# On the kernel of the surgery map restricted to the 1-loop part

Yuta Nozaki (Hiroshima Univ.)
joint work with
Masatoshi Sato (Tokyo Denki Univ.)
Masaaki Suzuki (Meiji Univ.)

The 17th East Asian Conference on Geometric Topology

2022/1/18

---

Last year (Abelian quotients of the $Y$-filtration via the LMO functor on the homology cylinders)

[NSS1] arXiv: 2001.09825 to appear in Geom. Topol.

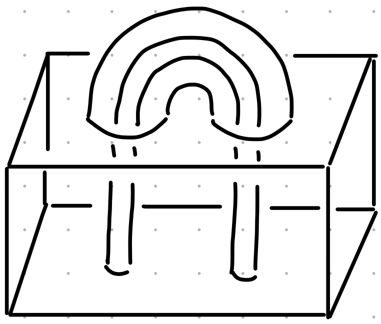
☆ [NSS2] arXiv: 2103.07086

Today

# §1. Introduction

$M$: connected oriented compact 3-mfd

$m : \partial(\Sigma_{g,1} \times [-1,1]) \xrightarrow{\cong} \partial M$ $\quad (m \rightsquigarrow m_+, m_-)$



$g=1$

$(M,m) \sim (M',m') \underset{\text{def}}{\iff}$

$$\begin{tikzcd} M \arrow[rr, "\exists \cong"] & & M' \\ & \partial(\Sigma_{g,1}\times[-1,1]) \arrow[ul, "m"] \arrow[ur, "m'"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

cobordism

<u>Def</u> $(M,m)$ is a homology cylinder over $\Sigma_{g,1}$

$\iff (m_+)_* = (m_-)_* : H_*(\Sigma_{g,1}) \xrightarrow{\cong} H_*(M)$.

$\mathcal{IC}_{g,1} := \{$homology cylinders over $\Sigma_{g,1}\}$

$\quad \parallel$

$\mathcal{IC}$ $\qquad$ is a monoid by $M \circ M' = \begin{array}{|c|}\hline M' \\ \hline M \\ \hline\end{array}$

---

◎ Our motivation for studying $\mathcal{IC}$

- monoid hom $c : \mathcal{I} \hookrightarrow \mathcal{IC}$

  Torelli group $\quad f \longmapsto (\Sigma_{g,1}\times[-1,1],\ f\times 1 \cup \mathrm{id}\times(-1))$

- $\mathcal{IC} = \left\{ \begin{array}{l} \text{homology cylinders obtained from} \\ \Sigma_{g,1}\times[-1,1] \text{ by clasper surgery} \end{array} \right\}$

- $\mathcal{IH} := \mathcal{IC}/\sim_H$ the homology cobordism group $(\mathcal{IH}_{0,1} \cong \Theta^3_{\mathbb{Z}})$

## §2. Preliminaries

◎ Clasper surgery ⊂ Dehn surgery $\left(\begin{array}{l}\text{Goussarov '99}\\ \text{Habiro '00}\end{array}\right)$

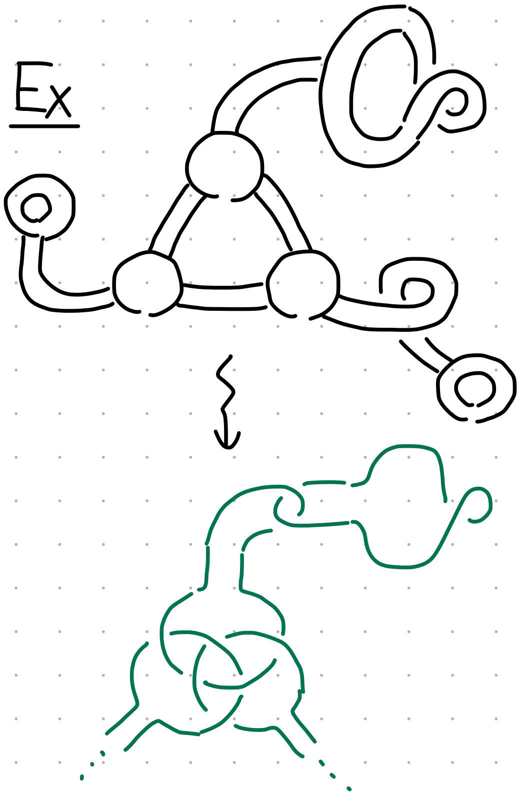
$M = (M, m) \in \mathcal{IC}$
$\cup$
$G$ : a graph clasper
$\left(\begin{array}{l}\text{a surface consisting of}\\ \text{disks, bands and annuli}\end{array}\right)$

