# Alexander pairs of quandles and generalizations of twisted Alexander polynomials

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$$G = \langle S | R \rangle_{\mathrm{Grp}} \xrightarrow[\text{abelianization}]{\text{Fox derivative}} \text{Alexander matrix} \longrightarrow \text{Alexander invariants}$$

$$(\rho = 1)$$

generalize

$$G = \langle S | R \rangle_{\mathrm{Grp}} \xrightarrow[\text{abelianization}]{\text{Fox derivative}} \text{Twisted Alexander matrix} \longrightarrow \text{Twisted Alexander invariants}$$

$$\rho : G \to H$$

generalize

$$X = \langle S | R \rangle_{\mathrm{Qnd}} \xrightarrow{(f_1, f_2)\text{-derivative}} (f_1, f_2)\text{-twisted Alexander matrix} \longrightarrow (f_1, f_2)\text{-twisted Alexander invariants}$$

$$\rho : X \to Q$$

## §Quandles

- A quandle is a set $Q$ with $\triangleleft : Q \times Q \to Q$ s.t.

(Q1) $\forall a \in Q$, $a \triangleleft a = a$,

(Q2) $\forall a, b \in Q$, $\exists 1 c \in Q$ s.t. $c \triangleleft b = a$,

(Q3) $\forall a, b, c \in Q$, $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$.

$c$ in (Q2) is denoted by $a \triangleleft^{-1} b$

$(Q_1, \triangleleft_1)$, $(Q_2, \triangleleft_2)$: quandles

- $f : Q_1 \to Q_2$ is a quandle homomorphism $:\overset{\text{def}}{\Leftrightarrow}$

$$f(a \triangleleft_1 b) = f(a) \triangleleft_2 f(b)$$

- $f : Q_1 \to Q_2$ is a quandle isom. $:\overset{\text{def}}{\Leftrightarrow}$ $f$: hom & bij.

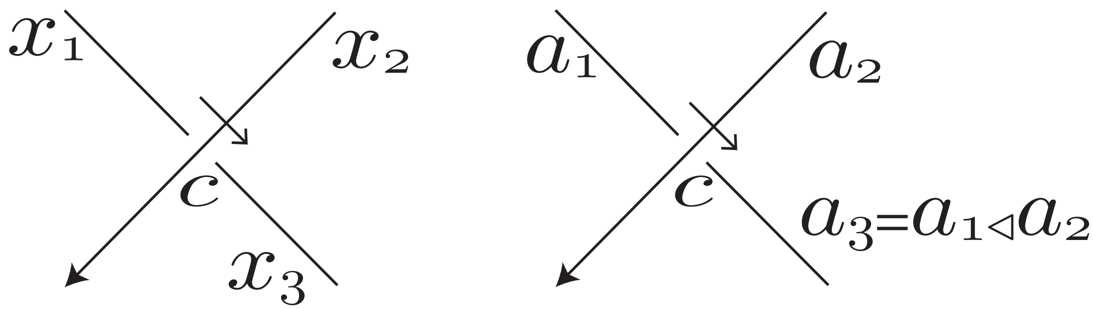
※ $\triangleleft a : Q \to Q; x \mapsto x \triangleleft a$ is a quandle isom.

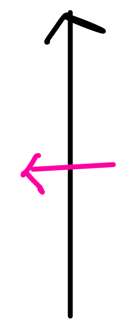
$(Q, \triangleleft)$: a quandle.

$D$: a diagram of an oriented knot $K$.

- $\rho : \{\text{arcs of } D\} \to Q$ is a $Q$-coloring of $D$ $:\overset{\text{def}}{\Leftrightarrow}$

  $\forall c$: crossing, $\rho(x_1) \triangleleft \rho(x_2) = \rho(x_3)$.

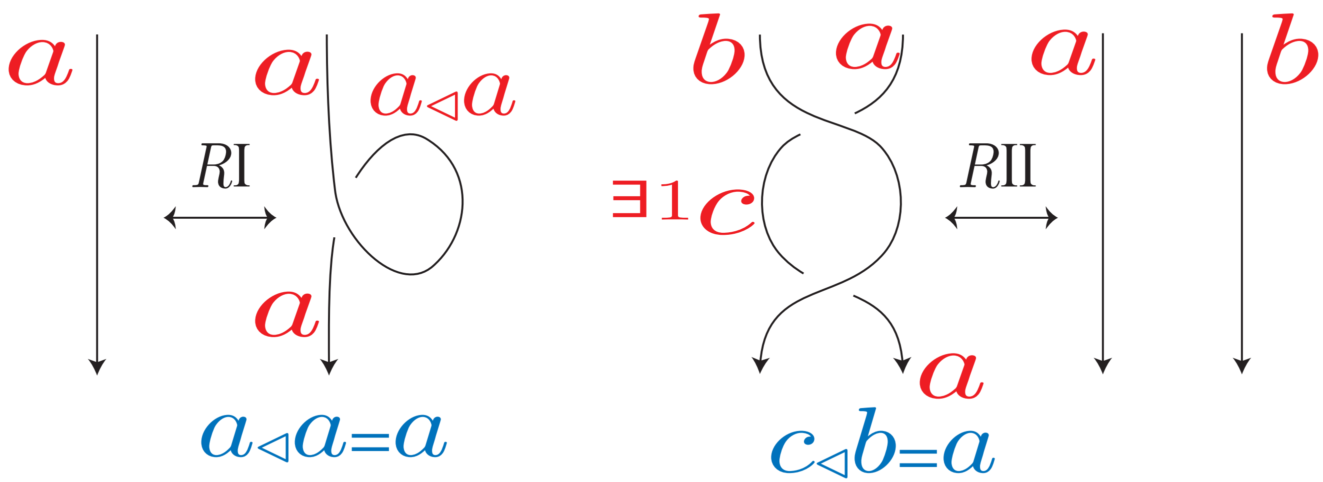


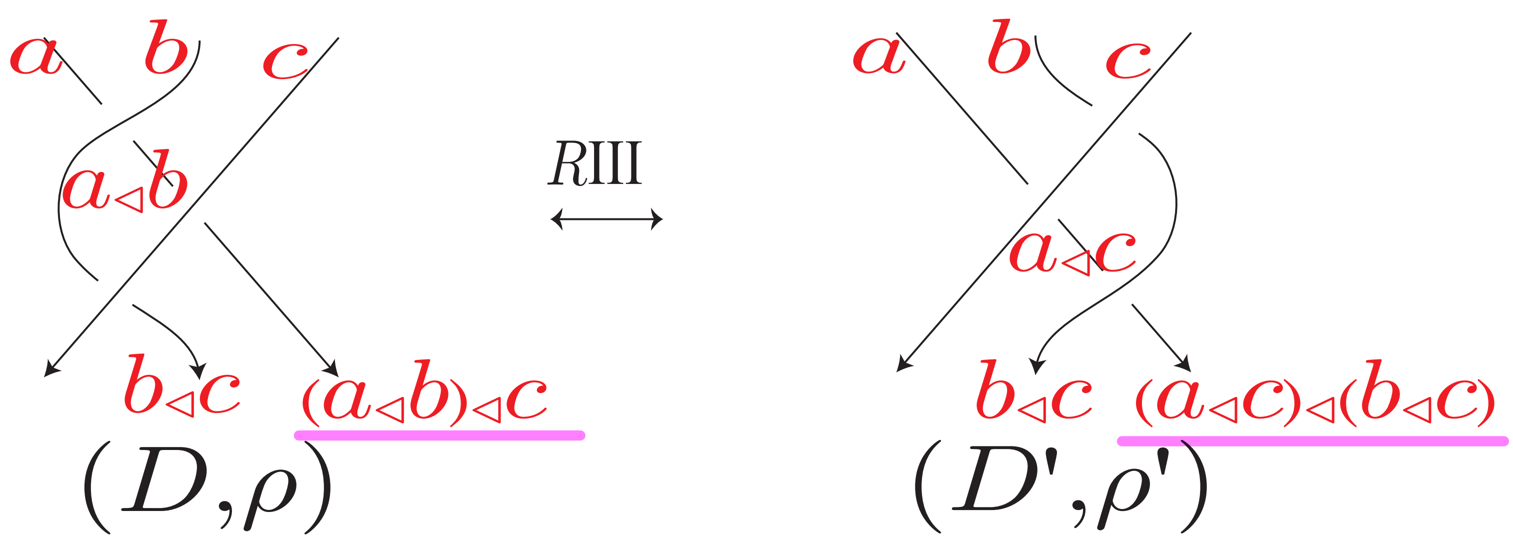



$\{Q\text{-colorings of } D\} \overset{\exists 1:1}{\longleftrightarrow} \mathrm{Hom}(Q(K), Q)$

