Computation of the Knot group and the Symmetric fundamental quandle of Surface-links using a Plat form

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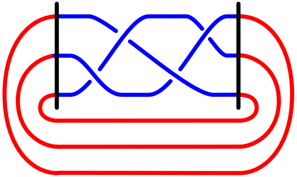
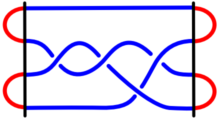
A plat form presentation for Links



The closure of braids



The plat closure of braids

Remark

The plat closure of braids is defined for braids of even degree.

Surface-links and Surface-knots

Definition

F: a *surface-knot* :$\Leftrightarrow$ F: a connected closed surface embedded in $\mathbb{R}^4$.
F: a *surface-link* :$\Leftrightarrow$ F: a disjoint union of surface-knots in $\mathbb{R}^4$.

Definition

F, F': two surface-links.
F and F' are *equivalent* :$\Leftrightarrow$ F is ambient isotopic to F' in $\mathbb{R}^4$.

Remark

In this talk, we discuss in PL category (or smooth category).

Braided surfaces and 2-dimensional braids

$D_1, D_2 \subset \mathbb{R}^2$: 2-dim disks, $y_0 \in \partial D_2$: a fixed based point of D_2.
$\mathrm{pr}_i : D_1 \times D_2 \to D_i$: the i-th factor projection ($i = 1, 2$).
$n \in \mathbb{Z}_{>0}$: a positive integer, $Q_n \subset D_1$: the n points subset of D_1.

Definition (Rudolph, 1983; Viro, 1990)

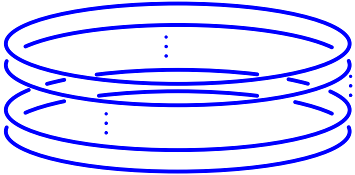
$S \subset D_1 \times D_2$: a *braided surface* of degree n.
:⇔ a surface S satisfies the following conditions.

1. $\pi_S := \mathrm{pr}_2|_S : S \to D_2$: a simple branched covering of degree n,
2. $\partial S \subset D_1 \times \partial D_2$: a closed braid of degree n,
3. $\mathrm{pr}_1(\pi_S^{-1}(y_0)) = Q_n$.

In particular, S: a 2-*dimensional braid* :⇔ $\partial S = Q_n \times \partial D_2$.

A plat form presentation for Surface-links

S: a 2-dim braid of degree $2m$ (i.e. $\partial S = Q_{2m} \times \partial D_2$).
$A \subset \mathbb{R}^4$: a disjoint union of m annuli.
$\widetilde{S} := S \cup A$: a surface-link.



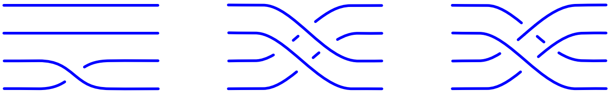
Definition

$\widetilde{S} = S \cup A$: the *plat closure* of a 2-dim braid S.

Theorem

Every orientable surface-link is equivalent to the plat closure of some 2-dim braid.

$K_{2m} := \left\langle \sigma_1,\ \sigma_2\sigma_1\sigma_3\sigma_2,\ \sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}^{-1}\sigma_{2i}^{-1}\ (i = 1, 2, \ldots, m-1) \right\rangle \subset B_{2m}$
: *Hilden's subgroup* of degree $2m$.



S : a braided surface of degree $2m$.
β_S : a braid obtained from ∂S by cutting along $\pi_S^{-1}(y_0)$.

Definition

A braided surface S is *adequate* $:\Leftrightarrow \beta_S \in K_{2m}$.

A plat form presentation for Surface-links

S: an adequate braided surface.
$A \subset \mathbb{R}^4$: a disjoint union of annuli and Möbius bands s.t. $\partial A = \partial S$ (A: of *wicket type*.)
$\widetilde{S} := S \cup A$: a surface-link.

Definition

$\widetilde{S} = S \cup A$: the *plat closure* of a braided surface S.

Definition

$F = \widetilde{S}$: a *plat form* of a surface-link F.

Theorem

Every surface-link is equivalent to the plat closure of some braided surface.

Computation of the Knot group of Surface-links

F: a surface-link, $G(F) = \pi_1(\mathbb{R}^4 \setminus F)$: the knot group of F.
S: a braided surface of degree n.
$G(S) := \pi_1(D_1 \times D_2 \setminus S)$: the knot group of S.
$x_1, \ldots, x_n \subset D_1 \setminus Q_n$: meridional loops of $D_1 \setminus Q_n = (D_1 \times \{y_0\}) \setminus S$.

