

# First order rigidity of homeomorphism groups of manifolds

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ABSTRACT. For every compact, connected manifold  $M$ , we prove the existence of a sentence  $\phi_M$  in the language of groups such that the homeomorphism group of another compact manifold  $N$  satisfies  $\phi_M$  if and only if  $N$  is homeomorphic to  $M$ . We prove an analogous statement for groups of homeomorphisms preserving Oxtoby–Ulam probability measures.

## CONTENTS

1. Introduction	1
2. Preliminaries	6
3. The AGAPE structure and basic observations	13
4. Interpretation of second-order arithmetic	27
5. Interpretation of points	30
6. Balls with definable parametrizations	36
7. Parametrization of collar neighborhoods	43
8. Completing the proof	49
9. Further questions	53
Acknowledgements	53
References	54

## 1. INTRODUCTION

By a *manifold*, we mean a second countable, metrizable topological space each point of which has a closed neighborhood homeomorphic to a fixed closed Euclidean ball. In particular, a manifold is allowed to have boundary. For a manifold  $M$  (possibly equipped with a measure  $\mu$ ), we let  $\text{Homeo}(M)$  and  $\text{Homeo}_\mu(M)$

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denote the homeomorphism group of  $M$  and its  $\mu$ -preserving subgroup, respectively. We denote by  $\text{Homeo}_0(M)$  and  $\text{Homeo}_{0,\mu}(M)$  the identity components of  $\text{Homeo}(M)$  and  $\text{Homeo}_\mu(M)$ , respectively. For topological spaces  $X$  and  $Y$ , We write  $X \cong Y$  if  $X$  and  $Y$  are homeomorphic.

Let us denote by  $\mathcal{M}$  the class of all pairs  $(M, G)$ , where  $M$  is a compact, connected manifold and  $G$  is a group satisfying

$$\text{Homeo}_0(M) \leq G \leq \text{Homeo}(M).$$

We also let  $\mathcal{M}_{\text{vol}}$  denote the class of all  $(M, G)$  where  $M$  is further assumed to be equipped with some *Oxtoby–Ulam measure*  $\mu$  (that is, a nonatomic Borel probability measure having full support and assigning measure zero to the boundary), and  $G$  is a group satisfying

$$\text{Homeo}_{0,\mu}(M) \leq G \leq \text{Homeo}_\mu(M).$$

Note that in this case, we have

$$\text{Homeo}_{0,\mu}(M) = \text{Homeo}_0(M) \cap \text{Homeo}_\mu(M);$$

cf. [17].

*Remark 1.1.* In statements that apply to both of the classes  $\mathcal{M}$  and  $\mathcal{M}_{\text{vol}}$ , we will often use the notation  $\mathcal{M}_{(\text{vol})}$ ; in such a statement, the choices of formulae may differ, even when the formulae share the same names.

We will later modify the definitions of the classes  $\mathcal{M}_{(\text{vol})}$  slightly so that only manifolds of dimension at least two are considered; see the remark at the end of Section 3.3.

The goal of this paper is to investigate, under the assumption that  $(M, G) \in \mathcal{M}_{(\text{vol})}$ , the extent to which the elementary theory of  $G$  determines the homeomorphism type of  $M$ .

**1.1. Elementary equivalence implies homeomorphism.** It is a classical result of Whittaker that the homeomorphism group of a compact manifold determines the homeomorphism type of the underlying manifold in the following sense.

**Theorem 1.2** (See [44]). *Let  $M$  and  $N$  be compact manifolds, and suppose*

$$\phi: \text{Homeo}(M) \longrightarrow \text{Homeo}(N)$$

*is an isomorphism of groups. Then there exists a homeomorphism*

$$\psi: M \longrightarrow N$$

*such that for all  $f \in \text{Homeo}(M)$ , we have  $\phi(f) = \psi \circ f \circ \psi^{-1}$ .*

Whittaker's result has been generalized by a number of authors; see Chapter 3 of [23] for a survey. For instance, combining the work of Bochner–Mongomery [5] on Hilbert's eighth problem and of Takens on smooth conjugation between diffeomorphisms [40] (cf. [18]), one obtains that if  $M$  and  $N$  are smooth and closed, and if the diffeomorphism groups  $\text{Diff}^k(M)$  and  $\text{Diff}^\ell(N)$  are isomorphic as groups, then  $k = \ell$  and each isomorphism between the groups is induced by some  $C^k$ -diffeomorphism between  $M$  and  $N$ .

In the continuous category, a different generalization was given by Rubin. We say a topological action of a group  $G$  on a topological space  $X$  is *locally dense* if for each pair  $(x, U)$  of a point  $x \in X$  and a neighborhood  $U \subseteq X$  of  $x$ , the orbit  $Z$  of  $x$  by the action of the group

$$G[U] := \{g \in G \mid g(y) = y \text{ for all } y \notin U\}$$

is somewhere dense; that is, the closure of  $Z$  has nonempty interior. Rubin's Theorem can be stated as follows.

**Theorem 1.3** ([38]). *Let  $X_1$  and  $X_2$  be perfect, locally compact, Hausdorff topological spaces, and let  $G_i \leq \text{Homeo}(X_i)$  be locally dense subgroups, for  $i \in \{1, 2\}$ . If there exists an isomorphism*

$$\phi: G_1 \longrightarrow G_2,$$

*then there exists a homeomorphism*

$$\psi: X_1 \longrightarrow X_2$$

*such that for all  $g \in G_1$ , we have  $\phi(g) = \psi \circ g \circ \psi^{-1}$ .*

An isomorphism between two groups of homeomorphisms is a rather unwieldy piece of data. Homeomorphism groups of manifolds are generally much too large to write down, and directly accessing homomorphisms between them is difficult. Therefore in this paper, we are interested in more finitary ways of investigating homeomorphism groups of manifolds, namely through their elementary theories.

We consider the language of groups, which consists of a binary operation (interpreted as the group operation) and a constant (interpreted as the identity element). Models of the theory of groups are just sets with interpretations of the group operation and identity element which satisfy the axioms of groups. We say that two groups  $G_1$  and  $G_2$  are *elementarily equivalent*, written  $G_1 \equiv G_2$ , if a first order sentence in the language of groups holds in  $G_1$  if and only if it holds in  $G_2$ ; this is sometimes expressed as saying that the theories of  $G_1$  and  $G_2$  agree, i.e.  $\text{Th}(G_1) = \text{Th}(G_2)$ . Here, *first order* refers to the scope of quantification, which is allowed to range over elements (as opposed to subsets, relations, or functions).

