Petal number of torus knots

Hyungkee Yoo[1]

(Joint work with Hyoungjun Kim[2] and Sungjong No[3])

[1]Research Institute for Natural Science, Hanyang University
[2]College of General Education, Kookmin University
[3]Department of Mathematics, Kyonggi University

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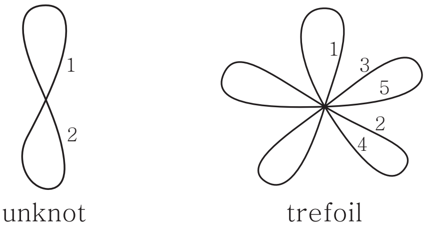
Petal projection

Definition

- *Petal projection* : a knot projection with single multi-crossing such that $\nexists$ nesting loops.



- *Petal number*: the smallest number of loops among all petal projections of given knot. The petal number of a knot K is denoted by $p(K)$.

Note: For any nontrivial knot K, $p(K)$ is always odd.

Main Theorems

Main Theorem 1

Let r and s be relatively prime integers with $1 < r < s$. If $r \equiv 1 \mod s - r$, then

$$p(T_{r,s}) = 2s - 1.$$

Proposition (Adams et al., 2012)

For any positive integer r with $r > 1$,

$$p(T_{r,r+1}) = 2r + 1.$$

Main Theorem 1 + Adams et al's result

Let r be a positive integer with $r > 1$, and let d be a divisor of $r - 1$. Then,

$$p(T_{r,r+d}) = 2r + 2d - 1.$$

Theorem (Lee-Jin, 2022)

For any odd number r with $r \geq 3$,

$$p(T_{r,r+2}) = 2r + 3.$$

Theorem (Adams et al, 2012)
Let $T_{r,s}$ be a torus knot with $s \equiv \pm 1 \mod r$. Then

$$p(T_{r,s}) \leq \begin{cases} 2s-1 & \textit{for} \quad s \equiv 1 \mod r, \\ 2s+3 & \textit{for} \quad s \equiv -1 \mod r. \end{cases}$$

We develop this result to the following theorem by using integral surgeries on curves around $T_{r,r+1}$.

Main Theorem 2
Let r and s be positive integers with $s \equiv \pm 1 \mod r$. Then

$$p(T_{r,s}) \leq 2s - 2\left\lfloor \frac{s}{r} \right\rfloor + 1.$$

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Bridge indices and superbridge indices

- $[K]$: an ambient isotopy class of a knot K
- $b_{\vec{v}}(K)$: the number of local maxima of K in the $\vec{v}$ direction

Definition

The *bridge index* $b(K)$ of K is defined by

$$b(K) = \min_{K' \in [K]} \min_{\vec{v} \in S^2} b_{\vec{v}}(K').$$

Definition

The *superbridge index* $sb(K)$ is defined by

$$sb(K) = \min_{K' \in [K]} \max_{\vec{v} \in S^2} b_{\vec{v}}(K').$$

Facts

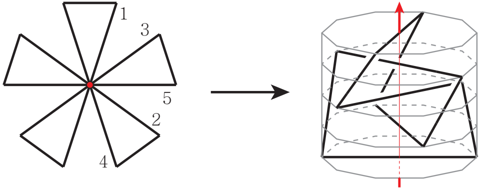
1. [Adams et al., 2015] For any nontrivial knot K, $2b(K) \leq p(K) - 1$
2. [Kuiper, 1987] For any knot K, $b(K) < sb(K)$.

Theorem (Kim-No-Y., 2022+)

For any nontrivial knot K,

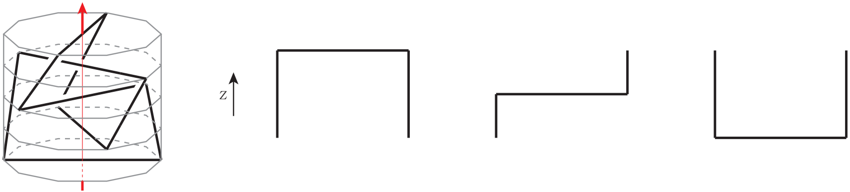
$$2sb(K) \leq p(K) + 1.$$

Sketch of proof.



Case 1: $\vec{v} = \vec{z} = (0, 0, 1)$

Then local maxima and minima are only appear at horizontal line segments.

