Asymptotic dimension of planes

Introduction to asymptotic dimension

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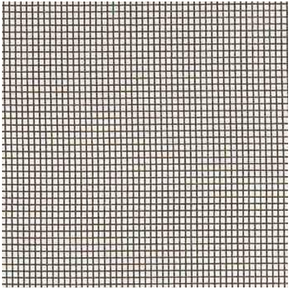
East Asia Geometric Topology conference. 19 Jan. 2022
(Zoom)

Dimension in the large scale



- What is the dimension of this object? 1, 2, 0 ?
- Gromov defined "dimension in the large scale", called asymptotic dimension.

Quasi-isometry

- Let $f : X \to Y$ be a map between two metric spaces. f is a (K, L)-quasi-isometric embedding if for all $x, y \in X$,

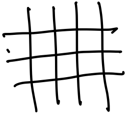
$$\frac{|x-y|}{K} - L \leq |f(x) - f(y)| \leq K|x-y| + L \qquad (1)$$

- In addition, if for any $y \in Y$ there is $x \in X$ such that $|y - f(x)| \leq L$, then f is called a (K, L)-quasi-isometry, and we say X is quasi-isometric (QI) to Y.

For example,

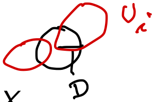
$f : X \to pt$
$L = diam\ X$

1. A bounded set X is QI to a point.
2. The $\mathbb{Z}^2$ grid graph (each edge has length 1) is QI to $\mathbb{E}^2$.



$\underset{QI}{\hookrightarrow} \mathbb{E}^2$

Asymptotic dimension



- Let X be a metric space. Let $\mathcal{U} = \{U_i\}$ be a cover of X.
- For $D > 0$, we say the *D-multiplicity* of the cover is (at most) n if any D-ball in X intersects at most n elements in $\mathcal{U}$.
- **Def.** The *asymptotic dimension* of X, $\operatorname{asdim} X$, is at most n if for any $D > 0$ there is a cover $\mathcal{U}$ of X such that
  (1) the D-multiplicity is at most $n + 1$.
  (2) There is B such that for all $U \in \mathcal{U}$, $\operatorname{diam} U \leq B$.
- If such n does not exist, then $\operatorname{asdim} X = \infty$. $\operatorname{asdim} X = n$ if $\operatorname{asdim} X \leq n$ and $\operatorname{asdim} X \not\leq n - 1$.

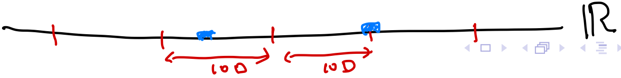
For example,

1. If X is bounded, then $\operatorname{asdim} X = 0$.
2. $\operatorname{asdim} \mathbb{R} = 1$.

$\mathcal{U} = \{X\}$
D-mult = 1
$B = \operatorname{diam} X$

D-mult = 2
$\Rightarrow \operatorname{asdim} \mathbb{R} \leq 1$

$D>0$



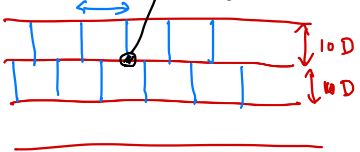
Examples

asdim $\mathbb{E}^2 \leq 2$

$D > 0$

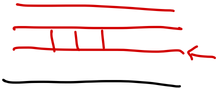
D-mult $= 3$

$\Rightarrow$ asdim ≤ 2

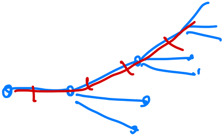


1. asdim $\mathbb{E}^n = n$.
2. asdim $\mathbb{H}^n = n$.
3. asdim of a tree is ≤ 1. ✓

asdim $\mathbb{H}^2$



$D > 0$



$\mathbb{R}$

Basic properties of asymptotic dimension

1. If X is QI to Y then $\operatorname{asdim} X = \operatorname{asdim} Y$.
2. If $Y \subset X$ then $\operatorname{asdim}(Y, d|_Y) \leq \operatorname{asdim}(X, d)$, more generally, if $f : Y \to X$ is a QI-embedding, then $\operatorname{asdim} Y \leq \operatorname{asdim} X$.
3. $\operatorname{asdim}(X \times Y) \leq \operatorname{asdim} X + \operatorname{asdim} Y$. On $X \times Y$, we put, for example, the ℓ^1-metric, ie, $d_X + d_Y$.

