# Coloring links by Symmetric group of order 3

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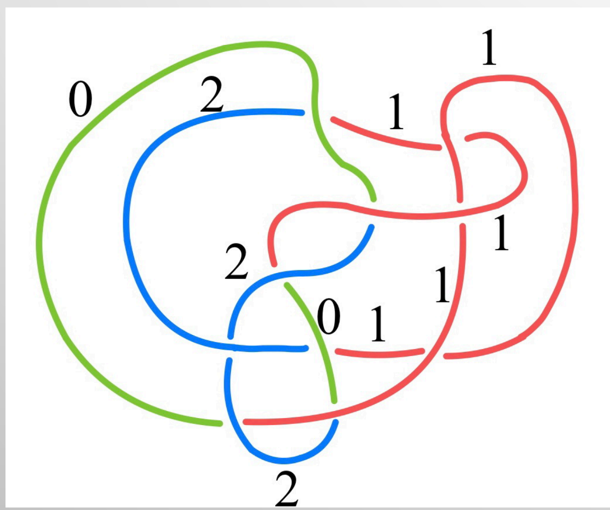
（Joint work with Eri Matsudo, Nihon Univ.）

# Introduction

One of the most well-known invariants of knots and links would be the Fox 3-coloring, originally introduced by R. Fox.

6. Let us say that a knot diagram has property $l$ if it is possible to color the projected overpasses in three colors, assigning a color to each edge in such a way that (a) the three overpasses that meet at a crossing are either all colored the same or are all colored differently; (b) all three colors are actually used.



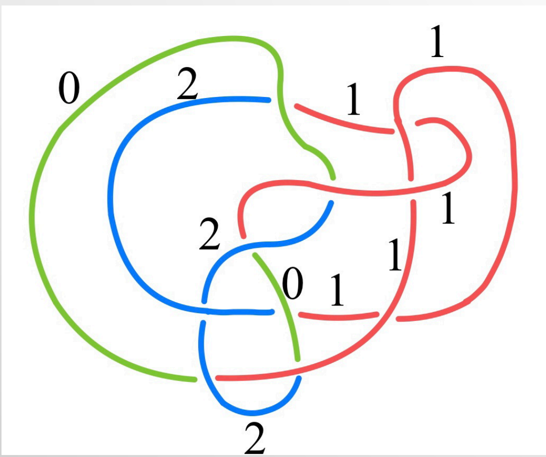

Chap.2 Presentation of a knot group, Exercise 6, p.92

**Crowell, Richard H.**; **Fox, Ralph H.** Introduction to knot theory. Based upon lectures given at Haverford College under the Philips Lecture Program *Ginn and Co., Boston, Mass.* 1963 {\rm x}+182 pp. **MR0146828**

# Introduction

EXERCISES 93

Show that a diagram of a knot $K$ has property $l$ if and only if the group of $K$ can be mapped homomorphically *onto* the symmetric group of degree 3.

