Some High-dimensional Compact Hyperbolic Coxeter Polytopes

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https://eacgt2022.cayley.kr/

Main Result

Theorem (Ma-Zheng, arxiv:2201.00154)

There are 348 4-dimensional compact Coxeter hyperbolic polytopes with eight facets.

Theorem (Ma-Zheng, appear soon)

There are 51 5-dimensional compact Coxeter hyperbolic polytopes with nine facets.

A. Burcroff (2022) also obtained the same results independently via different approaches (arxiv:2201.03437)

Presentation Outline

Hyperbolic Coxeter polytopes

Definition

A *hyperbolic (spherical, Euclidean) Coxeter polytope* $P \subset \mathbb{H}^n$ $(\mathbb{S}^n, \mathbb{E}^n)$ is a finite-volume convex polytope whose dihedral angles are of the form $\frac{\pi}{k}$, for some $k \in \{2, ..., \infty\}$

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co-finite volume, discrete isometry subgroup $\Gamma \subset Isom\,(\mathbb{H}^n)$ generated by finite many reflections

$$\Updownarrow$$

hyperbolic Coxeter polytope $P^n \subset \mathbb{H}^n$

- The **Gram matrix** $G(P) = (g_{ij})_{m\times m}$ of a n-dimensional Coxeter polytope P with m facets $F_1, F_2 \cdots F_m$ defined as follows:

$$g_{ij} = \begin{cases} 1 & \text{if} \quad j = i, \\ -\cos\frac{\pi}{m_{ij}} & \text{if} \quad \angle(F_i, F_j) = \frac{\pi}{m_{ij}}, \\ -1 & \text{if} \quad F_i \text{ is asymptotic to } F_j, \\ -\cosh(d(F_i, F_j)) & \text{if} \quad F_i \text{ and } F_j \text{ diverge by distance } d(F_i, F_j). \end{cases}$$

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✓ The **signature** of G $sgn(G)$ when the n-dimensional polytope P is spherical, Euclidean, or hyperbolic is $(n+1, 0)$, $(n, 0)$, or $(n, 1)$.

Coxeter diagram $\Gamma(P)$ of a Coxeter polytope P:

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⋆ vertices: v_i correspond to facets F_i of P.

⋆ edges:

- v_i is joined to v_j by an edge labelled k if

$$\angle F_i F_j = \frac{\pi}{k},\ 3 \le k < \infty$$

- v_i is joined to v_j by a bold edge if

$$F_i \text{ is asymptotic to} F_j$$

- v_i is joined to v_j by dotted edge labelled $h = cosh\ d(F_i, F_j)$ if F_i and F_j diverge at a hyperblic distance $d(F_i, F_j)$.

Example

✓ P is a hyperbolic quadrilateral with angles and distances

$$(\angle F_1F_2, \angle F_1F_4, \angle F_2F_3, \angle F_3F_4, d(F_1, F_3), d(F_2, F_4)) = (0, \frac{\pi}{7}, \frac{\pi}{2}, \frac{\pi}{6}, d_1, d_2)$$

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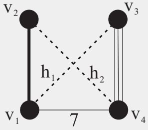
- $G(P) := \begin{pmatrix} 1 & -1 & -cosh\ d_1 & -cos\frac{\pi}{7} \\ -1 & 1 & 0 & -cosh\ d_2 \\ -cosh\ d_1 & 0 & 1 & -\frac{\sqrt{3}}{2} \\ -cos\frac{\pi}{7} & -cosh\ d_2 & -\frac{\sqrt{3}}{2} & 1 \end{pmatrix}$
- $\Gamma(P) :=$



