non-adjacent in Γ, for each $i = 1, 2, \ldots, l-1$, and

(iii) each β_i intersects with a segment in $u^{-1}v^m u$ and a segment in the interval $\tilde{w}^e w_2$.

Note that (iii) comes from the assumptions that no arc joins two segments in $u^{-1}v^m u$, and that the letters of w_1 are in $(Q \cap R)^{\pm 1}$.

As in (1), each segment of w_1 is joined to a segment in w''. In particular, $[b', w_1] = [\lambda(\beta_l), w_1] =_{A(\Gamma)} 1$. Since this is true for any $b' \in P$, $[w_1, \tilde{w}] =_{A(\Gamma)} 1$, and so, $w =_{A(\Gamma)} w_1 \tilde{w}^{m+e} w_2$. One has $\tilde{w}^{m+e} w_2 \in (\langle P_1, Q\rangle \cap \langle R\rangle) \setminus \langle Q \cap R\rangle$, since $w \notin \langle Q \cap R\rangle$ and $w_1 \in \langle Q \cap R\rangle$. By the minimality of w, we have $w_1 = 1$. This argument continues, and finally one can write $w =_{A(\Gamma)} \tilde{w}^{m'}$ for some $m' \neq 0$. In particular, any reduced word for w is a contraction word of P (Lemma 4.18). This is a contradiction to the assumption $w \in \langle R\rangle$, since $P \not\subseteq R$. □

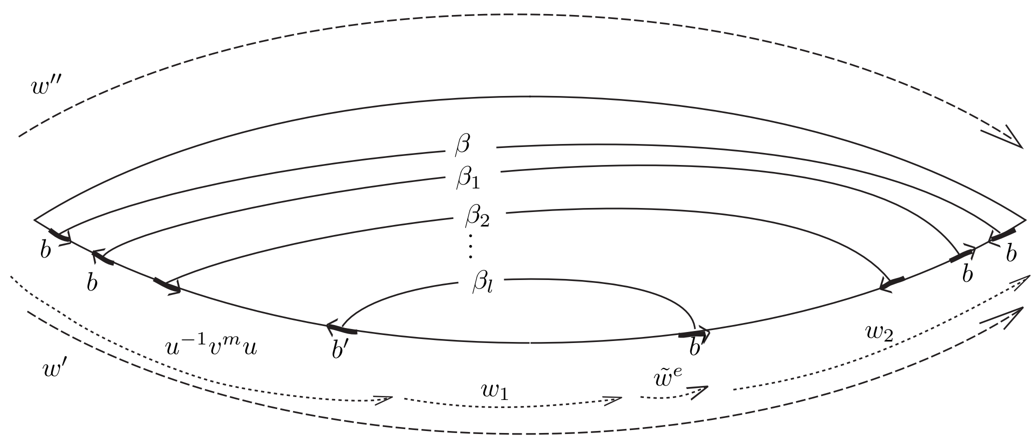


Figure 4.6: Δ in the proof of Lemma 4.21.

Lemma 4.22 (co-contraction induced embedding by contraction word)**.** *Let Γ be a graph, B be an anticonnected set of vertices of Γ and g be a contraction element of*

B. Then there exists an injective map $\phi : A(\overline{CO}(\Gamma, B)) \to A(\Gamma)$ *satisfying*

$$\phi(x) = \begin{cases} g & \textit{if } x = v_B \\ x & \textit{if } x \in V(\Gamma) \setminus B \end{cases}$$

Proof)

As in the proof of Lemma 4.9, let $\hat{\Gamma} = \overline{\mathrm{CO}}(\Gamma, B)$, $\hat{v} = v_B$ and $A = \{q : q \in V(\Gamma) \setminus B\}$. For $q \in A$, let $\hat{q}$ denote the corresponding vertex in $\hat{\Gamma}$, and $\hat{A} = \{\hat{q} : q \in A\}$. There exists a map $\phi : A(\hat{\Gamma}) \to A(\Gamma)$ satisfying

$$\phi(x) = \begin{cases} g & \text{if } x = \hat{v} \\ q & \text{if } x = \hat{q} \in \hat{A} \end{cases}$$

To prove that ϕ is injective, we use an induction on $|A|$.

If $A = \varnothing$, then $V(\Gamma) = B$ and $\hat{\Gamma}$ is the graph with one vertex $\hat{v}$. So, ϕ maps $\langle \hat{v} \rangle = A(\hat{\Gamma}) \cong \mathbb{Z}$ isomorphically onto $\mathbb{Z} \cong \langle g \rangle \leq A(\Gamma)$.

Assume the injectivity of ϕ for the case when $|A| = k$, and now let $|A| = k+1$.

Choose any $t \in A$. Let $A_0 = A \setminus \{t\}$ and $\hat{A}_0 = \{\hat{q} : q \in A_0\}$. Let Γ_0 be the induced subgraph on $A_0 \cup B$ in Γ, and $\hat{\Gamma}_0$ be the induced subgraph on $\hat{A}_0 \cup \{\hat{v}\}$ in $\hat{\Gamma}$. We consider $A(\Gamma_0)$ and $A(\hat{\Gamma}_0)$ as subgroups of $A(\Gamma)$ and $A(\hat{\Gamma})$, respectively, so that $A(\Gamma_0) = \langle A_0, B \rangle$ and $A(\hat{\Gamma}_0) = \langle \hat{A}_0, \hat{v} \rangle$. Let $K = \langle A_0, g \rangle = \phi(A(\hat{\Gamma}_0))$ and $J = \langle A, g \rangle = \phi(A(\hat{\Gamma}))$. By the inductive hypothesis, ϕ maps $A(\hat{\Gamma}_0)$ isomorphically onto K (Figure 4.7).

From Lemma 2.24, we can identify $A(\Gamma) = A(\Gamma_0)*_C$, where $C = \langle \mathrm{Link}_\Gamma(t) \rangle$ and t is the stable letter. Since $J = \langle A_0, g, t \rangle = \langle K, t \rangle$, Lemma 2.25 implies that we can also identify $J = (J \cap A(\Gamma_0))*_{J\cap C}$, where t is the stable letter again. Also, we identify $A(\hat{\Gamma}) = A(\hat{\Gamma}_0)*_D$, where $D = \langle \mathrm{Link}_{\hat{\Gamma}}(\hat{t})\rangle$ and $\hat{t}$ is the stable letter.

By Lemma 4.21 (2),

$$J \cap A(\Gamma_0) = \langle g, A\rangle \cap \langle A_0, B\rangle = \langle g, A \cap (A_0 \cup B)\rangle = \langle g, A_0\rangle = K = \phi(A(\hat{\Gamma}_0))$$

Applying Lemma 4.21 (3) for the case when $R = \mathrm{Link}_\Gamma(t)$,

$$\begin{aligned} J \cap C &= \langle g, A\rangle \cap \langle \mathrm{Link}_\Gamma(t)\rangle \\ &= \begin{cases} \langle \mathrm{Link}_\Gamma(t) \cap A, g\rangle & \text{if } B \subseteq \mathrm{Link}_\Gamma(t) \\ \langle \mathrm{Link}_\Gamma(t) \cap A\rangle & \text{otherwise} \end{cases} \end{aligned}$$

From the definition of co-contraction, we note that

$$D = \mathrm{Link}_{\hat{\Gamma}}(\hat{t}) = \begin{cases} \{\hat{q} : q \in \mathrm{Link}_\Gamma(t) \cap A\} \cup \{\hat{v}\} & \text{if } B \subseteq \mathrm{Link}_\Gamma(t) \\ \{\hat{q} : q \in \mathrm{Link}_\Gamma(t) \cap A\} & \text{otherwise} \end{cases}$$

Hence, $J \cap C = \phi(D)$. This implies that $\phi : A(\hat{\Gamma}) \to J$ is an isomorphism, as follows.

$$\begin{array}{ccccccc} D & \leq & A(\hat{\Gamma}_0) & \leq & A(\hat{\Gamma}_0)*_D & = & A(\hat{\Gamma}) \\ \downarrow \cong & & \downarrow \cong & & & & \downarrow \phi \\ J \cap C & \leq & K = J \cap A(\Gamma_0) & \leq & (J \cap A(\Gamma_0))*_{J\cap C} & = & J \end{array}$$

□

By an induction on m, one can prove the following generalization of Theorem 4.10.

Theorem 4.23. *Let Γ be a graph and $B_1, B_2, \ldots, B_m$ be disjoint subsets of $V(\Gamma)$ such that each B_i is anticonnected. For each i, let v_{B_i} denote the contracted vertex correponding to B_i in $\overline{CO}(\Gamma, (B_1, B_2, \ldots, B_m))$, and g_i be a contraction element of*

B_i. *Then there exists an injective map* $\phi : A(\overline{CO}(\Gamma, (B_1, B_2, \ldots, B_m))) \to A(\Gamma)$ *satisfying*

$$\phi(x) = \begin{cases} g_i & \textit{if } x = v_{B_i}\ ,\ \textit{for some } i \\ x & \textit{if } x \in V(\Gamma) \setminus \cup_{i=1}^{m} B_i \end{cases}$$

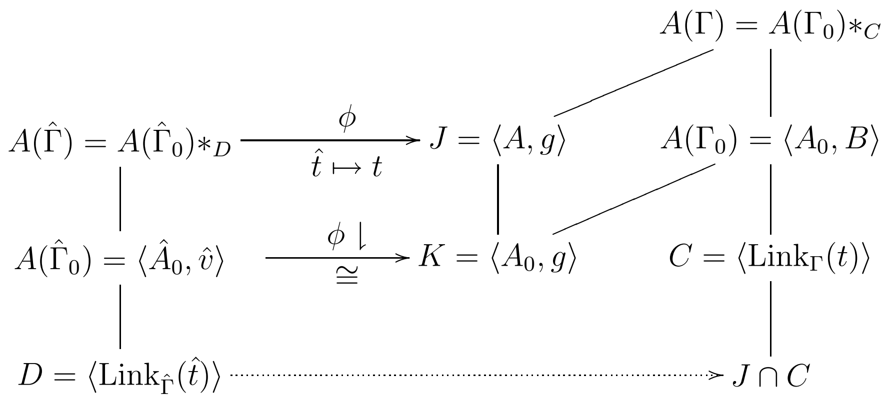


Figure 4.7: Proof of Lemma 4.22. Note that $V(\Gamma) = A \sqcup B = A_0 \cup \{t\} \cup B$ and $V(\hat{\Gamma}) = \hat{A} \sqcup \{\hat{v}\} = \hat{A}_0 \cup \{\hat{t}\} \cup \{\hat{v}\}$.

Appendix: Proof of Theorem 4.7

Lemma 4.24. *Let* Γ *be a graph and* $\{a, b\}$ *be an edge of* Γ*. If* $CO(\Gamma, \{a, b\})$ *contains an induced* C_n *for some* $n \geq 3$*, then* Γ *contains an induced* C_n *or* C_{n+1}*.*

Proof) Put $\hat{\Gamma} = \mathrm{CO}(\Gamma, \{a, b\})$. Let $\hat{v}$ denote the contracted vertex $v_{\{a,b\}}$ in $\hat{\Gamma}$. Let Γ' be the pre-image of an induced C_n in $\hat{\Gamma}$. By Remark 4.2 (2), a contraction of Γ' is C_n. So it suffices to consider only the case when $\hat{\Gamma}$ itself is C_n.

Label the vertices of the $\hat{\Gamma}$ as $\hat{v}, a_1, a_2, \ldots, a_{n-1}$, such that a_i and a_{i+1} are adjacent for $i = 1, 2, \ldots, n-2$, and $\hat{v}$ is adjacent to a_1 and a_{n-1}. By the definition of the contraction, neither a nor b is adjacent to any of $a_2, a_3, \ldots, a_{n-2}$. Also, each of a_1 and a_{n-1} is adjacent to at least one of a or b.

Case 1. a or b is adjacent to both of a_1 and a_{n-1}.

We may assume a is adjacent to both of a_1 and a_{n-1}. In this case, the vertices $a, a_1, a_2, a_3, \ldots, a_{n-1}$ span an induced C_n.

Case 2. Each of a and b is adjacent to exactly one of a_1 and a_{n-1}.