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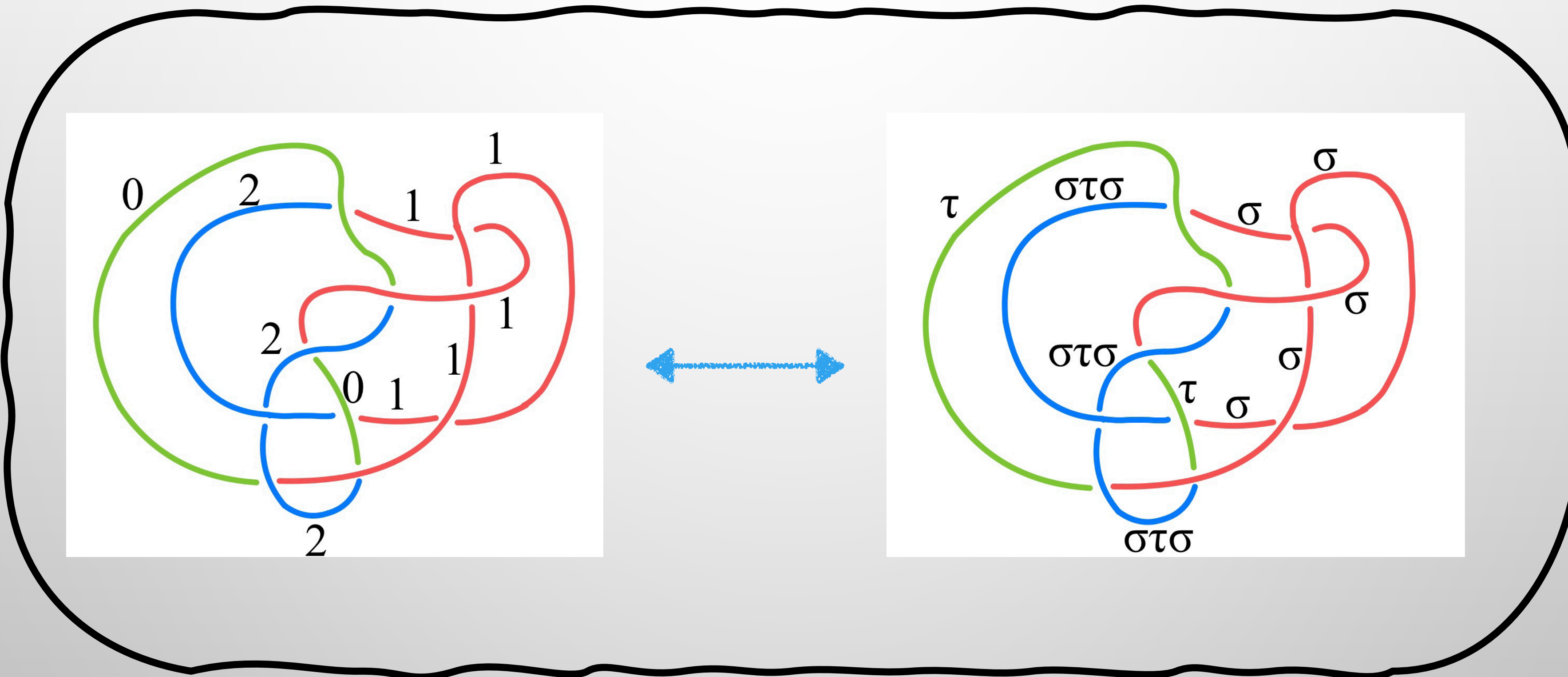
As an exercise, we see that a 3-coloring on a diagram of a knot $K$ corresponds to a representation of the knot group $\pi_1(S^3 - K)$ onto the symmetric group of degree 3.

# Introduction

We set; $S_3 = \langle \sigma, \tau \mid \sigma^2 = \tau^2 = e, \sigma\tau\sigma = \tau\sigma\tau \rangle$ ( $e$ : the identity element)
Note that $S_3 = \{e, \sigma, \tau, \sigma\tau\sigma, \sigma\tau, \tau\sigma\}$ as a set.



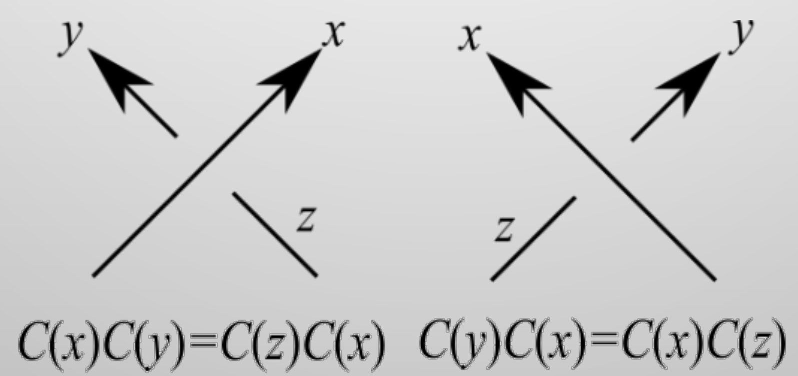
# Introduction

We consider colorings on links by the symmetric group $S_3$ of order 3.

**Definition 1**
Let $D$ be an oriented diagram of a link $L$.
A map $C$:{arcs of $D$} $\to \{S_3 - e\}$ is called an **$S_3$-coloring** on $D$ if it satisfies the following conditions at each crossings.