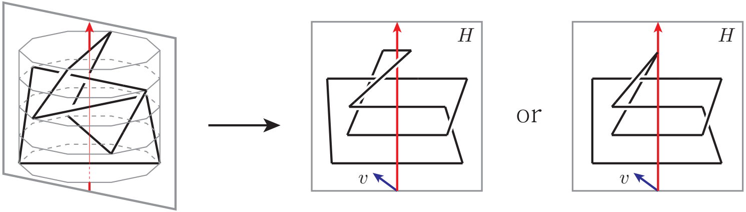


Note: There are the same number of local maxima and minima.

Since $p(K)$ is odd, there are at most $\frac{p(K)-1}{2}$ local maxima in this case.

Case 2: $\vec{v} \neq \vec{z} = (0, 0, 1)$

Let H be a plane spanned by two vectors $\vec{v}$ and $\vec{z} = (0, 0, 1)$.
Now we have a projection of the polygonal knot of K to H. Then there are two cases.



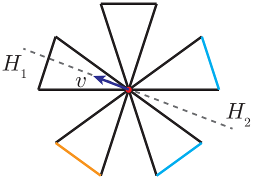
The image of line segment which is not horizontal segment is called a *vertical stick*.
Then there are exactly $p(K)$ vertical sticks.

Note: Each vertical stick has at most one local maximum for any direction.

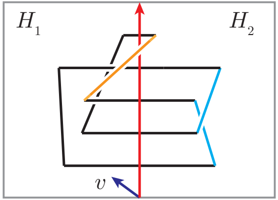
The z-axis divides H into two regions H_1 and H_2.

Without loss of generality, we assume that the terminal point of $\vec{v}$ is contained in H_1.

Key: Every vertical stick contained in H_2 which does not touch the z-axis cannot have a local maximum on $\vec{v}$ direction.

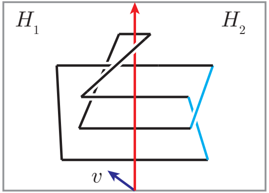


top view



side view

Subcase 1: $\nexists$ horizontal segment perpendicular to H



In this case, there are $\frac{p(K)-1}{2}$ vertical sticks which is contained the whole in H_2. This implies that the number of local maxima is at most $\frac{p(K)+1}{2}$.

Subcase 2: $\exists$ horizontal segment perpendicular to H

The number of vertical sticks contained in H_2 which does not touch the z-axis is either $\frac{p(K)-1}{2}$ or $\frac{p(K)-1}{2} - 1$.

In the last case, two vertical sticks which are connected by one point in the z-axis is contained in H_2. Even though interiors of these vertical sticks are contained in H_2, there is at most one local maximum in these sticks. Thus there are at most $\frac{p(K)+1}{2}$ local maxima in both cases.

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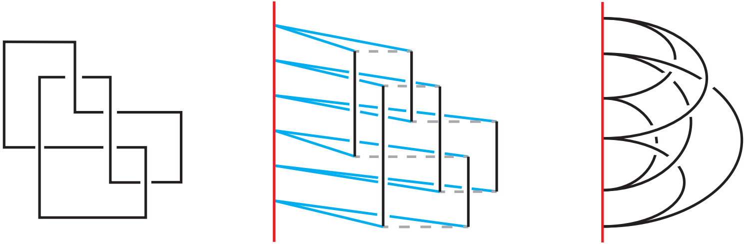
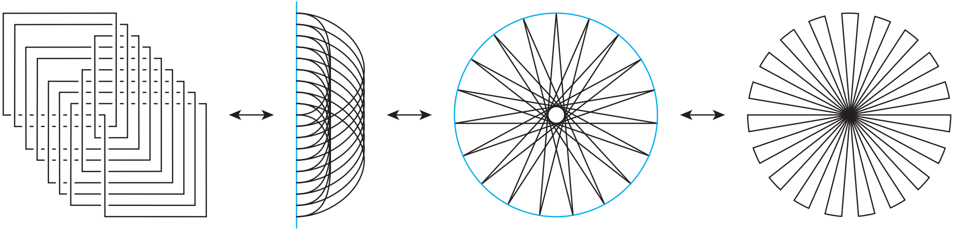
Grid diagrams and arc presentations



Note: A grid diagram can be expressed as an arc representation and vice versa. Thus the grid index is equal to the arc index for same knot type, i.e. $g(K) = \alpha(K)$.

Petal grid diagrams

Petal grid diagram : a grid diagram with $2n + 1$ vertical line segments such that every vertical line segments has length n or $n + 1$.



