Chord indices for knots in thickened surfaces

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This is a joint work with Hongzhu Gao and Mengjian Xu.

Chord index

- D an oriented knot diagram, $C(D)$ the set of crossing points.
- R_i $(1 \leq i \leq 3)$ denotes the i-th Reidemeister move
- It is quite easy to count the **crossing number**

$$c(D) = \sum_{c \in C(D)} 1.$$

  Unfortunately, it is not a knot invariant, since it is not preserved under R_1, R_2.
- Assigning crossing point c a sign $w(c) \in \{\pm 1\}$, the **writhe**

$$w(D) = \sum_{c \in C(D)} w(c)$$

  behaves much better. It is invariant under R_2, R_3, but not R_1.

In order to obtain a knot invariant by counting crossing points, we need to equip each crossing point with something extra.

Definition (C. 2016, inspired by Manturov's parity axioms)

For a set A, a **chord index** is a function

$$C(D) \to A,$$

which satisfies

1. If c is involved in R_1, then $f(c) = \tau$, a fixed element of A;
2. If c_1 and c_2 are involved in R_2, then $f(c_1) = f(c_2)$;
3. If c_1, c_2 and c_3 are involved in R_3, then $f(c_i)$ are preserved;
4. The index of any crossing point not involved in a Reidemeister move is invariant under this move.

Theorem (C. 2016)

Let D be a knot diagram and $f : C(D) \to A$ *a chord index, the formal sum*

$$\sum_{f(c) \neq \tau} w(c) f(c)$$

is invariant under Reidemeister moves, therefore it is a knot invariant.

Example (C. 2016)

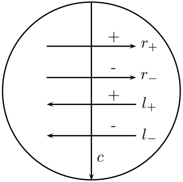
- D a knot diagram representing K, BQ_K the knot biquandle.
- $(BQ, *, \circ)$ a finite biquandle, G a group.
- Given $\phi : BQ \times BQ \to G$ satisfies $\forall x, y, z \in BQ$
    1. $\phi(x, x) = 1$;
    2. $\phi(x, y) = \phi(x * z, y * z)$;
    3. $\phi(y, z) = \phi(y \circ x, z \circ x)$;
    4. $\phi(x, z) = \phi(x * y, z \circ y)$.
- $\forall c \in C(D)$, the element

$$f(c) = \sum_{f : BQ_K \to BQ} \phi(f(x_c), f(y_c)) \in \mathbb{Z}G$$

is a chord index.

Example (CG2013, D2013, IKL2013, K2013, ST2014)

- D a virtual knot diagram, $G(D)$ the corresponding Gauss diagram.
- $\forall c \in C(D)$, we define four integers $r_+(c), r_-(c), l_+(c), l_-(c)$



- The integer

$$\mathrm{Ind}(c) = r_+(c) - r_-(c) - l_+(c) + l_-(c)$$

is a chord index.

- This leads to the virtual knot invariant **writhe polynomial**

$$W_K(t) = \sum_{\mathrm{Ind}(c) \neq 0} w(c) t^{\mathrm{Ind}(c)}.$$

Chord indices for knots in thickened surfaces

- Σ a closed oriented surface.
- K a knot embedded in $\Sigma \times [-1, 1]$, which provides a natural knot diagram D in Σ $(= \Sigma \times 0)$.
- $\mathcal{C}_{\Sigma}(D)$ the set of $\mathbb{Z}$-valued chord indices of D, which is an abelian group under the addition $(f_1 + f_2)(c) = f_1(c) + f_2(c)$.

Question

How to construct an element in $\mathcal{C}_{\Sigma}(D)$?

Theorem (C-Gao-Xu 2021)

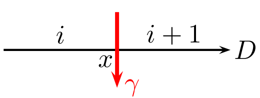
Let D be an oriented knot diagram on a closed oriented surface Σ, we define

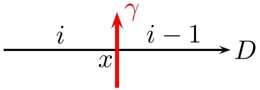
$$\mathcal{H}_1^D(\Sigma, \mathbb{Z}) = \{\alpha \in H_1(\Sigma, \mathbb{Z}) | \alpha \cdot [D] = 0\}.$$

Then there exists a homomorphism $h : \mathcal{H}_1^D(\Sigma, \mathbb{Z}) \to \mathcal{C}_\Sigma(D)$.

The construction of the homomorphism $h : \mathcal{H}_1^D(\Sigma, \mathbb{Z}) \to \mathcal{C}_\Sigma(D)$

- D an oriented knot diagram in Σ
- γ a closed oriented curve on Σ such that $[\gamma] \in \mathcal{H}_1^D(\Sigma, \mathbb{Z})$
- Perturb γ so that D meets γ transversally at n points, then $D \backslash (D \cap \gamma)$ consists of n parts.
- Assign an integer to each part as follows





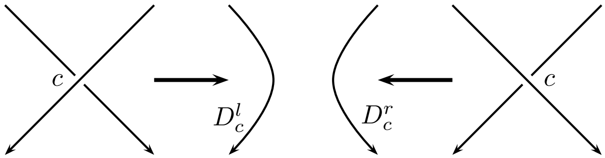
- The definition of $\mathcal{H}_1^D(\Sigma, \mathbb{Z})$ guarantees this coloring is well-defined modulo the addition of a constant integer.