① On a positive crossing; $C(x)C(y) = C(z)C(x)$

② On a negative crossing; $C(y)C(x) = C(x)C(z)$

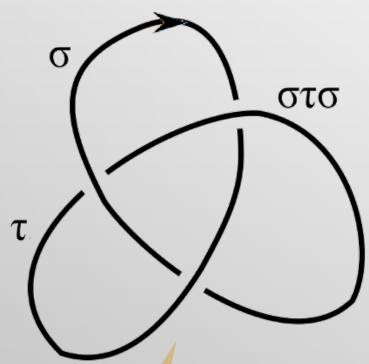


# Introduction

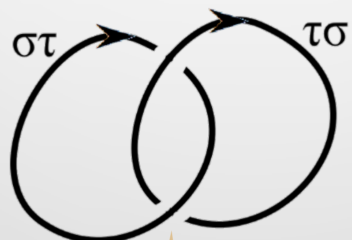
<u>**Definition 2**</u>

① A link $L$ is **$S_3$-colorable** if ∃diagram of $L$ admits a non-trivial $S_3$-coloring, i.e., an $S_3$-coloring with at least two colors.

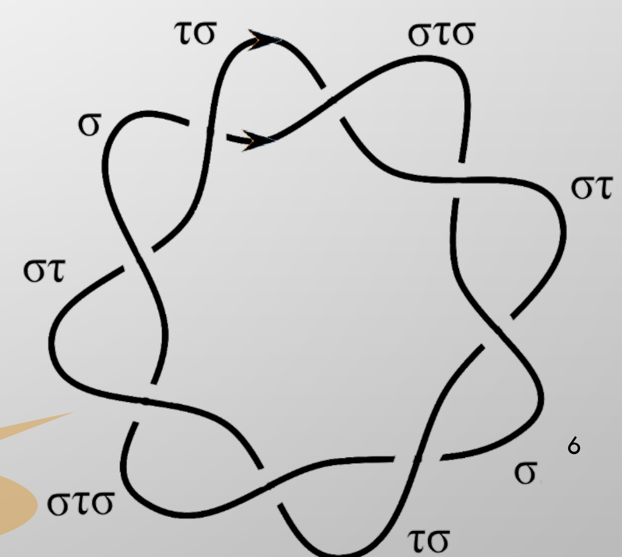
② A link $L$ is **$(S_3,\ n)$-colorable** if ∃diagram of $L$ admits an $S_3$-coloring with just $n$ colors.




$(S_3, 3)$-colorable




$(S_3, 2)$-colorable




$(S_3, 4)$-colorable

# Introduction

**Remark**

For a knot, $\exists$ one-to-one correspondence between

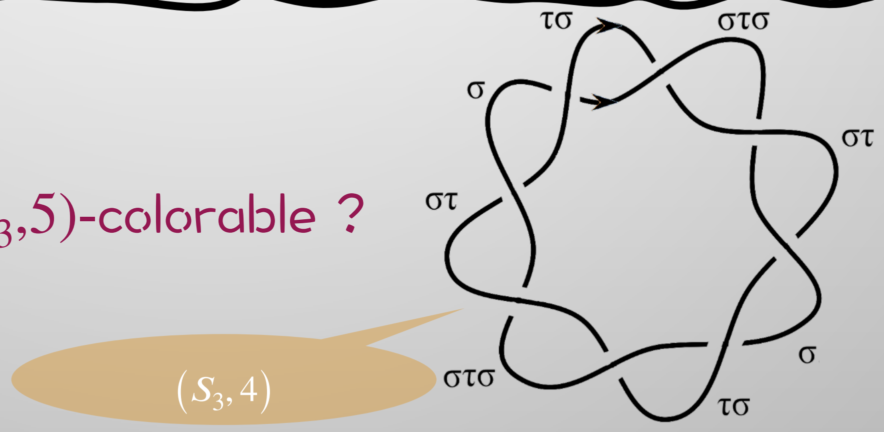
a non-trivial Fox 3-coloring & an $(S_3, 3)$-coloring.

$\Rightarrow$ a knot $K$ is $S_3$-colorable if and only if $K$ is Fox 3-colorable.

In particular, if a knot is $(S_3, n)$-colorable, then $n = 1$ or $3$.