Again, we may assume that a is adjacent to a_1, but not to a_{n-1}. This will imply that b is adjacent to a_{n-1}, but not to a_1. In this case, $a, a_1, a_2, \ldots, a_{n-1}, b$ span an induced C_{n+1}. □

Lemma 4.25. *Let* Γ *be a graph, and* $\{a, b\}$ *be a pair of non-adjacent vertices of* Γ*. Put* $\hat{\Gamma} = \overline{CO}(\Gamma, \{a, b\})$*.*

(1) If $\hat{\Gamma}$ *contains an induced* $\overline{C_n}$*, then* Γ *contains an induced* $\overline{C_n}$ *or* $\overline{C_{n+1}}$*.*

(2) If $\hat{\Gamma}$ contains an induced C_n for some $n \geq 6$, then Γ contains an induced C_m for some $5 \leq m \leq n$.

Proof)

Let $\hat{v}$ denote the contracted vertex $v_{\{a,b\}}$ in $\hat{\Gamma}$.

(1) Immediate from Lemma 4.24 by considering the complement graphs.

(2) As in the proof of Lemma 4.24, it suffices to consider the case $\hat{\Gamma} = C_n$ (Remark 4.2).

Label the vertices of the $\hat{\Gamma}$ as $\hat{v}, a_1, a_2, \ldots, a_{n-1}$, such that a_i and a_{i+1} are adjacent for $i = 1, 2, \ldots, n-2$, and $\hat{v}$ is adjacent to both a_1 and a_{n-1}. By the definition of the co-contraction, each of a and b is adjacent to both a_1 and a_{n-1}, and for $i = 2, 3, \ldots, n-2$, each a_i cannot be adjacent to both of a and b.

Now suppose that Γ does not have an induced C_m for any $5 \leq m \leq n$. We show the following claims.

Claim 1. *Let $2 \leq i \leq n-3$. Then each of a and b is adjacent to at least one of a_i and a_{i+1}.*

Assume neither a_i nor a_{i+1} is adjacent to a. Note that a_{i-1}, a_i, a_{i+1} and a_{i+2} span an induced path γ_1 of length 3 in Γ. Also, the induced subgraph of Γ on

$$\{a_{i+2}, a_{i+3}, \ldots, a_{n-1}, a, a_1, a_2, \ldots, a_{i-1}\}$$

contains an induced path γ_2 from a_{i+2} to a_{i-1}. Since a_{i-1} are a_{i+2} are not adjacent in Γ, the length of γ_2 is at least 2, and at most $n-3$. Then the subgraph $\gamma_1 \cup \gamma_2$ of Γ

is an induced C_m for some $5 \leq m \leq n$, which is a contradiction. The same argument applies for b in place of a.

Claim 2. *For $2 \leq i \leq n-2$, each a_i is adjacent to exactly one of a or b.*

We know that a_i cannot be adjacent to both of a and b. Suppose a_i is non-adjacent to both of a and b. If $i = 2, 3, \ldots, n-3$, then by Claim 1, a_{i+1} must be adjacent to both of a and b, which is a contradiction. If $i = n-2$, the same reasoning shows that a_{n-3} is adjacent to both of a and b, which is also a contradiction.

By the above claims, we may assume that $a_2, a_4, \ldots$ are adjacent to a, and $a_3, a_5, \ldots$ are adjacent to b. Since $n \geq 6$, the vertices a, a_1, b, a_3, a_4 span an induced C_5 (Figure 4.8).□

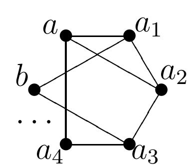


Figure 4.8: Proof of Lemma 4.25 (1).

Proof of Theorem 4.7. Let Γ be weakly chordal, B be anticonnected and $\hat{\Gamma} = \overline{\mathrm{CO}}(\Gamma, B)$. We show that $\hat{\Gamma}$ is weakly chordal. The case when $|B| = 2$ follows from Lemma 4.25. The general case follows by an induction, since $\hat{\Gamma}$ can be obtained from Γ by a succession of co-contractions (Lemma 4.4).

So we see that $\mathcal{W}$ is closed under co-contraction. $\mathcal{W}$ is closed under contraction also, since $\mathcal{W}$ is closed under taking complement graphs. □

Chapter 5

The Graph Classes $\mathcal{N}$ and $\mathcal{N}_\infty$

Recall that $\mathcal{N}$ is the class of graphs Γ such that $A(\Gamma)$ does not contain a hyperbolic surface group. In the previous chapters, we have seen that

(i) $\mathcal{N} \subseteq \mathcal{W}$, where $\mathcal{W}$ is the class of weakly chordal graphs (Corollary 4.12).

(ii) $\mathcal{N}$ is closed under co-contraction (Corollary 4.11).

In this chapter, we find a subclass $\mathcal{F}$ of $\mathcal{N}$, using graph-theoretic properties of $\mathcal{N}$ (Definition 5.11). $\mathcal{F}$ is large enough to contain any *chordal* graphs and *chordal-bipartite* graphs. This might be considered as a step toward combinatorial characterization of the graph class $\mathcal{N}$. For our purpose, it turns out to be more useful to consider another graph class $\mathcal{N}_\infty$, which is a "relative version" of $\mathcal{N}$.

5.1 Chordal and chordal bipartite graphs

In this section, we describe *chordal graphs* and *chordal bipartite graphs* ([Dir61], [GG78]). For the reader's convenience, we give proofs to their well-known properties.

Definition 5.1 ([Dir61, GG78]). (i) A graph Γ is *chordal* if Γ does not contain an induced cycle of length at least 4.

(ii) A graph Γ is *chordal bipartite*, if Γ is bipartite and Γ does not contain an induced cycle of length at least 6.

Remark 5.2. Any anti-cycle of length at least 6 contains an induced triangle and an induced square (Figure 4.1). So if a graph is chordal, then it is weakly chordal and does not contain an induced square. The converse is also true by the definition. Similarly, a graph is chordal bipartite, if and only if it is weakly chordal and does not contain an induced triangle. In particular, chordal graphs and chordal bipartite graphs are weakly chordal.

Definition 5.3 (bisimplicial edge). An edge $\{a, b\}$ of Γ is *bisimplicial*, if for any $a' \in \mathrm{Link}(a)$ and $b' \in \mathrm{Link}(b)$, either $a' = b'$ or $\{a', b'\} \in E(\Gamma)$.

One can easily verify that any induced subgraph of chordal (resp. chordal bipartite) graph is chordal (resp. chordal bipartite). Also, if Γ is chordal bipartite and e is a bisimplicial edge, then $\Gamma \setminus \mathring{e}$ is chordal bipartite. So the following theorem gives a recursive characterization of chordal and chordal bipartite graphs. We give a complete proof, following [CRS02] in the appendix to this chapter. A recursive formulation of the graph class $\mathcal{F}$ (Definition 5.11) is motivated by Theorem 5.4.

Theorem 5.4 ([Dir61],[GG78]). *Let Γ be a graph.*

(1) If Γ is chordal but not complete, then there exist $\Gamma_1, \Gamma_2 \lneq \Gamma$ such that $\Gamma_1 \cap \Gamma_2$ is complete, and $\Gamma = \Gamma_1 \cup \Gamma_2$.

(2) If Γ is chordal bipartite but not discrete, then Γ contains a bisimplicial edge.

5.2 A characterization of $\mathcal{N}_\infty$

In this section, we define and study various combinatorial properties of the graph class $\mathcal{N}_\infty$. As a result, we give a recursively defined graph class $\mathcal{F}$ as a lower bound for $\mathcal{N}$ and $\mathcal{N}_\infty$. This section essentially concludes the thesis, postponing the proofs of two crucial lemmas to Section 5.3 and 5.4.

Definition 5.5. $\mathcal{N}_\infty$ is the set of the graphs Γ such that there does not exist a π_1-injective map $f : S \to X_\Gamma$ from a compact hyperbolic surface S satisfying the following.

> For each boundary component $\partial_0 S$ of S, there exists $K \in \mathcal{K}(\Gamma)$, such that $f(\partial_0 S) \subseteq X_K$.

Lemma 5.6 (normalization)**.** *$\Gamma \notin \mathcal{N}$ ($\Gamma \notin \mathcal{N}_\infty$, respectively) if and only if there exists a normalized label-reading pair on a closed hyperbolic surface (a compact hyperbolic surface, respectively) with the underlying graph Γ.*

Proof) The statement for $\mathcal{N}$ is trivial from Lemma 2.39 ([CW04]).

Now suppose $(\mathcal{H}, \lambda)$ is a normalized label-reading pair on a compact hyperbolic surface S with the underlying graph Γ. By Remark 3.17, $(\mathcal{H}, \lambda)$ is cellular and so, induces a unique cubical map $f : X(S, \mathcal{H}) \to A(\Gamma)$. Let $\partial_0 S$ be a boundary component of S. Since $(\mathcal{H}, \lambda)$ is regular, there exists a complete graph $K \leq \Gamma$ such that $\partial_0 S$ intersects only with the curves labeled by vertices in $V(K)$. Hence, the edge-path $f(\partial_0 S)$ is contained in X_K.

Conversely, suppose Γ is a graph, and $f : S \to X_\Gamma$ is a π_1-injective map such that for each boundary component $\partial_0 S$, we can find a complete subgraph $K \leq \Gamma$ satisfying $f(\partial_0 S) \subseteq X_K$. By Proposition 2.41, f_* is a label-reading map associated with a label-reading pair $(\mathcal{H}, \lambda)$, such that no two curves of the same label intersect. We have only to show that $(\mathcal{H}, \lambda)$ can be chosen to be regular. Let d_i be an element of $\pi_1(S)$ corresponding to a boundary component $\partial_i S$ of S. There exists a complete graph $K \leq \Gamma$ such that $f(\partial_i S) \subseteq X_K$. So one can write $f_*(d_i) = w_i'^{-1} w_i w_i'$, where w_i is a word of the letters in $V(K) \cup V(K)^{-1}$. By Proposition 2.41 again, we may choose $(\mathcal{H}, \lambda)$ such that any curve in $\mathcal{H}$ intersecting with $\partial_i S$ is labeled by $V(K)$. So $(\mathcal{H}, \lambda)$ is regular.$\square$

Lemma 5.7. *$K_1 \in \mathcal{N}_\infty$.*

Proof) Note that $A(K_1) \cong \mathbb{Z}$ does not contain any hyperbolic surface group or non-abelian free group.$\square$

Recall that for two graphs Γ_1 and Γ_2,

$$\mathrm{Join}(\Gamma_1, \Gamma_2) = \overline{\overline{\Gamma_1} \sqcup \overline{\Gamma_2}}$$

Lemma 5.8 (closed under join)**.** *If $\Gamma_1, \Gamma_2 \in \mathcal{N}_\infty$, then Join$(\Gamma_1, \Gamma_2) \in \mathcal{N}_\infty$.*

Proof) Suppose $\Gamma = \mathrm{Join}(\Gamma_1, \Gamma_2) \notin \mathcal{N}_\infty$. There exists a normalized label-reading pair $(\mathcal{H}, \lambda)$ with the underlying graph Γ on a hyperbolic surface S, and an associated label-reading map $\phi : \pi_1(S) \to A(\Gamma) \cong A(\Gamma_1) \times A(\Gamma_2)$. Let $p_i : A(\Gamma) \to A(\Gamma_i)$ be the natural projection map.

We claim that $p_1 \circ \phi$ or $p_2 \circ \phi$ is injective. Suppose not, and choose $a_1 \in \ker(p_1 \circ \phi) \setminus \{1\}$ and $a_2 \in \ker(p_2 \circ \phi) \setminus \{1\}$. Write $\phi(a_1) = (1, b_2)$ and $\phi(a_2) = (b_1, 1)$ for some $b_1 \in A(\Gamma_1) \setminus \{1\}$ and $b_2 \in A(\Gamma_2) \setminus \{1\}$. Then

$$\phi[a_1, a_2] = [\phi(a_1), \phi(a_2)] = [(1, b_2), (b_1, 1)] = 1$$

Since S is hyperbolic, $a_1, a_2 \in \langle c \rangle$ for some $c \in \pi_1(S)$ (Lemma 2.32). This would imply

$$\mathbb{Z} \times \mathbb{Z} \cong \langle (1, b_2), (b_1, 1) \rangle = \langle \phi(a_1), \phi(a_2) \rangle \subseteq \langle \phi(c) \rangle$$

which is a contradiction. Hence, we may assume that $\ker(p_1 \circ \phi) = 1$, i.e. $p_1 \circ \phi$ is injective. This means that the label-reading map $\pi_1(S) \to A(\Gamma_1)$ obtained by removing curves in $\mathcal{H}$ labeled by $V(\Gamma_2)$ is injective. So $\Gamma_1 \notin \mathcal{N}_\infty$.$\square$

From the above lemmas, we note that any complete graphs and complete bipartite graphs are in $\mathcal{N}_\infty$. Now we state two key lemmas concerning properties of $\mathcal{N}_\infty$. Their proofs are given in Section 5.3 and 5.4, respectively.

