Computation of the symplectic derivation Lie algebra via classical representation theory

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General settings

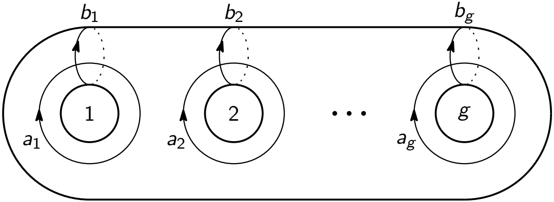
Let $g \geq 1$.

Σ_g : a closed oriented surface of genus g, $\quad H := H_1(\Sigma_g; \mathbb{Q}) \cong \mathbb{Q}^{2g}$.

$H \curvearrowleft \mathrm{Sp}(2g; \mathbb{Q})$: the canonical action,

$\mu \colon H \otimes H \to \mathbb{Q}$: the intersection form,

$a_1, \ldots, a_g, b_1, \ldots, b_g$: a symplectic basis w.r.t. μ.



What is the symplectic derivation Lie algebra?

The *symplectic derivation Lie algebra* $\mathfrak{g}_g$ is defined as

$$\mathfrak{g}_g := \{D \in \mathrm{Der}(A) \mid D \text{ trivially acts on } \omega_0\}$$

for an algebra A and an element ω_0 below:

world	Associative	Lie	Commutative
$\mathfrak{g}_g$	$\mathfrak{a}_g$	$\mathfrak{l}_g$	$\mathfrak{c}_g$
A	(free tensor algebra) $\bigoplus_{i\geq 1} H^{\otimes i}$	(free Lie algebra) $\bigoplus_{i\geq 1} \mathcal{L}_i(H)$	(free commutative algebra) $\bigoplus_{i\geq 1} S^i H$ $(\subset C^\infty(H))$
ω_0	$\sum_i (a_i \otimes b_i - b_i \otimes a_i)$ (i.e. $D(\omega_0) = 0$)	$\sum_i [a_i, b_i]$ (i.e. $D(\omega_0) = 0$)	$\sum_i da_i \wedge db_i$ $(\in \Omega^2(H))$ (i.e. $L_D\omega_0 = 0$)

Every $\mathfrak{g}_g$ has an ideal $\mathfrak{g}_g^+$ called *the positive weight part*.

Geometric correspondence

The homology group $H_\bullet(\mathfrak{g}_g)$ becomes a Hopf algebra when $g \to \infty$.

$\rightsquigarrow$ We can take "the primitive part" of the homology group.

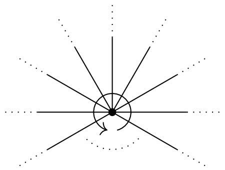
Kontsevich('93) has shown the following correspondence:

the primitive homology of		
Associative	Lie	Commutative
$\mathfrak{a}_g$	$\mathfrak{l}_g$	$\mathfrak{c}_g$
$\updownarrow$	$\updownarrow$	$\updownarrow$
the cohomology / homology of		
the moduli spaces of graphs	the moduli spaces of Riemann surfaces	commutative graphs

Relation to graph homology

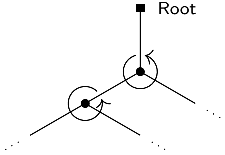
- Approaches from the commutative graph homology ($\approx H_\bullet(\mathfrak{c}_g^+)^{\mathrm{Sp}}$)
  - (Bar-Natan, McKay) Various graph homologies
  - (Conant-Gerlits-Vogtmann, 2005) Computation up to degree 12
  - (Willwacher-Živković, 2015)
    - The generating function of Euler characteristic
    - Displaying it up to weight 60

$\mathfrak{a}_g$



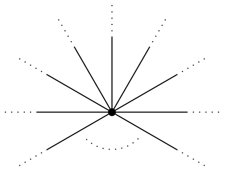
ribbon graphs

$\mathfrak{l}_g$



tree Jacobi diagrams

$\mathfrak{c}_g$



commutative graphs

Another description of $\mathfrak{c}_g$

Definition

For $w \geq 0$, $\mathfrak{c}_g(w) := S^{w+2}H$: the $(w+2)$-nd symmetric power,

$\mathfrak{c}_g$ and $\mathfrak{c}_g^+$ are identified as follows.

