Unperturbed weakly reducible non-minimal bridge positions

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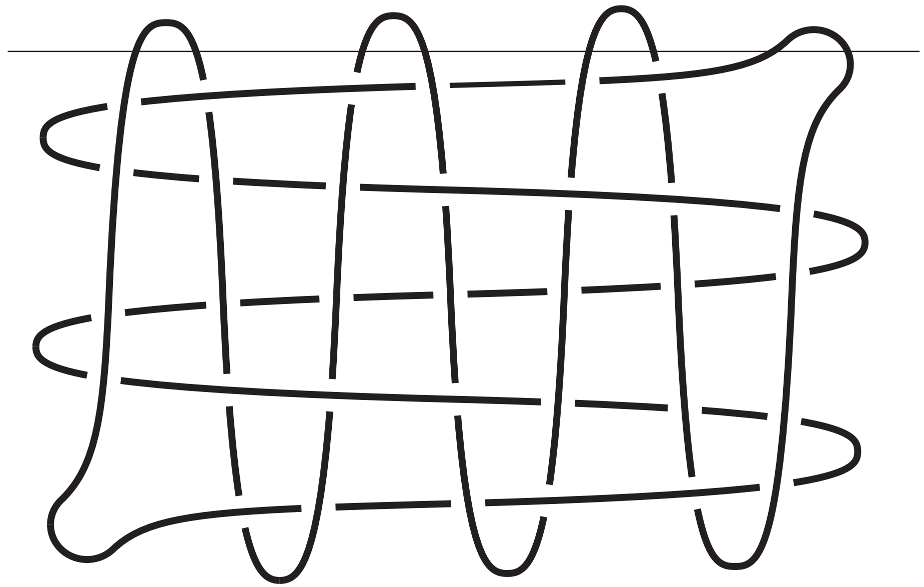
• Motivation

unperturbed strongly irreducible
4-bridge position of a 3-bridge knot [Ozawa-Takao]

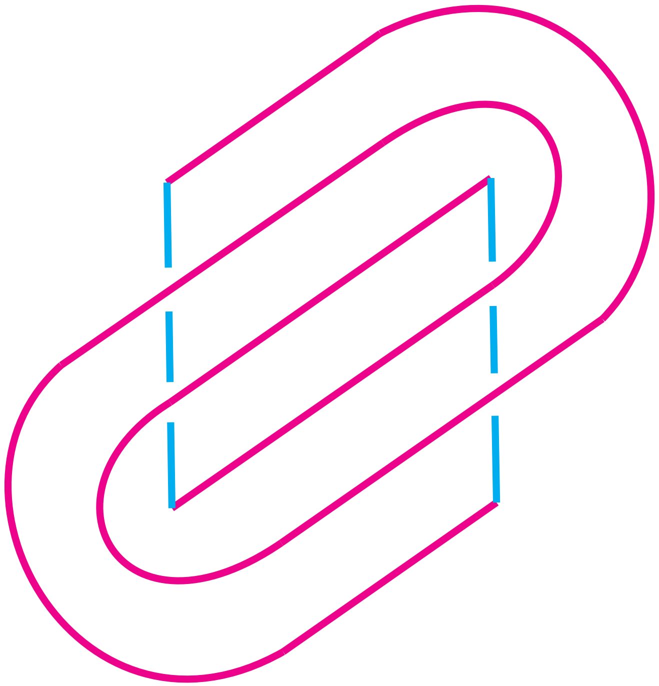


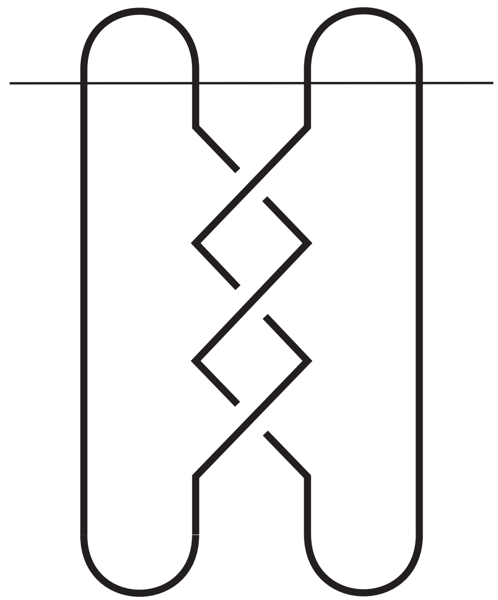
Question 1.
∃ unperturbed weakly reducible non-minimal bridge position ?