$\downarrow$

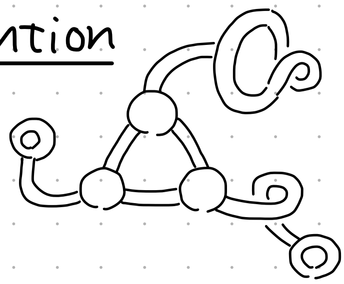
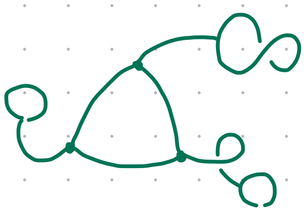
$L_G$ : framed link in $M$

$M_G := M_{L_G} \in \mathcal{IC}$

Ex



Convention

 $=:$ 

**Def** $M$ is $Y_n$-equivalent to $M'$

$\underset{\text{def}}{\Longleftrightarrow}$ $\exists\, G_1, \cdots, G_r \subset M$ s.t. $M_{\coprod_{j=1}^{r} G_j} = M'$ & $\deg G_j = n$ (# of disks)

$Y_n\mathcal{IC} := \{ M \in \mathcal{IC} \mid M \underset{Y_n}{\sim} \Sigma_{g,1} \times [-1,1] \}$ submonoid

$\mathcal{IC} = Y_1\mathcal{IC} \supset \cdots \supset Y_n\mathcal{IC} \supset \cdots$ : the $Y$-filtration

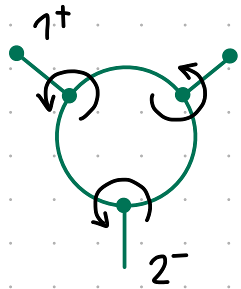
$\bigoplus_{i=1}^{\infty} Y_n\mathcal{IC}/\sim_{Y_{n+1}}$ is regarded as an approximation of $\mathcal{IC}$

↑ abelian group

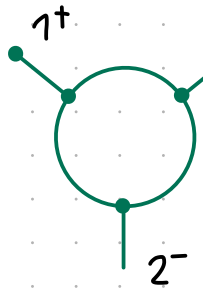
## Jacobi diagrams

**Def** A Jacobi diagram colored with $\{1^+,\cdots,g^+,1^-,\cdots,g^-\}$ is a uni-trivalent graph s.t. each $\begin{cases}\text{univalent vertex is colored}\\ \text{trivalent vertex has a cyclic order}\end{cases}$

**Ex**




$=:$



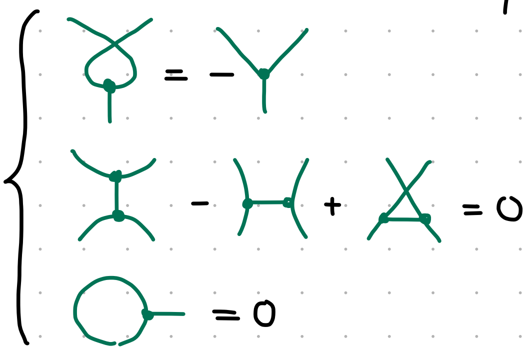

$\in \mathcal{A}^c_{3,1}$

$\mathcal{A}^c_n := \mathbb{Z}\{\text{conn. Jacobi diagrams of i-deg} = n\} \Big/ \begin{matrix}\text{AS}\\ \text{IHX}\\ \text{self-loop}\end{matrix}$

i-deg ~ # of trivalent vertices (internal)

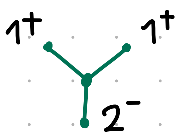
$\mathcal{A}^c_n = \bigoplus_{\ell \geq 0} \mathcal{A}^c_{n,\ell}$

$\ell$ ~ first Betti number




**Ex** $\mathcal{A}^c_1 = \langle$ $\underset{k}{\overset{i\quad j}{\text{Y}}}$ 's $\rangle$

$\cong (\mathbb{Z}^{2g})^{\otimes 3} \big/ x_1 \otimes x_2 \otimes x_3 \sim \mathrm{sgn}(\sigma)\, x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes x_{\sigma(3)}$