Our main result is to establish that two compact, connected manifolds have elementarily equivalent homeomorphism groups if and only if the two manifolds are homeomorphic to each other. More strongly, for each compact connected manifold

$M$  we prove the existence of a group theoretic sentence that asserts “I am homeomorphic to  $M$ ”:

**Theorem 1.4.** *For each compact, connected manifold  $M$ , there exists a sentence  $\phi_M^{(\text{vol})}$  in the language of groups such that when  $(N, H) \in \mathcal{M}_{(\text{vol})}$ , we have that*

$$\phi_M^{(\text{vol})} \in \text{Th}(H) \quad \text{if and only if} \quad N \cong M.$$

In other words, the theories of homeomorphism groups of manifolds are *quasi-finitely axiomatizable* within the class  $\mathcal{M}_{(\text{vol})}$ , a property that is stronger than first order rigidity.

In Theorem 1.4, we emphasize that  $M$  and  $N$  are not assumed to have any further structure, such as a smooth or piecewise-linear structure. We thus generalize Whittaker’s result without relying on it, and produce for each manifold a finite, group-theoretic sentence that certifies homeomorphism or non-homeomorphism with the manifold. The sentences  $\phi_M$  and  $\phi_M^{\text{vol}}$  are produced explicitly insofar as is possible, though in practice it would be a rather tedious task to record them. We also note that the connectedness hypothesis for  $N$  can also be dropped from the theorem, thus justifying the claim in the abstract; see Corollary 3.7, for instance.

We remark that Theorem 1.4 fits within a tradition of results proving first order rigidity of various structures, such as lattices in higher rank [2], function fields [13, 14, 42, 33], rings [26, 20], finite-by-abelian groups [29], and linear groups [32], and into the philosophy of distinguishing between objects that are difficult to access directly via finite syntactic objects.

Another perspective on Theorem 1.4 centers around the following question; a number of other motivating questions are enumerated in Section 9.

**Question 1.5.** *Let  $M$  be a compact, connected manifold. Under what conditions is there a finitely generated (or countable) group  $G_M \leq \text{Homeo}(M)$  such that if  $N$  is a compact manifold with  $\dim M = \dim N$  on which  $G_M$  acts faithfully with a dense orbit, then  $M \cong N$ ?*

Related results for actions of the full homeomorphism group of  $M$  are given by Chen–Mann [10]. They show that if the identity component of  $\text{Homeo}(M)$  acts transitively on a connected manifold or CW-complex  $N$ , then  $N$  is homeomorphic to a cover of a configuration space of points of  $M$ . In our context, we have the following immediate consequence of the downward Löwenheim–Skolem Theorem:

**Corollary 1.6.** *To each compact connected manifold  $M$  one can associate a countable group  $G_M \leq \text{Homeo}(M)$  which is elementarily equivalent to  $\text{Homeo}(M)$ , such that for two compact, connected manifolds  $M$  and  $N$  we have*

$$G_M \equiv G_N \quad \text{if and only if} \quad M \cong N.$$

*In particular  $G_M \cong G_N$  if and only if  $M \cong N$ .*

*Remark 1.7.* In [38], there is a cryptic announcement of a version of Theorem 1.4. In particular, Rubin claims that under the assumption  $V = L$  (i.e. Gödel constructibility) that two arbitrary connected manifolds are homeomorphic if and only if their homeomorphism groups are elementarily equivalent; it is likely that he implicitly made a few other assumptions to avoid trivial counterexamples such as  $\text{Homeo}(0, 1) \cong \text{Homeo}[0, 1]$ . To the knowledge of the authors, the paper bearing the title announced in [38] never appeared, and neither did any result (of any authors whatsoever) proving first order rigidity of homeomorphism groups of manifolds; cf. a related MathOverflow post [35]. We note that we only establish the main result for compact manifolds, in contrast to Rubin’s original announcement, and Rubin’s remark about the constructible universe remains puzzling to us; perhaps the goal was to use first order equivalence and promote it to second order equivalence, and then to use  $V = L$  to conclude the second order equivalent structures are isomorphic; cf. [1].

Our proof of Theorem 1.4 largely consists of two parts. The first part is an expansion of the group theory language to a seemingly more powerful language, called  $L_{\text{AGAPE}}$ . The universe of an  $L_{\text{AGAPE}}$  structure corresponding to  $(M, G) \in \mathcal{M}_{(\text{vol})}$  then contains the group  $G$ , the regular open sets  $\text{RO}$ , the real numbers  $\mathbb{R}$ , the maps  $C^0(\mathbb{R}^k, \mathbb{R}^\ell)$  for  $k, \ell \in \omega$  and the subsets of  $\text{int } M$  consisting of isolated points. Since the expansion is specified by first order definitions, we deduce the following, which roughly means that each sentence in the theory of  $\text{AGAPE}(M, G)$  can be translated, in a way that is independent of  $(M, G)$ , to a sentence in the theory of the group  $G$ ; see Section 2 for the precise definition of *uniform interpretation*.

**Theorem 1.8.** *Uniformly for  $(M, G) \in \mathcal{M}_{(\text{vol})}$ , the structure  $\text{AGAPE}(M, G)$  is interpretable in the group structure  $G$ .*

The second part is to show that the AGAPE language can express the sentence that “I am homeomorphic to  $M$ ”:

**Theorem 1.9.** *For each  $(M, G) \in \mathcal{M}_{(\text{vol})}$ , there exists an  $L_{\text{AGAPE}}$ -sentence  $\psi_{M,G}^{(\text{vol})}$  such that for all  $(N, H) \in \mathcal{M}_{(\text{vol})}$ , we have*

$$\psi_{M,G}^{(\text{vol})} \in \text{Th } \text{AGAPE}(N, H) \quad \text{if and only if} \quad N \cong M.$$

By Theorem 1.8, we can interpret  $L_{\text{AGAPE}}$ -sentences  $\psi_{M, \text{Homeo}(M)}$  and  $\psi_{M, \text{Homeo}_\mu(M)}$  to group theoretic sentence  $\phi_M$  and  $\phi_M^{\text{vol}}$  respectively, which distinguish  $M$  from all the other non-homeomorphic manifolds  $N$ ; see Lemma 2.11 for a more formal explanation. We thus obtain a proof of Theorem 1.4.