Note: Since every petal projection can be transformed to a petal grid presentation, $p(K) \geq \alpha(K)$.

Main Theorem 1

Facts

1. [Kuiper] For $1 < r < s$, $sb(T_{r,s}) = \min\{2r, s\}$.
2. [Kim-No-Y.] For any knot K, $2sb(K) \leq p(K) + 1$.

Main Theorem 1

Let r and s be relatively prime integers with $1 < r < s$. If $r \equiv 1 \mod s - r$, then

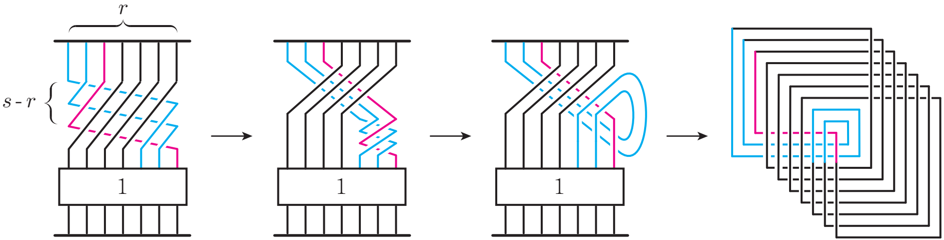
$$p(T_{r,s}) = 2s - 1.$$

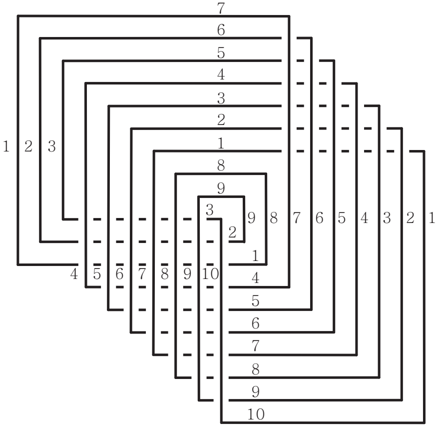
Sketch of proof.

Let r, s be relatively prime integers with $1 < r < s$ and $r \equiv 1 \mod s - r$. This implies that $1 < r < s < 2r$. By the facts, $2s - 1 \leq p(T_{r,s})$.

So it is sufficient to show that $p(T_{r,s}) \leq 2s - 1$.

Take a grid diagram D of $T_{r,s}$ which has $2s-1$ vertical and horizontal line segments each as drawn in the figure below.




7
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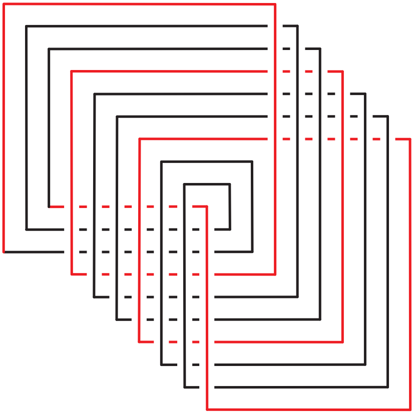

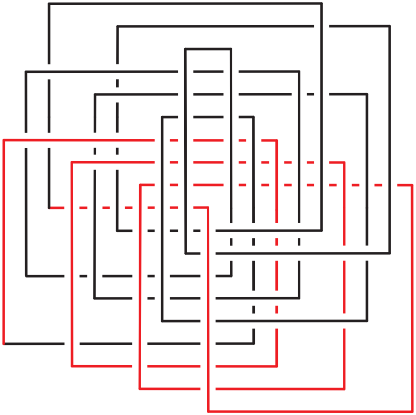

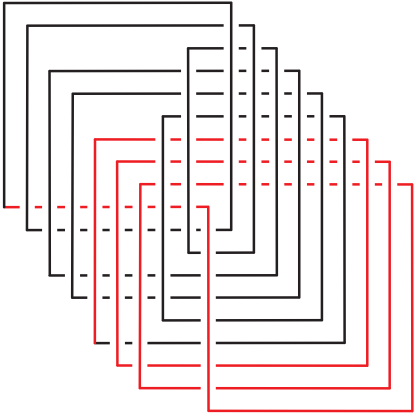

Thus we obtain a petal grid diagram of $T_{r,s}$ with $2s-1$ vertical line segments.

