Index polynomial invariants for ordered twisted links

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Today's contents

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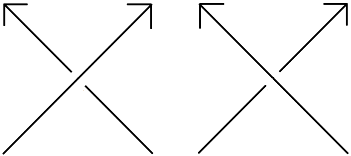
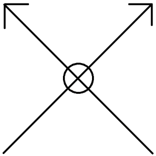
A definition of virtual link diagrams

Definition (Kauffman, 1999)

An m*-component virtual link diagram* is the image of an immersion of m ordered and oriented circles into $\mathbb{R}^2$, whose singular points are only transverse double points. Such double points are divided into *classical crossings* and *virtual crossings*.



classical crossing

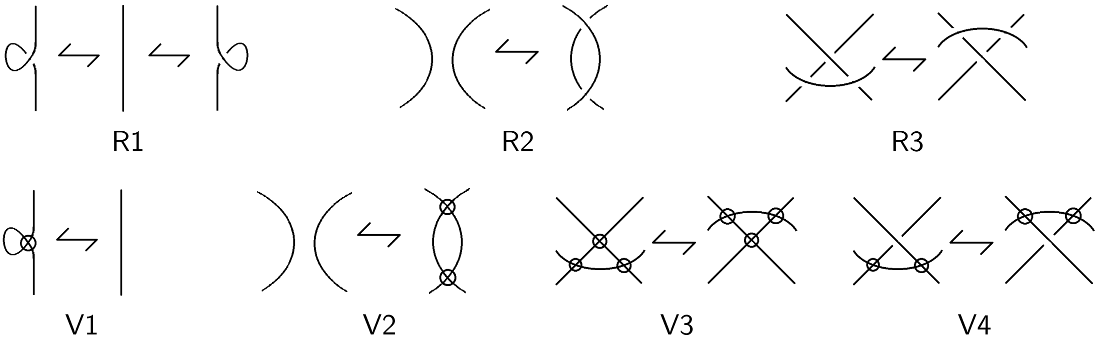


virtual crossing

Generalized Reidemeister moves

Definition (Kauffman, 1999)

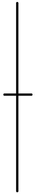
Generalized Reidemeister moves are local moves in the figure below.
$D \overset{v}{\sim} D' \Leftrightarrow D,\ D'$ are equivalent under generalized Reidemeister moves.



A definition of twisted link diagrams

Definition (Bourgoin, 2008)

An m*-component twisted link diagram* is an m-component virtual link diagram possibly *bars* on arcs.



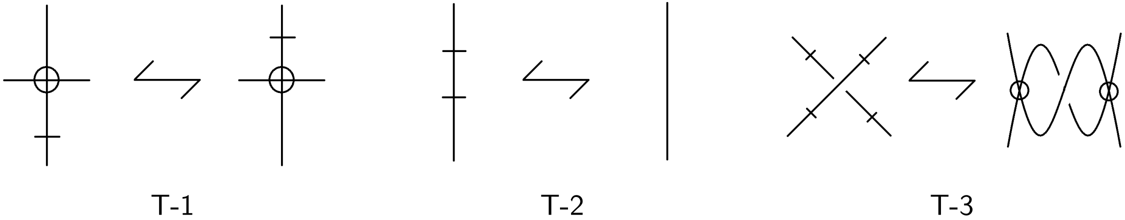
a bar on an arc

Extended Reidemeister moves

Definition (Bourgoin, 2008)

Extended Reidemeister moves are local moves together with generalized Reidemeister moves and the figure below.

$D \overset{t}{\sim} D' \Leftrightarrow D,\ D'$ are equivalent under extended Reidemeister moves.



An even type and odd type

Let $D = D_1 \cup \cdots \cup D_m$ be an m-comp. twisted link diagram.

Definition

For $i \in \{1, \ldots, m\}$,
(1) D_i is an *even type* if a number of bars on arcs of D_i is even.
(2) D_i is an *odd type* if a number of bars on arcs of D_i is odd.