**1.2. Outline of the paper.** The paper is devoted to proving Theorem 1.4 in several steps, each of which builds on the previous. Section 2 gathers basic results from

geometric topology and model theory, and fixes notation. In Section 3, we introduce the language and the structure of our primary concern, called AGAPE. In this structure, we interpret the regular open sets in  $G$ , and construct formulae that encode various topological properties of regular open sets. Section 4 interprets second order arithmetic using regular open sets and actions of homeomorphisms on them. Section 5 encodes individual points of a manifold, together with the exponentiation map. Section 6 interprets the dimension of a manifold, as well as certain definably parametrized embedded Euclidean balls in it. Section 7 definably parametrizes collar neighborhoods of the boundary of a compact manifold. Section 8 proves Theorem 1.4 by interpreting a result of Cheeger–Kister [9] and by encoding embeddings of manifolds into Euclidean spaces that are “sufficiently near”. We conclude with some closely related questions in Section 9.

## 2. PRELIMINARIES

In this section, we gather some notation, background, and generalities.

**2.1. Transitivity of balls in manifolds.** The high degree of transitivity of the action of homeomorphism groups on balls in manifolds is crucial for this paper. We begin with the following fundamental fact regarding Oxtoby–Ulam measures.

**Theorem 2.1** (von Neumann [43], Oxtoby–Ulam [31]). *If  $\mu$  and  $\nu$  are Oxtoby–Ulam measures on a compact connected manifold  $M$  then there exists a homeomorphism  $h$  of  $M$  isotopic to the identity and fixing the boundary such that  $h_*\mu = \nu$ .*

Thus, for Oxtoby–Ulam measures, the groups of measure-preserving homeomorphisms of  $M$  are all conjugate to each other. In particular, each  $(M, G) \in \mathcal{M}_{\text{vol}}$  corresponds to a measure that is unique up to topological conjugacy. We will therefore refer to groups of *measure-preserving homeomorphisms* without specifying a particular Oxtoby–Ulam measure. We refer the reader to [17, 3] for generalities about measure-preserving homeomorphisms.

A group theoretic interpretation of (certain) balls in a manifold will be a crucial step in this paper. By  $B^n(r)$ , we mean the compact ball of radius  $r > 0$  with the center at the origin in  $\mathbb{R}^n$ . The following lemma is originally due to Brown [7] for the first two parts, and to Fathi [17] for the rest. One may view this as a natural generalization of the fact that every compact connected 2-manifold can be obtained by gluing up the boundary of a polygon in a suitable way. See also [11, 25] for more details.

**Lemma 2.2** ([7, 17]; cf. [11, Chapter 17]). *For each compact connected  $n$ -manifold  $M$ , there exists a continuous surjection*

$$f: B^n(1) \longrightarrow M$$

*such that the following hold:*

- (i) the restriction of  $f$  on  $\text{int } B^n(1)$  is an embedding onto an open dense subset of  $M$ ;
  - (ii) we have  $f(\text{int } B^n(1)) \cap f(\partial B^n(1)) = \emptyset$ , and in particular,  $\partial M \subseteq f(\partial B^n(1))$ ;
- In the case when  $M$  is equipped with an Oxtoby–Ulam measure  $\mu$ , we can further require the following.
- (iii) we have  $\mu(f(\partial B^n(1))) = 0$ ;
  - (iv) The measure  $\mu$  is the pushforward of Lebesgue measure by  $f$ .

The conditions (i) and (ii) already imply that an Oxtoby–Ulam measure on  $M$  exists. For instance, one can pull back the Lebesgue measure on a ball using the surjection  $B^n(1) \rightarrow M$  as in Lemma 2.2. The condition (iv) is also easy to be attained from the previous conditions and Theorem 2.1; see also [19].

For a possibly non-compact manifold, we have the following variation also due to Fathi, which loosens the condition on the surjectivity of the map.

**Lemma 2.3** ([17]). *If a connected  $n$ -manifold  $M$  has nonempty boundary and if  $M$  is equipped with a nonatomic, fully supported Radon measure  $\mu$  that assigns zero measure to  $\partial M$ , then there exists an open embedding*

$$f: \mathbb{H}_+^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\} \rightarrow M$$

such that the following hold:

- (i)  $f(\text{int } \mathbb{H}_+^n) \subseteq \text{int } M$  and  $f(\partial \mathbb{H}_+^n) \subseteq \partial M$ ;
- (ii)  $M \setminus f(\mathbb{H}_+^n)$  is closed and of measure zero.

We call a topologically embedded image of  $B^n(1)$  in a manifold  $M^n$  a *ball*. The same goes for an *open ball* in  $M$ . If there exists an embedding  $h: B^n(2) \rightarrow M$ , then the image  $h(B^n(1))$  is called a *collared ball* [11, Chapter 17]. The same goes for a *collared open ball*. In the case when  $M$  is equipped with an Oxtoby–Ulam measure  $\mu$ , we say a collared ball  $B$  is  $\mu$ -good (or, simply *good*) if  $\partial B$  has measure zero. There exists an arbitrarily small covering of  $M$  by  $\mu$ -good balls [17]. For brevity of exposition, by a *good ball*, we mean both a collared ball in the context of  $(M, G) \in \mathcal{M}$  and a  $\mu$ -good ball in the context of  $(M, G) \in \mathcal{M}_{\text{vol}}$ . The same goes for a *good open ball*. Note that a good ball is always contained in the interior of  $M$ .

Recall the topological action of a group  $G$  on  $\text{int } M$  is *path-transitive* if for all path  $\gamma: I \rightarrow \text{int } M$  and for all neighborhood  $U$  of  $\gamma(I)$  there exists  $h \in G[U]$  such that  $h(\gamma(0)) = \gamma(1)$ . We say the action of  $G$  on  $\text{int } M$  is  *$k$ -transitive* if it induces a transitive action on the configuration space of  $k$  distinct points in  $\text{int } M$ . A path-transitive action on  $\text{int } M$  is always  $k$ -transitive whenever  $\dim M > 1$ ; see [3, Lemma 7.4.1]. Let us note the following fundamental facts on various notions of transitivity in manifolds.