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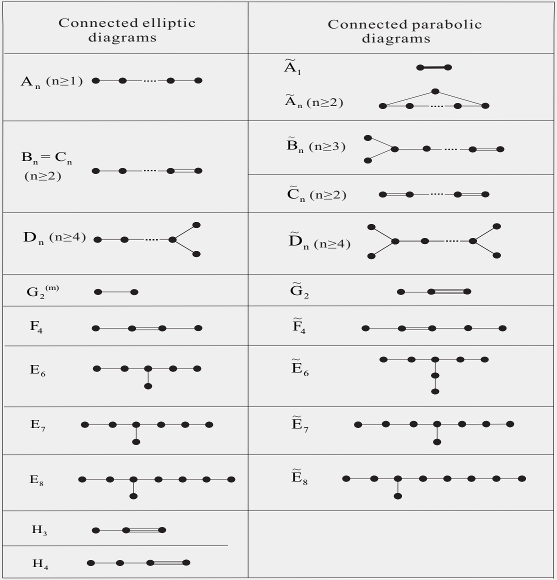
Theorem (Coxeter 1934')

There are complete classifications of spherical and Euclidean Coxeter polytopes, respectively.

Complete classification



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✓ In spherical and Euclidean settings, we only have finite many polytopes (exclude the dihedral G_2^m-type) up to isometry, whereas in hyperbolic case, the situation varies substantially.

- If P is compact $\Rightarrow$ P is simple

- k-faces $\leftrightarrow$ elliptic subdiagrams of rank $n-k$
  ideal-vertices $\leftrightarrow$ parabolic subdiagrams of rank $n-1$

- Indecomposible,symmetric matrix G with signature $(n,1)$ with natural geometric condition $\Rightarrow$ $\exists!\ P^n \subset \mathbb{H}^n,\ G = G(P)$.
- signature obstructions
- local determinants
- lannér diagrams / Esselmann list
- admissible sections
- lifting techniques
- configuration of missing faces

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- Martelli-Riolo-Slavich (2019): nontrivial plane bundle over surface covering a finite volume hyp 4-manifold via right-angled 120-cell. (Gromov-Lawson-Thurston Conjecture 1988)

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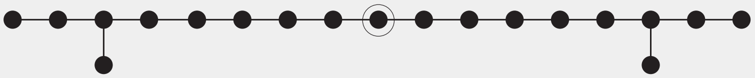
- Minimal arithmetic orbifold (Tumarkin, 2003, Emery, 2012)



- For a hyperbolic 4-manifold closed M, by Gauss-Bonnet formula, $vol(M) = \frac{4\pi^2}{3}\chi(M)$, it is not known whether there is a hyperbolic closed 4-manifold with $\chi(M) = 2$

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- Ratcliffe-Tschantz (2000): minimal volume cusped hyperbolic 4-manifolds via right-angled 24-cell.
- Everitt-Ratcliffe-Tschantz (2012): minimal volume cusped hyperbolic 6-manifolds via a right-angled polytope P^6.

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- $P \subset \mathbb{S}^n$ or $\mathbb{E}^n$, finitely many and classified [Coxeter,34']

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Question [Vinberg 1967]:"... find all the discrete groups of motions of Lobacevskii space generated by finitely many reflections and having finite fundamental regions...".

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Theorem (Allcock, 2005')

Compact Coxeter polytope: infinitely many in $\mathbb{H}^n$ *for all* $4 \leq n \leq 6$
Finite volume Coxeter polytope: infinitely many in $\mathbb{H}^n$ *for all* $4 \leq n \leq 19$.

2. Absence in large dimensions

Theorem (Vinberg' 84)

If $P \subset \mathbb{H}^n$ ***is compact Coxeter polytope, then*** $n \leq 29$

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Examples only known for $n \leq 8$.
Unique example for $n = 8$ [Bugaenko'92]:

- Vinberg's Absence Theorem:

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Theorem (Vinberg' 84)

*There are **No** arithmetic hyperbolic reflection group in dim $\geq$ 30.*

- $Conj$: Theorem above is true without the arithmeticity restrction
- Q: If $P \subset \mathbb{H}^n$ is compact, arithmetic then $n \leq 8$??

Theorem (Prokhorov' 85)

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- Example known for $n \leq 19$ [Vinberg, Kapliskaya '78], and $n = 21$ [Borcherds '87]
- If $P \subset \mathbb{H}^n$ is finite volume, arithmetic Coxeter polytope, then $n \leq 21, n \neq 20$ [Esselmann' 97].