Lemma 5.9 (complete graph amalgamation)**.** *Let Γ be a graph. Suppose $\Gamma_1, \Gamma_2 \leq \Gamma$ satisfies*

(i) $\Gamma_1 \cup \Gamma_2 = \Gamma$*,*

(ii) $\Gamma_1 \cap \Gamma_2$ *is complete, and*

(iii) $\Gamma_1, \Gamma_2 \in \mathcal{N}_\infty$*.*

Then $\Gamma \in \mathcal{N}_\infty$*.*

Lemma 5.10 (bisimplicial edge addition)**.** *Let Γ be a graph with a bisimplicial edge e. If $\Gamma \setminus \mathring{e} \in \mathcal{N}_\infty$, then $\Gamma \in \mathcal{N}_\infty$.*

Definition 5.11. We let $\mathcal{F}$ denote the smallest family of the graphs satisfying the following conditions.

(i) $K_1 = \bullet \in \mathcal{F}$.

(ii) $\Gamma_1, \Gamma_2 \in \mathcal{F}$, then $\text{Join}(\Gamma_1, \Gamma_2) \in \mathcal{F}$.

(iii) If $\Gamma_1, \Gamma_2 \in \mathcal{F}$ and $\Gamma_1 \cap \Gamma_2 = K_n$ for some $n \geq 0$, then $\Gamma_1 \cup \Gamma_2 \in \mathcal{F}$.

(iv) Suppose e is a bisimplicial edge of a graph Γ. If $\Gamma \setminus \mathring{e} \in \mathcal{F}$, then $\Gamma \in \mathcal{F}$.

(v) $\Gamma \in \mathcal{F}$ and B is an anticonnected subset of $V(\Gamma)$, then $\overline{\text{CO}}(\Gamma, B) \in \mathcal{F}$.

Remark 5.12. (i) Every complete graph is in $\mathcal{F}$, by (i) and (ii). Applying Theorem 5.4 and (iii), we see that every chordal graph is in $\mathcal{F}$.

(ii) By (i) and (iii) (with $n = 0$), every discrete graph is in $\mathcal{F}$. From Theorem 5.4 and (iv), it follows that every chordal bipartite graph is in $\mathcal{F}$.

From the lemmas in this section and Lemma 4.11, $\mathcal{N}_\infty$ satisfies the conditions (i) through (v) in Definition 5.11. By Corollary 4.12, we have $\mathcal{N} \subseteq \mathcal{W}$. So the following theorem summarizes the results on right-angled Artin groups included in this thesis.

Theorem 5.13 (bounds for $\mathcal{N}$)**.** $\mathcal{F} \subseteq \mathcal{N}_\infty \subseteq \mathcal{N} \subseteq \mathcal{W}$.

Corollary 5.14. *Any chordal graphs and chordal bipartite graphs are in $\mathcal{N}_\infty$. Hence, the right-angled Artin groups on such graphs do not contain hyperbolic surface groups.*

It is not hard to verify that $\mathcal{W}$ also satisfies the conditions of Definition 5.11 (see Theorem 4.7). This naturally leads to the following questions.

Question 5.15. *(1) Is $\mathcal{N}_\infty = \mathcal{W}$?*

(2) More weakly, is $\mathcal{N}_\infty = \mathcal{N}$?

(3) Note that $\mathcal{W}$ is closed under taking complement graphs. If $A(\Gamma)$ contains a hyperbolic surface group, does $A(\bar{\Gamma})$ necessarily contain a hyperbolic surface group?

Remark 5.16. $\overline{P_6}$ is weakly chordal (Figure 5.1). It is an interesting question whether $A(\overline{P_6})$ contains a hyperbolic surface group or not.

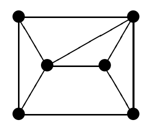


Figure 5.1: $\overline{P_6}$

In Section 5.3, we will see that an affirmative answer to the Question 5.15 (2) would follow from Conjecture 5.22, which is seemingly simpler.

5.3 Complete graph amalgamation

In this section, we prove Lemma 5.9.

Recall the convention that for a graph Γ, $X_\varnothing$ denotes the 0-skeleton of X_Γ.

Lemma 5.17 (disjoint union)**.** *Let* $\Gamma_1, \Gamma_2 \in \mathcal{N}$. *Then* $\Gamma_1 \sqcup \Gamma_2 \in \mathcal{N}$.

Proof) Suppose there exists an embedding of a hyperbolic surface group $\phi : \pi_1(S) \to A(\Gamma_1 \sqcup \Gamma_2) \cong A(\Gamma_1) * A(\Gamma_2)$. Note that $\pi_1(S)$ is freely indecomposable [LS77, Proposition 5.14]. By Kurosh subgroup theorem, $\pi_1(S)$ embeds into $A(\Gamma_1)$ or $A(\Gamma_2)$.$\square$

Lemma 5.18 (complete graph amalgamation, weaker version)**.** *Suppose* Γ *is a graph, such that* $\Gamma = \Gamma_1 \cup \Gamma_2$ *for some induced subgraphs* Γ_1 *and* Γ_2. *Furthermore, assume* *that* $\Gamma_1 \cap \Gamma_2 \cong K_n$ *for some* $n \geq 0$*. If* $\Gamma_1, \Gamma_2 \in \mathcal{N}_\infty$*, then* $\Gamma \in \mathcal{N}$

Proof) Let $K = \Gamma_1 \cap \Gamma_2$. X_Γ is homotopic to $X_{\Gamma_1} \cup X_K \times I \cup X_{\Gamma_2}$ amalgamated by the embeddings $X_K \times \{0\} \to X_{\Gamma_1}$ and $X_K \times \{1\} \to X_{\Gamma_2}$. Now suppose $\Gamma \notin \mathcal{N}$. Then there exists a π_1-injective map f from a closed surface S to X_Γ. By transversality (Lemma 2.40), we may assume that $f^{-1}(X_K \times \{\frac{1}{2}\})$ is a disjoint set of simple closed curves. f restricts to π_1-injective maps from the components of $S \setminus f^{-1}(X_K \times \{\frac{1}{2}\})$ into X_{Γ_i}, for $i = 1$ or 2. One of the components must be a hyperbolic surface and so, $\Gamma_i \notin \mathcal{N}_\infty$ for $i = 1$ or 2. $\square$

From Lemma 5.18, we see that it is more natural to consider $\mathcal{N}_\infty$ in studying hyperbolic surface subgroups of $A(\Gamma)$. Recall that for a given label-reading pair $(\mathcal{H}, \lambda)$ on S, we say that two curves α, β are *sufficiently close* if they are homotopic, transversely intersect $\mathcal{H}$ (or in $\mathcal{H}$), and any curve in $\mathcal{H}$ transversely intersecting with one of α and β transversely intersects with the other also.

Definition 5.19 ([Gol04])**.** A vertex q of a graph Γ is called *simplicial*, if Link(q) is a complete graph.

Lemma 5.20 (simplicial vertex addition, for $\mathcal{N}_\infty$)**.** *Let* Γ *be a graph. Suppose* $\{q_1, \ldots, q_r\}$ *is a set of pairwise non-adjacent simplicial vertices in* Γ*. Define* $\Gamma' = \Gamma \setminus \cup_{j=1}^r \mathring{Star}(q_j)$*. If* $\Gamma' \in \mathcal{N}_\infty$*, then* $\Gamma \in \mathcal{N}_\infty$*.*

Proof) Assume that $\Gamma \notin \mathcal{N}_\infty$, and we prove $\Gamma' \notin \mathcal{N}_\infty$.

First consider the case when $r = 1$, and let $q = q_1$.

Choose a normalized label-reading pair $(\mathcal{H}, \lambda)$ with the underlying graph Γ on a hyperbolic surface S, and let $\phi : \pi_1(S) \to A(\Gamma)$ be an associated injective label-reading map. Also, let $f : X(S, \mathcal{H}) \to X_\Gamma$ be the associated cubical map.

Put $\mathcal{H}_q = \lambda^{-1}(q)$.

Case 1. $\mathcal{H}_q$ consists of simple closed curves.

Choose a connected component S' of $S \setminus (\cup \mathcal{H}_q)$, so that S' is hyperbolic. The curves in the set $(\cup\mathcal{H}) \cap S'$ naturally inherit transverse orientations and labels from those of $(\mathcal{H}, \lambda)$, and so, determine a label-reading pair $(\mathcal{H}', \lambda')$ with the underlying graph Γ' on S'. Note that $\mathcal{H}'$ does not have any q-curve. Let ϕ' be an associated label-reading map with respect to $(\mathcal{H}', \lambda')$. We have the following commutative diagram with a suitable choice of the basepoint.

$$\begin{array}{ccc} \pi_1(S') & \xrightarrow{\phi'} & A(\Gamma') \\ \downarrow{\scriptstyle incl_*} & & \downarrow{\scriptstyle incl} \\ \pi_1(S) & \xrightarrow{\phi} & A(\Gamma) \end{array}$$

Since ϕ and $incl_* : \pi_1(S') \to \pi_1(S)$ are injective, so is ϕ'.

A simple closed curve in $\mathcal{H}_q$ intersects with a curve in $\mathcal{H}$ labeled by a vertex in $\mathrm{Link}(q)$. So each boundary component $\partial_i S'$ of S' either is a boundary component of S, or comes from a curve in $\mathcal{H}_q$. In the latter case, any curve in $\mathcal{H}'$ intersecting with $\partial_i S'$ must be labeled by a vertex in $\mathrm{Link}(q)$. Since $\mathrm{Link}(q)$ spans a complete graph in Γ', $\Gamma' \notin \mathcal{N}_\infty$.

Case 2. $\mathcal{H}_q$ contains a properly embedded arc γ. Suppose γ joins a boundary components $\partial_1 S$ to another $\partial_2 S$. By Lemma 3.16, $\partial_1 S \neq \partial_2 S$. Also, any curve intersecting with $\partial_i S$ has a label in $\mathrm{Link}(q) \cup \{q\}$, for $i = 1, 2$.

Choose $\gamma' \sim \gamma$, which transversely intersects $\mathcal{H}$, and assume that γ' is sufficiently close to γ. Then $w_{\gamma'} \in \langle \text{Link}(q) \rangle$.

Consider $\partial_i S$ as a loop based at $\gamma' \cap \partial_i S$, for $i = 1, 2$. Put $\alpha = \partial_1 S$, and $\beta = \gamma' \cdot \partial_2 S \cdot \gamma'^{-1}$ (Figure 5.2). Then $w_\alpha \in \langle \text{Link}(q) \cup \{q\} \rangle$, and $w_\beta = w_{\gamma'} w_{\partial_1 S} w_{\gamma'}{}^{-1} \in \langle \text{Link}(q) \cup \{q\} \rangle$. Hence $\phi([[\alpha], [\beta]]) = [w_\alpha, w_\beta] = 1$. Since ϕ is injective, $\alpha \sim \beta^{\pm 1}$, which is impossible unless S is an annulus (Lemma 2.32).

Now consider the case when $r > 1$. Note that q_r is a simplicial vertex of $\Gamma \setminus \cup \mathring{\text{Star}}_{i<r} q_i$. By induction, we see that $\Gamma' \notin \mathcal{N}_\infty$. $\square$

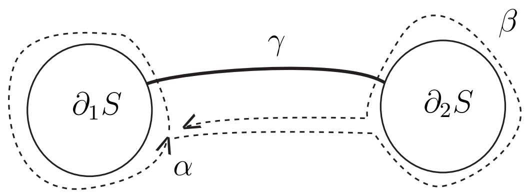


Figure 5.2: Proof of Lemma 5.20

Remark 5.21. Since every chordal graph has a simplicial vertex (Theorem 5.34), it already follows that all chordal graphs are in $\mathcal{N}_\infty$.

It is not known if Lemma 5.20 is still true if $\mathcal{N}_\infty$ is replaced by $\mathcal{N}$. Namely,

Conjecture 5.22 (simplicial vertex addition, for $\mathcal{N}$)**.** *Let q be a simplicial vertex of a graph Γ, and Γ' be the induced subgraph of Γ on $V(\Gamma) \setminus \{q\}$. If $\Gamma' \in \mathcal{N}$, then $\Gamma \in \mathcal{N}$.*

Lemma 5.23 (eventual nontriviality)**.** *Suppose we have a group presentation*

$$F = \langle x_1, \ldots, x_s, y_1, \ldots, y_n | X y_1 y_2 \cdots y_n = 1 \rangle$$

where $n \geq 1$ and X is a word of $\{x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_s^{\pm 1}\}$. Choose $u_1, \ldots, u_{2l+1}, v_1, \ldots, v_{2l} \in$

F such that

(i) $v_1, \ldots, v_{2l} \in \{y_1, \ldots, y_n\}$

(ii) If $v_i = v_{i+1}$, *then* $u_{i+1} \notin \langle v_i \rangle$.