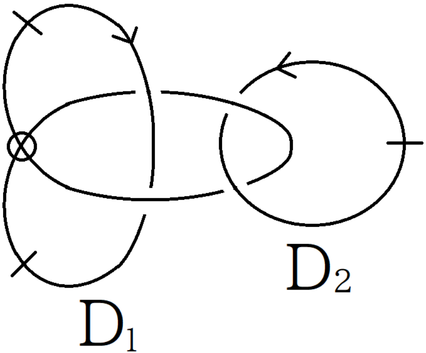
Remark

An even type, odd type is an invariant of twisted links.

An example of m-comp. twisted link diagrams

Example. Let $D = D_1 \cup D_2$ be a 2-comp. twisted link diagram in the figure below.

D_1 is an even type, and D_2 is an odd type.



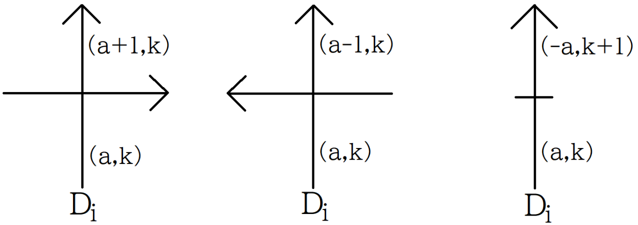
Today's contents

n-colorings for an even type D_i

After that, $i \in \{1, \ldots, m\}$ is fixed and we assume D_i is an even type. A *semi arc* of D_i is a segment along D which goes from a classical crossing or a bar to the next one, where virtual crossings are ignored. $\mathcal{A}(D_i) := \{$ semi arcs of D_i $\}$

Definition

Let n be an integer. Then, an *n-coloring* of D_i is a map $C_i : \mathcal{A}(D_i) \to \mathbb{Z}_n \times \mathbb{Z}_2$ satisfied the condition in the figure below, where double points in the figure are classical crossings.



$d_i(D)$-colorings for D_i

Definition

Let $\mathbb{1}_i : \mathcal{A}(D_i) :\to \mathbb{Z}_1 \times \mathbb{Z}_2$ be a 1-coloring of D_i.
Then, $d_i(D) := |\#\{$ figure (1) $\} - \#\{$ figure (2) $\} + \#\{$ figure (3) $\} - \#\{$ figure (4) $\}|$,
where double points in the figure are classical crossings.

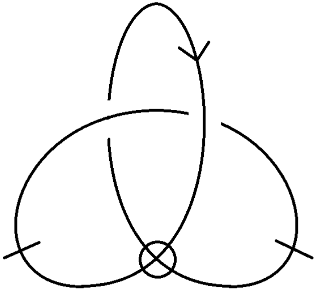
Figure (1) Figure (2) Figure (3) Figure (4)

Lemma

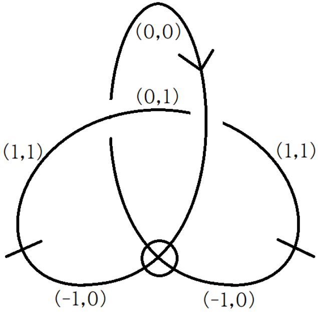
There exists a $d_i(D)$-coloring of D_i.

An example of $d_i(D)$-coloring of D_i

Let $D = D_1$ be a 1-comp. twisted link diagram shown below.
Then, $d_1(D) = 0$.
Let $C_1 : \mathcal{A}(D_1) \to \mathbb{Z} \times \mathbb{Z}_2$ be a map as shown in the figure, then C_1 is a 0-coloring of D_1.



D



0-coloring

The weight for self crossings of D_i

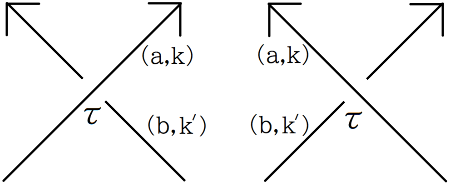
After that, C_i: $d_i(D)$-coloring of D_i is fixed.