Proposition (cf. Kamada, 2002)

The knot group $G(S)$ has a following group presentation:

$$G(S) = \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_k \rangle .$$

Theorem

For an adequate braided surface S of degree $2m$, the knot group $G(\widetilde{S})$ has a following group presentation:

$$G(\widetilde{S}) = \left\langle x_1, \ldots, x_{2m} \,|\, r_1, \ldots, r_k,\ x_{2i-1} = x_{2i}^{-1}\ (i = 1, \ldots, m) \right\rangle .$$

Computation of the Knot group of Surface-links

S: an adequate braided surface of degree $2m$, $\widetilde{S} = S \cup A$: the plat closure of S.

Theorem

The knot group $G(\widetilde{S})$ has a following group presentation:

$$G(\widetilde{S}) = \left\langle x_1, \ldots, x_{2m} \,|\, r_1, \ldots, r_k,\ x_{2i-1} = x_{2i}^{-1}\ (i = 1, \ldots, m) \right\rangle.$$

A scketch of an algebraic proof

1. By proposition, we have $G(S) = \langle x_1, \ldots, x_{2m} \,|\, r_1, \ldots, r_k \rangle$.
2. Compute a group presentation of $G(A) := \pi_1((\mathbb{R}^4 \setminus D_1 \times D_2) \setminus A)$ that contains the generators $x_1, \ldots x_{2m}$.
3. Since $\widetilde{S} = S \cup A$, the claim follows by Van Kampen's theorem.

Computation of the Knot group of Surface-links

S: an adequate braided surface of degree $2m$, $\widetilde{S} = S \cup A$: the plat closure of S.

Theorem

The knot group $G(\widetilde{S})$ has a following group presentation:

$$G(\widetilde{S}) = \left\langle x_1, \ldots, x_{2m} \,|\, r_1, \ldots, r_k,\ x_{2i-1} = x_{2i}^{-1}\ (i = 1, \ldots, m) \right\rangle.$$

A scketch of a geometric proof

1. Construct a diagram D of $\widetilde{S}$ by gluing a diagram of S and of A.
2. Consider meridional loops $x_1, \ldots, x_{2m}$ of S in D.
3. Then, they are naturally extended to meridional loops of A with relations $x_{2i-1} = x_{2i}^{-1}$ $(i = 1, \ldots, m)$.

Qunadles and Symmetric quandles

Definition (Joyce, Matveev, 1982)

X: a set, $\quad * : X \times X \to X$: a binary operation

$X = (X, *)$: a *quandle* $:\Leftrightarrow$ $*$ satisfies the following conditions:

1. $\forall x \in X,\ x * x = x$,
2. $\forall x, y \in X, \exists! z \in X$ s.t. $x * z = y$,
3. $\forall x, y, z \in X,\ (x * y) * z = (x * z) * (y * z)$.

Definition (Kamada, 2006)

X: a quandle, $\quad \rho : X \to X$: an involution map (i.e. $\rho \circ \rho = \mathrm{id}_X$).

$X = (X, \rho)$: a *symmetric quandle* $:\Leftrightarrow$ ρ satisfies the following conditions:

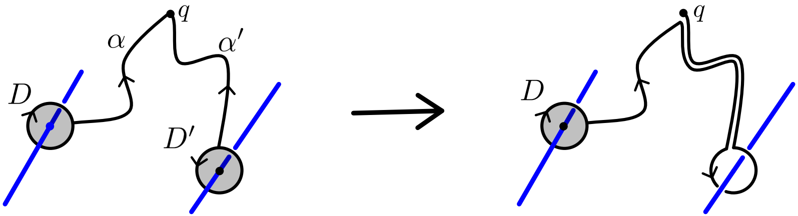
1. $\forall x, y \in X,\ \rho(x * y) = \rho(x) * y$,
2. $\forall x, y \in X,\ x * \rho(y) = x \bar{*} y$.

the Symmetric fundamental quandle of Surface-links

$F \subset \mathbb{R}^4$: a surface-link, $q \in \mathbb{R}^4 \setminus F$: a fixed based point.

$X(F) := \{(D, \alpha) \mid D$: a meridional disk of F, α: a path from ∂D to $q\}/\text{homotopy}$.
$[(D, \alpha)] * [(D', \alpha')] := [(D,\ \alpha \cdot \alpha'^{-1} \cdot \partial D' \cdot \alpha')]$: a binary operation of $X(F)$.
Then, $X(F) := (X(F), *)$ is a quandle (the *full knot quandle* of F).

$\rho : X(F) \to X(F)$; $\rho([(D, \alpha)]) := [(-D, \alpha)]$: an involution map.
$X(F) := (X(F), \rho)$: the *symmetric fundamental quandle* of F.



Main Theorem

F: a surface-link, $X(F)$: the symmetric fundamental quandle of F.

Theorem

S : an adequate braided surface of degree $2m$.
Then, $X(\widetilde{S})$ has a following symmetric quandle presentation:

$$X(\widetilde{S}) = \langle x_1, \dots, x_{2m} \,|\, r_1, \dots, r_k,\ x_{2i-1} = \rho(x_{2i})\ (i = 1, \dots, m) \rangle_{\mathrm{sq}} .$$