$= \{$representatins from $Q(K)$ to $Q\}$

$$D \cong D' \Longrightarrow \{Q\text{-colorings of } D'\} \overset{\exists 1:1}{\longleftrightarrow} \{Q\text{-colorings of } D'\}$$




$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$

$(X_1, \rho_1)$, $(X_2, \rho_2)$: quandles with quandle representations

$\rho_i : X_i \to Q$

- $(X_1, \rho_1) \cong (X_2, \rho_2) \overset{\text{def}}{:\Leftrightarrow} \exists f : X_1 \to X_2$: a quandle isom. s.t. $\rho_2 \circ f = \rho_1$

$$\begin{tikzcd} X_1 \arrow[rr, "f", "\cong"'] \arrow[dr, "\rho_1"'] & \arrow[d, phantom, "\circlearrowleft"] & X_2 \arrow[dl, "\rho_2"] \\ & Q & \end{tikzcd}$$

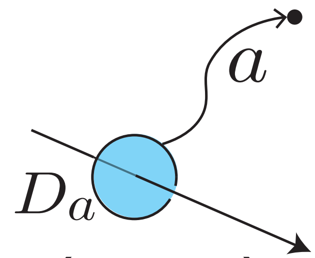
**Ex.** $\begin{cases} \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} \\ a \triangleleft b = 2b - a \end{cases}$ is a quandle.

$R_n := (\mathbb{Z}_n, \triangleleft)$ is called the dihedral quandle of order $n$.

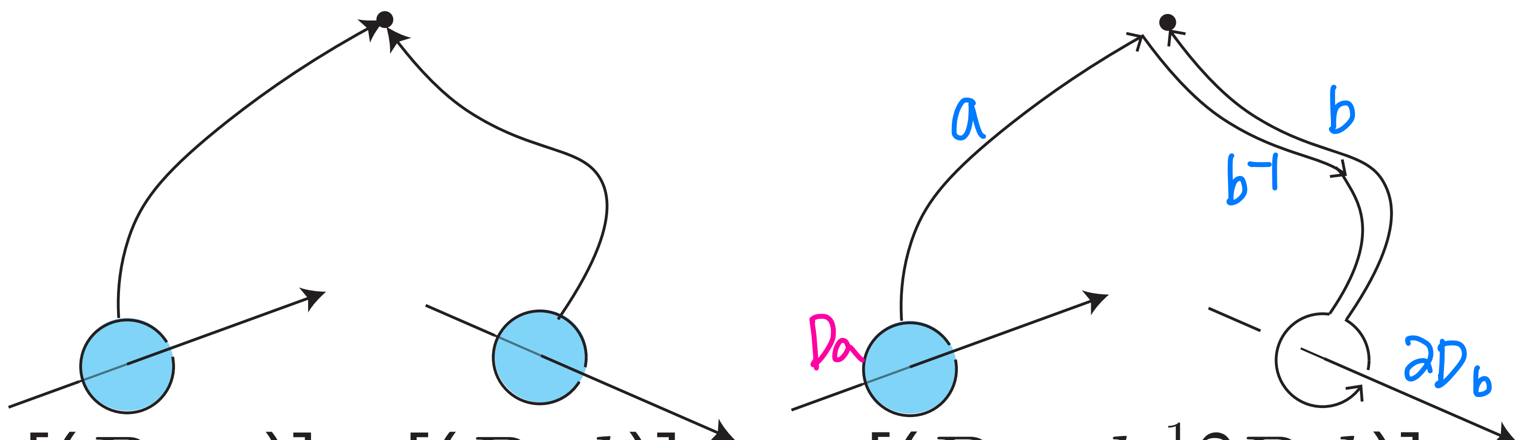
**Ex.** $\begin{cases} G : \text{ a group} \\ g \triangleleft h = h^{-1}gh \end{cases}$ is a quandle.

$G_{\text{conj}} := (G, \triangleleft)$ is called the conjugation quandle of $G$.

**Ex.** $\boldsymbol{K}$: an oriented knot.




$$\boldsymbol{Q}(\boldsymbol{K}) = \left\{ \left[ (D_a, a) \right]_{\text{homotopy class}} \right\}$$




$$[(D_a, a)] \triangleleft [(D_b, b)] = [(D_a, ab^{-1}\partial D_b b)]$$

$(\boldsymbol{Q}(\boldsymbol{K}), \triangleleft)$ is the knot quandle of $\boldsymbol{K}$

## §Quandle presentations

$S$: a finite set.

- The free quandle of $S$:

$$F_{\mathrm{Qnd}}(S) = S \times F_{\mathrm{Grp}}(S) / (a, a^{\pm 1}w) \sim (a, w)$$

$$[(a, w)] \triangleleft [(b, v)] = [(a, wv^{-1}bv)]$$

$$a \triangleleft^{\varepsilon_1} a_1 \triangleleft^{\varepsilon_2} a_2 \cdots \triangleleft^{\varepsilon_n} a_n := [(a, a_1^{\varepsilon_1} a_2^{\varepsilon_2} \cdots a_n^{\varepsilon_n})]$$

$$F_{\mathrm{Qnd}}(S) = \left\{ a \triangleleft^{\varepsilon_1} a_1 \cdots \triangleleft^{\varepsilon_n} a_n \;\middle|\; \cdots \right\} / a \triangleleft^{\pm 1} a = a$$

$$(a \underbrace{\triangleleft^{\varepsilon_1} a_1 \cdots \triangleleft^{\varepsilon_m} a_m}_{w}) \triangleleft (b \underbrace{\triangleleft^{\delta_1} b_1 \cdots \triangleleft^{\delta_n} b_n}_{v})$$

$$= a \underbrace{\triangleleft^{\varepsilon_1} a_1 \cdots \triangleleft^{\varepsilon_m} a_m}_{w} \underbrace{\triangleleft^{-\delta_n} b_n \cdots \triangleleft^{-\delta_1} b_1}_{v^{-1}} \triangleleft b \underbrace{\triangleleft^{\delta_1} b_1 \cdots \triangleleft^{\delta_n} b_n}_{v}$$

**Ex.** $(a \triangleleft c) \triangleleft (b \triangleleft c) = a \triangleleft c \triangleleft^{-1} c \triangleleft b \triangleleft c = a \triangleleft b \triangleleft c = (a \triangleleft b) \triangleleft c$

$R \subset F_{\mathrm{Qnd}}(S) \times F_{\mathrm{Qnd}}(S)$

"$r_1 = r_2$" $:= (r_1, r_2)$

- The quandle presentation with gen. $S$ and rel. $R$:
  $\langle S \mid R \rangle_{\mathrm{Qnd}} = \langle x_1, \ldots, x_n \mid r_{11} = r_{12}, \ldots, r_{m1} = r_{m2} \rangle_{\mathrm{Qnd}}$
  $:= F_{\mathrm{Qnd}}(S) / \sim_R$ $(r_{i1} \sim_R r_{i2},\ w \triangleleft r_{i1} \sim_R w \triangleleft r_{i2}, \cdots)$

$Q$: a quandle.