- **Bridge number**

[Schubert, 1954]

$b(K)$





[Schubert], [Schultens]

$b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$

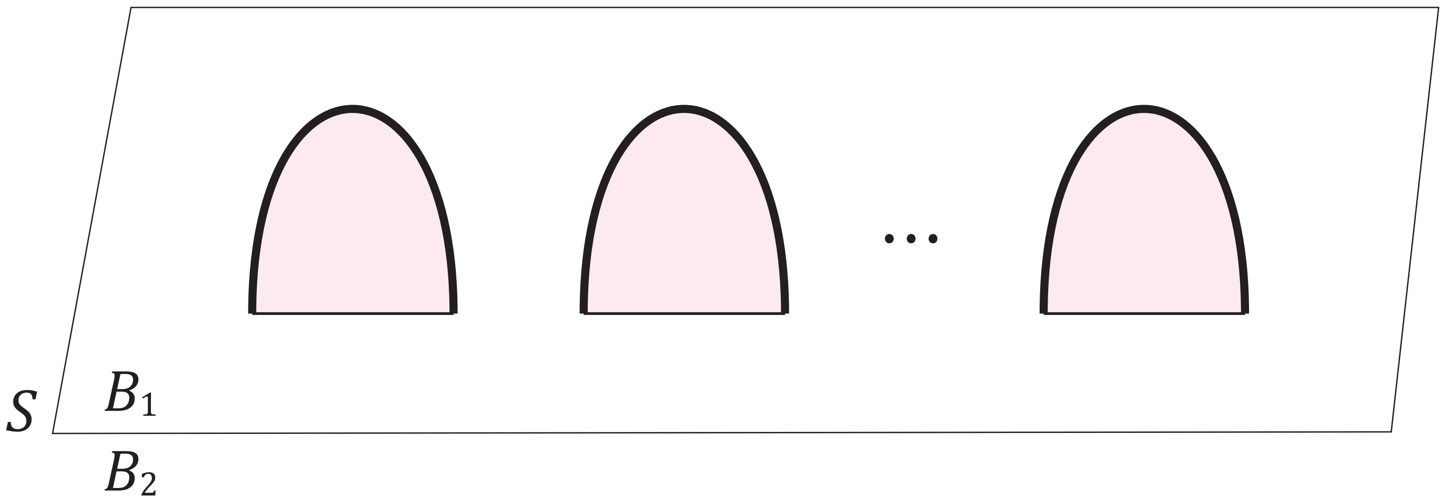
$K = (p, q)$-torus knot $(p, q > 0)$
$b(K) = \min\{p, q\}$

$K =$ a (p, q)-cable of a non-trivial knot J (longitudinally p times)
$b(K) = p \cdot b(J)$

• Bridge position

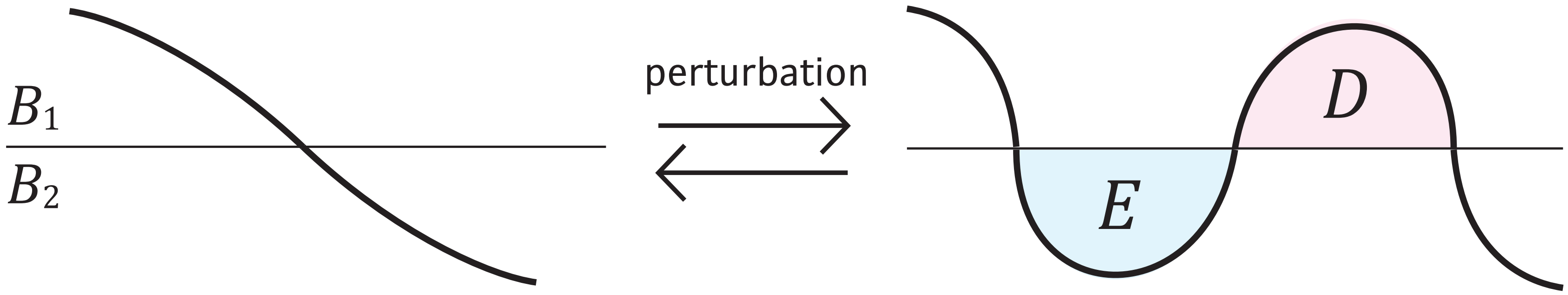
$S^3 = B_1 \cup_S B_2$, $\quad B_1, B_2$: 3-balls
$K \cap B_i$ = a collection of n ∂-parallel arcs

We say that K is in *n-bridge position* with respect to S.