3. Known Classifications

✓ **By dimension**

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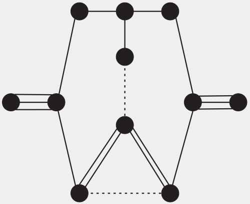
- dim=2 [Poincare, 1882]: $\sum \alpha_i \leq \pi(n-2)$;
- dim=3 [Andreev,'70]: necessary and sufficient condition for dihedral angles.
- dim≥ 4: Widely open

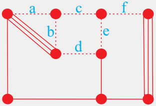
Compact hyperbolic Coxeter polytopes

- Lanner (1950): n-simplex
  (3, 4-dims, finite, $D(1) = 4$)
- Kaplinskaja (1974) Esselmann (1996), n-polytope with $n + 2$-facets
  (4, 5-dims, $D(2) = 5$)
- Esselmann (1996) Tumarkin 2007, n-polytope with $n + 3$-facets
  (4, 5, 6, 8-dims, $D(3) = 8$)
- Jacquemet-Tschantz (2018): n-cube
  (4-dim, 5-dim)
- Burcroff (2022): Some upper bounds for $D(5), D(6), \cdots, D(10)$.

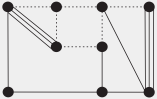
✓ Tumarkin-Flikson (2008), (7-dim $D(4) = 7$ [Bugaenko'92])



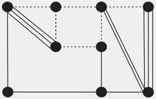
Ma-Zheng(2022), Burcroff(2022) (4-dim, 5-dim)



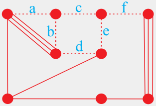
$P_{322,1}$



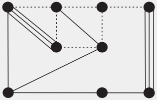
$P_{322,2}$



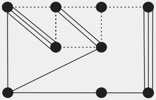
$P_{322,3}$



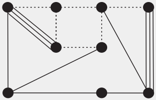
$P_{322,4}$



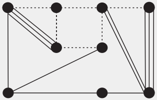
$P_{322,5}$



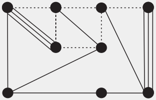
$P_{322,6}$



$P_{322,7}$

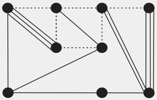


$P_{322,8}$

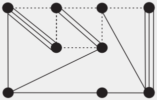


$P_{322,9}$

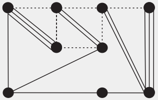
Results



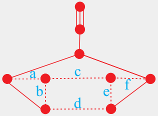
$P_{322,10}$



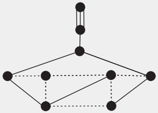
$P_{322,11}$



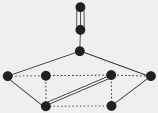
$P_{322,12}$



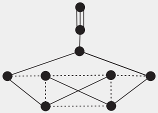
$P_{322,13}$



$P_{322,14}$



$P_{322,15}$



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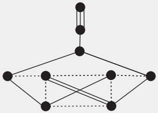
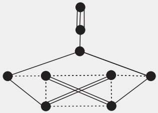
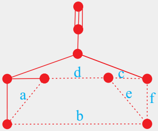


$P_{322,17}$



$P_{322,18}$



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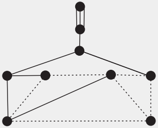
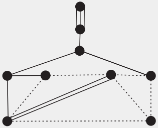


$P_{319,2}$



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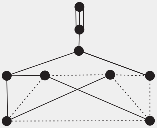

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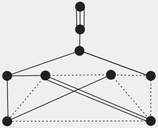
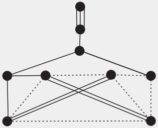


$P_{319,5}$



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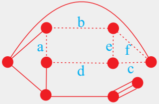
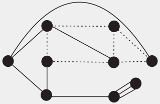
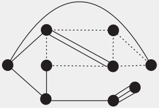