**Lemma 2.4.** [25, Corollaries 2.1 and 2.2] *For  $(M, G) \in \mathcal{M}_{(\text{vol})}$  with  $\dim M > 1$ , we have the following.*

- (1) *The action of  $G$  on  $\text{int } M$  is path-transitive and  $k$ -transitive for all  $k > 0$ .*  
(2) *If  $B_1$  and  $B_2$  are good balls of the same measure in an open connected set  $U \subseteq M$ , then there exists  $g \in G[U]$  such that  $g(B_1) = B_2$ .*

*Proof.* The path-transitivity of part (1) is well-known; see [3, Section 7.7] for  $G = \text{Homeo}_0(M)$ , and [17, p. 85] for  $G = \text{Homeo}_{0,\mu}(M)$ . The  $k$ -transitivity follows immediately.

The case when  $U = M$  in part (2) is precisely given in [25, Corollary 2.2] by Le Roux, based on the Annulus Theorem of Kirby [24] and Quinn [36] as well as the Oxtoby-Ulam theorem. In general, we can exhaust the topological manifold  $U$  by a sequence of compact bounded manifolds  $\{M_i\}$  so that some  $M_i$  contains  $B_1$  and  $B_2$  in its interior; this can be seen from [37], as explained in [34]. We can further require that  $\partial M_i$  has measure zero by countable additivity. Applying Le Roux's argument for  $M_i$ , we obtain the desired transitivity.  $\square$

**Lemma 2.5.** *Let  $M$  be a compact, connected  $n$ -manifold with  $n \geq 2$ , equipped with an Oxtoby-Ulam measure  $\mu$ . If  $U \subseteq M$  is an open connected subset, then for each positive real number  $r < \mu(U)$ , there exists a good ball of measure precisely  $r$  inside  $U$ . Moreover, we may require that  $U \setminus B$  is connected.*

*Proof.* Note the general fact that for a connected open subset  $U$  of  $M$  and for a collared ball  $B$  in  $U \cap \text{int } M$ , the set  $U \setminus B$  is connected; this can be seen from that a collared ball is cellular, and that each cellular set is pointlike [11, Chapter 17].

Pick sufficiently small good ball  $Q \subseteq U$  such that the connected  $n$ -manifold  $M' := U \setminus \text{int } Q$  has measure larger than  $r$  and has nonempty boundary. Applying Lemma 2.3 to  $M'$ , we have an open embedding

$$f: \mathbb{H}_+^n \longrightarrow M'$$

such that

$$f(\partial \mathbb{H}_+^n) \subseteq \partial M' = (\partial M \cap U) \cup \partial Q.$$

Since  $\text{int } \mathbb{H}_+^n$  is a countable increasing union of collared balls, we can find a collared ball  $\hat{B}$  in  $M'$  having measure larger than  $r$ ; moreover, we can further require that  $\hat{B}$  is good by countable additivity of  $\mu$ . Applying Theorem 2.1 to  $\hat{B}$ , we see that the restriction of  $\hat{B}$  is conjugate to a Lebesgue measure on a cube. It is then trivial to find a good ball  $B \subseteq \hat{B}$  with measure precisely  $r$ .  $\square$

**2.2. Regular open sets and homeomorphism groups.** Let  $X$  be a topological space. If  $A \subseteq X$  is a subset then we write  $\text{cl } A$  and  $\text{int } A$  for its closure and interior, respectively, and

$$\text{fr}(A) := \text{cl } A \setminus \text{int } A$$

for the *frontier* of  $A$ .



A set  $U \subseteq X$  is *regular open* if  $U = \text{int cl } U$ . For instance, a good ball is always regular open. The set of regular open subsets of  $X$  forms a Boolean algebra, denoted as  $\text{RO}(M)$ . In this Boolean structure, the minimal and maximal elements are the empty set and  $X$  respectively. The meet is the intersection, and the join of two regular open sets  $U$  and  $V$  is given by

$$U \oplus V := \text{int cl}(U \cup V).$$

The complement coincides with the exterior:

$$U^\perp := X \setminus \text{cl } U.$$

Consequently, the Boolean partial order  $U \leq V$  coincides with the inclusion  $U \subseteq V$  for  $U, V \in \text{RO}(X)$ . For each subcollection  $\mathcal{F} \subseteq \text{RO}(X)$  of regular open sets we can define its supremum as

$$\sup \mathcal{F} := \text{int cl} \left( \bigcup \mathcal{F} \right) \in \text{RO}(X).$$

In particular,  $\text{RO}(X)$  is a complete Boolean algebra. We remark briefly that the collection of open sets of a manifold (or indeed of an arbitrary topological space) is not a Boolean algebra in a natural way, but rather a Heyting algebra, since it is possible that  $U \subsetneq U^{\perp\perp}$ .

By a *regular open cover* of a space, we mean a cover consisting of regular open sets. We will repeatedly use the following straightforward fact, which implies that every finite open cover of a normal space can be refined by an open cover which consists of regular open sets.

**Lemma 2.6.** *If  $\mathcal{U} = \{U_1, \dots, U_m\}$  is an open cover of a normal space, then there exists a regular open cover  $\mathcal{V} = \{V_1, \dots, V_m\}$  such that  $\text{cl } V_i \subseteq U_i$  for each  $i$ .*

*Proof.* Under the given hypothesis, one can find an open cover  $\{W_i\}$  satisfying  $\text{cl } W_i \subseteq U_i$  for each  $i$ ; see [12, Corollary 1.6.4]. It then suffices for us to take  $V_i := \text{int cl } W_i$ , which is clearly a regular open set.  $\square$

Let  $g \in \text{Homeo}(X)$ . We denote its fixed point set by  $\text{fix } g$ , and define its (*open*) *support* as  $\text{supp } g := X \setminus \text{fix } g$ . We then define its *extended support* as

$$\text{supp}^e g := \text{int cl supp } g = \text{int cl}(X \setminus \text{fix } g).$$

Let  $G \leq \text{Homeo}(X)$ . We define the *rigid stabilizer (group)* of  $A \subseteq X$  as

$$G[A] := \{g \in G \mid \text{supp } g \subseteq A\}.$$

If  $U$  is regular open in  $X$ , we note that

$$G[U] = \{g \in G \mid \text{supp}^e g \subseteq U\}.$$

Recall from the introduction that the group  $G \leq \text{Homeo}(X)$  is *locally dense* if for each nonempty open set  $U$  and for each  $p \in U$  we have that

$$\text{int cl}(G[U].p) \neq \emptyset.$$

More weakly, we say  $G$  is *locally moving* if the rigid stabilizer of each nonempty open set is nontrivial.