Definition

For each self crossing τ of D_i,

$$W^{(C_i,D_i)}(\tau) := a - b \in \mathbb{Z}_{d_i(D)},$$

where a, b are values of semi arcs dipicted in the figure below.



Classification of self crossings of D_i

We classify self crossings of D_i into three sets $T^{(C_i,D_i)}$, $T_0^{(C_i,D_i)}$ and $T_1^{(C_i,D_i)}$.

Definition

$T^{(C_i,D_i)} := \{\tau$: self crossing of D_i | the condition in the figure (α) $\}$
$T_0^{(C_i,D_i)} := \{\tau$: self crossing of D_i | the condition in the figure (0) $\}$
$T_1^{(C_i,D_i)} := \{\tau$: self crossing of D_i | the condition in the figure (1) $\}$

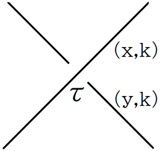


Figure (α)

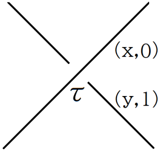


Figure (0)

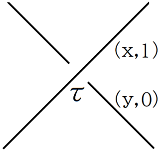


Figure (1)

A definition of index polynoials for D_i

Definition (I.)

We define index polynomials $\phi^{(C_i,D_i)}(t), \psi_0^{(C_i,D_i)}(t), \psi_1^{(C_i,D_i)}(t) \in \mathbb{Z}[t^{\pm 1}]/(t^{d_i(D)}-1)$ as

$$\phi^{(C_i,D_i)}(t) := \sum_{\tau \in T^{(C_i,D_i)}} \operatorname{sign}(\tau)(t^{W^{(C_i,D_i)}(\tau)}-1),$$

$$\psi_0^{(C_i,D_i)}(t) := \sum_{\tau \in T_0^{(C_i,D_i)}} \operatorname{sign}(\tau) t^{W^{(C_i,D_i)}(\tau)},$$

$$\psi_1^{(C_i,D_i)}(t) := \sum_{\tau \in T_1^{(C_i,D_i)}} \operatorname{sign}(\tau) t^{W^{(C_i,D_i)}(\tau)}.$$

Main results

For $f_1, f_2, g_1, g_2 \in \mathbb{Z}[t^{\pm 1}]/(t^n - 1)$,

$(f_1, f_2) \overset{(p)}{\equiv} (g_1, g_2) \overset{\text{def}}{\Longleftrightarrow}$ There exists an integer k such that $(f_1, f_2) = (g_1 \cdot t^{pk},\ g_2 \cdot t^{-pk})$ or $(f_1, f_2) = (g_2 \cdot t^{pk},\ g_1 \cdot t^{-pk})$.

Then, we have the following theorem.

Theorem (I.)

Let $D = D_1 \cup \cdots \cup D_m$, $D' = D'_1 \cup \cdots \cup D'_m$ be m-comp. twisted link diagrams that D_i is an even type. Then, $D \overset{t}{\sim} D' \Rightarrow$ the following is satisfied.

(1) $d_i(D) = d_i(D')$

For any C_i: $d_i(D)$-coloring of D_i and C'_i: $d_i(D')$-coloring of D'_i,

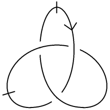
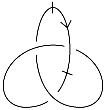
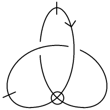
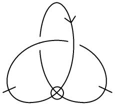
(2) $\phi^{(C_i, D_i)}(t) = \phi^{(C'_i, D'_i)}(t)$

(3) $(\psi_0^{(C_i, D_i)}(t),\ \psi_1^{(C_i, D_i)}(t)) \overset{(2)}{\equiv} (\psi_0^{(C'_i, D'_i)}(t),\ \psi_1^{(C'_i, D'_i)}(t))$

Therefore, $\phi^{(C_i, D_i)}(t)$ and $(\psi_0^{(C_i, D_i)}(t),\ \psi_1^{(C_i, D_i)}(t))$ are invariants of twisted links.