On the other hand, if a link $L$ has at least 2 components, then $L$ can be $(S_3, n)$-colorable with $n \geq 4$.

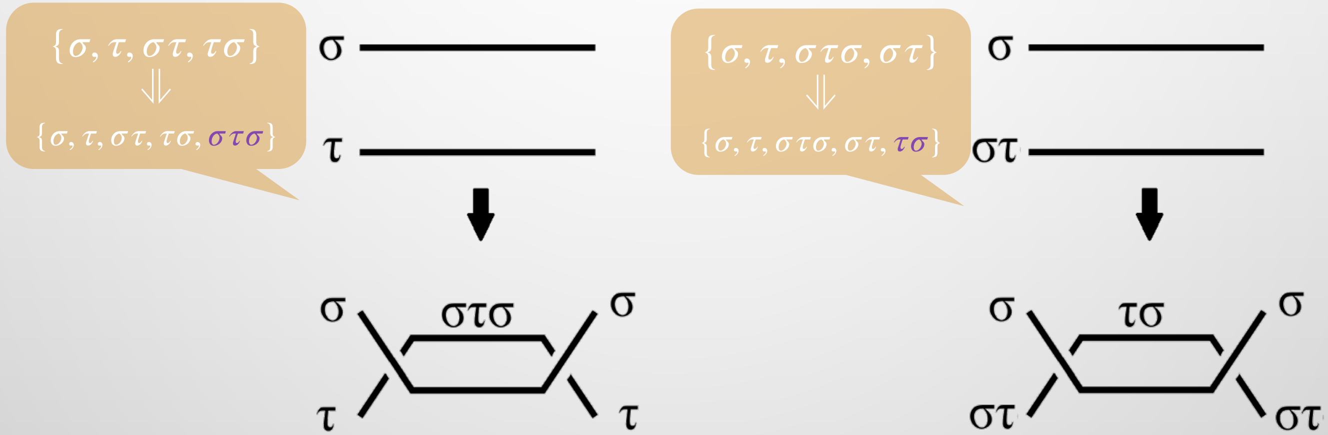
Q. Is this $(S_3,5)$-colorable ?



# Introduction

**Proposition**

Any $(S_3,\ 4)$-colorable link is also $(S_3,\ 5)$-colorable.




**Question**

Is an $(S_3,\ 5)$-colorable link $L$ always $(S_3,\ 4)$-colorable?

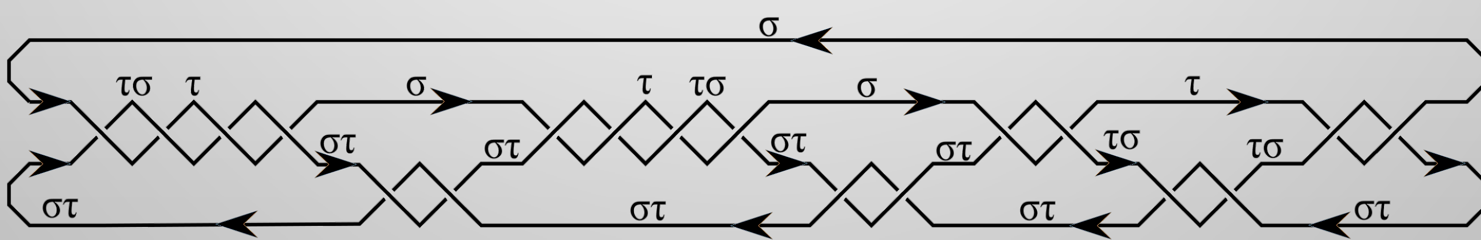
# Main theorem

**Theorem 1 [I. – Matsudo]**

Any $(S_3,\ 5)$-colorable 2-bridge link is also $(S_3,\ 4)$-colorable.

In fact, a 2-bridge link $L$ is $(S_3,\ 4)$-colorable if and only if $L$ has a Conway diagram $D = C(2a_1,\ 2b_1,\ 2a_2,\ 2b_2, \dots ,\ 2b_m\ ,\ 2a_{m+1})$ such that $D$ satisfies $\sum_{i=1}^{m+1} |\, a_i \,| \equiv 0 \pmod 2$.