a complete bridge disk system

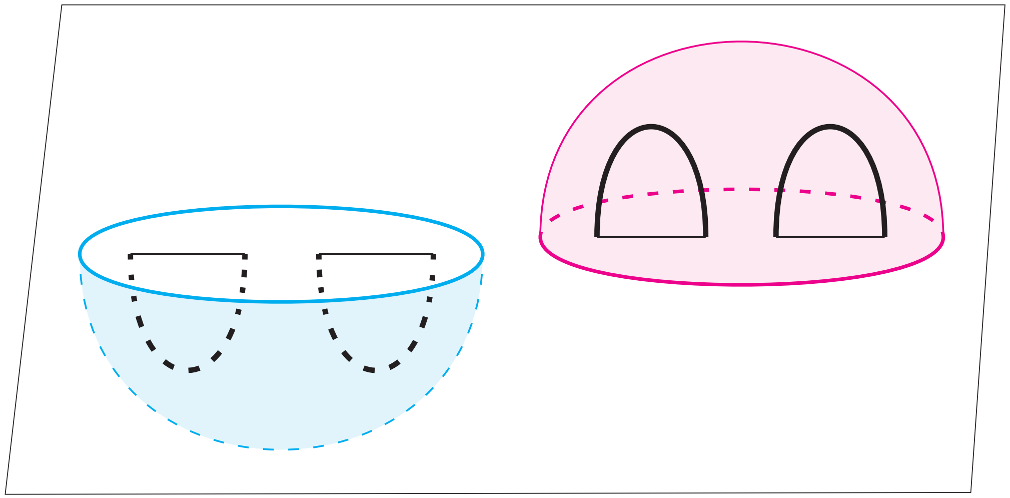
- **Perturbed bridge position**



A bridge position of a knot K is *perturbed* if
$\exists$ bridge disks $D \subset B_1, E \subset B_2$ such that $D \cap E =$ a point of K.
D, E : *cancelling disks*
(D, E) : a *cancelling pair*

Otherwise, it is *unperturbed*.

- **Strong irreducibility**



weakly reducible
Otherwise, *strongly irreducible*

n-bridge position ($n \geq 3$)
perturbed $\Longrightarrow$ weakly reducible
$\therefore$ strongly irreducible $\Longrightarrow$ unperturbed

- **Heegaard splitting**

$V \cup_F W$: a Heegaard splitting of a closed 3-manifold M
It is *stabilized* if $\exists$ $D \subset V$ and $E \subset W$ such that $|D \cap E| = 1$.
(D, E) : a *cancelling pair*
Otherwise, *unstabilized*.

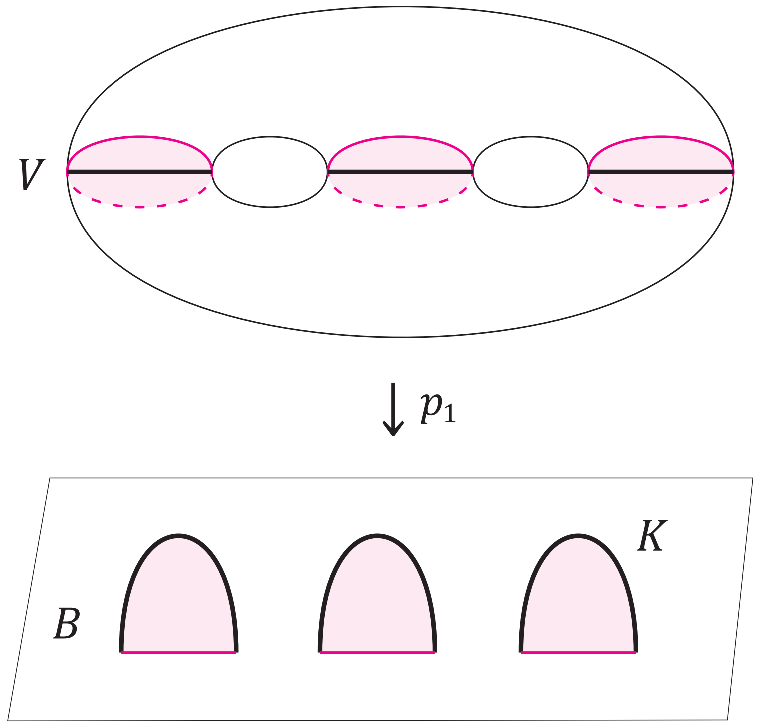
If $\exists$ essential disks $D \subset V$ and $E \subset W$ such that $\partial D = \partial E$,
then *reducible*.
Otherwise, *irreducible*.
If $\exists$ essential disks $D \subset V$ and $E \subset W$ such that $D \cap E = \emptyset$,
then *weakly reducible*.
Otherwise, *strongly irreducible*.

genus g Heegaard splitting ($g \geq 2$)
stabilized $\implies$ reducible $\implies$ weakly reducible