If  $G$  is a locally moving group of homeomorphisms of  $X$  then  $\text{RO}(X)$  has no atoms, and the set of extended supports

$$\{\text{supp}^e g \mid g \in G\}$$

is *dense* in the complete Boolean algebra  $\text{RO}(X)$ , i.e. for all  $U \in \text{RO}(X)$  there exists  $g \in G$  such that  $\text{supp}^e g \subseteq U$ ; see [39] and [23, Theorem 3.6.11]. When the ambient space is a manifold, we note a fundamental observation that every regular open set can actually be represented as the extended support of some homeomorphism. We write

$$U_1 \sqcup U_2 = V$$

when  $V$  is the disjoint union of two sets  $U_1$  and  $U_2$ .

**Proposition 2.7.** *Suppose that  $(M, G) \in \mathcal{M}$  with  $\dim M \geq 1$ , or that  $(M, G) \in \mathcal{M}_{\text{vol}}$  with  $\dim M > 1$ . Then each regular open set of  $M$  is the extended support of some element of  $G$ .*

*Proof.* Let us pick a countable dense subset  $\{x_i\}_{i \in \omega}$  of  $U \cap \text{int } M$ . Set  $j_1 := 1$ , and pick a good ball  $B_1$  containing  $x_{j_1} = x_1$  such that  $\text{diam } B_1 < 1$  and such that  $B_1 \subseteq U$ . Suppose we have constructed a sequence  $j_1 < j_2 < \dots < j_k$ , and a disjoint collection of good balls  $B_1, \dots, B_k$  such that  $x_{j_i} \in B_i$  and such that  $\text{diam } B_i < 1/i$  for each  $i$ ; furthermore, we require that

$$\{x_1, x_2, \dots, x_{j_k}\} \subseteq B_1 \cup \dots \cup B_k \subseteq U.$$

If  $U = \bigcup_{i=1}^k B_i$ , then we terminate the procedure; otherwise, we let  $j_{k+1}$  be the minimal index  $j$  such that

$$x_j \in W := U \setminus \bigcup_{i=1}^k B_i.$$

Pick a good ball  $B_{k+1} \subseteq W$  containing  $x_{j_{k+1}}$  such that  $\text{diam } B_{k+1} < 1/(k+1)$ . Thus, we build an infinite disjoint collection of good balls  $\{B_i\}_{i \in \omega}$  in  $U$  such that  $\{x_i\} \subseteq \bigcup_i B_i$ .

We claim that there exists  $h_i \in G$  for each  $i$  such that  $\text{supp}^e h_i = \text{int } B_i$ . In the case where there is no measure under consideration, this is clear from the definition of a good ball. In the case when a measure  $\mu$  is considered, we first pick a homeomorphism  $h$  in  $\text{Homeo}_{0, \text{Lcb}}(B^n(1), \partial B^n(1))$  whose fixed point set has empty interior; here, the condition that  $\dim M > 1$  is used. Let us also pick a homeomorphism

$u_i: B^n(1) \longrightarrow B_i$ . We see from Theorem 2.1 that the pullback measure of  $\mu$  on  $B^n(1)$  under the map  $u_i$  is conjugate to (a rescaling of) the Lebesgue measure by a homeomorphism. Hence, by conjugation and extension by the identity, we obtain a homeomorphism  $h_i \in \text{Homeo}_{0,\mu}(M)$  satisfying

$$\text{fix } h_i = (M \setminus \text{int } B_i) \sqcup Q_i$$

for some closed set  $Q_i \subseteq B_i$  with empty interior. This proves the claim.

Since we have

$$\sup_x d(x, h_i(x)) \leq \text{diam } B_i < 1/i$$

for all  $i$ , we see from the uniform convergence theorem that the infinite product  $g := \prod_i h_i$  converges in  $\text{Homeo}(M)$ , and is isotopic to the identity. We have that

$$\text{supp}^e g = \text{int cl}\left(\bigcup_i \text{int } B_i\right) = U.$$

Hence, this map  $g$  satisfies the conclusion.  $\square$

Let us note the measure-preserving homeomorphism groups of compact one-manifolds are highly restrictive.

**Proposition 2.8.** *For each compact connected one-manifold  $M$ , there exist a group theoretic formula  $\phi_M^{\text{vol}}$  such that when  $(N, H) \in \mathcal{M}_{\text{vol}}$ , we have that*

$$H \models \phi_M^{\text{vol}}$$

*if and only if  $N$  and  $M$  are homeomorphic.*

*Proof.* Since  $\text{Homeo}_\mu(I) \cong \mathbb{Z}/2\mathbb{Z}$ , the group theoretic sentence  $\phi_I^{\text{vol}}$  stating that there are at most two elements in the group is satisfied by a pair  $(N, H) \in \mathcal{M}_{\text{vol}}$  if and only if  $N \cong I$ . Since  $\text{Homeo}_\mu(S^1)$  contains the abelian group  $\text{Homeo}_{0,\mu}(S^1) \cong \text{SO}(2, \mathbb{R})$  as the index-two subgroup, a pair of the form  $(S^1, G) \in \mathcal{M}_{\text{vol}}$  satisfies the sentence

$$\phi_{S^1}^{\text{vol}} := (\forall \gamma_1, \gamma_2)[\gamma_1^2 \gamma_2^2 \gamma_1^{-2} \gamma_2^{-2} = 1] \wedge \neg \phi_I^{\text{vol}}.$$

Moreover, if  $(N, H) \in \mathcal{M}_{\text{vol}}$  with  $\dim N > 1$ , then  $H$  is not virtually abelian and hence  $H$  does not satisfy the above formulae.  $\square$

**2.3. First order logic.** Proposition 2.8 establishes the measure-preserving case of the main theorem with  $\dim M = 1$ . Our key strategy for all the other cases is to build a new language, which is powerful enough that it can distinguish a given manifold from the other ones, but which can still be “interpreted” to the language of groups. In order to do this, let us begin with a brief review of the basic terminology from multi-sorted first order logic. Details can be found in [28, 41] and also succinctly in [4].

On the syntactic side, a (multi-sorted, first order) *language*  $\mathcal{L}$  is specified by *logical symbols* and a *signature*. Logical symbols include quantifiers ( $\forall, \exists$ ), logical connectives ( $\wedge, \vee, \neg, \rightarrow$ ), the equality ( $=$ ) and a countable set of variables. We often write auxiliary symbols such as parentheses or commas for the convenience of the reader.