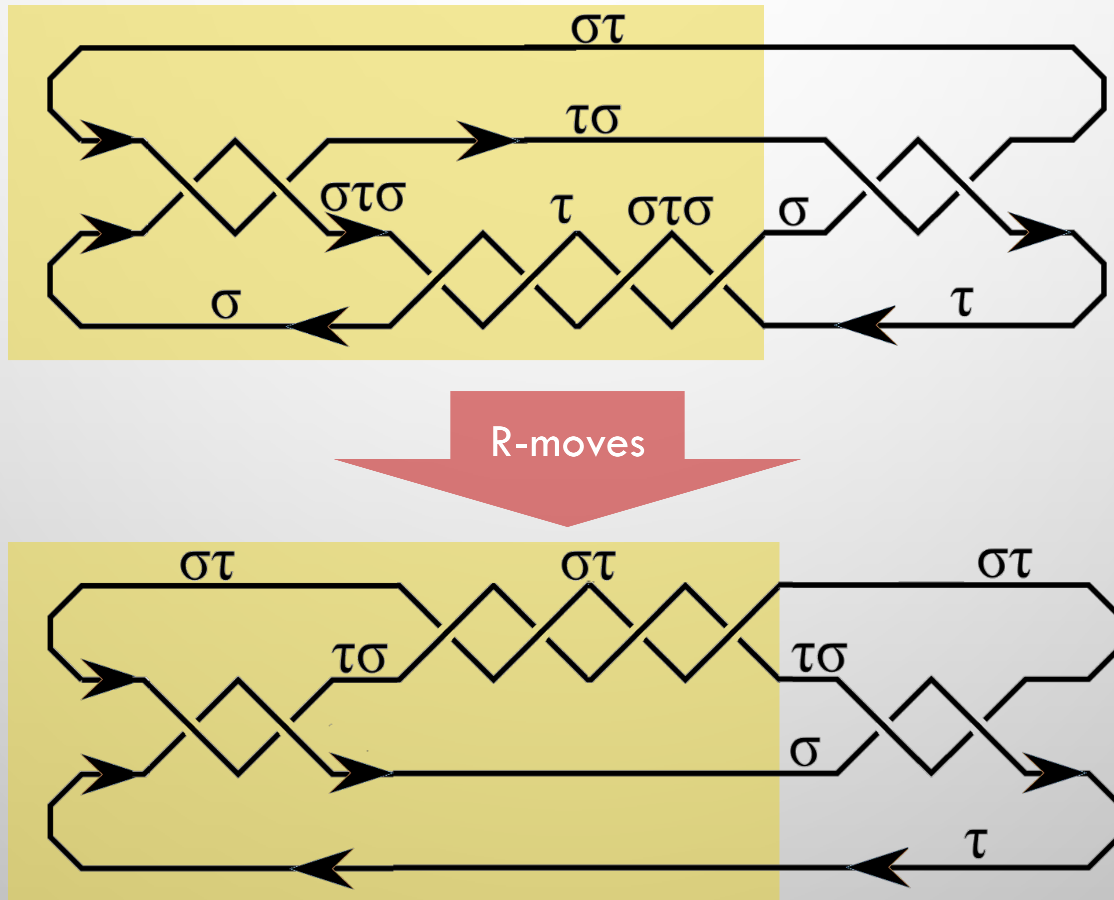
(Example 1) $C[4, -2, -4, 2, 2, -2, -2]$



# Main theorem

**[Key of proof ]** ($(S_3,\ 5)$-colorable $\Rightarrow$ $(S_3,\ 4)$-colorable)

$C[2,4,\ 2]$



# Examples

By Theorem 1, all the $(S_3,\ 5)$-colorable 2-bridge links are $(S_3,\ 4)$-colorable. Some of them actually are also $(S_3,\ 3)$-colorable, but some others are not.