$\therefore$ strongly irreducible $\implies$ unstabilized

• 2-Fold branched covering

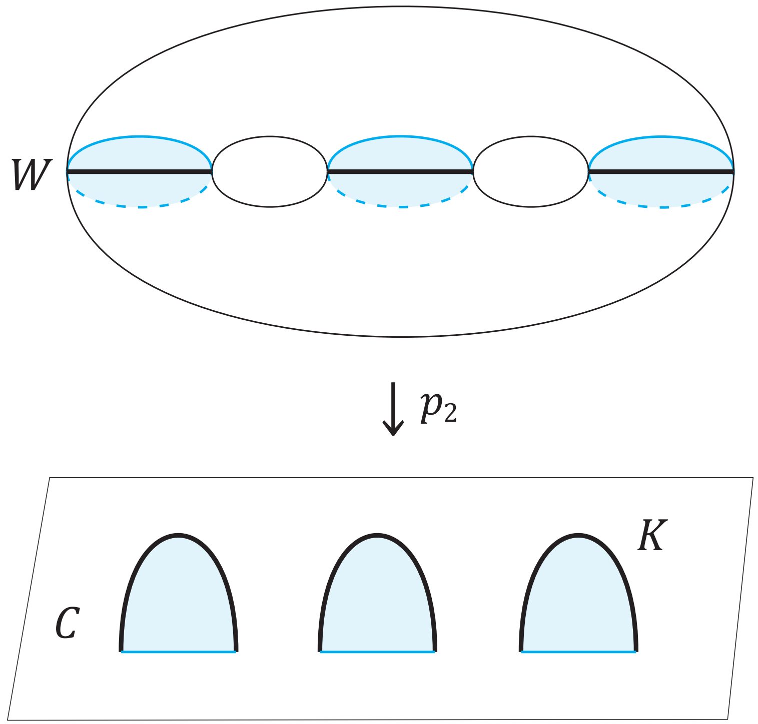
K : a knot in n-bridge position with respect to $S^3 = B \cup_S C$
V : a genus $n-1$ handlebody
2-fold covering $p_1 : V \to B$ branched along $B \cap K$



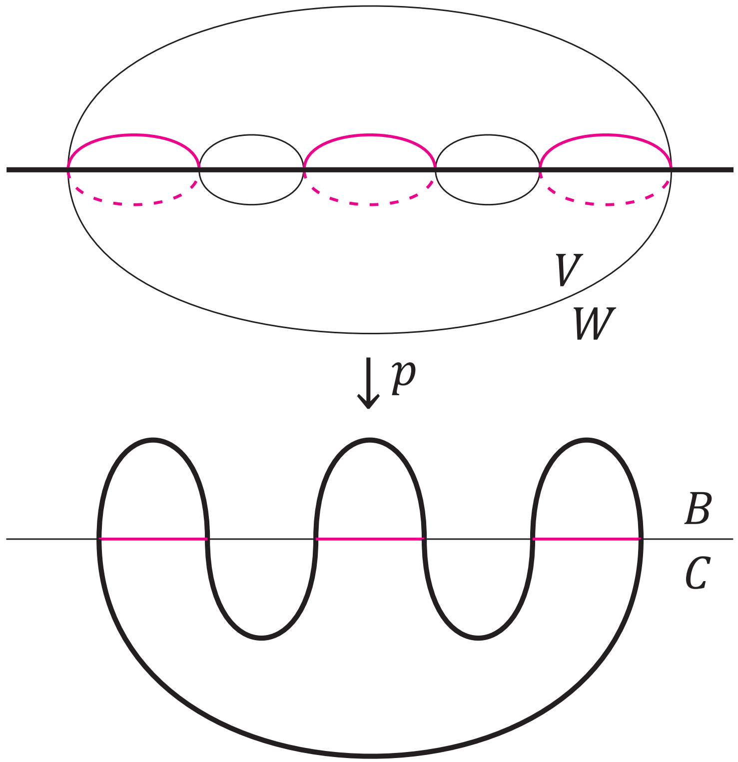
Similarly,

W : a genus $n-1$ handlebody

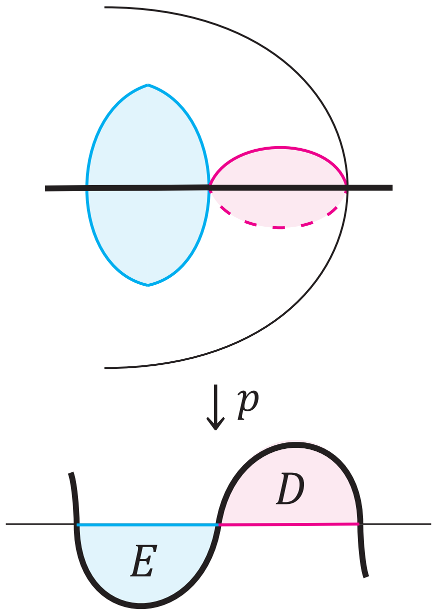
2-fold covering $p_2 : W \to C$ branched along $C \cap K$



2-fold covering $p : V \cup_F W \to B \cup_S C$ branched along K



(D, E) : a cancelling pair for $B \cup_S C$
$(p^{-1}(D), p^{-1}(E))$: a cancelling pair for $V \cup_F W$
$B \cup_S C$: perturbed $\Longrightarrow$ $V \cup_F W$: stabilized



Proposition.
$V \cup_F W$: unstabilized $\Longrightarrow$ $B \cup_S C$: unperturbed

$*$ The converse of the proposition does not hold.