- The presentation of $Q$: $\langle S \mid R \rangle_{\mathrm{Qnd}}$ s.t. $Q \cong \langle S \mid R \rangle_{\mathrm{Qnd}}$

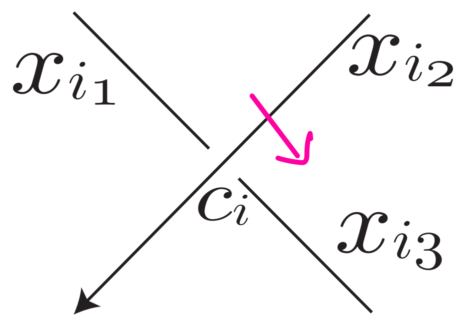
**Lem.**

$$(\langle S \mid R \rangle_{\mathrm{Qnd}}, \rho) \cong (\langle S' \mid R' \rangle_{\mathrm{Qnd}}, \rho')$$
$$\iff (\langle S \mid R \rangle_{\mathrm{Qnd}}, \rho) \overset{\text{Teize Trans.}}{\longleftrightarrow} (\langle S' \mid R' \rangle_{\mathrm{Qnd}}, \rho')$$

$\boldsymbol{D}$: a diagram of an oriented knot $\boldsymbol{K}$.

$\boldsymbol{x_1, \ldots, x_n}$: arcs of $\boldsymbol{D}$, $\boldsymbol{c_1, \ldots, c_n}$ : crossings of $\boldsymbol{D}$.

$$\boldsymbol{Q(K) \cong \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle_{\mathrm{Qnd}}}$$




$$r_i : x_{i_1} \lhd x_{i_2} = x_{i_3}$$

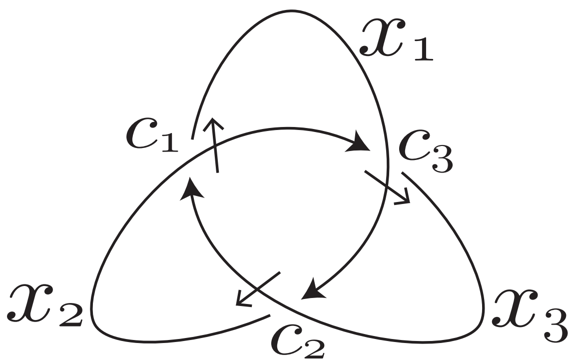
This presentation is called a Wirtinger presentation of $\boldsymbol{Q(K)}$.

※ repr. $\boldsymbol{\rho : Q(K) \to Q}$ satisfies $\boldsymbol{\rho(x_{i_1}) \lhd \rho(x_{i_2}) = \rho(x_{i_3})}$.

$\boldsymbol{\{Q}$-colorings of $\boldsymbol{D\}} \overset{\exists 1:1}{\longleftrightarrow}$ Hom$(\boldsymbol{Q(K), Q})$

$=$ {representatins from $\boldsymbol{Q(K)}$ to $\boldsymbol{Q}$}

## Ex.




$$r_1 : x_3 \triangleleft x_2 = x_1,$$
$$r_2 : x_1 \triangleleft x_3 = x_2,$$
$$r_3 : x_2 \triangleleft x_1 = x_3$$

$$Q(K) = \langle x_1, x_2, x_3 \mid x_3 \triangleleft x_2 = x_1, \;\; x_1 \triangleleft x_3 = x_2, \;\; x_2 \triangleleft x_1 = x_3 \rangle_{\mathrm{Qn}}$$

$$x \triangleleft y = z \rightsquigarrow y^{-1}xyz^{-1}$$

the knot quandle $K$ $\rightsquigarrow$ the knot group of $K$:

$$G(K) = \langle x_1, x_2, x_3 \mid x_2^{-1}x_3x_2x_1^{-1}, \; x_3^{-1}x_1x_3x_2^{-1}, \; x_1^{-1}x_2x_1x_3^{-1} \rangle$$

§Alexander pairs

$(\boldsymbol{Q}, \triangleleft)$: a quandle, $\quad \boldsymbol{R}$: a ring.

$\boldsymbol{f_1, f_2 : Q \times Q \to R}$: maps.

- $(\boldsymbol{f_1, f_2})$ is an Alexander pair $:\overset{\text{def}}{\Leftrightarrow}$
  - $\forall \boldsymbol{a} \in \boldsymbol{Q}$, $\boldsymbol{f_1(a, a) + f_2(a, a) = 1}$.
  - $\forall \boldsymbol{a, b} \in \boldsymbol{Q}$, $\boldsymbol{f_1(a, b)}$ is invertible.
  - $\forall \boldsymbol{a, b, c} \in \boldsymbol{Q}$,

$$\boldsymbol{f_1(a \triangleleft b, c) f_1(a, b) = f_1(a \triangleleft c, b \triangleleft c) f_1(a, c)},$$

$$\boldsymbol{f_1(a \triangleleft b, c) f_2(a, b) = f_2(a \triangleleft c, b \triangleleft c) f_1(b, c)}, \text{ and}$$

$$\boldsymbol{f_2(a \triangleleft b, c)}$$

$$\boldsymbol{= f_1(a \triangleleft c, b \triangleleft c) f_2(a, c) + f_2(a \triangleleft c, b \triangleleft c) f_2(b, c)}.$$

**Prop.** [cf. Andruskiewitsch-Graña] $M$: a left $R$-module.

$* : (Q \times M) \times (Q \times M) \to Q \times M$;

$$(a, x) * (b, y) = (a \triangleleft b, f_1(a, b)x + f_2(a, b)y).$$

$(f_1, f_2)$: an Alexander pair $\iff$ $(Q \times M, *)$: a quandle

$$\begin{aligned}
&((a, x) * (b, y)) * (c, z) \\
&= (a \triangleleft b, f_1(a, b)x + f_2(a, b)y) * (c, z) \\
&= ((a \triangleleft b) \triangleleft c, \\
&\quad f_1(a \triangleleft b, c) f_1(a, b)x + f_1(a \triangleleft b, c) f_2(a, b)y + f_2(a \triangleleft b, c)z)
\end{aligned}$$

$$\begin{aligned}
&((a, x) * (c, z)) * ((b, y) * (c, z)) \\
&= (a \triangleleft c, f_1(a, c)x + f_2(a, c)z) * (b \triangleleft c, f_1(b, c)y + f_2(b, c)z) \\
&= ((a \triangleleft c) \triangleleft (b \triangleleft c), \\
&\quad f_1(a \triangleleft c, b \triangleleft c) f_1(a, c)x + f_2(a \triangleleft c, b \triangleleft c) f_1(b, c)y \\
&\quad + (f_1(a \triangleleft c, b \triangleleft c) f_2(a, c) + f_2(a \triangleleft c, b \triangleleft c) f_2(b, c))z)
\end{aligned}$$

**Ex.** $Q$: a quandle, $R[t^{\pm 1}]$: the Laurent polynomial ring.
$f_1, f_2 : Q \times Q \to R[t^{\pm 1}]$;

$$f_1(a,b) = t, \qquad f_2(a,b) = 1 - t$$

give an Alexander pair, which is related to the Alexander polynomial.