Some motivation

- Method to show $\operatorname{asdim} X < \infty$ or obtain a bound on $\operatorname{asdim} X$ is sometimes interesting.
- One can define asymptotic dimension for a finitely generated group G using its Cayley graph, $Cay(G)$,

$Cay(\mathbb{Z}^2)$ 

$$\operatorname{asdim} G = \operatorname{asdim} Cay(G).$$

(1) If G is finitely presented and $\operatorname{asdim} G < \infty$ then the Novikov conjecture holds for G (Yu).
(2) $MCG(S)$, the mapping class group of a compact surface S has $\operatorname{asdim} MCG(S) < \infty$ (Bestvina-Bromberg-F).
(3) Unknown if $\operatorname{asdim} Out(F_n) < \infty$?

Sample results on graphs associated to surfaces

There are many graphs associated to a surface S. They tend to be hyperbolic in the sense of Gromov.
Let $S_{g,b}$ be the compact orientable surface of genus g with b boundary components.

$\partial\mathcal{C}(S)$

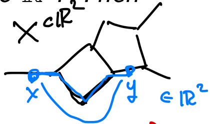
- Let $\mathcal{C}(S_{g,b})$ be the curve graph of $S_{g,b}$.
  asdim $\mathcal{C}(S) < \infty$ (Bell-F)
  asdim $\mathcal{C}(S_{g,b}) \leq 4g + b - 3$ (Bestvina-Bromberg)
- Let $\mathcal{A}(S_{g,b})$ be the arc graph of $S_{g,b}$.
  asdim $\mathcal{A}(S_{g,b}) \leq \frac{(4g+b)(4g+b-3)}{2} - 2$. (Schleimer-F).
- Let $\mathcal{D}(M, S_{g,b})$ be the disk graph for a compression body M with an upper boundary S (for example, M is a handlebody with $\partial M = S$).
  asdim $\mathcal{D}(M, S_{g,b}) \leq \frac{(4g+b)(4g+b-3)}{2} - 2$. (Schleimer-F).
  cf. let $\mathcal{D}(H_g)$ be the disc graph of the genus g handle body.
  asdim $\mathcal{D}(H_g) \leq 24(g-1)^2$ (Hamenstadt).

Asymptotic dimension of planes and planar graphs

Theorem (F-Papasoglu)

(1) *Let X be a geodesic space that is homeomorphic to $\mathbb{R}^2$. Then* $\operatorname{asdim} X \le 3$.

$X \subset \mathbb{R}^2$



(2) *Let X be a planar graph. Then* $\operatorname{asdim} X \le 3$.

$X \underset{\text{inj}}{\hookrightarrow} (\mathbb{R}^2, \exists d)$

QI-embed.

$\operatorname{asdim} X \le \operatorname{asdim} \mathbb{R}^2 \le 3$

Remark

(i) By now, Lang-Jorgensen improved it to $\operatorname{asdim} X \le 2$.

(ii) (2) easilly follows from (1) by embedding X to $\mathbb{R}^2$.

(iii) For each $n = 0, 1, 2, \cdots, \infty$, $\mathbb{R}^3$ has a (Riemannian) metric d such that $\operatorname{asdim}(\mathbb{R}^3, d) = n$.

$\operatorname{asdim}(\mathbb{R}^3, \exists d) \ge 100$

$\mathbb{E}^{100} \underset{QI}{\sim} \mathbb{Z}^{100} = \underset{\text{graph}}{\Gamma} \underset{\text{inj}}{\hookrightarrow} \mathbb{R}^3$ — QI-embed

asdim $= 100$

Proof of the theorem

asdm $(\mathbb{R}^2, d) \leq 3$

Proof of Thm (1). Given $D > 0$, we want to find a cover $\mathcal{U}$ of X by uniformly bounded sets, whose D-multiplicity ≤ 4.
Fix a base point $e \in X$. Define annuli

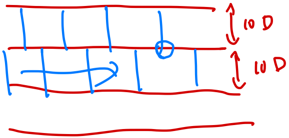
$$A_n = \{x \in X | 10D(n-1) \leq d(e,x) < 10Dn\}$$

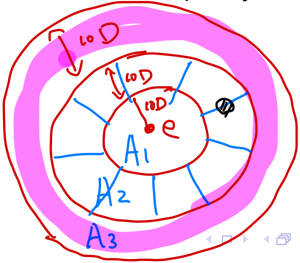
Then $X = \cup_{n \in \mathbb{N}} A_n$.