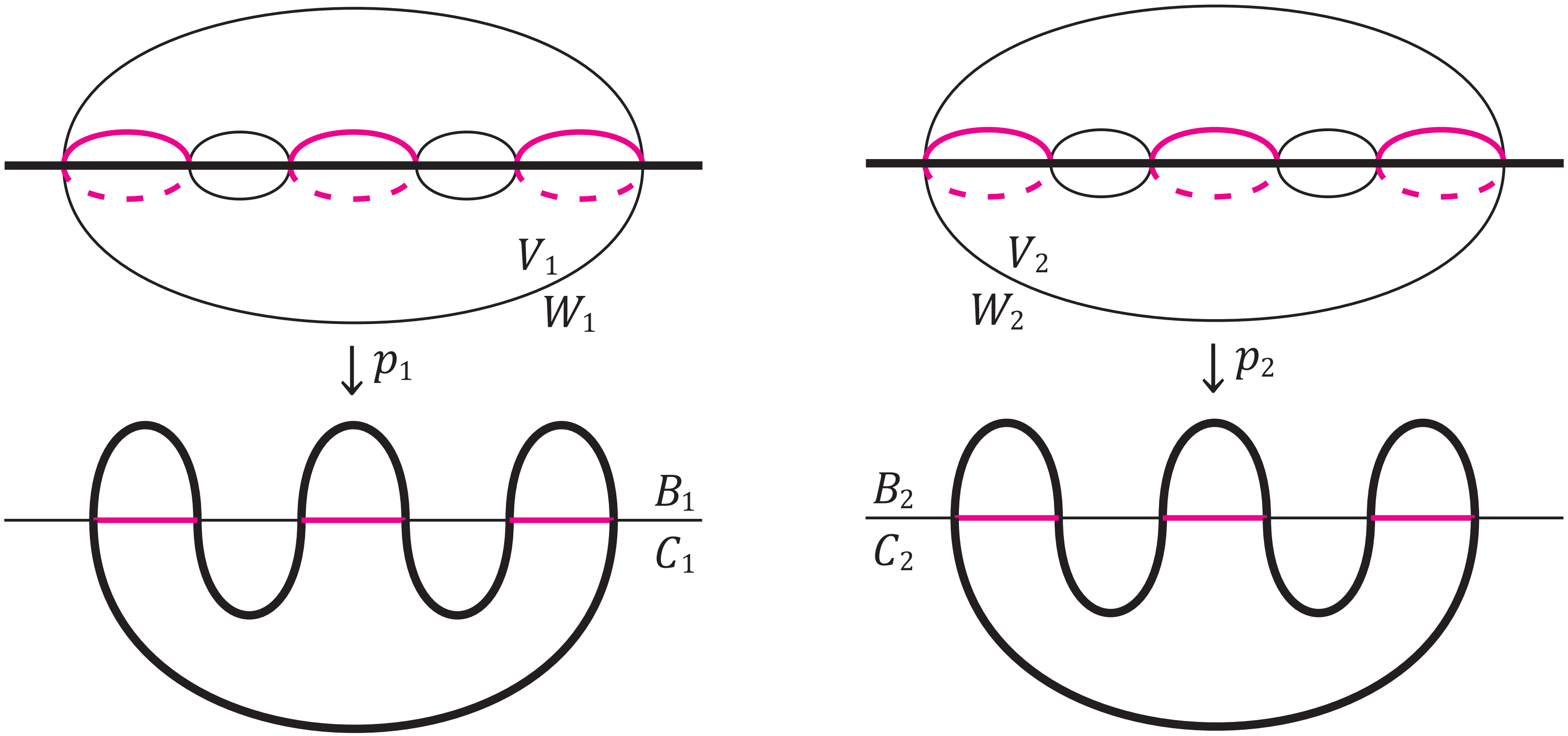
$K = (p, q)$-torus knot $(p < q)$, $b(K) = p$
The p-bridge position $B \cup_S C$ is unperturbed.

$M =$ 2-fold covering branched along K is a small Seifert fibered manifold.
An irreducible Heegaard splitting is either vertical or horizontal.
vertical: genus ≤ 2
horizontal: genus is even.

The 2-fold covering $V \cup_F W$ is of genus $p - 1$.
$\therefore$ For example, if $(p, q) = (4, 5)$,
then $V \cup_F W$ is reducible, hence stabilized.