Then there exists M_0, such that for any $r > M_0$, the word

$$f(r) = u_1 v_1^r u_2 v_2^{-r} \cdots u_{2l} v_{2l}^{-r} u_{2l+1}$$

is non-trivial in F.

Proof)

For the convenience of notations, we prove the lemma for $(n+1)$ in place of n. That is, we let $F = \langle x_1, \ldots, x_s, y_1, \ldots, y_{n+1} | X y_1 y_2 \cdots y_{n+1} = 1 \rangle$, for some $n \geq 0$, and $v_1, \ldots, v_{2l} \in \{y_1, \ldots, y_{n+1}\}$. We consider F as the free group $\langle x_1, \ldots, x_s, y_1, \ldots, y_n | - \rangle$ and y_{n+1} as the word $y_{n+1} = Y = (X y_1 y_2 \cdots y_n)^{-1}$, so that $v_1, \ldots, v_{2l} \in \{y_1, \ldots, y_n, Y\}$.

Regard X, Y, u_i and v_i as (freely) reduced words. Note that $v_i^{\pm r}$ is a cyclically reduced words for each i. Let $l : F \to \mathbb{N} \cup \{0\}$ be the length fuction. Put $M = \max_i \{l(u_i)\}$ and choose any $M_0 > 2M$. Suppose $r > M_0$ and $f(r) = 1$. Consider a dual van Kampen diagram $\Delta = (\mathcal{H}, \lambda)$ of the word $f(r) = u_1 v_1^r \cdots u_{2l} v_{2l}^{-r} u_{2l+1}$ in F, where F is considered as a right-angled Artin group on the discrete graph with vertices $\{x_1, \ldots, x_s, y_1, \ldots, y_n\}$. All the curves in $\mathcal{H}$ are disjoint, and hence, any subword between a cancelling pair is trivial in F.

Consider any subword $v_i^{\pm r}$ of $f(r)$. Write $v_i^{\pm r} = w_1 \cdot w_2 \cdot w_3$, where $w_1 = w_3 = v_i^{\pm M}$, and $w_2 = v_i^{\pm(r-2M)}$. Any letter $y_j^{\pm 1}$ in the interval w_2 is called a *mid-word letter of* $f(r)$. Since $r > M_0 > 2M$, every $v_i^{\pm r}$ in $f(r)$ contains a mid-word letter.

Let γ be an arc in $\mathcal{H}$, intersecting with a mid-word letter, say y_j or ${y_j}^{-1}$ for some j. May assume γ is innermost among the arcs intersecting with mid-word letters. Without loss of generality, that mid-word letter y_j or ${y_j}^{-1}$ is assumed to be contained in a subword v_i^r, rather than v_i^{-r}, for some i. From now on, these i and j are fixed.

Case 1. $v_i = y_j$.

In this case, the mid-word letter in v_i^r is a y_j-segment.

Claim 1. *The ${y_j}^{-1}$-segment, joined by γ, is contained in some $v_{i'}^{\pm r}$.*

Suppose it is in some $u_{i'}$. Since γ is chosen to be innermost, and each $v_k^{\pm r}$ contains mid-word letters, $i' = i$ or $i' = i+1$. We may assume $i' = i+1$. One can write $u_{i+1} = u \cdot {y_j}^{-1} \cdot u'$ such that for some $p \geq 0$, the first and the last letter in the following interval (which is a subword of $v_i^r u_{i+1} = y_j^r u {y_j}^{-1} u'$) on $\partial\Delta$ are joined by γ:

$$y_j y_j^{M+p} u y_j^{-1}$$

Since the word between a cancelling pair is trivial in F, we have $y_j^{M+p} u = 1$, i.e. $u = y_j^{-(M+p)}$. This is a contradiction to the assumption that $l(u) \leq l(u_{i+1}) < M/2$. The claim is proved.

So the ${y_j}^{-1}$-segment that γ intersects is in some $v_{i'}^{\pm r}$. Moreover, we see from the above argument that $i' = i$ or $i' = i+1$. We may set $i' = i+1$ (Figure 5.3 (a)). We have $v_{i+1} = y_j$ or $v_{i+1} = Y$. Since that the word $Y^{-r} = (Xy_1 \cdots y_n)^r$ does not contain the letter y_j^{-1}, for $r > 0$, we have $v_{i+1} = y_j$. Then for some nonnegative integers p and q, the cancelling pair that γ joins consists of the first and the last letter in the following interval (which is a subword of $v_i^r u_{i+1} v_{i+1}^{-r} = y_j^r u_{i+1} y_j^{-r}$) in $\partial\Delta$ (Figure 5.3):

$$y_j y_j^{M+p} u_{i+1} y_j^{-q} y_j^{-1}$$

Since the word between a cancelling pair is trivial in F, $y_j^{M+p} u_{i+1} y_j^{-q} = 1$ But that would imply $v_i = y_j = v_{i+1}$ and $u_{i+1} = y_j^{q-(M+p)}$, which violates the restriction on $u_{i+1} \notin \langle v_i \rangle = \langle y_j \rangle$. This complete the proof for the *Case 1*.

Case 2. $v_i = Y$.

The proof for this case is similar, as follows. Let γ intersects with a mid-word letter ${y_j}^{-1}$ in the word $v_i^r = Y^r = (Xy_1 \cdots y_n)^{-r}$.

Claim 2. *The other letter joined by γ is in some $v_{i'}^{\pm r}$.*

Suppose not. Then γ intersects with u_i or u_{i+1}, say u_{i+1}, for γ is an innermost arc among those intersecting with a mid-word letter. As in *Case 1*, for some nonnegative integer p, one can write $u_{i+1} = u \cdot y_j \cdot u'$ such that the first and the last letter of the following interval (which is a subword of $v_i^r u_{i+1} = Y^r u y_j u'$) in $\partial\Delta$ is joined by γ.

$${y_j}^{-1} {y_{j-1}}^{-1} \cdots {y_1}^{-1} X^{-1} Y^{M+p} u y_j$$

So ${y_{j-1}}^{-1} {y_{j-2}}^{-1} \cdots X^{-1} Y^{M+\alpha} u = 1$. Since $X \in \langle x_1, \ldots, x_s \rangle$, ${y_{j-1}}^{-1} {y_{j-2}}^{-1} \cdots X^{-1} Y^{M+\alpha}$ has the length at least $M + \alpha$. This contradicts to $l(u) \leq l(u_{i+1}) \leq \frac{M}{2}$.

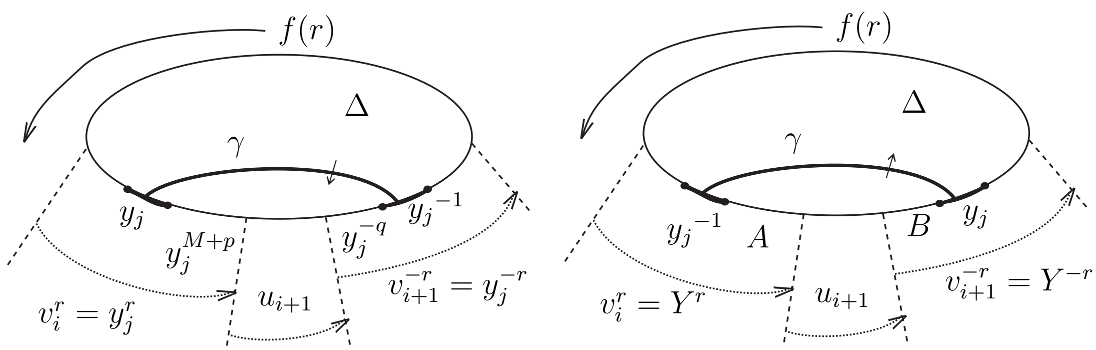
Now we may assume that the other letter y_j that γ intersects is contained in v_i or v_{i+1}, say v_{i+1}, without loss of generality (Figure 5.3 (b)). $v_{i+1} = y_j$ or $v_{i+1} = Y$. Since v_{i+1}^{-r} should contain the letter y_j and $r > 0$, we have $v_{i+1} = Y$. Again, the first and the last letter in the following interval (which is a subword of $v_i^r u_{i+1} v_i^{-r} = Y^r u_{i+1} Y^{-r}$) on $\partial\Delta$ are the cancelling pair that γ intersects with, for some $p, q \geq 0$:

$$y_j{}^{-1} y_{j-1}{}^{-1} \cdots y_1{}^{-1} X^{-1} Y^{M+p} u_{i+1} Y^{-q} X y_1 y_2 \cdots y_j$$

hence $y_{j-1}{}^{-1} \cdots y_1{}^{-1} X^{-1} Y^{M+p} u_{i+1} Y^{-q} X y_1 y_2 \cdots y_{j-1} = 1$ This implies $u_{i+1} = Y^{-(M+p)+q}$ As in Case 1, this is again a contradiction, since $v_i = Y = v_{i+1}$.

So we have $f(r) \neq 1$. □

Remark 5.24. When $v_i = Y$ for each i, this lemma becomes a special case of a proposition in [Bau62].



(a) Case 1. (b) Case 2.

Figure 5.3: Proof of Lemma 5.23. In (b), we let $A = y_{j-1}{}^{-1} \cdots y_1{}^{-1} X^{-1} Y^{M+p}$ and $B = Y^{-q} X y_1 y_2 \cdots y_{j-1}$.

For a compact surface S, recall our convention that the boundary components $\partial_1 S, \partial_2 S, \ldots, \partial_m S$ are oriented so that $\sum[\partial_i S] = 0$ in $H_1(S)$. We let $D(S)$ denote the double of S along its boundary.

Lemma 5.25 (eventual injectivity)**.** *Let S be a surface with the boundary components $\partial_1 S, \ldots, \partial_m S$, for some $m \geq 1$. Let $q : D(S) \to S$ be the natural quotient map. Define $T_i : D(S) \to D(S)$ to be the full Dehn twist about $\partial_i S$, and let $T = T_1 \circ \cdots \circ T_m$.*

Then, for any non-trivial $x \in \pi_1(D(S))$, there exists $r \geq 0$ such that $(q \circ T^r)_*(x)$ is non-trivial in $\pi_1(S)$.

Proof)

Fix a homeomorphism $g : S \to S'$, and write $D(S) = S \cup S'$, where each $\partial_i S$ is glued to $g(\partial_i S)$ for each i. Fix a basepoint $c \in \mathrm{Int}(S)$ of $\pi_1(D(S))$, a point $c_i \in \partial_i S$ for each i, and a path e_i from c to c_i for each i, such that $e_i \cap \partial S = \{c_i\}$ (Figure 5.4). Let $c' = g(c)$ be the basepoint of $\pi_1(S')$, and f_i be the path from c to c' obtained by juxtaposing e_i and $g(e_i)^{-1}$. We consider $\partial_i S$ as a loop based at c_i, and let δ_i be the loop in $\mathrm{Int}(S)$, homotopic to the concatenation $e_i \cdot \partial_i S \cdot e_i^{-1}$.

Now fix a non-trivial $x \in \pi_1(D(S))$. Let $i : S \to D(S)$ be the natural inclusion. If $x = i_*(y)$ for some $y \in \pi_1(S)$, then $q_*(x) = (q \circ i)_*(y) = y \neq 1$. So we may assume $x \notin i_*(\pi_1(S))$.

Claim 1. *For some $l \geq 1$, there exist loops $\gamma_1, \ldots, \gamma_{2l+1}$ and indices $i_1, \ldots, i_{2l}$ such that*

(i) γ_{2i-1} is a loop in Int(S), based at c, for each $i = 1, \ldots, l+1$,

(ii) γ_{2i} is a loop in Int(S'), based at c', for each $i = 1, \ldots, l$,

(iii) x is the homotopy class of the concatenated paths

$$\alpha = \gamma_1 \cdot f_{i_1} \cdot \gamma_2 \cdot f_{i_2}^{-1} \cdot \gamma_3 \cdot f_{i_3} \cdots \gamma_{2l} \cdot f_{i_{2l}}^{-1} \cdot \gamma_{2l+1}$$

Moreover, by choosing l to be minimal, we may assume that

(iv) if $i_{2j} = i_{2j+1}$, then $[\gamma_{2j+1}] \notin < [\delta_{i_{2j}}] > = < [\delta_{i_{2j+1}}] >$ in $\pi_1(S)$.