We will find a cover of each A_n by uniformly (over n as well) bounded sets, of D-multiplicity 2, ie, asdim $A_n \leq 1$, "uniformly".
This gives a desired cover of X, of D-multiplicity 4.

asdm $\mathbb{E}^2 \leq 2$

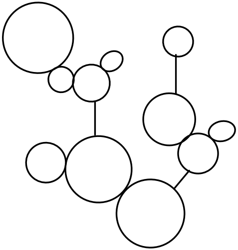
$D > 0$





Cactus

- A cactus is a graph/geodesic space such that a segment is contained in at most one circle.



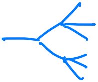
- Similar to a tree, if X is a cactus, then $\operatorname{asdim} X \leq 1$, uniformly (over all cacti).
- If X is QI to a cactus, then $\operatorname{asdim} X \leq 1$.

Lemma (Annuli are cacti)

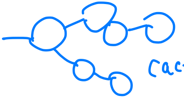
The annuli A_n are uniformly QI to cacti, so that $\operatorname{asdim} A_n \leq 1$*, uniformly.*

Remark. Strictly speaking we are looking at a connected component of each annulus.

The lemma implies $\operatorname{asdim} X \leq 3$. $\square$



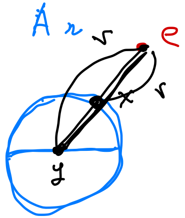
asdm ≤ 1

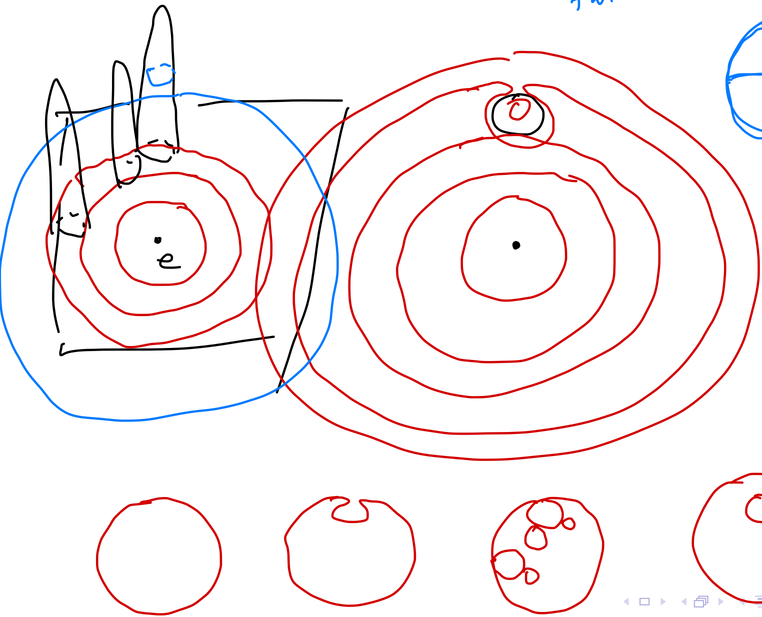


asdm ≤ 1

cactus

What do annuli look like?





Some questions

1. Can we characterize a geodesic space X with $\operatorname{asdim} X = 1$?
   If in addition, $H_1(X)$ is uniformly generated (ie, there is $L > 0$ such that $H_1(X)$ is generated by a loop of length $\leq L$), then X is QI to an unbounded tree (F-Whyte).
2. It follows that if a f.p. group G has $\operatorname{asdim} G = 1$, then it is virtually free. But there is a f.g. group G of $\operatorname{asdim} G = 1$, which is not virtually free, for example, the "lamplighter group, which is the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}$. Known examples are with torsions.
   Is it true that a torsion free, f.g. group G of $\operatorname{asdim} G = 1$ is always a free group?
3. Let K_m be the complete graph with m vertices. If a graph Γ with finite degrees excludes K_m as a minor, then $\operatorname{asdim} \Gamma \leq 4^m - 1$ (Ostrovskii-Rosenthel). In particular a planar graph satisfies asdim $\leq 4^5 - 1$.
   Can we find a QI-invariant condition for a graph Γ to have $\operatorname{asdim} \Gamma < \infty$?