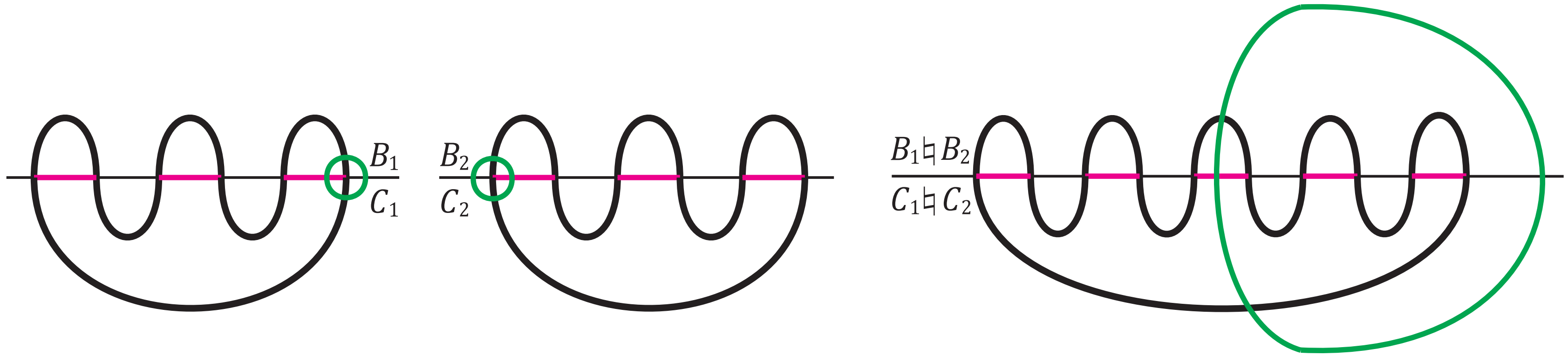
2-fold branched coverings

$p_1 : V_1 \cup_{F_1} W_1 \to B_1 \cup_{S_1} C_1, \quad p_2 : V_2 \cup_{F_2} W_2 \to B_2 \cup_{S_2} C_2$



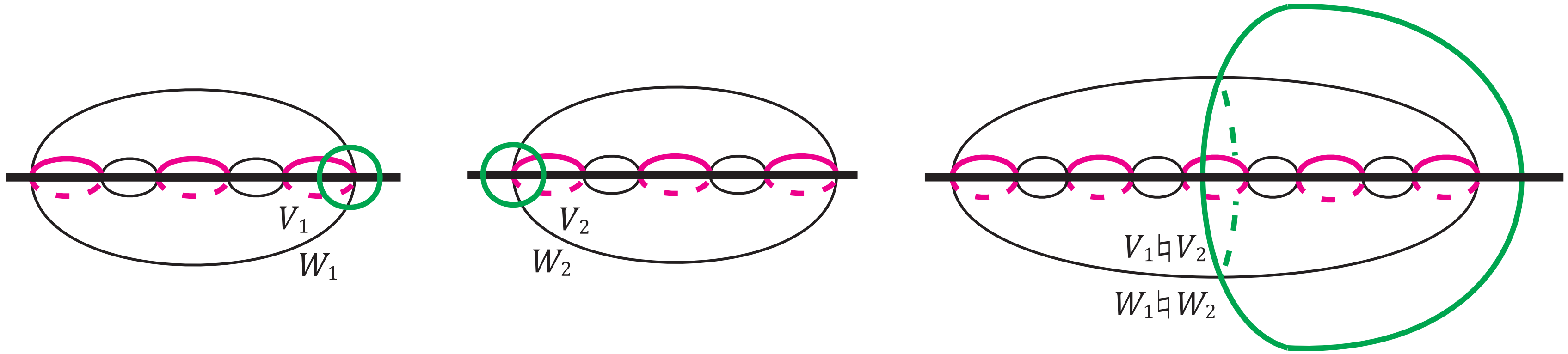
- **Connected sum**

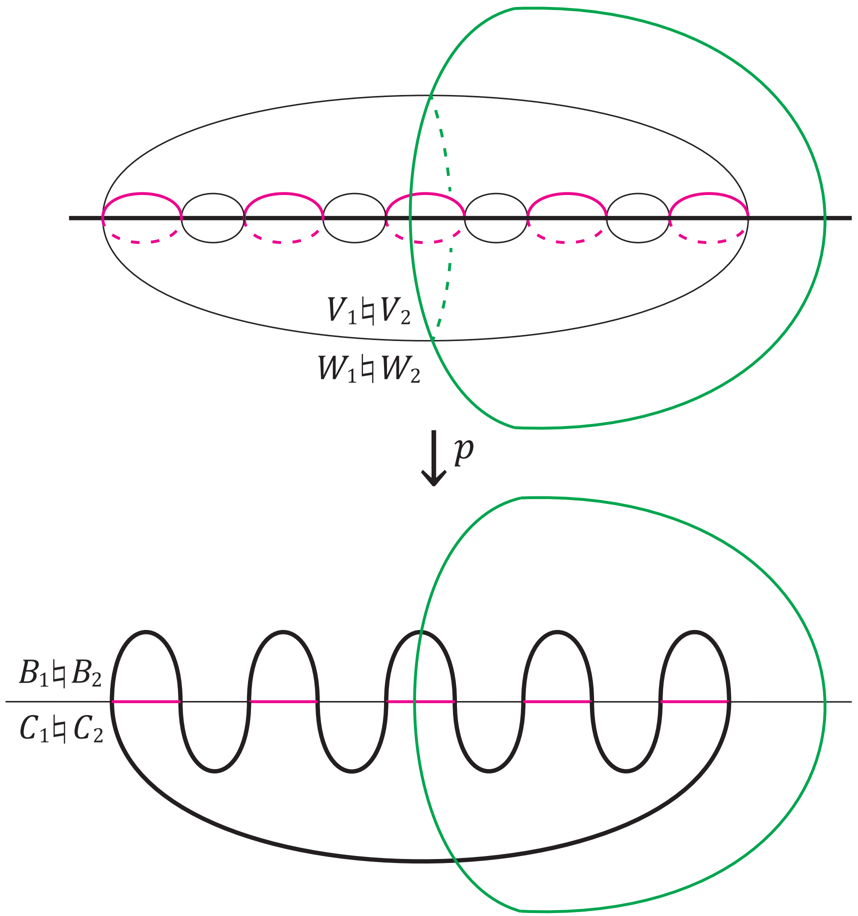
$$(B_1 \cup_{S_1} C_1)\#(B_2 \cup_{S_2} C_2) = (B_1 \natural B_2) \cup_{S_1 \# S_2} (C_1 \natural C_2)$$