(v) *if* $i_{2j-1} = i_{2j}$, *then* $[\gamma_{2j}] \notin < [g(\delta_{i_{2j-1}})] > = < [g(\delta_{i_{2j+1}})] >$ *in* $\pi_1(S')$.

Finding a representative α for x satisfying (i),(ii) and (iii) is immediate, by making α transversely intersect ∂S. Note that $l \geq 1$, since we assume that $x \notin i_*(\pi_1(S))$. For (iv) and (v), note that $f_j^{-1} \cdot \delta_j^k \cdot f_j$ is homotopic, with the basepoint at c', to a loop $g(\delta_j)^k$, and similarly $f_j \cdot g(\delta_j)^k \cdot f_j^{-1}$ is homotopic, with the basepoint at c, to a loop δ_j^k. The claim follows from the minimality of l.

Now set $d_i = [\delta_i]$, for each i. We have set a presentation

$$\pi_1(S) = < x_1, \ldots, x_{2g}, d_1, \ldots, d_m | \prod_{i=1}^{g} [x_{2i-1}, x_{2i}] d_1 d_2 \cdots d_m = 1 >$$

From Claim 1, we can write

$$(q \circ T^r)_*(x) = [\gamma_1 \cdot \delta_{i_1}^r \cdot g^{-1}(\gamma_2) \cdot \delta_{i_2}^{-r} \cdot \gamma_3 \cdot \delta_{i_3}^r \cdots g^{-1}(\gamma_{2l}) \cdot \delta_{i_{2l}}^{-r} \cdot \gamma_{2l+1}]$$

So it follows that:

Claim 2. *There exist* $u_1, \ldots, u_{2l+1} \in \pi_1(S)$ *and* $v_1, \ldots, v_{2l} \in \{d_1, \ldots, d_m\}$ *for some* $l \geq 1$, *such that*

(i)

$$(q \circ T^r)_*(x) = u_1 v_1^r u_2 v_2^{-r} \cdots u_{2l} v_{2l}^{-r} u_{2l+1}$$

(ii) *If* $v_i = v_{i+1}$, *then* $u_{i+1} \notin < v_i > = < v_{i+1} >$.

Apply Lemma 5.23 for $F = \pi_1(S)$ and $X = \prod_{i=1}^{g} [x_{2i-1}, x_{2i}]$, to conclude that $(q \circ T^r)_*(x) \neq 1$ for sufficiently large r. □

Definition 5.26. Let Γ be a graph. Define a graph $D_\infty(\Gamma)$ by

(i) $V(D_\infty(\Gamma)) = V(\Gamma) \sqcup \{v_{K,u} | K \in \mathcal{K}(\Gamma),\ u \in V(K)\}$

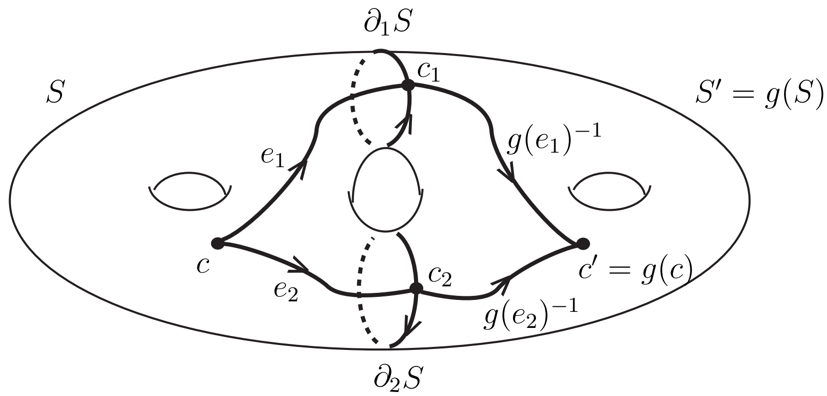


Figure 5.4: The double of a surface.

(ii) $E(D_\infty(\Gamma)) = E(\Gamma) \sqcup \{\{v_{K,u}, u'\} | K \in \mathcal{K}(\Gamma),\ u, u' \in V(K)\}$

$D_\infty(\Gamma)$ is obtained from Γ by adding a simplicial vertex (denoted by $v_{K,u}$) for each maximal complete subgraph K and a vertex u of K. For examples of $D_\infty(\Gamma)$, see Figure 5.5.

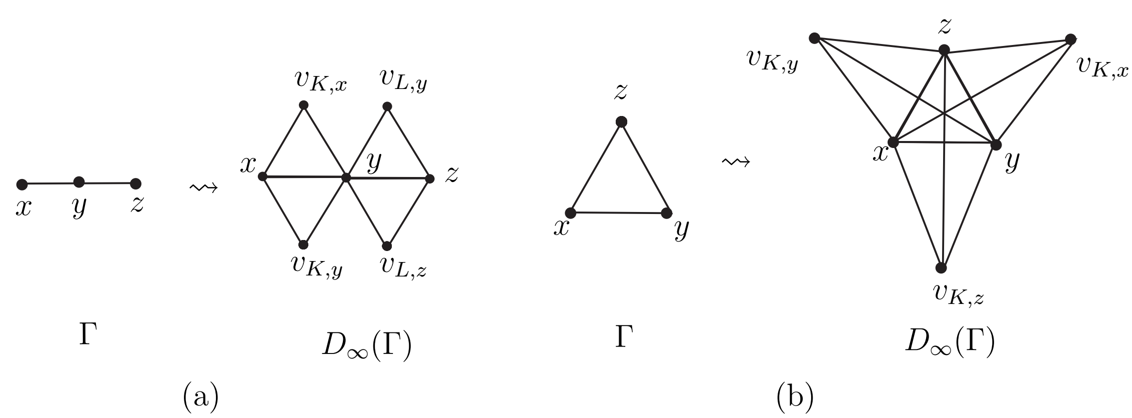


Figure 5.5: Examples of $D_\infty(\Gamma)$. In (a), the maximal complete subgraphs are, $K = \{x, y\}$ and $L = \{y, z\}$. In (b), it is $K = \Gamma$.

Lemma 5.27 (decomposing $D_\infty(\Gamma)$)**.** *Suppose Γ is a graph, such that $\Gamma = \Gamma_1 \cup \Gamma_2$ for some induced subgraphs Γ_1 and Γ_2. Furthermore, assume that $\Gamma_1 \cap \Gamma_2$ is complete.*

Then there exists an induced subgraphs Γ_1^* and Γ_2^* of $D_\infty(\Gamma)$, such that

(i) $D_\infty(\Gamma) = \Gamma_1^* \cup \Gamma_2^*$.

(ii) $\Gamma_1^* \cap \Gamma_2^*$ *is complete.*

(iii) For each i, Γ_i is obtained from Γ_i^* by removing a set of pairwise non-adjacent simplicial vertices in Γ_i^*.

Proof) Let $\bar{K} = \Gamma_1 \cap \Gamma_2$. We may set $\bar{K} \neq \varnothing$, otherwise the conclusion is obvious.

Claim 1. *For any complete subgraph K of Γ, either $K \leq \Gamma_1$ or $K \leq \Gamma_2$.*

This is because there does not exist an edge between a vertex in $V(\Gamma_1) \setminus V(\Gamma_2)$ and a vertex in $V(\Gamma_2) \setminus V(\Gamma_1)$.

Claim 2. $\mathcal{K}(\Gamma) \subseteq \mathcal{K}(\Gamma_1) \cup \mathcal{K}(\Gamma_2)$

This follows from Claim 1.

Claim 3. *, $\mathcal{K}(\Gamma_1) \cap \mathcal{K}(\Gamma_2) \subseteq \{\bar{K}\}$. The equality holds if and only if $\bar{K} \in \mathcal{K}(\Gamma)$.*

If $K \in \mathcal{K}(\Gamma_1) \cap \mathcal{K}(\Gamma_2)$, then $K \leq \Gamma_1 \cap \Gamma_2 = \bar{K}$. By maximality, $K = \bar{K}$. If $\bar{K} \in \mathcal{K}(\Gamma)$, the other inclusion is immediate.

Claim 4. *If for some i, $K \in \mathcal{K}(\Gamma_i) \setminus \mathcal{K}(\Gamma)$, then $K = \bar{K}$. Therefore, if $\bar{K} \in \mathcal{K}(\Gamma)$, then $\mathcal{K}(\Gamma_1) \cup \mathcal{K}(\Gamma_2) = \mathcal{K}(\Gamma)$.*

If $K \in \mathcal{K}(\Gamma_1) \setminus \mathcal{K}(\Gamma)$, then $K \lneq K' \in \mathcal{K}(\Gamma)$, for some K' containing a vertex in Γ_2. By Claim 1, $K \lneq K' \leq \Gamma_2$, and so $K \leq \Gamma_1 \cap \Gamma_2 = \bar{K}$. Since K is maximal in Γ_1, $K = \bar{K}$. If $\bar{K} \in \mathcal{K}(\Gamma)$, then $\mathcal{K}(\Gamma_i) \setminus \mathcal{K}(\Gamma) = \varnothing$ for $i = 1, 2$. Hence $\mathcal{K}(\Gamma_1) \cup \mathcal{K}(\Gamma_2) \subseteq \mathcal{K}(\Gamma)$.

Now we consider four cases.

Case 1. $\bar{K} \in \mathcal{K}(\Gamma)$.

Let Γ_1^* be the induced subgraph of $D_\infty(\Gamma)$ on the vertex set $V(\Gamma_1) \cup \{v_{K,u} | K \in \mathcal{K}(\Gamma_1) \setminus \{\bar{K}\},\ u \in V(K)\}$. Put $\Gamma_2^* = D_\infty(\Gamma_2) \leq D_\infty(\Gamma)$. By Claim 3 and 4, $\mathcal{K}(\Gamma_1) \cap \mathcal{K}(\Gamma_2) = \{\bar{K}\}$ and $(\mathcal{K}(\Gamma_1) \setminus \{\bar{K}\}) \sqcup \mathcal{K}(\Gamma_2) = \mathcal{K}(\Gamma)$. This implies that $\Gamma_1^* \cap \Gamma_2^* = \Gamma_1 \cap \Gamma_2 = \bar{K}$ and $\Gamma_1^* \cup \Gamma_2^* = D_\infty(\Gamma)$.

Case 2. $\bar{K} \notin \mathcal{K}(\Gamma)$ *and* $\bar{K} \in \mathcal{K}(\Gamma_1)$.

By Claim 3, $\bar{K} \notin \mathcal{K}(\Gamma_2)$. Since $\mathcal{K}(\Gamma)$ is the disjoint union of $\mathcal{K}(\Gamma_1) \setminus \{\bar{K}\}$ and $\mathcal{K}(\Gamma_2)$, Γ_1^* and Γ_2^* as defined in Case 1 satisfy the desired properties.

Case 3. $\bar{K} \notin \mathcal{K}(\Gamma)$ *and* $\bar{K} \in \mathcal{K}(\Gamma_2)$.

Similar to Case 2, by symmetry.

Case 4. $\bar{K} \notin \mathcal{K}(\Gamma_1) \cup \mathcal{K}(\Gamma_2)$.

Put $\Gamma_1^* = D_\infty(\Gamma_1)$ and $\Gamma_2^* = D_\infty(\Gamma_2)$. Note that $\mathcal{K}(\Gamma)$ is the disjoint union of $\mathcal{K}(\Gamma_1)$ and $\mathcal{K}(\Gamma_2)$. So $\Gamma_1^* \cap \Gamma_2^* = \Gamma_1 \cap \Gamma_2 = \bar{K}$, and $\Gamma_1^* \cup \Gamma_2^* = D_\infty(\Gamma)$.$\square$

The following lemma is a key step to the proof of Lemma 5.9.

Lemma 5.28 (from bounded to closed)**.** *Let* Γ *be a graph. If* $D_\infty(\Gamma) \in \mathcal{N}$, *then* $\Gamma \in \mathcal{N}_\infty$.

Proof) Assume $\Gamma \notin \mathcal{N}_\infty$. We prove that $D_\infty(\Gamma) \notin \mathcal{N}$.

Choose a normalized label-reading pair $(\mathcal{H}, \lambda)$ on a hyperbolic surface S with the underlying graph Γ. Let $\partial_1 S, \ldots, \partial_m S$ denote the boundary components of S.