**Ex.** $Q := \mathrm{Conj}\, G$, where $G := GL(k; R)$.
$f_1, f_2 : Q \times Q \to R[t^{\pm 1}][G]$;

$$f_1(a,b) = tb^{-1}, \qquad f_2(a,b) = b^{-1}a - tb^{-1}$$

give an Alexander pair, which is related to the twisted Alexander polynomial.

**Ex.** $R_3 = (\mathbb{Z}_3, a \triangleleft b = 2b - a)$.

$f_1, f_2 : Q \times Q \to \mathbb{Z}$;

$$f_1(a,b) = -1, \qquad f_2(a,b) = 3\delta_{ab} - 1$$

give an Alexander pair.

**Prop.** $\rho : X \to Q$: a quandle representation.

$(f_1, f_2)$: an Alexander pair of $f_1, f_2 : Q^2 \to R$. Then

$$f \circ \rho^2 := (f_1 \circ (\rho \times \rho),\ f_2 \circ (\rho \times \rho))$$

is an Alxander pair.

$f_1 \circ (\rho \times \rho)(x, y) = f_1(\rho(x), \rho(y))$

## § $(f_1, f_2)$-derivatives and quandle twisted Alex. inv.

Recall: Twisted Alexander invariant (Case 1)

$G(K) = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$

$\rho : G(K) \to \mathrm{GL}_k(R)$: a group representation.

- Fox calculus:

$$\mathbb{Z}[F(x)] \xrightarrow{\frac{\partial_{\mathsf{Grp}}}{\partial x_i}} \mathbb{Z}[F(x)] \xrightarrow{\mathrm{pr}} \mathbb{Z}[G(K)] \xrightarrow{\rho\otimes\alpha} \mathbb{Z}[t^{\pm 1}]$$

$$\frac{\partial_{\mathsf{Grp}}}{\partial x_i}(pq) = \frac{\partial_{\mathsf{Grp}}}{\partial x_i}(p) + p\frac{\partial_{\mathsf{Grp}}}{\partial x_i}(q) \quad \text{and} \quad \frac{\partial_{\mathsf{Grp}}}{\partial x_j}(x_i) = \delta_{ij}.$$

$\alpha$ : Wirtinger generator $\mapsto t^{-1}$

$A = \left( (\rho \otimes \alpha) \circ \mathrm{pr} \left( \frac{\partial_{\mathsf{Grp}}}{\partial x_j}(r_i) \right) \right) \rightsquigarrow \Delta_d(K, \rho)$: twist. Alex. poly

$Q = \langle \underbrace{x_1, \ldots, x_n}_{=S} \mid r_1, \ldots, r_m \rangle_{\mathrm{Qnd}}$: quandle.

$f = (f_1, f_2)$: an Alexander pair of $f_1, f_2 : Q^2 \to R$

- The $f$-derivative w.r.t. $x_j$:
  a map $\frac{\partial_f}{\partial x_j} : F_{\mathrm{Qnd}}(S) \to R$ satisfying

$$\frac{\partial_f}{\partial x_j}(a \triangleleft b) = f_1(a, b)\frac{\partial_f}{\partial x_j}(a) + f_2(a, b)\frac{\partial_f}{\partial x_j}(b),$$

$$\frac{\partial_f}{\partial x_j}(x_i) = \delta_{ij}$$

**Th.** $\dfrac{\partial_f}{\partial x_j}$ is well-defined.

Indeed, $(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c) \in F_{\mathrm{Qnd}}(S)$.

$$
\begin{aligned}
&\frac{\partial_f}{\partial x_j}((a \lhd b) \lhd c) \\
&= f_1(a \lhd b, c)\frac{\partial_f}{\partial x_j}(a \lhd b) + f_2(a \lhd b, c)\frac{\partial_f}{\partial x_j}(c) \\
&= \underline{f_1(a \lhd b, c) f_1(a, b)}\frac{\partial_f}{\partial x_j}(a) + \underline{f_1(a \lhd b, c) f_2(a, b)}\frac{\partial_f}{\partial x_j}(b) + \underline{f_2(a \lhd b, c)}\frac{\partial_f}{\partial x_j}(c) \\
&= \underline{f_1(a \lhd c, b \lhd c) f_1(a, c)}\frac{\partial_f}{\partial x_j}(a) + \underline{f_2(a \lhd c, b \lhd c) f_1(b, c)}\frac{\partial_f}{\partial x_j}(b) \\
&\quad + \underline{f_1(a \lhd c, b \lhd c) f_2(a, c)}\frac{\partial_f}{\partial x_j}(c) + \underline{f_2(a \lhd c, b \lhd c) f_2(b, c)}\frac{\partial_f}{\partial x_j}(c) \\
&= f_1(a \lhd c, b \lhd c)\frac{\partial_f}{\partial x_j}(a \lhd c) + f_2(a \lhd c, b \lhd c)\frac{\partial_f}{\partial x_j}(b \lhd c) \\
&= \frac{\partial_f}{\partial x_j}((a \lhd c) \lhd (b \lhd c)).
\end{aligned}
$$

**Lem.**

$$\begin{aligned}
&\frac{\partial_f}{\partial x_j}(a \triangleleft^{-1} b)\\
&= f_1(a \triangleleft^{-1} b, b)^{-1}\frac{\partial_f}{\partial x_j}(a)\\
&\quad - f_1(a \triangleleft^{-1} b, b)^{-1} f_2(a \triangleleft^{-1} b, b)\frac{\partial_f}{\partial x_j}(b).
\end{aligned}$$

**Proof**

$$\begin{aligned}
&\frac{\partial_f}{\partial x_j}(a) = \frac{\partial_f}{\partial x_j}((a \triangleleft^{-1} b) \triangleleft b)\\
&= f_1(a \triangleleft^{-1} b, b)\frac{\partial_f}{\partial x_j}(a \triangleleft^{-1} b) + f_2(a \triangleleft^{-1} b, b)\frac{\partial_f}{\partial x_j}(b).
\end{aligned}$$

$Q$: a quandle, $R$: a ring.

$f = (f_1, f_2)$: an Alexander pair of $f_1, f_2 : Q^2 \to R$.

$X = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$: a quandle,

$\rho : X \to Q$: a quandle representation.

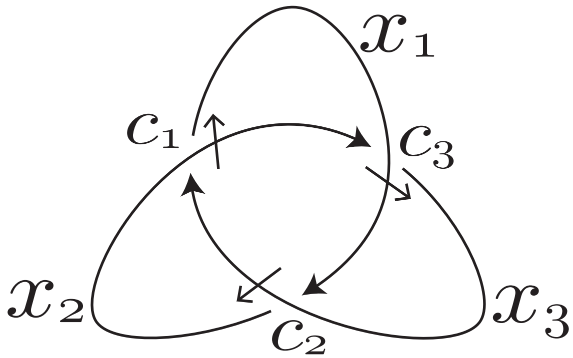
- The $f$-Alexander matrix of $(X, \rho)$:

$$A(X, \rho; f_1, f_2) = \begin{pmatrix} \frac{\partial_{f \circ \rho^2}}{\partial x_1}(r_1) & \cdots & \frac{\partial_{f \circ \rho^2}}{\partial x_n}(r_1) \\ \vdots & \ddots & \vdots \\ \frac{\partial_{f \circ \rho^2}}{\partial x_1}(r_m) & \cdots & \frac{\partial_{f \circ \rho^2}}{\partial x_n}(r_m) \end{pmatrix}$$

※ $\frac{\partial_{f \circ \rho^2}}{\partial x_j}(r_i) = \frac{\partial_{f \circ \rho^2}}{\partial x_j}(r_{i1} = r_{i2}) = \frac{\partial_{f \circ \rho^2}}{\partial x_j}(r_{i1}) - \frac{\partial_{f \circ \rho^2}}{\partial x_j}(r_{i2})$