$$\mathfrak{c}_g = \bigoplus_{w\geq 0} \mathfrak{c}_g(w) \supset \bigoplus_{w \geq 1} \mathfrak{c}_g(w) = \mathfrak{c}_g^+.$$

We regard $\mathfrak{c}_g$ and $\mathfrak{c}_g^+$ as sets of polynomial functions on H.

$[,]$: the classical Poisson bracket on H, i.e.

$$[f,h] = \sum_{i=1}^{g} \left(\frac{\partial f}{\partial a_i}\frac{\partial h}{\partial b_i} - \frac{\partial f}{\partial b_i}\frac{\partial h}{\partial a_i} \right) \quad (f, h \in \mathfrak{c}_g).$$

$\mathfrak{c}_g^+ \subset \mathfrak{c}_g$ becomes a Lie subalgebra.

Lie algebra homology

$\wedge^\bullet \mathfrak{c}_g = \bigoplus_{k\geq 0} \wedge^k \mathfrak{c}_g$: the Chevalley-Eilenberg chain space

$\partial : \wedge^{k+1}\mathfrak{c}_g \to \wedge^k \mathfrak{c}_g$: the differential of the CE chain complex,

i.e.

$$\partial(f_1 \wedge \cdots \wedge f_{k+1}) := \sum_{1\leq i<j\leq k+1} (-1)^{i+j+1}[f_i, f_j] \wedge f_1 \wedge \cdots \wedge \widehat{f_i} \wedge \cdots \cdots \wedge \widehat{f_j} \wedge \cdots \wedge f_{k+1}.$$

$\wedge^\bullet \mathfrak{c}_g^+ \subset \wedge^\bullet \mathfrak{c}_g$ becomes a chain subcomplex.

Definition

$H_\bullet(\mathfrak{c}_g) := H_\bullet((\wedge^\bullet \mathfrak{c}_g, \partial)), \quad H_\bullet(\mathfrak{c}_g^+) := H_\bullet((\wedge^\bullet \mathfrak{c}_g^+, \partial)).$

Weight

$\wedge^\bullet \mathfrak{c}_g$ has another $\mathbb{Z}_{\geq 0}$-grading than the homological degree.

Definition

- For $f_1 \in \mathfrak{c}_g(w_1), \ldots, f_k \in \mathfrak{c}_g(w_k)$, we say that

$$f_1 \wedge \cdots \wedge f_k \in \wedge^k \mathfrak{c}_g \text{ is of } \textit{weight } w_1 + \cdots + w_k.$$

- $\left(\wedge^k \mathfrak{c}_g^+\right)_w := \mathrm{Span}\left\{\omega \in \wedge^k \mathfrak{c}_g^+ \mid \omega \text{ is of weight } w\right\}$

The differential ∂, weights, and the Sp-action are compatible.

Definition

$$H_\bullet(\mathfrak{c}_g^+)_w := H_\bullet(((\wedge^\bullet \mathfrak{c}_g^+)_w, \partial))$$

Hence, $H_n(\mathfrak{c}_g^+) = \bigoplus_{w \geq 1} H_n(\mathfrak{c}_g^+)_w$.

Homology group of $\mathfrak{c}_g^+$

Proposition

$H_1(\mathfrak{c}_g^+) = (\wedge^1 \mathfrak{c}_g^+)_1 = \mathfrak{c}_g(1) = S^3 H.$

Proof. $\partial_2 = [,]$ is surjective on the weight $w \geq 2$ part.

Theorem (H., 2020)

If $g, w \geq 4$, then $H_2(\mathfrak{c}_g^+)_w = 0$.