We will define a closed surface $\hat{D}(S)$, which is homeomorphic to $D(S)$.

Fix a homeomorphism $g : S \to S'$ and put $\partial_i S' = g(\partial_i S)$. Let $D(S) = S \cup S'$ where $\partial_i S$ is glued to $\partial_i S'$ by g. Now consider a disjoint union $S \sqcup S'$. For each i, we glue an annulus $A_i = S^1 \times I$ to $S \sqcup S'$, so that one boundary component, say $\partial_0 A_i$, is identified with $\partial_i S$, and the other one, say $\partial_1 A_i$, with $\partial_i S'$. $\hat{D}(S)$ is defined to be the resulting quotient space of $S \cup S' \cup A_1 \cup A_2 \cup \cdots A_m$. In this proof, it is more convenient to consider $\hat{D}(S)$ instead of $D(S)$. We choose a properly embedded arc of the form $\{\text{a point}\} \times I$ in each A_i, which we will call the *principal arc* of A_i.

Write $\mathcal{H} = \mathcal{B} \sqcup \mathcal{C}$ where $\mathcal{B}$ is a set of the properly embedded arcs and $\mathcal{C}$ is a set of simple closed curves. We now describe a natural way of constructing a label-reading pair $(\hat{\mathcal{H}}, \hat{\lambda})$ on $\hat{D}(S)$ with the underlying graph $D_\infty(\Gamma)$ (Figure 5.6).

For each $\beta \in \mathcal{B}$, there is a unique way of constructing a simple closed curve, say $\hat{\beta}$, on $\hat{D}(S)$, such that $\hat{\beta}$ is the union of β, $g(\beta)$ and two arcs parallel to principal arcs. We let $\hat{\beta}$ inherit the transverse orientation and labeling, denoted by $\hat{\lambda}$, from those of β, so that $\hat{\lambda}(\hat{\beta}) = \lambda(\beta)$. Let $\hat{\mathcal{B}} = \{\hat{\beta} : \beta \in \mathcal{B}\}$. Also, let $\hat{\mathcal{C}} = \mathcal{C} \cup g(\mathcal{C})$, where each element of $\hat{\mathcal{C}}$ inherits the transverse orientation and labeling of the corresponding element in $\mathcal{C}$ and $g(\mathcal{C})$.

Consider any boundary component $\partial_i S$. There exists $K \in \mathcal{K}(\Gamma)$ such that each $\alpha \in \mathcal{B}$ intersecting with $\partial_i S$ is labeled by a vertex in $V(K)$. Let $\{\beta \in \mathcal{B} : \beta \cap \partial_i S \neq \varnothing\} = \{\beta_1, \beta_2, \ldots, \beta_s\}$. We choose disjoint essential simple closed curves $\alpha_1, \ldots, \alpha_s$ in the interior of A_i, and let $\hat{\lambda}(\alpha_j) = v_{K,\lambda(\beta_j)} \in V(D_\infty(\Gamma))$, for each j. Moreover, we let the transverse orientation of α_j be from $\partial_0 A_i$ to $\partial_1 A_i$, if the transverse orientation of β_j

coincides with the orientation of $\partial_i S$, and from $\partial_1 A_i$ to $\partial_0 A_i$ otherwise. Let $\hat{\mathcal{C}}_0$ be the union of all such α_j's, where the union is taken over all the boundary components (Figure 5.6 (a)).

Let $\hat{\mathcal{H}} = \hat{\mathcal{B}} \cup \hat{\mathcal{C}} \cup \hat{\mathcal{C}}_0$, with the labeling given by $\hat{\lambda} : \hat{\mathcal{H}} \to V(D_\infty(\Gamma))$. The label-reading pair $(\hat{\mathcal{H}}, \hat{\lambda})$ defines a label-reading map, say, $\hat{\phi} : \pi_1(D(S)) \cong \pi_1(\hat{D}(S)) \to A(D_\infty(\Gamma))$. Now it suffices show that $\hat{\phi}$ is injective.

Using the notation in Lemma 5.25, we define $q_r = (q \circ T^r)_* : \pi_1(\hat{D}(S)) \to \pi_1(S)$. We also define $p_r : A(D_\infty) \to A(\Gamma)$ by $p(v_{K,u}) = u^r$, for $K \in \mathcal{K}(\Gamma)$ and $u \in V(K)$.

We have a diagram

$$\begin{array}{ccc} \pi_1(\hat{D}(S)) & \xrightarrow{\hat{\phi}} & A(D_\infty(\Gamma)) \\ \downarrow{\scriptstyle q_r} & & \downarrow{\scriptstyle p_r} \\ \pi_1(S) & \xrightarrow{\phi = f_*} & A(\Gamma) \end{array}$$

Claim 1. *The above diagram commutes, with a suitable choice of the basepoints.*

Let $\bar{\mathcal{H}} = \hat{\mathcal{B}} \cup \hat{\mathcal{C}} \subseteq \hat{\mathcal{H}}$ and $\bar{\lambda} = \hat{\lambda} \lfloor \bar{\mathcal{H}}$. Then for each $[\gamma] = x \in \pi_1(\hat{D}(S))$, $\phi \circ q_r(x) = \phi[(q \circ T^r)(\gamma)]$ is equal to the image of $[\gamma]$ by the label-reading map with respect to the pair $(T^{-r}(\bar{\mathcal{H}}), \bar{\lambda} \circ T^r)$ on $\hat{D}(S)$.

Consider a label-reading pair $(\tilde{\mathcal{H}}, \tilde{\lambda})$ defined as follows. We replace each simple closed curve $\hat{\alpha} \in \hat{\mathcal{C}}_0$ inside an annulus, say A_i, by r copies of disjoint essential simple closed curves $\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_r$ on A_i with the same transverse orientation as $\hat{\alpha}$. If $\hat{\lambda}(\hat{\alpha}) = v_{K,u}$, then let $\tilde{\lambda}(\tilde{\alpha}_j) = u$ for each j. Then define $\tilde{\mathcal{C}}_0$ to be the union of all such $\tilde{\alpha}_j$'s, where the union is taken over all the annuli A_i, $i = 1, 2, \ldots, m$.

Then define $\widetilde{\mathcal{H}} = \hat{\mathcal{B}} \cup \hat{\mathcal{C}} \cup \tilde{\mathcal{C}}_0$, with the labeling $\tilde{\lambda}$ which coincide with $\hat{\lambda}$ on $\hat{\mathcal{B}} \cup \hat{\mathcal{C}}$. Note that two curves of the same label are not necessarily disjoint in this construction (Figure 5.6 (b)).

The image of $[\gamma]$ by the label-reading map with respect to the pair $(\widetilde{\mathcal{H}}, \widetilde{\lambda})$ is $p_r \circ \hat{\phi}(x)$. Note that $(\widetilde{\mathcal{H}}, \widetilde{\lambda})$ is also equivalent to the pair $(T^{-r}(\bar{\mathcal{H}}), \bar{\lambda} \circ T^r)$ on $\hat{D}(S)$ (Figure 5.6 (c)). Therefore, $p_r \circ \hat{\phi}(x) = \phi \circ q_r(x)$ (with a suitable choice of the basepoints). The claim is proved.

Now suppose x is a non-trivial element in $\pi_1(\hat{D}(S))$. By Lemma 5.25, there exists $r \geq 0$ such that $q_r(x) \neq 0$. The injectivity of ϕ and the commutativity of the diagram imply that $\hat{\phi}(x)$ is non-trivial. This proves that $\hat{\phi}$ is injective, and so $D_\infty(\Gamma) \notin \mathcal{N}_\infty$.□

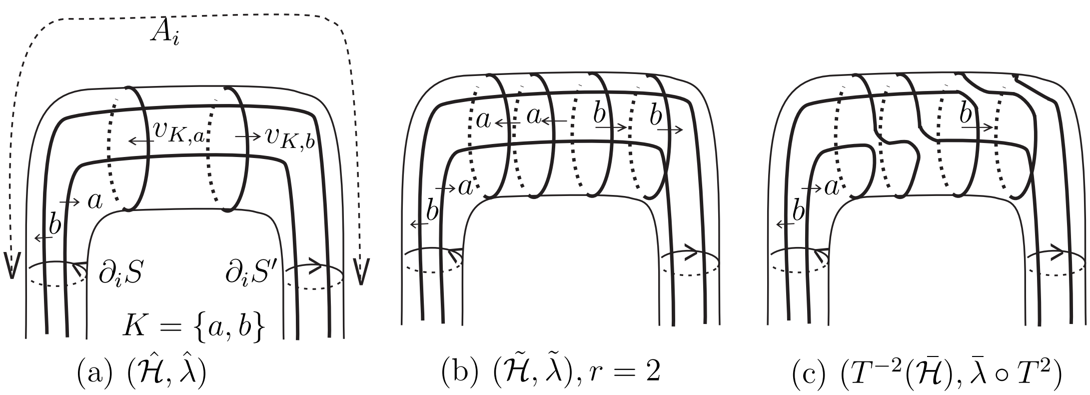


Figure 5.6: Inducing a label-reading pair on $\hat{D}(S)$. (b) and (c) are equivalent label-reading pairs.

Proof of Lemma 5.9

Assume $\Gamma \notin \mathcal{N}_\infty$, and we will show that $\Gamma_i \notin \mathcal{N}_\infty$ for $i = 1$ or 2. By Lemma 5.28, $D_\infty(\Gamma) \notin \mathcal{N}$. Write $D_\infty(\Gamma) = \Gamma_1^* \cup \Gamma_2^*$ as in Lemma 5.27. By Lemma 5.18, $\Gamma_i^* \notin \mathcal{N}_\infty$

for $i = 1$ or 2. Then by Lemma 5.20, $\Gamma_i \notin \mathcal{N}_\infty$, for $i = 1$ or 2. □

Remark 5.29. We can now prove that Conjecture 5.22 implies an affirmative answer to Question 5.15, as follows. Suppose $\Gamma \notin \mathcal{N}_\infty$. By Lemma 5.28, $D_\infty(\Gamma) \notin \mathcal{N}$. Note that $D_\infty(\Gamma)$ is obtained by adding vertices to Γ, such that the new vertices are simplicial in $D_\infty(\Gamma)$. Hence, if Conjecture 5.22 is true, we have $\Gamma \notin \mathcal{N}$. This would prove $\mathcal{N} \subseteq \mathcal{N}_\infty$. $\mathcal{N}_\infty \subseteq \mathcal{N}$ is trivial.

5.4 Adding a bisimplicial edge

We prove Lemma 5.10, by the steps described in Lemma 5.31.

Notation 5.30. Throughout this section, we fix the following notations.

(1) Let S be a hyperbolic surface, with the boundary components of S denoted by $\partial_1 S, \partial_2 S, \ldots, \partial_m S$. Let Γ be a graph, and $(\mathcal{H}, \lambda)$ be a normalized label-reading pair with the underlying graph Γ on S. $\phi : \pi_1(S) \to A(\Gamma)$ will denote an associated label-reading map. $\mathcal{B}$ and $\mathcal{C}$ denote the set of properly embedded arcs and the set of simple closed curves, respectively, in $\mathcal{H}$.

(2) Consider a vertex a of Γ. We let $\mathcal{B}_a = \mathcal{B} \cap \lambda^{-1}(a)$ and $\mathcal{C}_a = \mathcal{C} \cap \lambda^{-1}(a)$. We call each curve in $\mathcal{B}_a$ as an a-arc. $\bar{\mathcal{B}}_a$ denotes a set of a-arc representatvies. Also, let $\partial^a S$ be the union of the boundary components of S that intersects with an a-arc. For each $\alpha \in \mathcal{B}_a$, we have chosen a strip, denoted by $\eta_\alpha : I \times [-1, 1] \to S$, so that the strip im η_α contains any a-arc homotopic to α. Moreover, any two strips in $\{\eta_\beta : \beta \in \mathcal{B}_a\}$, are disjoint or identical.

(3) For $\alpha \in \mathcal{B}$, chan(α) denotes the channel of α with respect to the set of the arcs $\mathcal{B}_{\lambda(\alpha)}$, and $\widetilde{\text{chan}}(\alpha)$ denotes a channel surface of α. We also let $\hat{\alpha}$ be the induced simple closed curve of α. Recall that this means $\hat{\alpha}$ is the unique component of the frontier of chan(α) in S.