※ $f \circ \rho^2 = (f_1 \circ (\rho \times \rho), f_2 \circ (\rho \times \rho))$

**Ex.**




$$r_1 : x_3 \triangleleft x_2 = x_1,$$
$$r_2 : x_1 \triangleleft x_3 = x_2,$$
$$r_3 : x_2 \triangleleft x_1 = x_3$$

$$X = \langle x_1, x_2, x_3 \mid x_3 \triangleleft x_2 = x_1,\quad x_1 \triangleleft x_3 = x_2,\quad x_2 \triangleleft x_1 = x_3 \rangle$$

$$= \frac{\partial_{f\circ\rho^2}}{\partial x_1}(x_3 \triangleleft x_2) - \frac{\partial_{f\circ\rho^2}}{\partial x_1}(x_1)$$

$$\frac{\partial_{f\circ\rho^2}}{\partial x_1}(r_1) = f_1(a_3,a_2)\frac{\partial_{f\circ\rho^2}}{\partial x_1}(x_3) + f_2(a_3,a_2)\frac{\partial_{f\circ\rho^2}}{\partial x_1}(x_2) - \frac{\partial_{f\circ\rho^2}}{\partial x_1}(x_1) = -1$$
$$\frac{\partial_{f\circ\rho^2}}{\partial x_2}(r_1) = f_1(a_3,a_2)\frac{\partial_{f\circ\rho^2}}{\partial x_2}(x_3) + f_2(a_3,a_2)\frac{\partial_{f\circ\rho^2}}{\partial x_2}(x_2) - \frac{\partial_{f\circ\rho^2}}{\partial x_2}(x_1) = f_2(a_3,a_2)$$
$$\frac{\partial_{f\circ\rho^2}}{\partial x_3}(r_1) = f_1(a_3,a_2)\frac{\partial_{f\circ\rho^2}}{\partial x_3}(x_3) + f_2(a_3,a_2)\frac{\partial_{f\circ\rho^2}}{\partial x_3}(x_2) - \frac{\partial_{f\circ\rho^2}}{\partial x_3}(x_1) = f_1(a_3,a_2)$$

$$A(X,\rho; f_1, f_2) = \begin{pmatrix} -1 & f_2(a_3,a_2) & f_1(a_3,a_2) \\ f_1(a_1,a_3) & -1 & f_2(a_1,a_3) \\ f_2(a_2,a_1) & f_1(a_2,a_1) & -1 \end{pmatrix}$$

Suppose that $R$ is a GCD domain.

- The $d$th Alexander invariant $\Delta_d(A)$ of $A \in M(m, n; R)$:

$$\Delta_d(A) = \begin{cases} 1 & \text{if } n - d < 1, \\ \gcd\{(n-d)\text{-minors of } A\} & \text{if } 1 \leq n - d \leq m, \\ 0 & \text{if } m < n - d. \end{cases}$$

- The $d$th quandle twisted Alexander invariant:

$$\Delta_d(X, \rho; f_1, f_2) = \Delta_d(A(X, \rho; f_1, f_2))$$

When $R$ is a matrix ring of consisting of $k \times k$ matrices,

- The $d$th quandle twisted Alexander invariant :

$\Delta_d(X, \rho; f_1, f_2) = \gcd\{\det \overline{B} \mid B : \text{ submatrix of order } n - d\}$,

$\overline{B}$ is the $k(n-d) \times k(n-d)$-matrix.

**Th.**

$$(X_1,\rho_1)\cong(X_2,\rho_2)\Longrightarrow\Delta_d(X_1,\rho_1;f_1,f_2)\doteq\Delta_d(X_2,\rho_2;f_1,f_2)$$

※ $\Delta_d(X,\rho;f_1,f_2)$ is determined up to unit multiple

When $X=Q(K)$ for an ori. knot $K$,

$$\Delta_d(K,\rho;f_1,f_2):=\Delta_d(X,\rho;f_1,f_2).$$

**Cor.** (1) $\Delta_d(K,\rho;f_1,f_2)$ is an invariant of $(K,\rho)$.
(2) The multiset $\{\Delta_d(K,\rho;f_1,f_2)\mid\rho\in\mathrm{Hom}(Q(K),Q)\}$ is an invariant of $K$.

## § Quandle twisted Alexander invariants (ver2)

Recall: Twisted Alexander invariant (Case 2)

$G(K) = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_{n-1} \rangle$

$\rho : G(K) \to \mathrm{GL}_k(R)$: a group representation.

- Fox calculus:

$$\mathbb{Z}[F(x)] \xrightarrow{\frac{\partial_{\mathsf{Grp}}}{\partial x_i}} \mathbb{Z}[F(x)] \xrightarrow{\mathrm{pr}} \mathbb{Z}[G(K)] \xrightarrow{\rho \otimes \alpha} \mathbb{Z}[t^{\pm 1}]$$

$$A = \left( (\rho \otimes \alpha) \circ \mathrm{pr} \left( \frac{\partial_{\mathsf{Grp}}}{\partial x_j}(r_i) \right) \right)$$

$\rightsquigarrow \Delta(K, \rho) = \det \bar{A}_j / \det(\underbrace{\Phi(x_j - 1)}_{t^{-1}\rho(x_j) - E}))$: twisted Alex. poly

$(f_1, f_2)$: an Alexander pair of $f_1, f_2 : Q \times Q \to R$.

- $f_{\mathrm{col}} : Q \to R$ is a column relation map $:\overset{\mathrm{def}}{\Leftrightarrow}$

$$f_{\mathrm{col}}(a \triangleleft b) = f_1(a,b) f_{\mathrm{col}}(a) + f_2(a,b) f_{\mathrm{col}}(b)$$

<u>**Ex.**</u> $Q := \mathrm{Conj}\, G$, where $G := GL(k; R)$.

$f_1, f_2 : Q \times Q \to R[t^{\pm 1}][G]$;

$$f_1(a,b) = tb^{-1}, \qquad f_2(a,b) = b^{-1}a - tb^{-1}.$$

$$f_{\mathrm{col}} : Q \to R[t^{\pm 1}][G]; \quad f_{\mathrm{col}}(x) = t^{-1}x - 1$$

is a column relation map. ($\rightsquigarrow$ the twisted Alex. poly.)

**Prop.** $\forall c \in Q$,

$$f_{\mathrm{col}} : Q \to R; \quad f_{\mathrm{col}}(x) = f_2(x \triangleleft^{-1} c, c)$$

is a column relation map.

**Proof**

$$\begin{aligned} f_{\mathrm{col}}(a \triangleleft b) &= f_2((a \triangleleft b) \triangleleft^{-1} c, c) \\ &= f_2((a \triangleleft^{-1} c) \triangleleft (b \triangleleft^{-1} c), c) \\ &= f_1(a, b) f_2(a \triangleleft^{-1} c, c) + f_2(a, b) f_2(b \triangleleft^{-1} c, c) \\ &= f_1(a, b) f_{\mathrm{col}}(a) + f_2(a, b) f_{\mathrm{col}}(b). \end{aligned}$$

**Ex.** $Q = R_3 = (\mathbb{Z}_3, a \triangleleft b = 2b - a)$.
$f_1, f_2 : Q \times Q \to \mathbb{Z}; \quad f_1(a, b) = -1, \quad f_2(a, b) = 3\delta_{ab} - 1$.

$$f_{\mathrm{col},c} : Q \to \mathbb{Z}; \quad f_{\mathrm{col},c}(x) = 3\delta_{xc} - 1$$

is a column relation map for $c \in Q$.