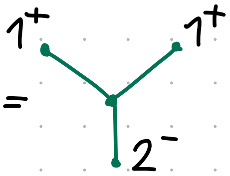

$(\neq 0)$ is a 2-torsion

symmetry
⇝
torsion

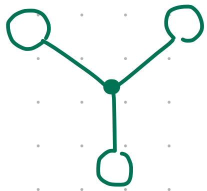
## ⊘ Surgery map

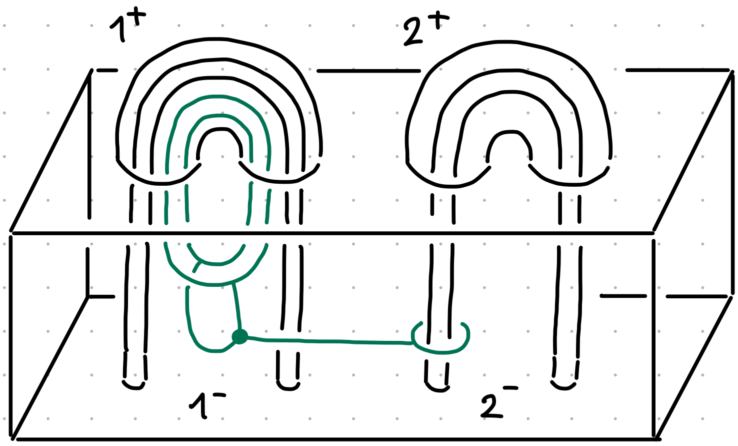
The surgery map $\mathfrak{s}_n : \mathcal{A}_n^c \xrightarrow{\text{hom}} Y_n\mathcal{IC}/\sim_{Y_{n+1}}$ is defined by…

<u>Ex</u> $(n=1,\ g=2)$

$J =$ 


$\rightsquigarrow$ $\Sigma_{2,1}\times[-1,1] \supset G_J$






$\mathfrak{s}_1(J) = (\Sigma_{2,1}\times[-1,1])_{G_J}$ is well-defined up to $\sim_{Y_2}$

<u>Fact</u> Jacobi diag. homology cyl.

$$\begin{tikzcd} \mathcal{A}_n^c \arrow[r, "\mathfrak{s}_n"] \arrow[d] & Y_n\mathcal{IC}/Y_{n+1} \arrow[d] \\ \mathcal{A}_n^c\otimes\mathbb{Q} \arrow[r, "\cong"'] & Y_n\mathcal{IC}/Y_{n+1}\otimes\mathbb{Q} \end{tikzcd}$$

is surjective if $n\geq 2$

Cheptea-Habiro-Massuyeau '08

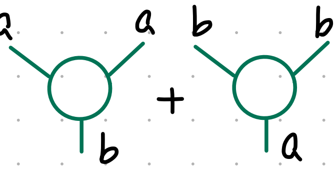
$$\text{tor}\,\mathcal{A}_n^c \supset$$

Let us focus on $0 \to \operatorname{Ker}\mathfrak{s}_n \to \mathcal{A}_n^c \xrightarrow{\mathfrak{s}_n} Y_n\mathcal{IC}/Y_{n+1} \to 0$

## Related results

① $\operatorname{Ker} s_n$ & $Y_n\mathcal{IC}/Y_{n+1}$ was determined for

$$\begin{cases} n=1,2 & \text{Massuyeau-Meilhan '03 '13} \\ n=3,4 & \text{[NSS 1\&2]} \end{cases}$$

<u>Ex</u> $0 \neq$ 

$\overset{s_3}{\longmapsto} 0$

pair of torsions
$\Updownarrow$
kernel

② $\operatorname{Ker} s_{n,0}$ was investigated by Conant-Schneiderman-Teichner '16 [NSS 1]

$$s_n = \bigoplus s_{n,\ell} : \bigoplus_{\ell \geq 0} \mathcal{A}^c_{n,\ell} \longrightarrow Y_n\mathcal{IC}/Y_{n+1}$$

---

**On the kernel of the surgery map restricted to the 1-loop part**

<u>Q.</u> How about $\operatorname{Ker}(s_{n,1} : \mathcal{A}^c_{n,1} \longrightarrow Y_n\mathcal{IC}/Y_{n+1})$ ?

