Introduction to hyperbolic geometry - exercises (July 2019)

SANG-HYUN KIM

1. Hyperbolic plane

We consider the Poincaré's upper-half plane model of $\mathbb{H}^2$ as

$$\mathbb{H}^2 := H_+ := \{z \in \mathbb{C} \mid \mathrm{Im}\,[z] > 0\}.$$

Denoting $\overline{\mathbb{H}^2}$ as the one-point compactification, we may write

$$\overline{\mathbb{H}^2} = \mathbb{H}^2 \coprod \partial\mathbb{H}^2 = \mathbb{H}^2 \coprod \mathbb{R} \coprod \{\infty\}.$$

Using the euclidean length element d_e, we defined a hyperbolic length element d_h as

$$d_h = d_e/y$$

at each point $(x, y) \in \mathbb{H}^2$. This defines the *length* $\ell(\alpha)$ of a piecewise C^1–curve $\alpha\colon [a, b] \to \mathbb{H}^2$ as

$$\ell(\alpha) := \int_a^b \frac{|\alpha'|}{\mathrm{Im}\,[\alpha]} dt.$$

The induced *length metric* is given as

$$d(x, y) := \inf\{\ell(\alpha)\colon \alpha \text{ is a } C^1 \text{ cuve joining } x \text{ and } y\}.$$

Problem 1.1. Compute the lengths of the following curves in $\mathbb{H}^2$:

(1) The length of a line segment:

$$\alpha(t) = tz + c \text{ for } z \in \mathbb{C}, c \in \mathbb{R}, t \in [a, b].$$

p:csc (2) semi-circular arc with the center on $\mathbb{R}$:

$$\alpha(t) = re^{it} + v \text{ for } r > 0, v \in \mathbb{R}, t \in [a, b] \subseteq (0, \pi).$$

Hint. For (2), search for `integrate csc x`.

p:metric **Problem 1.2.** (1) Prove that $(\mathbb{H}^2, d)$ is a metric space.

p:psl-isom (2) Prove that $\mathrm{PSL}_2(\mathbb{R}) \leqslant \mathrm{Isom}_+(\mathbb{H}^2)$. (This will be strengthened below)

p:uniq (3) Prove that $\mathbb{H}^2$ is uniquely geodesic. Classify all geodesics.

Date: July 17, 2019.

2010 *Mathematics Subject Classification.* Primary: 57M60; Secondary: 20F36, 37C05, 37C85, 57S05.

Key words and phrases. hyperbolic plane, Fuchsian group, bi-holomorphic map.

Hint. For (1), one needs to prove $d(x,y) \neq 0$ for all $x \neq y$. For (2), pick $g \in \mathrm{PSL}_2(\mathbb{R})$ and $\alpha\colon J \to \mathbb{H}^2$; then prove

$$\frac{|(g\circ\alpha)'|}{\mathrm{Im}\,[g\circ\alpha]} = \frac{|\alpha'|}{\mathrm{Im}\,[\alpha]}.$$

For (3), prove first that the vertical line is the unique geodesic joining i and ai. Then use (2) to move an arbitrary pair of points to the pair (i, ai).

It is often useful the use Poincaré disk model

$$\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}.$$

Note that $\varphi\colon z \mapsto (z-i)/(z+i)$ maps conformally $\mathbb{H}^2$ to $\mathbb{D}$.

Problem 1.3. (1) Prove that φ induces an isometry

$$(\mathbb{H}^2, d_h) \to (\mathbb{D}, d_d = 2d_e/(1-|z|^2)).$$

(2) In the Poincaré disk model, compute the distinct between 0 and $r \in (0,1)$. Compute also the length of radial arc

$$\{re^{i\theta} \colon \theta \in [\alpha, \beta]\}.$$

(3) Using the polar coordinate in $\mathbb{D}$, prove that the hyperbolic area element is given by

$$\frac{4r}{(1-r^2)^2} dr d\theta.$$

Deduce that the disk $\{z \in \mathbb{C} \colon |z| < a\} \subseteq \mathbb{D}$ has the area

$$\frac{4\pi a^2}{1-a^2}.$$

(4) Prove that the area of a hyperbolic triangle with angles A, B, C is

$$\pi - A - B - C.$$

(5) Using Gauss–Bonnet theorem, deduce that $\mathbb{D}$ (and hence, $\mathbb{H}^2$) has the Gaussian curvature -1.

Hint. For (4), first explain why this is an elementary Euclidean geometry plus a double integral exercise. The computation can be rough, though; see [1] for a simpler calculation trick in this approach.

For (5), compute K by taking the limit of the average

$$[K]_\Delta = \int_\Delta K dA / \operatorname{Area}(\Delta),$$

where Δ is a triangle becoming extremely small.