A *signature* consists of *sort symbols*, *relation symbols* (also called as *predicate symbols*), *function symbols* and *constant symbols*. For the brevity of exposition we often regard a function or constant symbol as a special case of a relation symbol. An *arity function* is also in the signature, which assigns a finite tuple of sort symbols to each relation symbol. The arity function for each constant symbol is further required to assign only a single (i.e. 1-tuple of) sort symbol.

A (*well-formed*)  $\mathcal{L}$ -*formula* is a juxtaposition of the above symbols which is “valid”; the precise meaning of this validity requires a recursive definition [28], although it is intuitively clear. For instance, if  $P$  is a relation symbol with the arity value  $(s, t)$  for some sort symbols  $s$  and  $t$ , and if  $x$  and  $y$  are variables with sort values  $s$  and  $t$ , respectively, then  $Pxy$  is a formula. We write  $P(x, y)$  instead of  $Pxy$  for the ease of reading. The *language*  $\mathcal{L}$  specified by the above information is the collection of all formulae. Unquantified variables in a formula are called *free*, and a *sentence* is a formula with no free variables.

On the semantic side, we have an  $\mathcal{L}$ -*structure* (or, an  $\mathcal{L}$ -*model*)  $\mathcal{X}$ , which is specified by a set  $|\mathcal{X}|$  called the *universe*, a sort function  $\sigma$  from  $|\mathcal{X}|$  to the set of sort symbols, and an *assignment* that is a correspondence from each relation symbol  $P$  to an actual relation  $P^{\mathcal{X}}$  among tuples of the elements in the universe. For each sort symbol  $s$ , we call  $s^{\mathcal{X}} := \sigma^{-1}(s)$  the *domain* of  $s$  in  $\mathcal{X}$ . It is required that the relation  $P^{\mathcal{X}}$  respects the arity value of  $P$ . For instance, if  $P$  is as in the previous paragraph, then  $P^{\mathcal{X}}$  will be a subset of  $s^{\mathcal{X}} \times t^{\mathcal{X}}$ . A function symbol is assigned the graph of some function, and often written as a function notation such as  $f(x) = y$ . A constant symbol is fixed as an element in the universe by an assignment. An assignment (for relations) naturally extends to an assignment  $\phi^{\mathcal{X}}$  for each formula  $\phi$ . We sometimes omit  $\mathcal{X}$  from  $\phi^{\mathcal{X}}$  when the meaning is clear.

For an  $\mathcal{L}$ -formula  $\phi$  with a tuple of free variables  $\underline{x}$ , and for a tuple  $\underline{a}$  of elements in  $|\mathcal{X}|$ , we write  $\mathcal{X} \models \phi(\underline{a})$  if  $\phi^{\mathcal{X}}$  holds after  $\underline{a}$  has been substituted for  $\underline{x}$ . We define  $\text{Th}(\mathcal{X})$  as the set of all  $\mathcal{L}$ -sentences  $\phi$  such that  $\mathcal{X} \models \phi$ .

Let  $p, q \geq 0$ , and let  $\underline{b}$  be a  $q$ -tuple of elements of  $|\mathcal{X}|$ . A subset  $A$  of  $X^p$  is *definable (by  $\phi$ ) with parameters  $\underline{b} = (b_1, \dots, b_q)$*  if for some formula  $\phi$  with  $p + q$  free variables, the set  $A$  coincides with the set

$$\mathcal{X}(\phi; \underline{b}) := \{\underline{a} \in |\mathcal{X}|^p : \phi(\underline{a}, \underline{b})\}.$$

If  $q = 0$  we simply say  $A$  is *definable*, in which case we denote the above set as  $\mathcal{X}(\phi)$ . We now formalize the concept of “interpreting” a new language.

**Definition 2.9.** Let  $\mathcal{L}_1$  and  $\mathcal{L}_2$  be languages. Suppose we have a class  $\mathcal{X}$  of ordered pairs in the form  $(X_1, X_2)$  with  $X_i$  being an  $\mathcal{L}_i$ -structure. We say  $X_2$  is *interpretable in  $X_1$  uniformly for  $(X_1, X_2)$  in  $\mathcal{X}$*  if there exist some  $\mathcal{L}_1$ -formulae  $\phi_{\text{dom}}$  and  $\phi_{\text{eq}}$ , and there also exists a map  $\alpha$  from the set of  $\mathcal{L}_2$ -formulae to the set of  $\mathcal{L}_1$ -formulae such that the following hold.

for each  $(X_1, X_2) \in \mathcal{X}$ , we have a surjection

$$\rho: X_1(\phi_{\text{dom}}) \longrightarrow |X_2|$$

with its fiber uniformly defined by  $\phi_{\text{eq}}$  in the sense that

$$X_1(\phi_{\text{eq}}) = \{(x, y) \in X_1(\phi_{\text{dom}}) \times X_1(\phi_{\text{dom}}) \mid \rho(x) = \rho(y)\}.$$

Furthermore, it is required for each  $\mathcal{L}_2$ -formula  $\psi$  that

$$\rho^{-1}(X_2(\psi)) = X_1(\alpha(\psi)).$$

The bijection

$$\rho^{-1}: |X_2| \longrightarrow X_1(\phi_{\text{dom}})/X_1(\phi_{\text{eq}})$$

along with the map  $\alpha$  is called a *uniform interpretation* of  $X_2$  in  $X_1$ .

*Remark 2.10.* (1) In the above, if  $\psi$  is  $m$ -ary (as a relation) and  $\phi_{\text{dom}}$  is  $n$ -ary, then  $\alpha(\psi)$  is  $mn$ -ary. In practice, we only need to consider relation symbols (in a broad sense, including function and constant symbols)  $\psi$  rather than all possible  $\mathcal{L}_2$ -formulae.

(2) In various instances of this paper, it will be the case that  $\mathcal{L}_1 \subseteq \mathcal{L}_2$  and that the interpretation restricts to the identity on  $\mathcal{L}_1$ . As a consequence of such interpretability, we will have that  $\text{Th}(X_2)$  is a conservative extension of  $\text{Th}(X_1)$  for each  $(X_1, X_2) \in \mathcal{X}$ . Also, we will often add a function symbol in  $\mathcal{L}_2$  corresponding to the surjection  $\rho$ , which is clearly justified.

The following lemma explains how the combination of Theorems 1.8 and 1.9 implies Theorem 1.4.