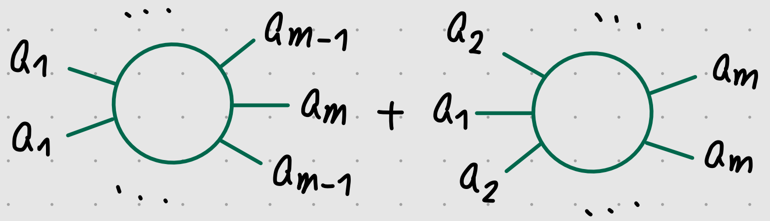
<u>Fact</u> (NSS 1) $\operatorname{Ker} s_{2m,1} = 0$.

$\longrightarrow$ Let us focus on $\operatorname{Ker} s_{2m-1,1}$ $(m \geq 2)$

## §3. Main result and corollaries

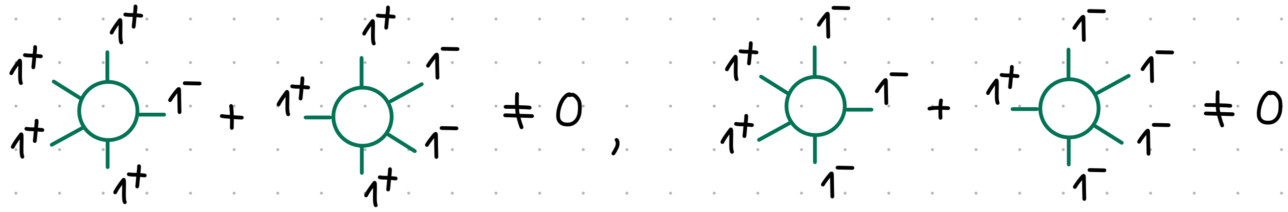
**Thm 1** (NSS 2)

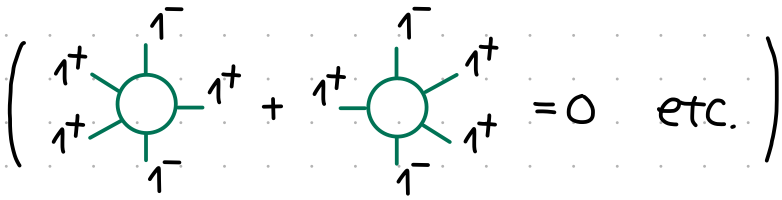
$$\mathrm{Ker}\left(\mathcal{A}_{2m-1,1} \xrightarrow{S_{2m-1,1}} Y_{2m-1}\mathcal{IC}/Y_{2m} \overset{``\pi"}{\twoheadrightarrow} \left(Y_{2m-1}\mathcal{IC}/Y_{2m}\right)/\sim\right)$$

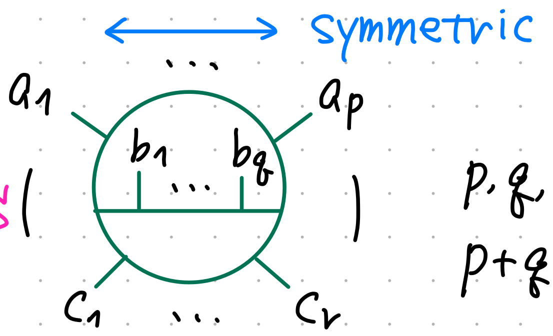
is
- generated by 

- free $\mathbb{Z}/2$-module of rank $\frac{1}{2}\left((2g)^m-(2g)^{\lceil \frac{m}{2}\rceil}\right)$

**Ex** ($g=1$, $m=3$)




$\sim$ means [ modulo $S$( 

) $p, q, r \geq 1$, $p+q+r+2=2m-1$ ]

**Rem** $\mathrm{Ker}\, \pi \circ S_{2m-1,1} = \mathrm{Ker}\, S_{2m-1,1}$ if $m=2,3$

# Proof of Thm 1

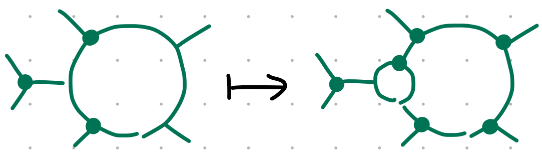
Part 1 : ① & ② give elements not in $\mathrm{Ker}\, \pi \circ \mathfrak{s}_{2m-1,1}$