Definition (Motivated by Boden-Rushworth's $\mathscr{C}$-parity)

Choose a closed curve $\gamma \subset \Sigma$ with $[\gamma] \in \mathcal{H}_1^D(\Sigma, \mathbb{Z})$, we define a map $f_\gamma : C(D) \to \mathbb{Z}$ which assigns the difference between the integer on the over-arc and the integer on the under-arc to each crossing point.

Another interpretation

- $\forall c \in C(D)$, by smoothing c according to the orientation we obtain two knot diagrams D_c^l and D_c^r.



- Now we have

$$f_\gamma(c) = w(c)([\gamma] \cdot [D_c^r]) = -w(c)([\gamma] \cdot [D_c^l]).$$

Proposition

The assignment f_γ *is a chord index, and* $h([\gamma])(c) = f_\gamma(c)$ *is the desired homomorphism.*

Corollary

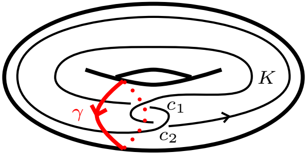
Let K *be an oriented knot in* $\Sigma \times [-1,1]$*, the writhe polynomial of* K *with respect to* γ

$$W_K^\gamma(t) = \sum_{f_\gamma(c) \neq 0} w(c) t^{f_\gamma(t)}$$

is a knot invariant.

Example

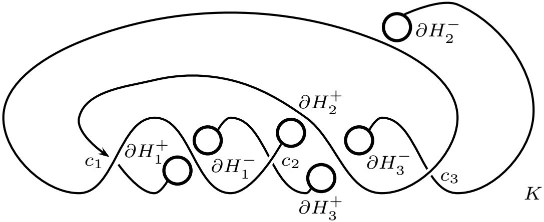
Consider a knot $K \subset T^2 \times [-1,1]$ and $\gamma \in \mathcal{H}_1^K(T^2, \mathbb{Z})$,



- $f_\gamma(c_1) = -1$ and $f_\gamma(c_2) = 1$;
- $W_K^\gamma(t) = -t - t^{-1}$.

Example

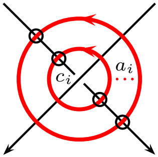
Consider a knot $K \subset \Sigma_3 \times [-1,1]$ and $\gamma \in \mathcal{H}_1^K(\Sigma_3, \mathbb{Z})$,



- $f_\gamma(c_1) = f_\gamma(c_2) = 1$ and $f_\gamma(c_3) = -2$;
- $W_K^\gamma(t) = 2t + t^{-2}$;
- K is non-invertible and chiral.

Applications in virtual knot theory

- Recall that {virtual knots}={knots in $\Sigma \times [-1, 1]$}/{surface homeomorphisms+(de)stabilizations}.
- Due to the stabilization, for each crossing point c_i and arbitrary integer a_i, there exists an element $[\gamma] \in \mathcal{H}_1^D(\Sigma, \mathbb{Z})$ such that $f_\gamma(c_i) = a_i$.



Theorem (Kuperberg 2003)

Every stable equivalence class of knots in thickened surfaces has a unique irreducible representative.

As a corollary, we have

Corollary

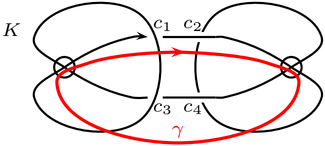
Let K be a virtual knot which possesses a irreducible representative in thickened surface $\Sigma \times I$, then the set of writhe polynomials $\{W_K^\gamma(t) | [\gamma] \in \mathcal{H}_1^K(\Sigma, \mathbb{Z})\}$ is a virtual knot invariant.

Remark

For an oriented knot K in $\Sigma \times I$, choose a closed curve γ on Σ such that $[\gamma] = [K]$, then $W_K^\gamma(t) = W_K(t)$.

Example

- Consider the Kishino knot $K \subset \Sigma_2 \times [-1, 1]$, which has supporting genus 2 (Dye-Kauffman 2005);
- Choose $\gamma \in \mathcal{H}_1^K(\Sigma_3, \mathbb{Z})$;



- $f_\gamma(c_1) = f_\gamma(c_2) = 0$, $f_\gamma(c_3) = -1$ and $f_\gamma(c_4) = 1$;
- $W_K^\gamma(t) = -t - t^{-1}$.

Thank you!