Sketch of the proof.

$$\left(\begin{array}{l} \bullet \text{ weight} \\ \bullet\ \mathrm{Sp}(2g;\mathbb{Q})\text{-irreducible} \end{array} \right) \text{ decomposition for the chain space } \wedge^2 \mathfrak{c}_g^+$$

$\rightsquigarrow$ Enough to consider each generator of such irreducible components.

Representation theory of $\mathrm{Sp}(2g;\mathbb{Q})$

Theorem

$$\left\{ \begin{array}{c} \textit{Finite dim. poly.} \\ \textit{irred. } \mathrm{Sp}(2g;\mathbb{Q}) \textit{ rep.} \end{array} \right\} \Big/ \cong \quad \overset{1:1}{\longleftrightarrow} \quad \left\{ \begin{array}{c} \textit{Young diag. with} \\ \textit{at most } g \textit{ rows} \end{array} \right\}$$

e.g.

$$\left[\begin{array}{l} \left\langle \begin{array}{l} (a_1 \wedge a_2 \wedge a_3) \\ \otimes (a_1 \wedge a_2) \otimes a_1 \otimes a_1 \end{array} \right\rangle_{\text{Sp-module}} \\ \subset (\wedge^3 H) \otimes (\wedge^2 H) \otimes H \otimes H \end{array} \right] \longleftarrow\!\shortmid$$

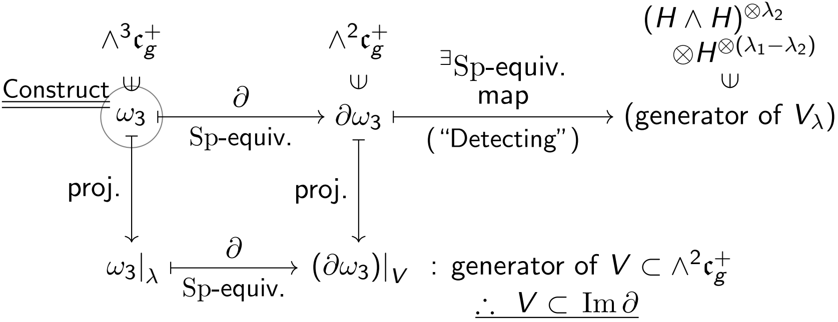
$$\lambda = [421] = {}^t[3211]$$

a_1	a_1	a_1	a_1
a_2	a_2		
a_3			

Sketch of the proof

Let $V (\cong V_\lambda = V_{[\lambda_1 \lambda_2]})$ be an irreducible component of $\wedge^2 \mathfrak{c}_g^+$.

We want $\omega_3 \in \wedge^3 \mathfrak{c}_g^+$ satisfying the following diagram.



i.e. we reduce our argument to the generators.

Corollary

Theorem (Recall)

If $g, w \geq 4$*, then* $H_2(\mathfrak{c}_g^+)_w = 0$.

Corollary

If $g \geq 4$*, then* $H_2(\mathfrak{c}_g^+) = [51] + [33] + [22] + [11] + [1] + [0]$ *as an* $\mathrm{Sp}(2g;\mathbb{Q})$*-module.*

Corollary

If $g \geq 4$*, then*

$$H_3(\mathfrak{c}_g^+)_3 = [711]+[63]+[531]+[333]+[52]+[421]+[322]+[41]+2[311]+2[3]$$

as an $\mathrm{Sp}(2g;\mathbb{Q})$*-module.*

References

[1] J. Conant, F. Gerlits, and K. Vogtmann. "Cut vertices in commutative graphs". In: *The Quarterly Journal of Mathematics* 56.3 (Sept. 2005), pp. 321–336.

[2] S. Harako. *The second homology group of the commutative case of Kontsevich's symplectic derivation Lie algebra*. preprint. 2020. arXiv: 2006.06064 [math.AT].

[3] M. Kontsevich. "Rozansky–Witten Invariants via Formal Geometry". In: *Compositio Mathematica* 115.1 (Jan. 1999), pp. 115–127.