Part 2 : ③ shows that the rest of the elements are in Ker

---

① $\bar{z}_{n+1} : Y_n\mathcal{IC}/Y_{n+1} \xrightarrow{\text{hom}} \mathcal{A}^c_{n+1} \otimes \mathbb{Q}/\mathbb{Z} = \bigoplus_{\ell \geq 0} \mathcal{A}^c_{n+1,\ell} \otimes \mathbb{Q}/\mathbb{Z}$ [NSS1]

② bu : $\mathcal{A}^c_{n,\ell} \xrightarrow{\text{hom}} \mathcal{A}^c_{n+2,\ell+1}$



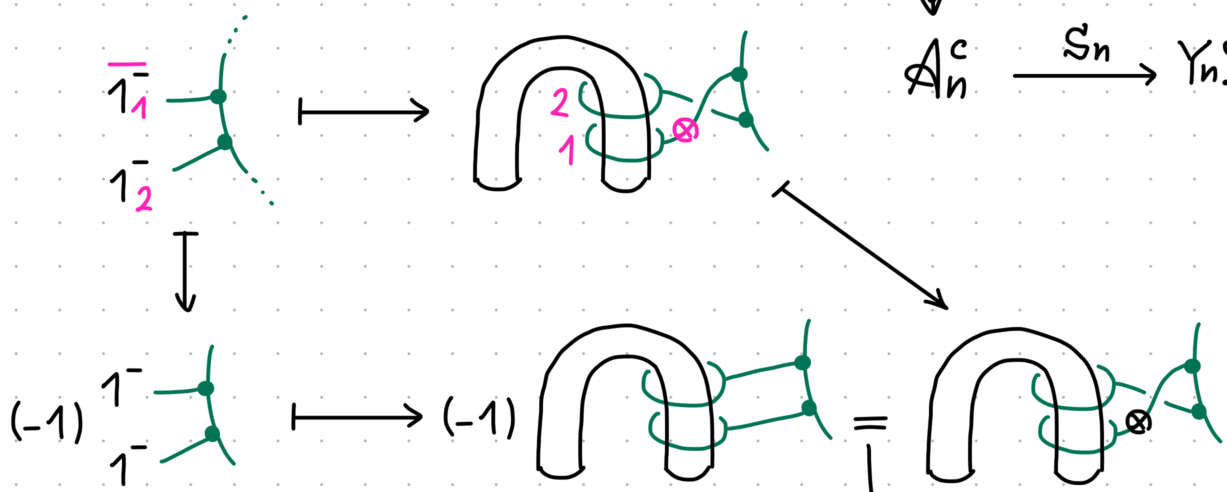
"blowing up"

**Thm 2** (NSS2) bu : $\mathcal{A}^c_{n-2,1} \xrightarrow{\cong} \mathcal{A}^c_{n,2} / \langle \Theta_n^{\geq 1} \rangle$

③ refined surgery map $(n \geq 2)$

abelian group

$$\begin{tikzcd} \mathbb{Z}\tilde{\mathcal{J}}^c_n \arrow[r, "\tilde{\mathfrak{s}}_n"] \arrow[d] & Y_n\mathcal{IC}/Y_{n+2} \arrow[d] \\ \mathcal{A}^c_n \arrow[r, "\mathfrak{s}_n"] & Y_n\mathcal{IC}/Y_{n+1} \end{tikzcd}$$



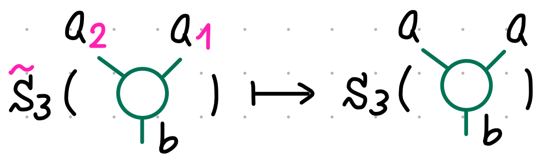

AS relation etc. ▨

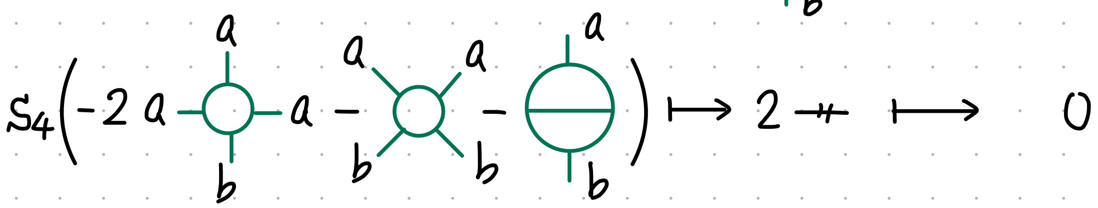
## Application 1

$$0 \to Y_{n+1}IC/Y_{n+2} \to Y_nIC/Y_{n+2} \to Y_nIC/Y_{n+1} \to 0 \quad \text{(exact)}$$