Main results

Example. Let $K_1, \dots, K_4$ be 1-comp. twisted link diagrams in the figure below.

	 K_1	 K_2	 K_3	 K_4
$d_1(D)$	0	2	2	0
$\phi^{(C_1,D_1)}(t)$	0	0	$t-1$	0
$(\psi_0^{(C_1,D_1)}(t),\ \psi_1^{(C_1,D_1)}(t))$	$(1,1)$	$(1,0)$	$(t,0)$	(t,t^{-1})

By the table, $t-1 \neq 0$, $(1,1) \overset{(2)}{\not\equiv} (t,t^{-1})$. Therefore, $K_1, \dots, K_4$ are not equivalent.

Main results

Moreover, let $D = D_1$ be a 1-comp. virtual link diagram and let $P_{D_1}(t)$ be an affine index polynomial invariant defined by Kauffman in 2013. Then, we have the following proposition.

Proposition (I.)

If $D = D_1$ be a 1-comp. virtual link diagram, then the following is satisfied.
For any C_1: $d_1(D)$-coloring of D_1,
(1) $\phi^{(C_1,D_1)}(t) = P_{D_1}(t)$
(2) $\psi_0^{(C_1,D_1)}(t) = \psi_1^{(C_1,D_1)}(t) = 0$

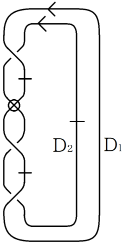
Therefore, the restriction of our index polynomial invariants to virtual knots coincides with the affine index polyomial invariant.

Today's contents

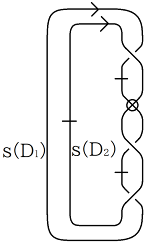
A definition of double coverings

For an m-comp. twisted link diagram $D = D_1 \cup \cdots \cup D_m$, we compose a double covering diagram of D denoted by $\widetilde{D} = \widetilde{D}_1 \cup \cdots \cup \widetilde{D}_m$ in the following steps.

(step 1) $s(D) = s(D_1) \cup \cdots \cup s(D_m)$ is a m-comp. twisted link diagram that is horizontal mirror image of D and changed over-under information at every classical crossing. And we draw the diagram $s(D) = s(D_1) \cup \cdots \cup s(D_m)$ on the right side of D.



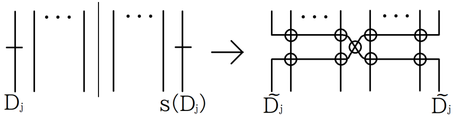
$D = D_1 \cup D_2$

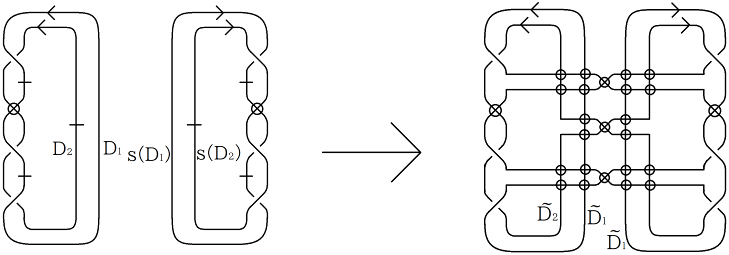


$s(D) = s(D_1) \cup s(D_2)$

A definition of double coverings

(step 2) All bars of D_j and bars corresponding to them of $s(D_j)$ cut open as shown in the figure to the right, and we connected D_j and $s(D_j)$ denoted by $\widetilde{D}_j$.





$D \cup s(D)$ $\qquad$ $\widetilde{D} = \widetilde{D}_1 \cup \widetilde{D}_2$

Then, double covering diagram of D denoted by $\widetilde{D} = \widetilde{D}_1 \cup \cdots \cup \widetilde{D}_m$ is a virtual link diagram composed in the step 1 and step 2.