**Lem.** $\rho : X \to Q$ a quande representation.

$f_{\mathrm{col}} : Q \to R$: a column relation map.

Then $f_{\mathrm{col}} \circ \rho : X \to R$ is a col. rel. map of $(f_1 \circ \rho^2, f_2 \circ \rho^2)$.

**Proof**

$$
\begin{aligned}
&(f_1 \circ \rho^2)(a,b)(f_{\mathrm{col}} \circ \rho)(a) + (f_2 \circ \rho^2)(a,b)(f_{\mathrm{col}} \circ \rho)(b) \\
&= f_1(\rho(a),\rho(b)) f_{\mathrm{col}}(\rho(a)) + f_2(\rho(a),\rho(b)) f_{\mathrm{col}}(\rho(b)) \\
&= f_{\mathrm{col}}(\rho(a) \triangleleft \rho(b)) \\
&= (f_{\mathrm{col}} \circ \rho)(a \triangleleft b)
\end{aligned}
$$

$$R_{\mathrm{col}}(X,\rho;f_{\mathrm{col}}) := \begin{pmatrix} (f_{\mathrm{col}}\circ\rho)(x_1) \\ \vdots \\ (f_{\mathrm{col}}\circ\rho)(x_n) \end{pmatrix}.$$

**Prop.** $A(X,\rho;f_1,f_2)R_{\mathrm{col}}(X,\rho;f_{\mathrm{col}}) = 0$

**Proof**

$$\sum_{j=1}^{n} \frac{\partial_{f\circ\rho^2}}{\partial x_j}(r_i)(f_{\mathrm{col}}\circ\rho)(x_j)$$
$$= \sum_{j=1}^{n} \frac{\partial_{f\circ\rho^2}}{\partial x_j}(r_{i1})(f_{\mathrm{col}}\circ\rho)(x_j) - \sum_{j=1}^{n} \frac{\partial_{f\circ\rho^2}}{\partial x_j}(r_{i2})(f_{\mathrm{col}}\circ\rho)(x_j)$$
$$= (f_{\mathrm{col}}\circ\rho)(r_{i1}) - (f_{\mathrm{col}}\circ\rho)(r_{i2}) = 0.$$

※ $f_{\mathrm{col}}(w) = \sum_{j=1}^{n} \frac{\partial_f}{\partial x_j}(w) f_{\mathrm{col}}(x_j)$ (fundamental fomula)

For $A = (a_1, \ldots, a_n)$, $A_{\hat{s}} := (a_1, \ldots, \hat{a}_s, \ldots, a_n)$.

When $A \in M(m, n; R)$ and $R$ is a GCD domain,

$$\Delta(A) = \begin{cases} \gcd\{n\text{-minors of } A\} & \text{if } n \le m, \\ 0 & \text{if } m < n. \end{cases}$$

- The quandle twisted Alex, inv. ass. w/ $(f_1, f_2)$, $f_{\mathrm{col}}$:

  $\Delta(X, \rho; f_1, f_2; f_{\mathrm{col}}) := \Delta(A(X, \rho; f_1, f_2)_{\hat{j}}) / \det(f_{\mathrm{col}} \circ \rho)(x_j)$

  for $j$ s.t. $\det(f_{\mathrm{col}} \circ \rho)(x_j) \neq 0$.

**Th.**

$(X_1, \rho_1) \cong (X_2, \rho_2)$

$\Longrightarrow \Delta(X_1, \rho_1; f_1, f_2; f_{\mathrm{col}}) \doteq \Delta(X_2, \rho_2; f_1, f_2; f_{\mathrm{col}}) \in \mathrm{Quot}(R)$

(up to unit multiple of $R$)

**Lem.**

$$\Delta(A_{\hat{s}}) \det(f_{\mathrm{col}} \circ \rho(x_t)) \doteq \Delta(A_{\hat{t}}) \det(f_{\mathrm{col}} \circ \rho(x_s))$$

$$\Delta(A_{\hat{s}}) / \det(f_{\mathrm{col}} \circ \rho(x_s)) \doteq \Delta(A_{\hat{t}}) / \det(f_{\mathrm{col}} \circ \rho(x_t)) \in \mathrm{Quot}(R)$$

**Proof** $s = 1, t = 2$. Very rougthly!

$$\det \begin{pmatrix} \frac{\partial_{f \circ \rho^2}}{\partial x_2}(r_i) & \dots & \frac{\partial_{f \circ \rho^2}}{\partial x_n}(r_i) \end{pmatrix} \det(f_{\mathrm{col}} \circ \rho(x_2))$$

$$= \det \begin{pmatrix} \frac{\partial_{f \circ \rho^2}}{\partial x_2}(r_i) \det(f_{\mathrm{col}} \circ \rho(x_2)) & \dots & \frac{\partial_{f \circ \rho^2}}{\partial x_n}(r_i) \end{pmatrix}$$

$$= \det \begin{pmatrix} \sum_{j \neq 2} \frac{\partial_{f \circ \rho^2}}{\partial x_j}(r_i) \det(f_{\mathrm{col}} \circ \rho(x_j)) & \dots & \frac{\partial_{f \circ \rho^2}}{\partial x_n}(r_i) \end{pmatrix}$$

$$= \det \begin{pmatrix} \frac{\partial_{f \circ \rho^2}}{\partial x_1}(r_i) \det(f_{\mathrm{col}} \circ \rho(x_1)) & \frac{\partial_{f \circ \rho^2}}{\partial x_3}(r_i) & \dots & \frac{\partial_{f \circ \rho^2}}{\partial x_n}(r_i) \end{pmatrix}$$

$$= \det \begin{pmatrix} \frac{\partial_{f \circ \rho^2}}{\partial x_1}(r_i) & \frac{\partial_{f \circ \rho^2}}{\partial x_3}(r_i) & \dots & \frac{\partial_{f \circ \rho^2}}{\partial x_n}(r_i) \det(f_{\mathrm{col}} \circ \rho(x_1)) \end{pmatrix}$$

More generally, we can increase column relations:

$f_{\mathrm{col},1}, f_{\mathrm{col},2} : Q \to R$: column relation maps.

We define

$$R_{\mathrm{col}}(X, \rho; f_{\mathrm{col},1}, f_{\mathrm{col},2}) := \begin{pmatrix} (f_{\mathrm{col},1} \circ \rho)(x_1) & (f_{\mathrm{col},2} \circ \rho)(x_1) \\ \vdots & \vdots \\ (f_{\mathrm{col},1} \circ \rho)(x_n) & (f_{\mathrm{col},2} \circ \rho)(x_n) \end{pmatrix}.$$

**Prop.** $A(X, \rho; f_1, f_2) R_{\mathrm{col}}(X, \rho; f_{\mathrm{col},1}, f_{\mathrm{col},2}) = 0$

$$R_{\mathrm{col}}(X, \rho; f_{\mathrm{col},1}, f_{\mathrm{col},2})_{(i,j)_{\mathrm{row}}} := \begin{pmatrix} (f_{\mathrm{col},1} \circ \rho)(x_i) & (f_{\mathrm{col},2} \circ \rho)(x_i) \\ (f_{\mathrm{col},1} \circ \rho)(x_j) & (f_{\mathrm{col},2} \circ \rho)(x_j) \end{pmatrix}$$

$$A(X, \rho; f_1, f_2)_{(\hat{i},\hat{j})_{\mathrm{col}}} = \begin{pmatrix} a_1 & \cdots & \hat{a}_i & \cdots & \hat{a}_j & \cdots & a_n \end{pmatrix}$$