$P_{319,7}$

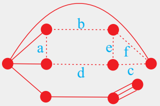


$P_{319,8}$

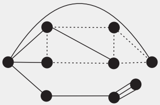


$P_{319,9}$

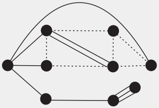
Results



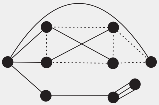
$P_{319,10}$



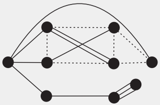
$P_{319,11}$



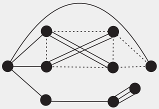
$P_{319,12}$



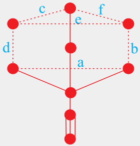
$P_{319,13}$



$P_{319,14}$

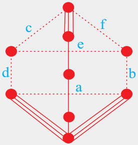


$P_{319,15}$

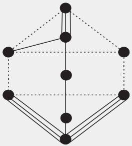


$P_{319,16}$

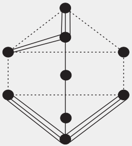
RESULTS



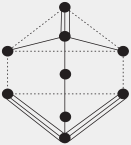
$P_{319,17}$



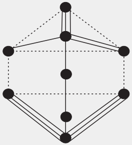
$P_{319,18}$



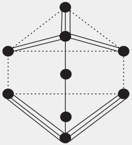
$P_{319,19}$



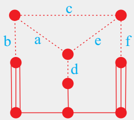
$P_{319,20}$



$P_{319,21}$

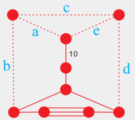


$P_{319,22}$

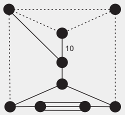


$P_{312,1}$

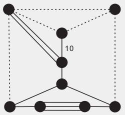
Results



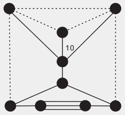
$P_{302,1}$



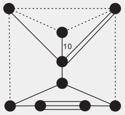
$P_{302,2}$



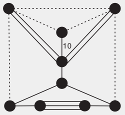
$P_{302,3}$



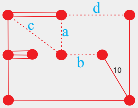
$P_{302,4}$



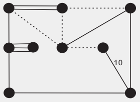
$P_{302,5}$



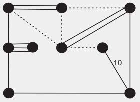
$P_{302,6}$



$P_{313,1}$

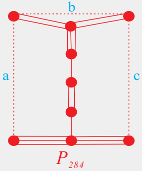


$P_{313,2}$



$P_{313,3}$

Results



	a	b	c	d	e
$P_{302,1}$	$\frac{1}{2}\sqrt{4+\sqrt{5}}$	$\frac{1}{2}\sqrt{30+13\sqrt{5}}$	$\frac{1}{4}(31+15\sqrt{5})$	$\frac{1}{2}\sqrt{30+13\sqrt{5}}$	$\frac{1}{2}\sqrt{4+\sqrt{5}}$
$P_{313,1}$	$\frac{1}{2}\sqrt{3+\sqrt{5}}$	$\frac{1}{2}\sqrt{5+\sqrt{5}}$	$\frac{1}{2}(3+\sqrt{5})$	$\sqrt{5+2\sqrt{5}}$	
P_{284}	$\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1+\sqrt{5})$		