A property of double coverings

Proposition (N.Kamada, S.Kamada, 2016)

$D = D_1 \cup \cdots \cup D_m$, $D' = D'_1 \cup \cdots \cup D'_m$: m-comp. twisted link diagram.
Then, $D \overset{t}{\sim} D' \Rightarrow \widetilde{D} \overset{v}{\sim} \widetilde{D}'$.

An application

Let $D = D_1 \cup \cdots \cup D_m$, $D' = D'_1 \cup \cdots \cup D'_m$ be m-comp. twisted link diagrams that D_i is an even type.

Theorem A (I.)

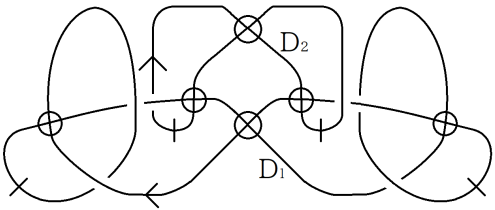
$\widetilde{D} \overset{v}{\sim} \widetilde{D}' \Rightarrow$ the following is satisfied.
(1) $d_i(D) = d_i(D')$.
For any C_i: $d_i(D)$-coloring of D_i and C'_i: $d_i(D')$-coloring of D'_i,
(2) $\phi^{(C_i,D_i)}(t) = \phi^{(C'_i,D'_i)}(t)$.
(3) $(\psi_0^{(C_i,D_i)}(t),\ \psi_1^{(C_i,D_i)}(t)) \overset{(1)}{\equiv} (\psi_0^{(C'_i,D'_i)}(t),\ \psi_1^{(C'_i,D'_i)}(t))$.

Theorem B (I.)

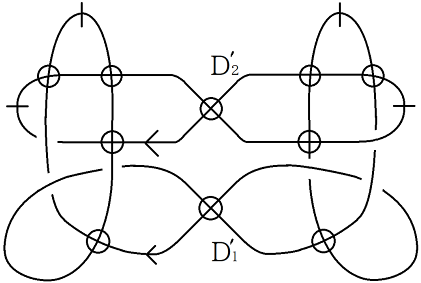
There exists D, D' such that $\widetilde{D} \overset{v}{\sim} \widetilde{D}'$ and $D \overset{t}{\nsim} D'$.

Proof of theorem B

Proof of theorem B. Let $D = D_1 \cup D_2$, $D' = D'_1 \cup D'_2$ be 2-comp. twisted link diagrams in the figure below. Then, D_1, D'_1 are even types.



$D = D_1 \cup D_2$



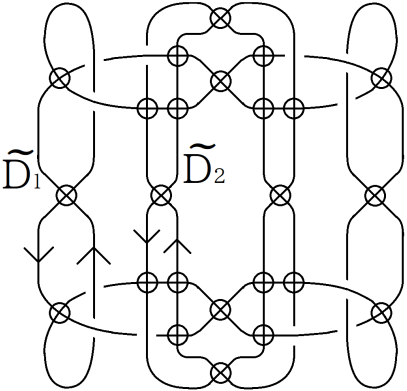
$D' = D'_1 \cup D'_2$

Proof of theorem B

Double covering diagrams of D, D' are equivalent to a virtual link diagram in the figure to the right. So, $\widetilde{D} \overset{v}{\sim} \widetilde{D}'$.

We give C_1: $d_1(D)$-coloring of D_1 and
C_1': $d_1(D')$-coloring of D_1'.
$d_1(D) = d_1(D') = 2$,
$(\psi_0^{(C_1,D_1)}(t),\ \psi_1^{(C_1,D_1)}(t)) = (2t, 0)$,
$(\psi_0^{(C_1',D_1')}(t),\ \psi_1^{(C_1',D_1')}(t)) = (2, 0)$.

So,
$(\psi_0^{(C_1,D_1)}(t),\ \psi_1^{(C_1,D_1)}(t)) \overset{(2)}{\not\equiv} (\psi_0^{(C_1',D_1')}(t),\ \psi_1^{(C_1',D_1')}(t))$.
Therefore, $D \overset{t}{\nsim} D'$. □



Thank you for your attention.