- The quandle twisted Alex, inv. ass. w/ $(\boldsymbol{f}_1, \boldsymbol{f}_2)$, $\boldsymbol{f}_{\mathrm{col},1}, \boldsymbol{f}_{\mathrm{col},2}$:

$$\boldsymbol{\Delta}(\boldsymbol{X}, \boldsymbol{\rho}; \boldsymbol{f}_1, \boldsymbol{f}_2; \boldsymbol{f}_{\mathrm{col},1}, \boldsymbol{f}_{\mathrm{col},2}) :=$$

$$\boldsymbol{\Delta}(\boldsymbol{A}(\boldsymbol{X}, \boldsymbol{\rho}; \boldsymbol{f}_1, \boldsymbol{f}_2)_{(\hat{i},\hat{j})_{\mathrm{col}}}) / \det \boldsymbol{R}_{\mathrm{col}}(\boldsymbol{X}, \boldsymbol{\rho}; \boldsymbol{f}_{\mathrm{col},1}, \boldsymbol{f}_{\mathrm{col},2})_{(i,j)_{\mathrm{row}}},$$

where $\det \boldsymbol{R}_{\mathrm{col}}(\boldsymbol{X}, \boldsymbol{\rho}; \boldsymbol{f}_{\mathrm{col},1}, \boldsymbol{f}_{\mathrm{col},2})_{(i,j)_{\mathrm{row}}} \neq \boldsymbol{0}$.

**<u>Th.</u>**

$$(\boldsymbol{X}_1, \boldsymbol{\rho}_1) \cong (\boldsymbol{X}_2, \boldsymbol{\rho}_2) \Longrightarrow$$

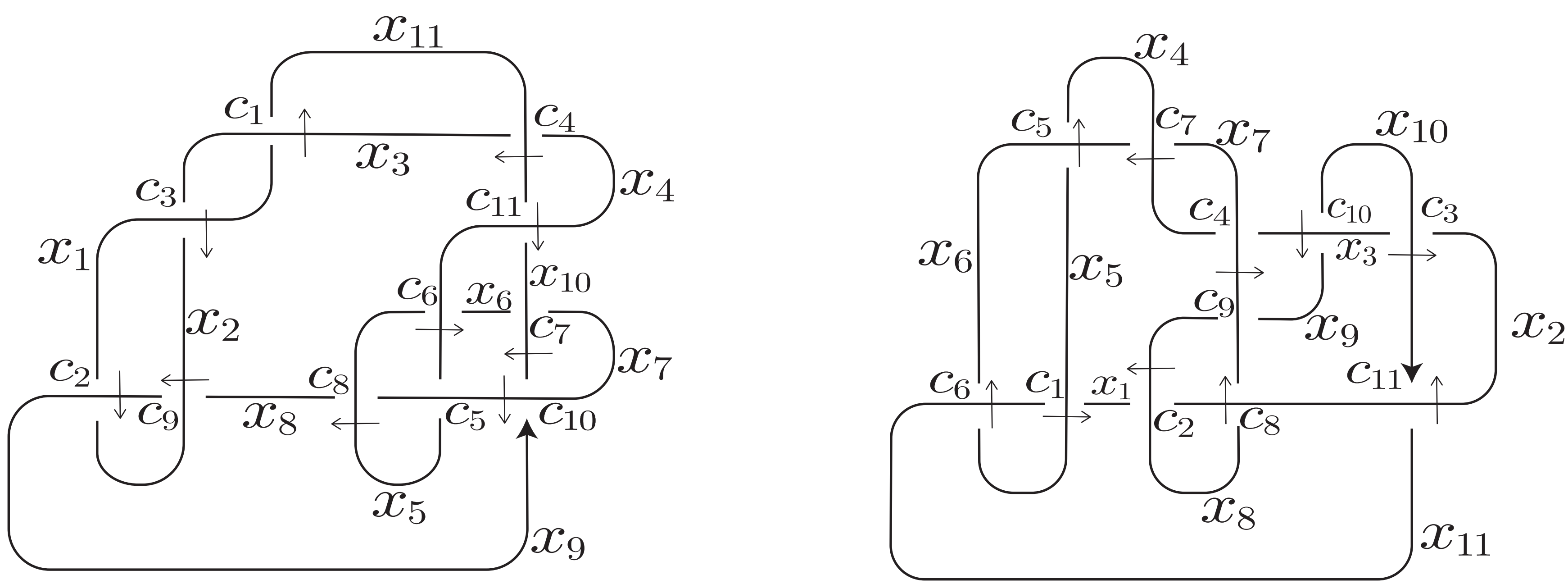
$$\boldsymbol{\Delta}(\boldsymbol{X}_1, \boldsymbol{\rho}_1; \boldsymbol{f}_1, \boldsymbol{f}_2; \boldsymbol{f}_{\mathrm{col},1}, \boldsymbol{f}_{\mathrm{col},2}) \doteq \boldsymbol{\Delta}(\boldsymbol{X}_2, \boldsymbol{\rho}_2; \boldsymbol{f}_1, \boldsymbol{f}_2; \boldsymbol{f}_{\mathrm{col},1}, \boldsymbol{f}_{\mathrm{col},2})$$

$\in \mathrm{Quot}(R)$

**Ex.** $Q = R_3 = (\mathbb{Z}_3, a * b = 2b - a)$.

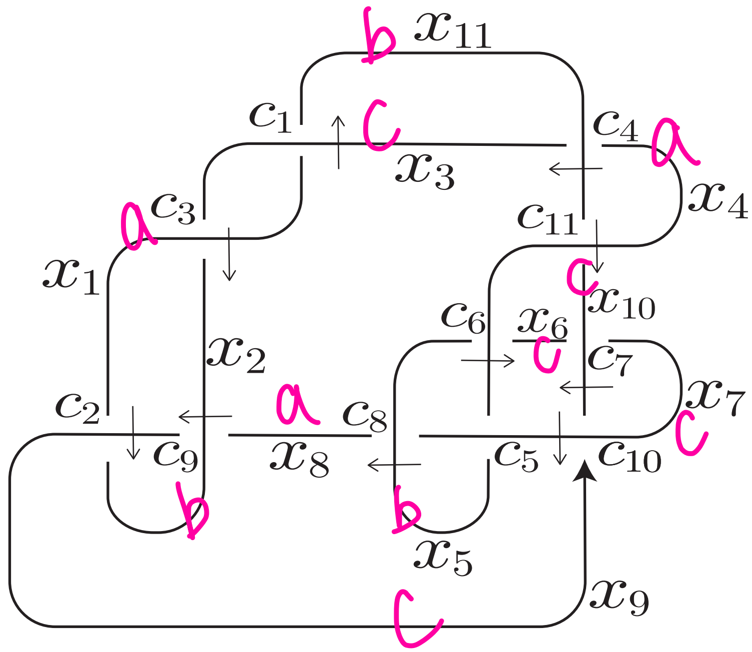
$f_1, f_2 : Q \times Q \to \mathbb{Z}$; $f_1(a,b) = -1$, $f_2(a,b) = 3\delta_{ab} - 1$.

$f_{\mathrm{col},c} : Q \to \mathbb{Z}$; $f_{\mathrm{col},c}(x) = 3\delta_{xc} - 1$ for $c \in Q$.