Lemma 5.31. *Following Notation 5.30, suppose $(\mathcal{H}, \lambda)$ is a normalized label-reading pair with the underlying graph Γ on a hyperbolic surface S. Let $e = \{a, b\}$ be a bisimplicial edge of Γ. Assume further that $\lambda(\mathcal{H}) \not\subseteq Link(a) \cup \{a\}$ and $\lambda(\mathcal{H}) \not\subseteq Link(b) \cup \{b\}$. Then the following are true.*

(1) Suppose α and β are essential closed curves on S such that $w_\alpha \in \langle Link(a) \rangle$ and $w_\beta \in \langle Link(b) \rangle$. If α and β intersect, then they are homotopic up to orientation.

(2) $(\cup \mathcal{C}_a) \cap (\cup \mathcal{C}_b) = \varnothing$.

(3) If $\alpha \in \mathcal{B}_a$ and $\beta \in \mathcal{B}_b$, then $\alpha \not\sim \beta$.

(4) If a b-arc β joins two components in $\partial^a S$, then β intersects a curve $\gamma \in \mathcal{H}$ that is not labeled by a vertex in $Link(a) \cup \{a\}$.

(5) Let $\hat{\alpha}$ be an induced simple closed curve of an a-arc α, and $\hat{\alpha}'$ be the boundary component of $\widetilde{chan}(\alpha)$, homotopic to $\hat{\alpha}$. Then $w_{\hat{\alpha}'} \in \langle Link(a) \rangle$.

(6) An induced simple closed curve of an a- or b-arc is essential.

(7) $(\cup \mathcal{B}_a) \cap (\cup \mathcal{C}_b) = \varnothing$.

(8) Let $\alpha \in \mathcal{B}_a$ and $\beta \in \mathcal{B}_b$. Choose the induced simple closed curves $\hat{\alpha}$ and $\hat{\beta}$ of α and β, respectively. Suppose either

(i) α *and* β *intersect, or*

(ii) $\hat{\alpha}$ *and* $\hat{\beta}$ *intersect, and there exists a boundary component which intersects with both* α *and* β.

Then $\hat{\alpha}$ *is homotopic to* $\hat{\beta}$ *up to orientation,* $\widetilde{chan}(\alpha) \cap \partial S = \widetilde{chan}(\beta) \cap \partial S$, *and moreover, there exists a homotopy from* $\widetilde{chan}(\alpha)$ *onto* $\widetilde{chan}(\beta)$ *fixing* $\widetilde{chan}(\alpha) \cap \partial S$.

(9) Suppose a b-arc β *joins two components of* $\partial^a S$ *that are contained in* $\widetilde{chan}(\alpha)$, *for some a-arc* α. *Let* $\hat{\alpha}$ *and* $\hat{\beta}$ *be the induced simple closed curves of* α *and* β, *respectively. Then* $\hat{\alpha} \cap \hat{\beta} = \varnothing$.

(10) $(\cup \mathcal{B}_a) \cap (\cup \mathcal{B}_b) = \varnothing$.

(11) $\partial^a S \cap \partial^b S = \varnothing$.

For (4),(5),(6),(7) and (9), the statements with the roles of a and b being interchanged, are also true.

Proof)

Note that the given condition is symmetric for a and for b. The condition that $\lambda(\mathcal{H}) \not\subseteq \text{Link}(a) \cup \{a\}$ and $\lambda(\mathcal{H}) \not\subseteq \text{Link}(b) \cup \{b\}$ will be used only for (6) through (11).

(1)

We may choose the basepoint of $\pi_1(S)$ in $\alpha \cap \beta$. We have $\phi[[\alpha],[\beta]] = [w_\alpha, w_\beta] = 1$, since any vertex in Link(a) is adjacent to any vertex in Link(b). By Lemma 2.32,

$\alpha \sim \beta^{\pm 1}$.

(2)

Suppose $p \in \alpha \cap \beta$, for some $\alpha \in \mathcal{C}_a$ and $\beta \in \mathcal{C}_b$. Choose the basepoint x_0 in one of the disk components of $S \setminus (\cup\mathcal{H})$, so that the disk component contains p on its boundary. Choose a closed curve α', based at x_0, such that α' is sufficiently close to, but disjoint from, α. Similarly we choose $\beta' \sim \beta$. Then $w_{\alpha'} \in \text{Link}(a)$, since any curve intersecting with α is labeled by a vertex in $\text{Link}(a)$. Also, $w_{\beta'} \in \text{Link}(b)$. By (1), we have $\alpha' \sim \beta'^{\pm 1}$, and so $\alpha \sim \beta^{\pm 1}$. This contradicts to the fact that the curves in $\mathcal{H}$ are minimally intersecting (Lemma 3.16).

(3)

Suppose an a-arc α and a b-arc β are homotopic to each other, intersecting with $\partial_1 S$ and $\partial_2 S$. Note that $\alpha \cap \beta = \varnothing$ (Lemma 3.16 (2)). Consider an embedding of a strip $\eta : I \times [-1,1] \to S$ such that $\eta(I \times \{0\}) = \alpha$ and $\beta \subseteq \eta(I \times (-1,1))$. As in (2), consider a properly embedded arc $\beta' \neq \beta$, which is sufficiently close to β.

Now suppose $\gamma \in \mathcal{H}$ intersects with β'. Then $\gamma \cap \beta \neq \varnothing$ and $\lambda(\gamma) \in \text{Link}(b)$. If γ does not intersect α, then as in the proof of Lemma 3.10, one can find $\gamma' \sim \gamma$ such that $\gamma' \cap \text{im}\ (\eta) = \varnothing$. This implies that $\gamma' \cap \beta = \varnothing$, which is a contradiction to the assumption that β and γ are minimally intersecting (Lemma 3.16). So γ intersects with α also, and $\lambda(\gamma) \in \text{Link}(a) \cap \text{Link}(b)$.

Let the loops $\partial_1 S$ and $\partial_2 S$ based at $\beta' \cap \partial_1 S$ and $\beta' \cap \partial_2 S$, respectively, and let the arc β' travel from $\partial_1 S$ to $\partial_2 S$. We have,

$$w_{\partial_1 S}, w_{\partial_2 S}, w_{\beta'} \in \langle \{a,b\} \cup (\mathrm{Link}(a) \cap \mathrm{Link}(b)) \rangle$$

Choose the basepoint of $\pi_1(S)$ at $\partial_1 S \cap \beta'$. Then $\phi[\partial_1 S] = w_{\partial_1 S}$ and $\phi[\beta' \cdot \partial_2 S \cdot \beta'^{-1}] = w_{\beta'} w_{\partial_2 S} {w_{\beta'}}^{-1}$ are commuting, since $\{a,b\} \cup (\mathrm{Link}(a) \cap \mathrm{Link}(b))$ span a complete graph in Γ. This leads to a contradiction again, for in a hyperbolic surface group, two boundary components do not commute (Lemma 2.32).

(4)

This is similar to (3). Choose β' sufficiently close to β. Assume that each curve $\gamma \in \mathcal{H}$ that is intersecting with β' is labeled by a vertex in $\mathrm{Link}(a) \cup \{a\}$. Note that γ is labeled by a vertex in $\mathrm{Link}(b)$ also, since β' is close to β.

Let $\partial_1 S, \partial_2 S$ be the boundary components joined by β. Since $\partial_i S \subseteq \partial^a S \cap \partial^b S$ for $i = 1, 2$, $w_{\partial_1 S}, w_{\partial_2 S} \in \langle \{a,b\} \cup (\mathrm{Link}(a) \cap \mathrm{Link}(b)) \rangle$. By assumption, $w_{\beta'} \in \langle (\mathrm{Link}(a) \cup \{a\}) \cap \mathrm{Link}(b) \rangle = \langle \{a\} \cup (\mathrm{Link}(a) \cap \mathrm{Link}(b)) \rangle$. As in the proof of (3), $[[\partial_1 S], [\beta' \cdot \partial_2 S \cdot \beta'^{-1}]] = 1$, which is a contradiction.

(5)

This is obvious, since any curve in $\mathcal{H}$ transversely intersecting with $\hat{\alpha}$ has the label in $\mathrm{Link}(a)$.

(6)

Suppose an induced simple closed curve $\hat{\alpha}$ of α is null-homotopic. By Lemma 3.10 (2), any properly embedded arc is labeled by a vertex in $\mathrm{Link}(a) \cup \{a\}$, and any simple closed curve in $\mathcal{H}$ is labeled by a vertex in Link(a). By the assumption that $\lambda(\mathcal{H}) \notin \text{Link}(a) \cup \{a\}$, this is forbidden.

(7)

Supose $\alpha \in \mathcal{B}_a$ and $\beta \in \mathcal{C}_b$ intersect at p, and choose the basepoint x_0 of $\pi_1(S)$ as in (2). Let $\hat{\alpha}$ be the induced simple closed curve of α with respect to $(\mathcal{H}, \lambda)$, and $\alpha' \sim \hat{\alpha}$ be the unique boundary component of $\widetilde{\text{chan}}(\alpha)$ that is not in ∂S (Lemma 3.9). $\alpha' \not\sim 0$ by (6). By (5), $w_{\alpha'} \in \langle \text{Link}(a) \rangle$. Since $\alpha \cap \beta \neq \varnothing$, $\alpha' \cap \beta \neq \varnothing$. By (1), $\alpha' \sim \beta^{\pm 1}$. So, $i(\alpha, \beta) = (\alpha, \alpha') = 0$. This contradicts to the assumption that α and β are minimally intersecting.

(8)

Note that in the case when $\alpha \cap \beta \neq \varnothing$, $\hat{\alpha} \cap \hat{\beta} \neq \varnothing$.

As in (7), we choose $\alpha' \sim \hat{\alpha}$ and $\beta' \sim \hat{\beta}$, which are boundary components of $\widetilde{\text{chan}}(\alpha)$ and $\widetilde{\text{chan}}(\beta)$, respectively. Since $\hat{\alpha}$ and $\hat{\beta}$ are intersecting, so are α' and β'. By (1) and (5), we have $\alpha' \sim \beta'^{\pm 1}$. By Lemma 3.9, both α' and β' are separating simple closed curves of S. So either $\widetilde{\text{chan}}(\alpha) \sim \widetilde{\text{chan}}(\beta)$ or $\widetilde{\text{chan}}(\alpha) \sim \overline{S \setminus \widetilde{\text{chan}}(\beta)}$. I claim that $\widetilde{\text{chan}}(\alpha) \not\sim \overline{S \setminus \widetilde{\text{chan}}(\beta)}$. Suppose not. Then any boundary component of S contained in $\widetilde{\text{chan}}(\alpha)$ will not be contained in $\widetilde{\text{chan}}(\beta)$. So no boundary component of S can intersect both α and β. Moreover, $\alpha \rightsquigarrow S \setminus \widetilde{\text{chan}}(\beta)$ and $i(\alpha, \beta) = 0$. So neither (i) nor (ii) of the given conditions cannot hold. Hence $\widetilde{\text{chan}}(\alpha) \sim \widetilde{\text{chan}}(\beta)$.

(9)

Assume a b-arc β joins $\partial_1 S$ and $\partial_2 S$ which are contained in $\widetilde{\text{chan}}(\alpha)$, and $\hat{\alpha} \cap \hat{\beta} \neq \varnothing$.

By (4), there exists $\gamma \in \mathcal{H}$, intersecting with β, such that $\lambda(\gamma) \notin \mathrm{Link}(a) \cup \{a\}$. Then $\gamma \cap (\cup \mathcal{B}_a) = \varnothing$ and $\gamma \cap \partial^a S = \varnothing$. By (8) and Lemma 3.10,

$$\gamma \rightsquigarrow S \setminus \widetilde{\mathrm{chan}}(\alpha) \sim S \setminus \widetilde{\mathrm{chan}}(\beta)$$

So $i(\beta, \gamma) = 0$, which is a contradiction.

(10)

Suppose $\alpha \in \mathcal{B}_a$ and $\beta \in \mathcal{B}_b$ intersect. By (8), $\hat{\alpha} \sim \hat{\beta}$ and $\widetilde{\mathrm{chan}}(\alpha) \sim \widetilde{\mathrm{chan}}(\beta)$. This implies that $\widetilde{\mathrm{chan}}(\alpha) \cap \partial S = \widetilde{\mathrm{chan}}(\beta) \cap \partial S$, and $\beta \subseteq \widetilde{\mathrm{chan}}(\beta)$ joins two boundary components contained in $\widetilde{\mathrm{chan}}(\alpha)$. We have a contradiction by (9).