Ex $(n=3)$

$$\begin{array}{ccccccccc} & & \text{tor}=0 & & & & \text{tor}\neq 0 & & \\ 0 & \to & Y_4IC/Y_5 & \to & Y_3IC/Y_5 & \to & Y_3IC/Y_4 & \to & 0 \end{array}$$




**Cor (NSS 2)** The abelian group $Y_3IC/Y_5$ is free

## Application 2

$I\mathcal{H} = IC/\sim_H$ the homology cobordism group

**Cor (NSS 2)** The abelian group $Y_3I\mathcal{H}/Y_5$ is free

Summary:

$$\begin{array}{ccccccccc} & & \text{free} & & \text{free} & & \exists\,\text{torsion} & & \\ 0 & \to & Y_4IC/Y_5 & \to & Y_3IC/Y_5 & \to & Y_3IC/Y_4 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & Y_4I\mathcal{H}/Y_5 & \to & Y_3I\mathcal{H}/Y_5 & \to & Y_3I\mathcal{H}/Y_4 & \to & 0 \\ & & \text{free} & & \text{free} & & \exists\,\text{torsion} & & \end{array}$$

**Thm 1** (NSS 2)

$$\mathrm{Ker}\left(\mathcal{A}_{2m-1,1} \xrightarrow{\mathfrak{s}_{2m-1,1}} Y_{2m-1}\mathcal{IC}/Y_{2m} \xrightarrow{``\pi"} \left(Y_{2m-1}\mathcal{IC}/Y_{2m}\right)/\sim\right) \cong \mathbb{Q}$$

— Future perspective —

We would like to

- determine the kernel without $\pi$.
- investigate the 2-loop part and more.
- develope clasper calculus in $Y_n\mathcal{IC}/Y_{n+2}$.
- determine the structures of $Y_n\mathcal{IC}/Y_{\substack{n+1\\n+2}}$ & $Y_n\mathcal{IH}/Y_{\substack{n+1\\n+2}}$.

Recently, we have constructed
a "non-commutative Reidemeister-Turaev torsion" on $\mathcal{IC}$,
and described a relation with
the Enomoto-Satoh trace and the 1-loop part of the LMO functor.

[NSS, in preparation]

## ◎ Goussarov-Habiro Conjecture

<u>Conj</u>

$M \underset{Y_{n+1}}{\sim} M' \iff f(M) = f(M')$ for $\forall f$ "finite type inv. of deg $\leq n$"

<u>Cor</u> (NSS 2) Conj is true for $n = 4$

cf. $n = 2$ Massuyeau-Meilhan '13

$n = 3$ [NSS 1]

---

$\mathcal{IH} = \mathcal{IC}/\sim_H$ the homology cobordism group of homology cylinders

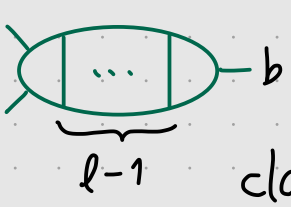
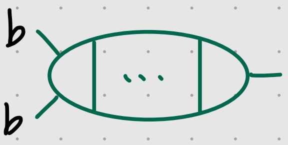
$(M, m) \sim_H (N, n) \overset{\text{def}}{\iff} \exists W^4$: cpt ori smooth s.t.

$\partial W = M \underset{m \circ n^{-1}}{\cup} (-N)$ & $H_*(M) \xrightarrow{\cong} H_*(W) \xleftarrow{\cong} H_*(N)$

$m : \partial(\Sigma_{g,1} \times [-1,1]) \xrightarrow{\cong} \partial M$

---

<u>Prop</u> (NSS 2)

$\mathcal{S}($  $) = \mathcal{S}($  $) \neq 0$



↑ clasper calculus

↑ $\mathbb{Z}_{2\ell+2}$ & weight system