Results

	a	b	c
	d	e	f
$P_{322,1}$	$\frac{1}{2}\sqrt{\frac{1}{2}(6+\sqrt{5})}$	$\frac{1}{4}\sqrt{23+8\sqrt{5}+\sqrt{5(63+26\sqrt{5})}}$	$\frac{1}{4}\sqrt{946+423\sqrt{5}+\sqrt{1749835+782550\sqrt{5}}}$
	$\frac{1}{4}(5+2\sqrt{5}+\sqrt{63+26\sqrt{5}})$	$\frac{1}{2}\sqrt{\frac{1}{2}(1257+562\sqrt{5}+3\sqrt{349967+156510\sqrt{5}}}$	$\frac{1}{2}\sqrt{\frac{1}{2}(9+\sqrt{5})}$
$P_{322,4}$	$\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{4}(1+\sqrt{5}+2\sqrt{8+3\sqrt{5}})$	$\frac{9}{2}+2\sqrt{5}+\frac{1}{4}\sqrt{1382+618\sqrt{5}}$
	$\frac{1}{4}(3+\sqrt{5}+2\sqrt{8+3\sqrt{5}})$	$\frac{1}{2}(9+4\sqrt{5}+\sqrt{132+59\sqrt{5}})$	$\frac{1}{2}\sqrt{\frac{1}{2}(9+\sqrt{5})}$
$P_{322,13}$	$\sqrt{\frac{23}{8}+\frac{9\sqrt{5}}{8}}$	$\frac{1}{2}\sqrt{6+\sqrt{5}}$	$\frac{1}{4}(15+7\sqrt{5})$
	$\frac{1}{4}(3+\sqrt{5})$	$\frac{1}{2}\sqrt{6+\sqrt{5}}$	$\sqrt{\frac{23}{8}+\frac{9\sqrt{5}}{8}}$
$P_{319,1}$	$\frac{1}{4}(2+\sqrt{5})$	$\frac{1}{4}\sqrt{\frac{5}{2}(23+9\sqrt{5})}$	$\sqrt{\frac{23}{8}+\frac{9\sqrt{5}}{8}}$
	$\frac{1}{4}\sqrt{\frac{5}{2}(23+9\sqrt{5})}$	$\frac{1}{8}(7+5\sqrt{5})$	$\sqrt{\frac{23}{8}+\frac{9\sqrt{5}}{8}}$

Results

$P_{319,7}$	$\frac{1}{4}(5+3\sqrt{5}+2\sqrt{\frac{57}{2}+\frac{25\sqrt{5}}{2}})$	$\frac{5+3\sqrt{5}+\sqrt{46+18\sqrt{5}}}{4\sqrt{2}}$	$\frac{1}{2}(1+\sqrt{5})$
	$\frac{5}{2}+\sqrt{5}+\frac{1}{4}\sqrt{114+50\sqrt{5}}$	$\frac{1}{4}\sqrt{47+17\sqrt{5}+2\sqrt{570+250\sqrt{5}}}$	$\frac{1}{2}\sqrt{\frac{1}{2}(6+\sqrt{5})}$
$P_{319,10}$	$2+\sqrt{5}$	$3+\sqrt{5}$	$\frac{1}{2}(1+\sqrt{5})$
	$3+\sqrt{5}$	$2+\sqrt{5}$	$\frac{1}{2}(1+\sqrt{5})$
$P_{319,16}$	$\frac{1}{4}(19+9\sqrt{5})$	$\frac{1}{4}\sqrt{5(57+25\sqrt{5})}$	$\frac{1}{2}\sqrt{\frac{1}{2}(6+\sqrt{5})}$
	$\frac{1}{4}\sqrt{5(57+25\sqrt{5})}$	$\frac{1}{8}(7+5\sqrt{5})$	$\frac{1}{2}\sqrt{\frac{1}{2}(6+\sqrt{5})}$
$P_{319,17}$	$\frac{1}{4}(5+\sqrt{5})$	$\sqrt{\frac{691}{8}+\frac{309\sqrt{5}}{8}}$	$\frac{1}{2}\sqrt{\frac{1}{2}(9+\sqrt{5})}$
	$\sqrt{\frac{691}{8}+\frac{309\sqrt{5}}{8}}$	$\frac{1}{4}(119+55\sqrt{5})$	$\frac{1}{2}\sqrt{\frac{1}{2}(9+\sqrt{5})}$
$P_{312,1}$	$\frac{1}{2}\sqrt{\frac{5}{2}+\sqrt{5}}$	$\frac{1}{4}\sqrt{15+\sqrt{5}}$	$1+\frac{\sqrt{5}}{2}$
	$\frac{1}{2}\sqrt{4+\sqrt{5}}$	$\frac{1}{2}\sqrt{\frac{5}{2}+\sqrt{5}}$	$\frac{1}{4}\sqrt{15+\sqrt{5}}$

Thank You for your listening!