$T(2,q)$ : a torus link

$T(2,q)$ is $(S_3,\ 4)$-colorable but not $(S_3,\ 3)$-colorable

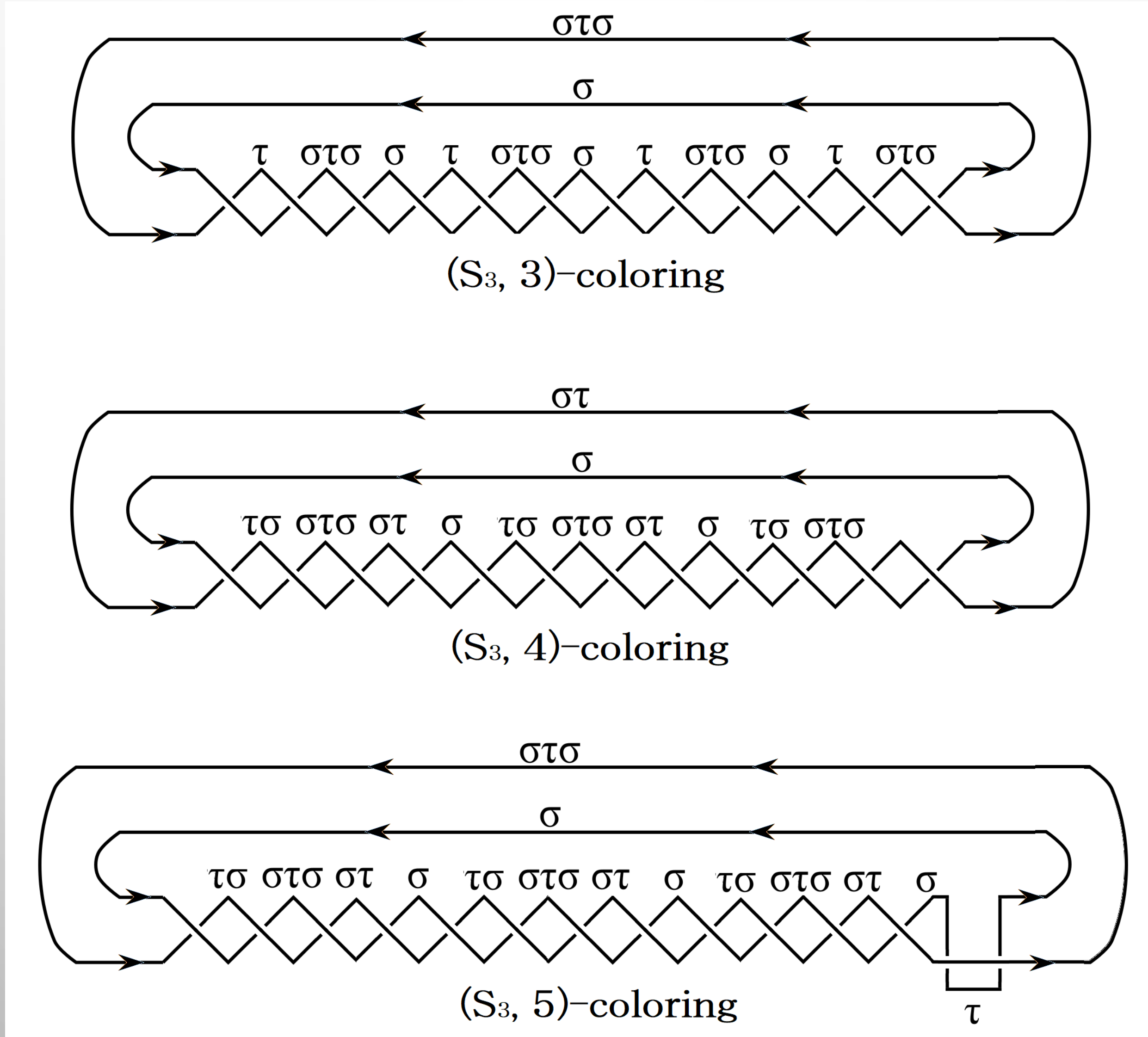
$\Leftrightarrow q \equiv 0\ (mod\ 4)$ and $q \not\equiv 0\ (mod\ 3)$

$J(k,l)$ : a double twist link

$J(k,l)$ is $(S_3,\ 4)$-colorable but not $(S_3,\ 3)$-colorable

$\Leftrightarrow kl \equiv 3 (mod\ 4)$ and $kl \not\equiv 2\ (mod\ 3)$

# Examples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(S3, 4)–coloring
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(S3, 5)–coloring
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Thank you
for your attention.