

THE ARENS SPACE S_2 IS A PARACOMPLEX: A NOTE ON NAGATA'S PROBLEM 3

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ABSTRACT. In 1974, J. Nagata posed the problem of finding a paracomplex that is not the closed continuous image of any metric space. In this note, we observe that the standard quotient presentation of the classical Arens space S_2 explicitly realizes it as a 1-paracomplex. Because S_2 is known not to be a Fréchet–Urysohn space, it cannot be a Lašnev space (a closed continuous image of a metric space). This observation affirmatively resolves Nagata's Problem 3. Combined with a prior remark by T. Miwa, it establishes that the classes of paracomplexes and Lašnev spaces are mutually incomparable.

1. INTRODUCTION

In the search for a topological category that suitably generalizes both metric spaces and CW-complexes for algebraic topology, D. M. Hyman [3] introduced a class of spaces originally termed *M-spaces*. To avoid confusion with Morita's previously established *M-spaces*, C. J. R. Borges [1] proposed renaming Hyman's spaces to *paracomplexes*, a convention adopted by J. Nagata [7] in his investigation of their dimensional and topological properties.

Following Nagata's framework, all topological spaces in this note are assumed to be at least regular. Nagata proved that every paracomplex is an M_1 -space (and thus stratifiable). A central theme in general topology is understanding the precise relationships between various generalizations of metric spaces, particularly those defined via mappings. Nagata concluded his 1974 paper with three open problems to clarify these boundaries. Problem 1 asked whether every closed continuous image of a paracomplex is an M_1 -space; this was later answered affirmatively by T. Mizokami [6]. The present note is concerned with Nagata's Problem 3:

Problem 1.1 (Nagata [7], Problem 3). Give an example of a paracomplex which is not the closed continuous image of any metric space.

Spaces that are the closed continuous images of metric spaces are classically known as *Lašnev spaces* [4]. The relationship between paracomplexes and Lašnev spaces is of natural historical interest. T. Miwa [5], who placed paracomplexes into a broader setting by introducing \mathcal{D} -complexes, observed that there exist Lašnev spaces which are not paracomplexes. Nagata's Problem 1.1 essentially asked for the reverse separation: does there exist a paracomplex that is not a Lašnev space?

In this note, we provide this missing separation. While the topological properties of the canonical Arens space S_2 and its quotient presentations are classical, we explicitly observe that its standard quotient presentation precisely satisfies Hyman's inductive definition of a 1-paracomplex. Because the Arens space is a standard example of a space that is not Fréchet–Urysohn, it cannot be a Lašnev space. This resolves Problem 1.1 and establishes that the classes of paracomplexes and Lašnev spaces are strictly incomparable.

2. THE ARENS SPACE AS A PARACOMPLEX

We briefly recall Hyman's inductive adjunction characterization of paracomplexes, exactly as utilized by Nagata [7].

Definition 2.1 (Hyman [3]). The class of n -paracomplexes is defined inductively:

- (1) A 0-paracomplex is a metric space.
- (2) Assume that n -paracomplexes have been defined. A space Z is an $(n+1)$ -paracomplex if it is homeomorphic to the adjunction space $X \cup_f Y$, where X is a metric space, Y is an n -paracomplex, A is a closed subset of X , and $f: A \rightarrow Y$ is a continuous map.

A space W is a *paracomplex* if there exists a sequence $(W_i)_{i=1}^{\infty}$ of closed subsets such that $W = \bigcup_{i=1}^{\infty} W_i$, each W_i is an n_i -paracomplex, and W is dominated by $\{W_i\}$. In particular, every finite-order paracomplex is a paracomplex.

As described by M. Sakai [8] (citing Engelking [2, Example 1.6.19]), the Arens space S_2 is commonly modeled as the quotient of a topological sum of countably many nontrivial convergent sequences, obtained by identifying the limit of the n -th sequence with the n -th point of a distinguished convergent sequence.

We formalize this standard construction to cleanly align with Hyman's definition. Let $Y = \{0\} \cup \{1/n \mid n \in \mathbb{N}\}$ be equipped with the standard subspace topology from \mathbb{R} . As a metrizable space, Y is a 0-paracomplex.

For each $n \in \mathbb{N}$, we define a convergent sequence space X_n . To ensure the branches remain distinct, we tag the points by n :

$$X_n = \{(n, 0)\} \cup \left\{ \left(n, \frac{1}{m} \right) \mid m \in \mathbb{N} \right\},$$

endowed with the obvious metric topology such that the sequence $(n, 1/m)_{m=1}^{\infty}$ converges to $(n, 0)$. Let $X = \bigoplus_{n=1}^{\infty} X_n$ be their topological sum (disjoint union). Because X is the topological sum of countably many metrizable spaces, it is canonically a metric space (for example, by giving each component a bounded metric and declaring the distance between distinct components to be 1). Thus, X is a 0-paracomplex.

Define the distinguished set $A = \{(n, 0) \mid n \in \mathbb{N}\} \subset X$, which consists of the limit points of the individual branches. In the disjoint union topology, a subset of X is closed if and only if its intersection with every component X_n is closed in X_n . Since $A \cap X_n = \{(n, 0)\}$ is a singleton, it is closed in X_n , and thus A is closed in X . Furthermore, the subspace topology on A inherited from X is discrete.

Define the map $f: A \rightarrow Y$ by $f(n, 0) = 1/n$. Because A is a discrete space, f is automatically continuous.

We now form the adjunction space $Z = X \cup_f Y$. By Definition 2.1, because X is a metric space, Y is a 0-paracomplex, A is closed in X , and f is continuous, Z is explicitly a 1-paracomplex. By the aforementioned classical quotient presentation, Z is homeomorphic to the Arens space S_2 .

Theorem 2.2. *The Arens space S_2 is a 1-paracomplex.*

3. RESOLUTION OF PROBLEM 3

The fundamental obstruction to a space being Lašnev is the Fréchet–Urysohn property. A space S is *Fréchet–Urysohn* if for every $B \subseteq S$ and $y \in \overline{B}$, there exists a sequence in B converging to y .

Lemma 3.1 (Lašnev [4]). *Every closed continuous image of a metric space (i.e., every Lašnev space) is a Fréchet–Urysohn space.*

Corollary 3.2. *The Arens space S_2 is a paracomplex that is not the closed continuous image of any metric space. Consequently, the classes of paracomplexes and Lašnev spaces are mutually incomparable.*

Proof. By Theorem 2.2, S_2 is a 1-paracomplex, and therefore a paracomplex. It is a classical topological fact that S_2 is a sequential space that is *not* a Fréchet–Urysohn space (see [2, Example 1.6.19]).

By Lemma 3.1, because S_2 lacks the Fréchet–Urysohn property, it cannot be a Lašnev space. This resolves Nagata’s Problem 1.1 by providing a paracomplex that is not the closed continuous image of any metric space.

Combined with Miwa’s observation [5] that there exist Lašnev spaces which are not paracomplexes, we conclude that neither class contains the other. The two classes of spaces are strictly incomparable. \square

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