(11)

Suppose $\partial_i S$ intersects with an a-arc α and a b-arc β. By choosing a nearest pair of such arcs on $\partial_i S$, we can find induced simple closed curves $\hat{\alpha}$ and $\hat{\beta}$ of α and β such that $\hat{\alpha} \cap \hat{\beta} \neq \varnothing$ (Figure 5.7). By (8) again, we see that β joins two boundary components of $\widetilde{\mathrm{chan}}(\alpha)$. This is a contradiction by (9). ☐

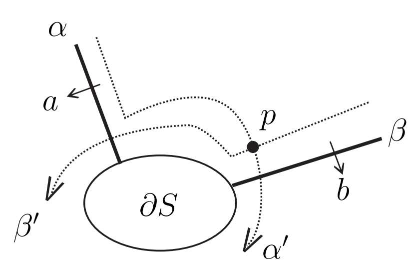


Figure 5.7: Proof of Lemma 5.31 (11). Here, α' denotes a homotopic curve to the induced arc $\hat{\alpha}$ of α, and similarly define $\beta' \sim \hat{\beta}$.

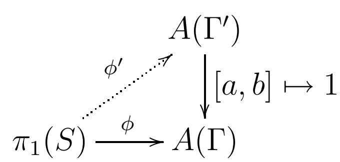
Proof of Lemma 5.10

Suppose $\Gamma \notin \mathcal{N}_\infty$ and let $e = \{a, b\}$ be a bisimplicial edge of Γ. Set $\Gamma' = \Gamma \setminus \mathring{e}$. We will prove that $\Gamma' \notin \mathcal{N}_\infty$.

Choose a normalized label-reading pair $(\mathcal{H}, \lambda)$ on a hyperbolic surface S with the underlying graph Γ, and let ϕ denote an associated label-reading map.

First consider the case when $\lambda(\mathcal{H}) \subseteq \text{Link}(a) \cup \{a\}$. Let $\Gamma'' = \Gamma_{\text{Link}(a)\cup\{a\}}$. Then $\text{im}\,\phi \subseteq A(\Gamma'')$, and hence $\Gamma'' \notin \mathcal{N}_\infty$. Put $\Gamma''' = \Gamma_{\text{Link}(a)}$. Then $\Gamma'' = \text{Join}(\{a\}, \Gamma''')$. By Lemma 5.8, $\Gamma''' \notin \mathcal{N}_\infty$. Since $\Gamma''' \leq \Gamma'$, we have $\Gamma' \notin \mathcal{N}_\infty$. The case when $\lambda(\mathcal{H}) \subseteq \text{Link}(b) \cup \{b\}$ is similar.

Now assume $\lambda(\mathcal{H}) \nsubseteq \text{Link}(b) \cup \{b\}$ and $\lambda(\mathcal{H}) \nsubseteq \text{Link}(b) \cup \{b\}$. By Lemma 5.31 (2),(7),(10) and (11), $(\mathcal{H}, \lambda)$ is also a label-reading pair with the underlying graph Γ'. From the following commutative diagram, an associated label-reading map ϕ' is also injective.



Hence $\Gamma' \notin \mathcal{N}_\infty$. $\square$

Remark 5.32. Suppose Γ is a graph with a bisimplicial edge e, such that $\Gamma \notin \mathcal{N}$. Let $\Gamma' = \Gamma \setminus \mathring{e}$. Then there exists a normalized label-reading pair $(\mathcal{H}, \lambda)$ with the underlying graph Γ on a *closed* surface S. Note that the statements of Lemm 5.31 (3) through (11) are trivial when S is closed. Hence, from Lemma 5.31 (2) only, we already see that $(\mathcal{H}, \lambda)$ is a normalized label-reading pair with the underlying graph Γ' also, and hence, $\Gamma' \notin \mathcal{N}$. It again follows that any chordal bipartite graphs are in $\mathcal{N}$ (Theorem 5.4).

Appendix: Proof of Theorem 5.4

Definition 5.33 ([CRS02])**.** An induced P_k of a graph Γ is *simplicial*, if it is not contained in the (topological) interior of any induced P_{k+2}.

Note that a simplicial vertex, as defined in Definition 5.19, is a simplicial P_1, and vice versa. A simplicial P_2 is called a *simplicial edge*.

Theorem 5.34. *([CRS02]) Let k be a positive integer and Γ be a graph without any induced cycle of length at least $k+3$. If Γ contains an induced P_k, then it has a simplicial P_k.*

Proof) We use an induction on $|V(\Gamma)|$.

For any $A \subseteq V(\Gamma)$, we let $N(A) = \{q \in V(\Gamma) : q \text{ is a neighbor of } A\} \cup A$, and $M(A) = V(\Gamma) \setminus N(A)$.

Let $\mathcal{S}$ be the set of subsets $A \subseteq V(\Gamma)$ such that

(i) Γ_A is connected, and

(ii) the induced subgraph on $M(A)$ contains an induced P_k.

Note that $\varnothing \in \mathcal{S}$, so we may choose a maximal $A \in \mathcal{S}$.

First, consider the case when $A = \varnothing$. If an induced P_k, say $(q_1, q_2, \ldots, q_k)$, is not simplicial, one can find an induced P_{k+2}, $(q_0, q_1, \ldots, q_k, q_{k+1})$, for some vertices q_0 and q_{k+1}. Then $(q_2, q_3, \ldots, q_{k+1})$ is an induced P_k in $\Gamma_{M(q_0)}$. Then $\{q_0\} \in \mathcal{S}$, and this contradicts to the maximality of A. So any induced P_k is simplicial, and we are done.

Now we may assume that $A \neq \varnothing$. By the inductive hypothesis, $\Gamma_{M(A)}$ contains an induced subgraph Γ_B, which is a simplicial P_k in $\Gamma_{M(A)}$. We will prove that Γ_B is simplicial in Γ.

Let $\Gamma_B = (q_1, q_2, \ldots, q_k)$. Suppose Γ_B is contained in the interior of induced P_{k+2}, say $(q_0, q_1, \ldots, q_k, q_{k+1})$. q_0 and q_{k+1} do not belong to A, for $q_1, q_k \in M(A)$. Since Γ_B is simplicial in $\Gamma_{M(A)}$, at least one of q_0 and q_{k+1} is not in $M(A)$, and so, is adjacent to a vertex in A. If both are adjacent to some vertices in A, then let γ be a shortest path from q_0 to q_{k+1} in the connected graph $\Gamma_{A \cup \{q_0, q_{k+1}\}}$. Then the union of γ and $(q_0, q_1, \ldots, q_k, q_{k+1})$ will be an induced cycle of length at least $k+3$ in Γ, which is a contradiction (Figure 5.8 (a)). So exactly one of q_0 and q_{k+1}, say q_{k+1}, is adjacent to a vertex in A. Let $A' = A \cup \{q_{k+1}\}$. Then $\Gamma_{A'}$ is connected, and $(q_0, q_1, \ldots, q_{k-1})$ is an induced P_k in $\Gamma_{M(A')}$ (Figure 5.8 (b)). This contradicts to the maximality of A. □

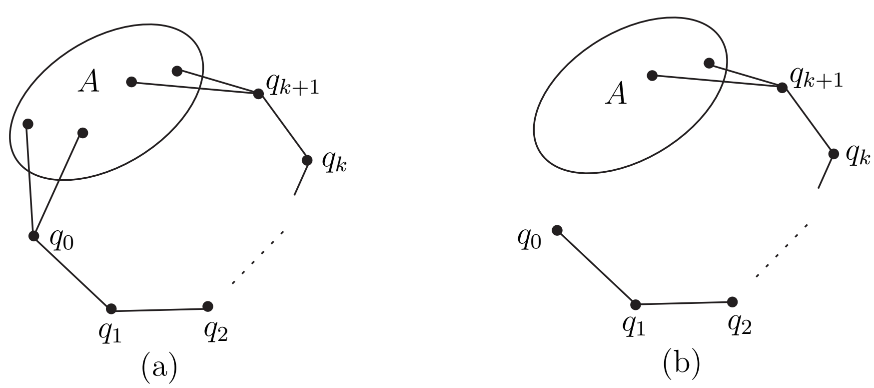


Figure 5.8: Proof of Theorem 5.34

Proof of Theorem 5.4 from Theorem 5.34) (1) If Γ is chordal, then Theorem 5.34 applies with $k = 1$, and Γ has a simplicial vertex q. By simpliciality, any two vertices in $\mathrm{Link}(q)$ are adjacent. Let Γ_1 and Γ_2 be the induced subgraphs on $\{q\} \cup \mathrm{Link}(q)$ and $V(\Gamma) \setminus \{q\}$, respectively. Then $\Gamma_1 \cap \Gamma_2 = \Gamma_{\mathrm{Link}(q)}$ is complete, and $\Gamma = \Gamma_1 \cup \Gamma_2$. We note that Γ_1 is also complete.

(2) Suppose Γ is chordal bipartite. By Theorem 5.34 for $k = 2$, there exists a simplicial edge $e = \{a, b\}$ in Γ. Let $c \in \mathrm{Link}(a)$ and $d \in \mathrm{Link}(b)$. Since Γ is bipartite $d \notin \mathrm{Link}(a)$ and $c \notin \mathrm{Link}(b)$. If c and d are not adjacent, then (c, a, b, d) will be an induced P_4 containing $\{a, b\}$ in its interior. By simpliciality of e, we see that c and d are adjacent. It follows that e is bisimplicial. $\square$

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List of Notations

$\Gamma \in \mathcal{N}$	$A(\Gamma)$ does not contain a hyperbolic surface group (p.12)
$\mathcal{N}_\infty$	a "relative version" of $\mathcal{N}$ (p.12)
$\Gamma_1 \cong \Gamma_2$	Γ_1 is (combinatorially) isomorphic to Γ_2 (p.16)
$V(\Gamma)$	the set of vertices of Γ (p.16)
$E(\Gamma)$	the set of edges of Γ (p.16)
$\overline{\Gamma}$	the complement graph of Γ (p.16)
$\mathrm{Link}_\Gamma(q), \mathrm{Link}(q)$	the set of vertices adjacent to q in Γ (p.16)
$\mathring{\mathrm{Star}}_\Gamma(q), \mathring{\mathrm{Star}}(q)$	the open star of q in Γ (p.16)
$\mathrm{d}_\Gamma(q), \mathrm{d}(q)$	the degree of q in Γ (p.16)
$\partial\Gamma$	$\{v \in V(\Gamma) : \mathrm{d}_\Gamma(v) = 1\}$ (p.16)
$\sqcup$	a disjoint union (p.17)
$\mathrm{Join}(\cdot,\cdot)$	the join of two graphs (p.17)
K_n	a complete graph on n vertices (p.17)
$K_{m,n}$	a complete bipartite graph on m and n vertices (p.17)
D_n	a discrete graph on n vertices (p.17)
P_n	a path on n vertices (p.17)
C_n	a cycle on n vertices (p.17)

$\overline{C_n}$	an anti-cycle on n vertices (p.17)
$\mathcal{K}(\Gamma)$	the set of maximal complete subgraphs of Γ (p.17)
Γ_S	the induced subgraph of Γ on S (p.18)
w_γ	the label-reading of γ (p.21)
$\mathrm{GP}(\Gamma, \{G_q\})$	the graph product of $\{G_q\}$ with the underlying graph Γ (p.23)
$A(\Gamma)$	the right-angled Artin group on Γ (p.24)
$C(\Gamma)$	the right-angled Coxeter group on Γ (p.24)
$\mathrm{Link}_X(v)$	the link of v in X (p.41)
f_v	the simplicial map on the links induced by f (p.41)
X_Γ	the standard Eilenberg-Maclane space of $A(\Gamma)$ (p.42)
$X_\varnothing$	the set of the unique vertex of X_Γ (p.43)
$D(\Gamma)$	the double of Γ (p.43)
$\alpha \sim \beta$	α and β are homotopic (p.45)
$\alpha \rightsquigarrow A$	α can be homotoped into A (p.45)
$i(\cdot,\cdot)$	the geometric intersection number (p.45)
$\mathrm{chan}(\alpha)$	the channel of α (p.60)
$\widetilde{\mathrm{chan}}(\alpha)$	a channel surface of α (p.60)
$\mathcal{W}$	the class of weakly chordal graphs (p.80)
$o_G(g), o(g)$	the order of g in a group G (p.80)
$\mathcal{F}$	a recursively defined class of graphs contained in $\mathcal{N}_\infty$ (p.105)
$D(S)$	the double of a surface S along its boundary (p.113)

$D_\infty(\Gamma)$	Γ, along with certain simplicial vertices attached (p.115)

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