Problem 1.4. We let $\mathrm{Homeo}_+(S^1)$ denote the group of orientation preserving homeomorphisms on S^1. Prove that

$$\mathrm{PSL}_2(\mathbb{R}) = \mathrm{Isom}_+(\mathbb{H}^2) = \mathrm{Conf}(\mathbb{H}^2) = \mathrm{biHol}(\mathbb{H}^2) \leqslant \mathrm{Homeo}_+(S^1).$$

1.1. **Fuchsian groups.**

Problem 1.5. (1) Prove that for all distinct triple $a, b, c \in \mathbb{C}$ there uniquely exists $S_{a,b,c} \in \mathrm{PSL}_2(\mathbb{C})$ such that $S_{a,b,c}(a, b, c) = (1, 0, \infty)$. We then define the *cross ratio* of distinct points $z, a, b, c \in \mathbb{C}$ as the complex number

$$[z : a : b : c] := S_{a,b,c}(z).$$

p:cr-preserve

(2) Prove that every $g \in \mathrm{PSL}_2(\mathbb{C})$ preserves the cross ratio.

(3) Prove that $[a : b : c : d] \in \mathbb{R}$ iff a, b, c, d all lie in a circle or a line.

(4) Conclude that $\mathrm{PSL}_2(\mathbb{C})$ maps circles or lines to circles or lines.

(5) For distinct points $a, b, c, d \in \mathbb{C}$, prove that

$$[a : b : c : d] = [b : a : d : c] = [c : d : a : b] = 1 - \frac{1}{[c : a : b : d]}.$$

Moreover, if $a < b < c < d$ are real, then show that

$$\log[a : b : c : d] := \iint_{[a,b]\times[c,d]} \frac{dx\,dy}{(x-y)^2}.$$

(6) For $z \in \mathbb{C}$, we let the *reflection of z along the real line* as

$$R_{\mathbb{R}}(z) := \bar{z}.$$

More generally, let C be a circle or a line in $\mathbb{C}$. Then the *reflection of z along C* is defined as the point

$$R_C(z) := g^{-1}\overline{g(z)},$$

where $g \in \mathrm{PSL}_2(\mathbb{C})$ is an element that maps C to the real line. Prove that R_C does not depend on the choice of g; moreover, show that whenever a, b, c are distinct points on C, we have that

$$[R_C(z) : a : b : c] := \overline{[z : a : b : c]}.$$

Hint. Almost no computation is involved in (2).

Problem 1.6. Prove that $\pi_1(\mathbb{R}P^2\#\mathbb{R}P^2\#\mathbb{R}P^2)$ embeds into the right-angled pentagon reflection group as an index–four subgroup.

Problem 1.7. Let G be a nonempty subgroup of $\mathrm{Homeo}_+(S^1)$ such that every orbit is infinite. Prove that the *limit set*

$$\Lambda(G) := \bigcap_{x\in S^1} \overline{G.x}$$

is either S^1 or a Cantor set.

Hint. Use Zorn's lemma to see that there exists a minimal closed nonempty G–invariant set $L \subseteq S^1$. If $L = S^1$, then every orbit must be dense and we are done. If not, the minimality implies that

$$L = L' = \partial L.$$

This implies that L is a Cantor set. To see $L = \Lambda(G)$, we pick an arbitrary $y \in J = (a, b) \subseteq S^1 \setminus L$, where J is a maximal open interval with this condition. Using $(G.a)' = L$, we can find a sequence $\{g_n\} \subseteq G$ such that $g_n(a) \to a$ and $g_n(a)$ are all distinct. This implies $g_n(y) \to a$ and we are done. See [2], for instance.

Problem 1.8. A Fuchsian group $G \leqslant \mathrm{PSL}_2(\mathbb{R})$ is *elementary* if it has a finite orbit. Prove that an elementary group G satisfies exactly one of the following.

(i) G is finite cyclic.

(ii) G is virtually cyclic and conjugate into the group

$$\langle z \mapsto -1/z \rangle \ltimes \{z \mapsto e^t z \colon t \in \mathbb{R}\}.$$

(iii) G is cyclic and conjugate into the group

$$\langle z \mapsto z + 1 \rangle.$$

Hint. Prove first that every finite subgroup of $\mathrm{Homeo}_+(S^1)$ is finite cyclic.

References

1. Andrew J. Casson and Steven A. Bleiler, *Automorphisms of surfaces after Nielsen and Thurston*, London Mathematical Society Student Texts, vol. 9, Cambridge University Press, Cambridge, 1988. MR964685
2. A. Navas, *Groups of circle diffeomorphisms*, Spanish ed., Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2011. MR2809110

School of Mathematics, Korea Institute for Advanced Study (KIAS), Seoul, 02455, Korea
Email address: skim.math@gmail.com
URL: http://cayley.kr