$K_1$: 11n38 $\qquad$ $K_2$: 11n102

They have the same $E_d(K)$ and $\Delta_d(K)$.




$$Q(K_1) = \left\langle x_1, \dots, x_{11} \;\middle|\; \begin{array}{l} x_1 \triangleleft x_3 = x_{11},\ x_1 \triangleleft x_9 = x_2,\ x_3 \triangleleft x_1 = x_2, \\ x_4 \triangleleft x_{11} = x_3,\ x_4 \triangleleft x_7 = x_5,\ x_5 \triangleleft x_4 = x_6, \\ x_7 \triangleleft x_{10} = x_6,\ x_7 \triangleleft x_5 = x_8,\ x_8 \triangleleft x_2 = x_9, \\ x_{10} \triangleleft x_7 = x_9,\ x_{11} \triangleleft x_4 = x_{10} \end{array} \right\rangle.$$

$\rho_1 : Q(K_1) \to Q$ ;

$$\rho_1(x_1) = \rho_1(x_4) = \rho_1(x_8) = a,$$

$$\rho_1(x_2) = \rho_1(x_5) = \rho_1(x_{11}) = b,$$

$$\rho_1(x_3) = \rho_1(x_6) = \rho_1(x_7) = \rho_1(x_9) = \rho_1(x_{10}) = c,$$

where $a \neq b$, $c = 2a + 2b$

※ All nontrivial repr. can be obtained by setting $a$ and $b$.

Then, $\boldsymbol{A(Q(K_1), \rho_1; f_1, f_2)}$ is

$$\begin{pmatrix}
f_1^{\bullet} & 0 & f_2^{\neq} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
f_1^{\bullet} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & f_2^{\neq} & 0 & 0 \\
f_2^{\neq} & -1 & f_1^{\bullet} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & f_1^{\bullet} & 0 & 0 & 0 & 0 & 0 & 0 & f_2^{\neq} \\
0 & 0 & 0 & f_1^{\bullet} & -1 & 0 & f_2^{\neq} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & f_2^{\neq} & f_1^{\bullet} & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & f_1^{\bullet} & 0 & 0 & f_2^{=} & 0 \\
0 & 0 & 0 & 0 & f_2^{\neq} & 0 & f_1^{\bullet} & -1 & 0 & 0 & 0 \\
0 & f_2^{\neq} & 0 & 0 & 0 & 0 & 0 & f_1^{\bullet} & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & f_2^{=} & 0 & -1 & f_1^{\bullet} & 0 \\
0 & 0 & 0 & f_2^{\neq} & 0 & 0 & 0 & 0 & 0 & -1 & f_1^{\bullet}
\end{pmatrix},$$

where $\boldsymbol{f_1^{\bullet} = -1}$, $\boldsymbol{f_2^{=} = 2}$ and $\boldsymbol{f_2^{\neq} = -1}$.

$$
\begin{aligned}
&R_{\mathrm{col}}(Q(K_1), \rho_1; f_{\mathrm{col},0}, f_{\mathrm{col},1}) \\
&= \begin{pmatrix}
f_{\mathrm{col},0}(a) & f_{\mathrm{col},1}(a) \\
f_{\mathrm{col},0}(b) & f_{\mathrm{col},1}(b) \\
f_{\mathrm{col},0}(c) & f_{\mathrm{col},1}(c) \\
f_{\mathrm{col},0}(a) & f_{\mathrm{col},1}(a) \\
f_{\mathrm{col},0}(b) & f_{\mathrm{col},1}(b) \\
f_{\mathrm{col},0}(c) & f_{\mathrm{col},1}(c) \\
f_{\mathrm{col},0}(c) & f_{\mathrm{col},1}(c) \\
f_{\mathrm{col},0}(a) & f_{\mathrm{col},1}(a) \\
f_{\mathrm{col},0}(c) & f_{\mathrm{col},1}(c) \\
f_{\mathrm{col},0}(c) & f_{\mathrm{col},1}(c) \\
f_{\mathrm{col},0}(b) & f_{\mathrm{col},1}(b)
\end{pmatrix}
= \begin{pmatrix}
3\delta_{a0} - 1 & 3\delta_{a1} - 1 \\
3\delta_{b0} - 1 & 3\delta_{b1} - 1 \\
3\delta_{c0} - 1 & 3\delta_{c1} - 1 \\
3\delta_{a0} - 1 & 3\delta_{a1} - 1 \\
3\delta_{b0} - 1 & 3\delta_{b1} - 1 \\
3\delta_{c0} - 1 & 3\delta_{c1} - 1 \\
3\delta_{c0} - 1 & 3\delta_{c1} - 1 \\
3\delta_{a0} - 1 & 3\delta_{a1} - 1 \\
3\delta_{c0} - 1 & 3\delta_{c1} - 1 \\
3\delta_{c0} - 1 & 3\delta_{c1} - 1 \\
3\delta_{b0} - 1 & 3\delta_{b1} - 1
\end{pmatrix}.
\end{aligned}
$$

$R_{\mathrm{col}}(Q(K_1), \rho_1; f_{\mathrm{col},0}, f_{\mathrm{col},1})_{(1,2)_{\mathrm{row}}}$ is

$$
\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ -1 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix},
$$

we have $\det R_{\mathrm{col}}(Q(K_1), \rho_1; f_{\mathrm{col},0}, f_{\mathrm{col},1})_{(1,2),\bar{2}} \doteq 3$.

We have

$$\begin{aligned}
&\Delta(Q(K_1), \rho_1; f_1, f_2; f_{\mathrm{col},0}, f_{\mathrm{col},1}) \\
&= \Delta(A(Q(K_1), \rho_1; f_1, f_2)_{(\hat{1},\hat{2})_{\mathrm{col}}}) / \det R_{\mathrm{col}}(Q(K_1), \rho_1; f_{\mathrm{col},0}, f_{\mathrm{col},1})_{(1,2)_{\mathrm{row}}} \\
&\doteq 2/3.
\end{aligned}$$

Similarly for any nontrivial repr. $\rho_2 : Q(K_2) \to \mathbb{Z}_3$,

$$\begin{aligned}
&\Delta(Q(K_2), \rho_2; f_1, f_2; f_{\mathrm{col},0}, f_{\mathrm{col},1}) \\
&= \Delta(A(Q(K_2), \rho_2; f_1, f_2)_{(\hat{1},\hat{2})_{\mathrm{col}}}) / \det R_{\mathrm{col}}(Q(K_2), \rho_1; f_{\mathrm{col},0}, f_{\mathrm{col},1})_{(1,2)_{\mathrm{row}}} \\
&\doteq 7/3.
\end{aligned}$$

Since

$$\begin{aligned}
&\{\Delta(Q(K_1), \rho_1; f_1, f_2; f_{\mathrm{col},0}, f_{\mathrm{col},1}) \mid \rho_1 : \text{nontri. repr.}\} \\
\neq &\{\Delta(Q(K_2), \rho_2; f_1, f_2; f_{\mathrm{col},0}, f_{\mathrm{col},1}) \mid \rho_2 : \text{nontri. repr.}\},
\end{aligned}$$

$K_1 \not\cong K_2$.

11n38 11n102

We can also define a quandle twisted Alexander invariant by using

- an Alexander pair
- column relations
- row relations with $\cdots$
- more informations obtained from knot diag.

$$\boldsymbol{\Delta}(\boldsymbol{L}, \boldsymbol{\rho}; \boldsymbol{f}_1, \boldsymbol{f}_2; (\boldsymbol{f}_{\mathbf{row}}, \boldsymbol{y}); \boldsymbol{f}_{\mathbf{col}}) = \cdots$$

The invariant is exactly determined ~~up to unit multiple~~

Correction: For my answer of a question, I answered that this stronger invariant $\boldsymbol{\Delta(L,\rho;f_1,f_2;(f_{\mathrm{row}},y);f_{\mathrm{col}})}$ does not distinguish $\boldsymbol{K}$ and $\boldsymbol{-K^*}$ because their quandles are the same. But we used more information which is coming from oriented diagrams for this invariant, and hence, this invariants might be able to distinguish $\boldsymbol{K}$, $\boldsymbol{-K}$, $\boldsymbol{K^*}$, and $\boldsymbol{-K^*}$.

Thank you for your attention!