- **Connected sum**

$$(V_1 \cup_{F_1} W_1)\#(V_2 \cup_{F_2} W_2) = (V_1 \natural V_2) \cup_{F_1 \# F_2} (W_1 \natural W_2)$$





$p : (V_1 \cup_{F_1} W_1)\#(V_2 \cup_{F_2} W_2) \to (B_1 \cup_{S_1} C_1)\#(B_2 \cup_{S_2} C_2)$ is a 2-fold branched covering.

- **Gordon's Conjecture**

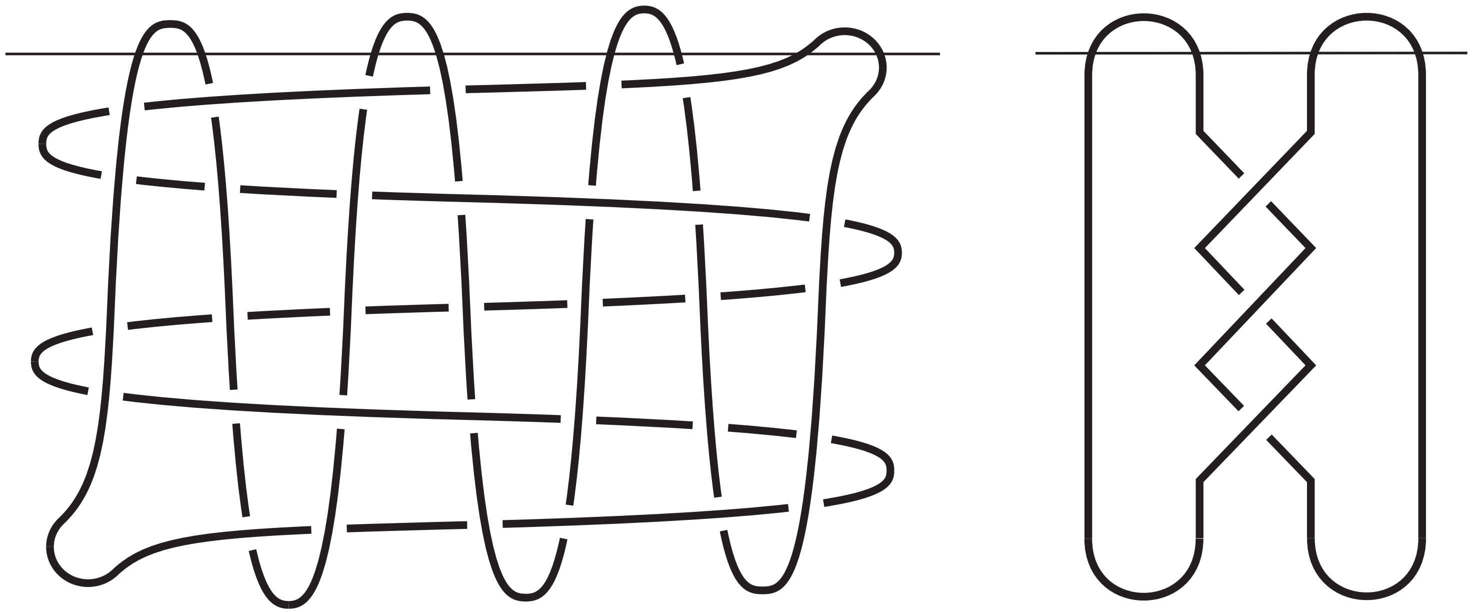
[Bachman], [Qiu-Scharlemann]

The connected sum of two unstabilized Heegaard splittings is unstabilized.