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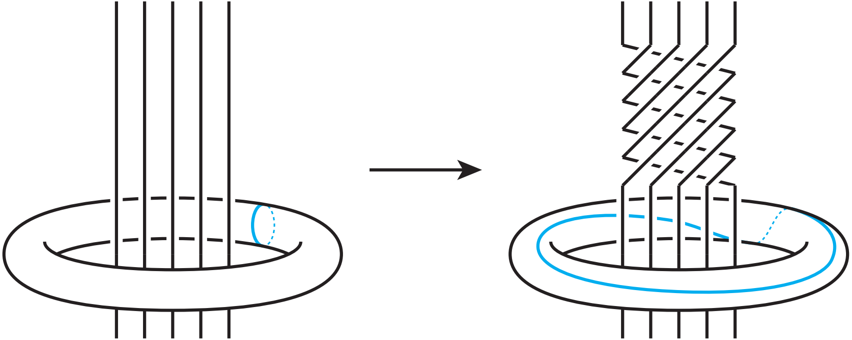
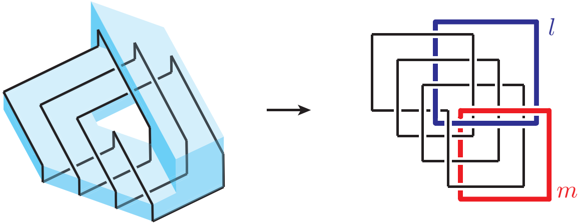
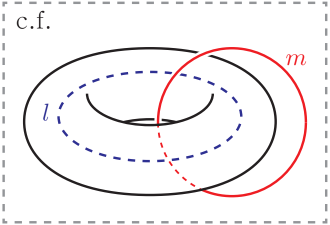
Integral surgeries and twists



Let V be a standard solid torus in S^3 and let T be the boundary of V.





Consider the torus knot $T_{r,r+1}$ on T for any positive integer r.
For $n \geq 2$,

- $T_{r,nr+1}$ is obtained by a $\frac{1}{n-1}$-surgery on m.
- $T_{nm-1,m}$ is obtained by a $\frac{1}{n-1}$-surgery on l, where $m = r + 1$.

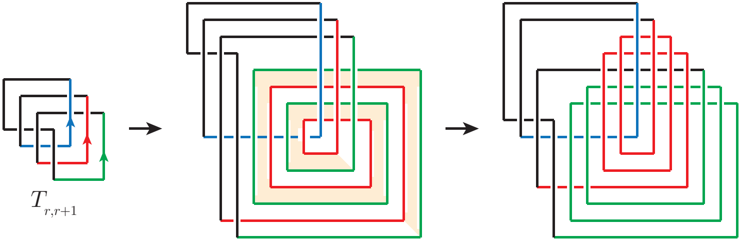
Lemmas

Lemma 1

For any positive integers r and n, $p(T_{r,nr+1}) \leq 2(r-1)n+3$.

Sketch of proof.

We first remark that $T_{r,nr+1}$ is obtained from $T_{r,r+1}$ by using a $\frac{1}{n-1}$-surgery on m.



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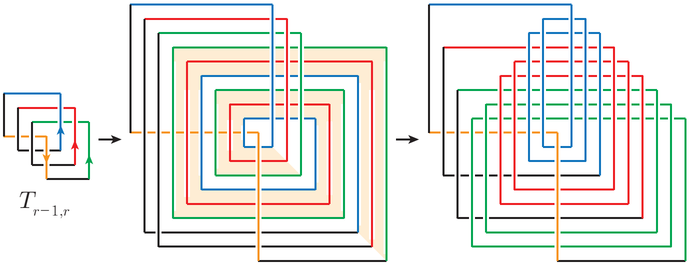
Lemma 2

For any positive integers r and n, $p(T_{r,nr-1}) \leq 2(r-1)n+1$.

Sketch of proof.

Since $T_{r,nr-1}$ and $T_{nr-1,r}$ have the same knot type, we regard that $T_{r,nr-1}$ is obtained from $T_{r-1,r}$ by using a $\frac{1}{n-1}$-surgery on l.



Main Theorem 2

Lemma 1 and Lemma2 give Main Theorem 2.

Main Theorem 2

Let r and s be positive integers with $s \equiv \pm 1 \mod r$. Then

$$p(T_{r,s}) \leq 2s - 2\left\lfloor \frac{s}{r} \right\rfloor + 1.$$

Conjecture

Main Thoerem 2 is hold for every torus knot.

Thank you for your attention!