**Lemma 2.11.** *Suppose  $\mathcal{L}_1, \mathcal{L}_2$  and  $\mathcal{X}$  are as in Definition 2.9 so that  $X_2$  is interpretable in  $X_1$  uniformly for  $(X_1, X_2) \in \mathcal{X}$ . Let  $(X_1, X_2)$  and  $(Y_1, Y_2)$  be in  $\mathcal{X}$ . Then for each sentence  $\psi$  belonging to  $\text{Th}(X_2) \setminus \text{Th}(Y_2)$ , the interpretation  $\alpha(\psi)$  belongs to  $\text{Th}(X_1) \setminus \text{Th}(Y_1)$ . In particular, if  $X_1 \equiv Y_1$ , then  $X_2 \equiv Y_2$ .*

### 3. THE AGAPE STRUCTURE AND BASIC OBSERVATIONS

The fundamental universe that we work in will be the group of homeomorphisms of a manifold. Objects such as regular open sets, real numbers, points in the manifold, continuous functions, etc. will all be constructed as definable equivalence classes of definable subsets of finite tuples of homeomorphisms.

**3.1. The language  $L_{\text{AGAPE}}$  and the structure  $\text{AGAPE}(M, G)$ .** The ultimate language we will work in will be called AGAPE, which stands for ‘‘Action of a Group, Analysis, Points and Exponentiation’’. This language is denoted as  $L_{\text{AGAPE}}$  and contains the following different sort symbols for  $k, \ell \in \omega$ :

$$\mathbf{G}, \text{RO}, \mathbf{N}, \mathcal{P}(\mathbf{N}), \mathbb{R}, \mathbf{M}, \mathbf{M}^{\text{disc-int}}, \text{Cont}_{k,\ell}.$$

The above sorts come with some symbols that are intrinsic to the sort (such as a group operation), and others which relate the sorts to each other, as we spell out below. There will be a countable set of variables for each sort, as is typically required. We also describe an AGAPE structure assigned to each pair  $(M, G)$  in the class  $\mathcal{M}$  or  $\mathcal{M}_{\text{vol}}$ . In this structure, we give the ‘‘intended’’ choice of the domain of each sort symbol.

**The group sort.** The domain of the sort symbol  $\mathbf{G}$  will be the group  $G$ , under our standing assumption that  $(M, G) \in \mathcal{M}_{(\text{vol})}$ . The signatures only relevant for this sort are

$$1, \circ, {}^{-1},$$

which are respectively assigned with the natural meanings in the group theory. These symbols, along with variables, form the *language of groups*  $L_{\text{Act}}^0 = L_{\mathbf{G}}$ . The group  $G$  is regarded an  $L_{\mathbf{G}}$ -structure  $\text{Act}^0(M, G) = \text{Act}_{\mathbf{G}}(M, G)$ . We will usually not write the  $\circ$  symbol.

**The sort of regular open sets.** The domain of the sort symbol  $\text{RO}$  is the set  $\text{RO}(M)$  of the regular open sets in  $M$ . The newly introduced signatures for this sort are

$$\subseteq, \cap, {}^{\perp}, \oplus, \emptyset, \mathbf{M}, \text{supp}^e, \text{appl}.$$

The symbol  $\mathbf{M}$  means the manifold  $M$  in the structure. By the natural assignment as before, we have Boolean symbols

$$\subseteq, \cap, \oplus, {}^{\perp}, \mathbf{M}, \emptyset$$

for the Boolean algebra  $\text{RO}(M)$ . We let the function symbol  $\text{supp}^e$  mean the map  $G \rightarrow \text{RO}(M)$  defined as

$$g \mapsto \text{supp}^e g.$$

We have an assignment for  $\text{appl}$  so that

$$\text{appl}(g, U) = g(U)$$

with  $g \in G$  and  $U \in \text{RO}(M)$ . The symbols introduce so far (along with countably many variables for each sort) form the *language of a group action on a Boolean algebra*  $L_{\text{Act}}^1 = L_{\mathbf{G}, \text{RO}}$ . The  $L_{\text{Act}}^1$ -structure described above on the universe  $G \sqcup \text{RO}(M)$  is denoted as  $\text{Act}^1(M, G) = \text{Act}_{\mathbf{G}, \text{RO}}(M, G)$ .

**The sorts from the analysis** We then introduce new sort symbols, which are  $\mathbb{N}$ ,  $\mathcal{P}(\mathbb{N})$ ,  $\mathbb{R}$  and  $\cong_{k,\ell}$  for  $k, \ell \in \omega$ . The signatures introduced here are

$$0, 1, <, +, \times, \in, \subseteq, \#\pi_0, \text{norm}.$$

The usual second order arithmetic

$$\text{Arith}_2 = (\mathbb{N}, \mathcal{P}(\mathbb{N}), 0, 1, <, +, \times, \in, \subseteq)$$

is given the sort symbols  $\mathbb{N}$  and  $\mathcal{P}(\mathbb{N})$ , as well as with relevant non-logical symbols. We note the ambiguity of our notation that the sort symbols  $\mathbb{N}$  and  $\mathcal{P}(\mathbb{N})$  will be assigned with the set of the natural numbers  $\mathbb{N}$  and its power set  $\mathcal{P}(\mathbb{N})$ , respectively. The symbol  $\#\pi_0$  is interpreted so that

$$\#\pi_0(U) = k$$

means  $U \in \text{RO}(M)$  has  $k$  connected components. See Section 4 for details. The ordered ring of the real numbers

$$\{0, 1, +, \times, <, =\}$$

is assigned with the sort symbol  $\mathbb{R}$  and the signatures above. Note that, as is usual,  $\mathbb{N}$  is considered as a subsort of  $\mathbb{R}$ , by identifying each integer as a real number.