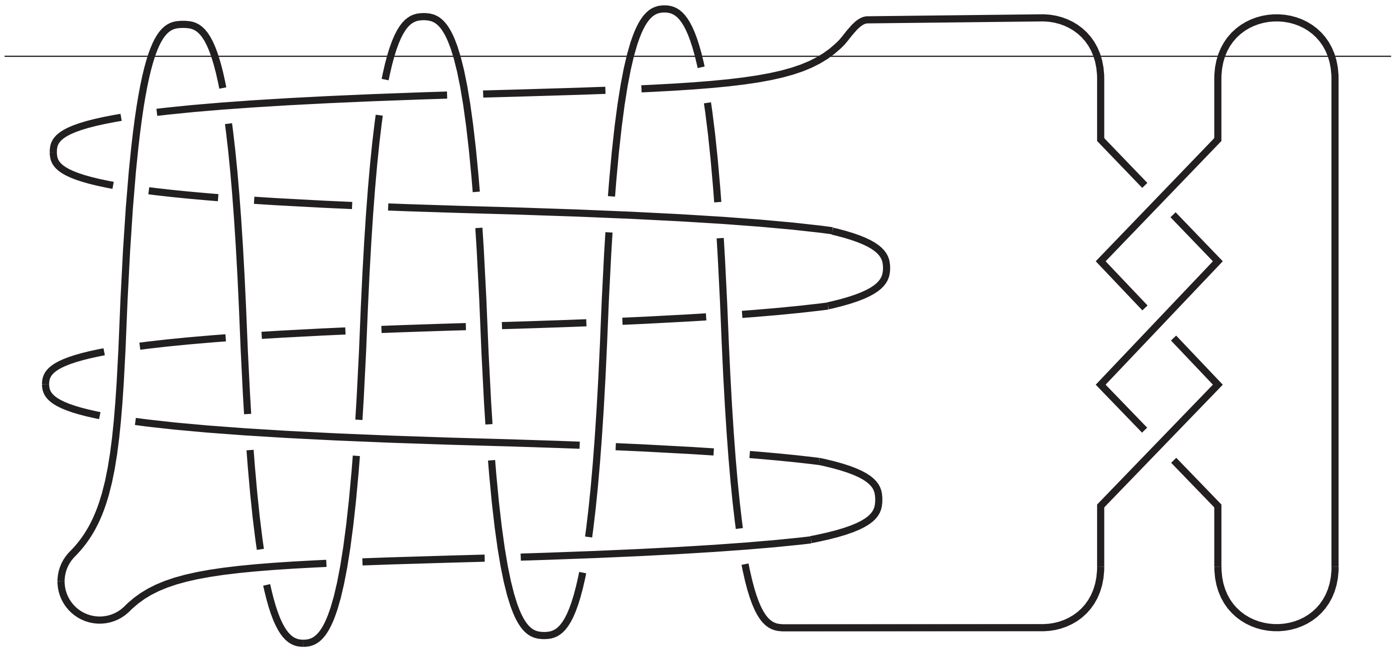
• unperturbed. ($\because$ strongly irreducible.)

Moreover,

• admits a 2-fold branched covering whose Heegaard splitting is strongly irreducible, hence unstabilized.



unperturbed weakly reducible
5-bridge position of a (composite) 4-bridge knot



• A bridge version of Gordon’s Conjecture

The connected sum of two unperturbed bridge positions is unperturbed.

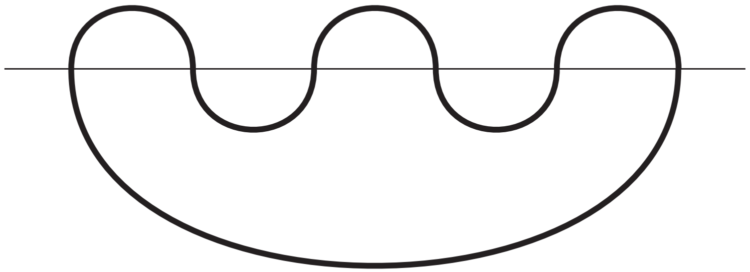
Question 1.
$\exists$ unperturbed weakly reducible non-minimal bridge position ?
Yes. (composite knots)

Question $1'$. What about prime knots?

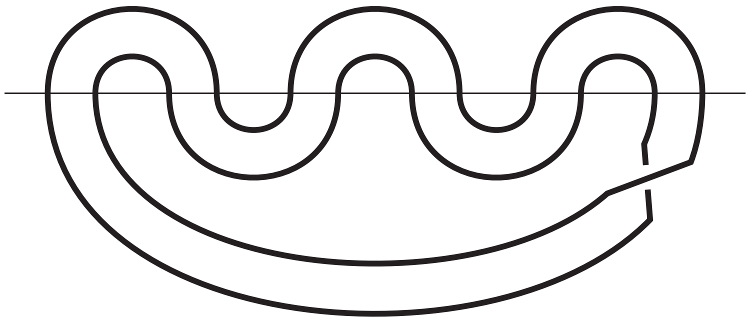
- **2-Cable position**

J : a non-trivial knot in n-bridge position

K : a $(2, q)$-cable knot of J for some odd q, where
K is in $2n$-bridge position, called a *2-cable position*.



J



K

Question 2.

Suppose that a bridge position of J is unperturbed.

Then a 2-cable position of K is unperturbed.
($K =$ a $(2, q)$-cable knot of J)

Thank you for your attention.