The domain of the sort symbol  $\text{Cont}_{k,\ell}$  will be the set  $C(\mathbb{R}^k, \mathbb{R}^\ell)$  of continuous functions. We also have a formula  $\text{appl}(f, x) = y$  when the sort value of  $f$  is  $\text{Cont}_{k,\ell}$ , and when  $x$  and  $y$  are tuples of variables assigned with the sort symbol  $\mathbb{R}$ . We have the  $C^0$ -norm  $f \mapsto \text{norm}(f) := \|f\|$ , which will be also a part of the language. Combining these symbols with  $L_{\text{Act}}^1$ , we obtain the language  $L_{\text{Act}}^2 = L_{\mathbf{G}, \text{RO}, \mathbb{R}}$ . An  $L_{\text{Act}}^2$ -structure  $\text{Act}^2(M, G) = \text{Act}_{\mathbf{G}, \text{RO}, \mathbb{R}}$  is assigned to each  $(M, G) \in \mathcal{M}_{(\text{vol})}$  having the universe

$$G \sqcup \text{RO}(M) \sqcup \mathcal{P}(\mathbb{N}) \sqcup \mathbb{R} \sqcup \bigsqcup_{k,\ell} C(\mathbb{R}^k, \mathbb{R}^\ell).$$

**The point and the discrete subset sorts  $\mathbf{M}$  and  $\mathbf{M}^{\text{disc-int}}$ .** The domain of the sort symbol  $\mathbf{M}$  will be the set of the points in a manifold. We also introduce the sort symbol  $\mathbf{M}^{\text{disc-int}}$  to mean a subset  $A$  of  $\text{int } M$  every point of which is isolated in  $A$ . By abuse of notation, the symbols  $\in$  and  $\subseteq$  introduced above will have multiple meanings (depending on the context), so that they have the arity values  $(\mathbf{M}, \text{RO})$ ,  $(\mathbf{M}, \mathbf{M}^{\text{disc-int}})$  and  $(\mathbf{M}^{\text{disc-int}}, \text{RO})$ . We can naturally form a singleton from a point. We also have a cardinality function

$$\#A = m$$

meaning that the cardinality of  $A \subseteq \text{int } M$  is  $m$ , assuming that every point in  $A$  is isolated.

The interpretation of points of the manifold will allow us to include symbols such as  $\text{cl}$  and  $\text{fr}$ , the closure and frontier of a regular open set, together with membership

relations into these sets. These symbols will simply be abbreviations for formulae which impose the intended meaning. We will be able to separate out boundary points of  $M$  from the interior ones, and hence justified to use the notations

$$\pi \in \partial \mathbf{M}, \pi' \in \text{int } \mathbf{M}$$

for point sort variables  $\pi$  and  $\pi'$ . The function symbol  $\text{appl}$  has a natural additional meaning as below:

$$\text{appl}: G \times M \longrightarrow M.$$

In all contexts, we abbreviate  $\text{appl}(\gamma, x)$  by  $\gamma(x)$  when the sort of  $\gamma$  is either  $\mathbf{G}$  or  $\text{Cont}_{k,\ell}$  and when the sort of  $x$  is (tuples of)  $\mathbb{R}$ ,  $\mathbf{M}$ ,  $\mathbf{M}^{\text{disc-int}}$  or  $\text{RO}$ .

The omnibus language, combining all of the previous sorts and relevant symbols, is denoted by

$$L_{\text{Act}}^3 = L_{\mathbf{G}, \text{RO}, \mathbb{R}, \mathbf{M}} = L_{\text{AGAPE}},$$

or simply as AGAPE. We have so far described the  $L_{\text{AGAPE}}$ -structure  $\text{Act}^3(M, G) = \text{AGAPE}(M, G)$  corresponding to  $(M, G) \in \mathcal{M}_{(\text{vol})}$ .

Dealing with these structures, we often make use of functions or relations defined by fixed formulae that are not explicitly specified. The following terminology will be handy when we need to avoid ambiguity in such situations. See Remark 1.1 for our notation for  $\mathcal{M}_{(\text{vol})}$ .

**Definition 3.1.** Let  $i = 0, 1, 2, 3$ , and let  $\phi_{(\text{vol})}$  be a formula in  $L_{\text{Act}}^i$ . Suppose for each  $(M, G) \in \mathcal{M}_{(\text{vol})}$  that a function or relation  $f_{M,G}$  is defined by  $\phi$  in  $\text{Act}^i(M, G)$ . Then the collection

$$\{f_{M,G} \mid (M, G) \in \mathcal{M}_{(\text{vol})}\}$$

is said to be *uniformly defined over*  $\mathcal{M}_{(\text{vol})}$ .

*Remark 3.2.* In dealing with the sorts in Subsection 2.3, we will distinguish notationally between variables referring to a particular sort and elements of that sort. For the convenience of the reader, we will record a table summarizing the notation. In general, we will write an underline to denote an arbitrary finite tuple of variables or objects.

Sort	variable	object
Group elements	$\gamma, \delta, \underline{\gamma}, \underline{\delta}$	$g, h$
Regular open sets	$u, v, w, \underline{u}, \underline{v}, \underline{w}$	$U, V, W$
Natural numbers	$\alpha, \beta, \underline{\alpha}, \underline{\beta}$	$k, m, n$
Sets of natural numbers	$\Lambda, \underline{\Lambda}$	$A$
Real numbers	$\rho, \sigma, \underline{\rho}, \underline{\sigma}$	$r, s$
Sets of points	$\pi, \underline{\pi}, \underline{\tau}, \underline{\tau}$	$p, q, T$
Functions	$\chi, \theta, \underline{\chi}, \underline{\theta}$	$f$



From now on, we will reserve the letters in this table for exclusive use as variables or objects of a particular sort, unless specified otherwise. In the ambient metalanguage, we will use  $i, j \in \omega$  to denote indices. The symbols  $M$  and  $N$  will be reserved for manifolds.

**3.2. Interpreting action structures in homeomorphism groups.** Since the uniform interpretability (Definition 2.9) is transitive, the following proposition would trivially imply Theorem 1.8.

**Proposition 3.3.** *For each  $i = 0, 1, 2$ , and uniformly for  $(M, G) \in \mathcal{M}_{(\text{vol})}$ , the  $L_{\text{Act}}^{i+1}$ -structure  $\text{Act}^{i+1}(M, G)$  is interpretable in the  $L_{\text{Act}}^i$ -structure  $\text{Act}^i(M, G)$ .*

The proof of this proposition will require construction of  $L_{\text{Act}}^i$ -formulae  $\phi_{\text{dom}}^i$  and  $\phi_{\text{eq}}^i$ , and a surjection

$$\rho_i: \text{Act}^i(M, G)(\phi_{\text{dom}}^i) \longrightarrow |\text{Act}^{i+1}(M, G)|$$

for all  $(M, G) \in \mathcal{M}_{(\text{vol})}$  satisfying the conditions of Definition 2.9. Such a construction will span over Sections 4 and 5, as well as this section.

Rubin's Theorem [38, 39] stated in the introduction can be used to prove various *reconstruction theorems*, by which we mean the group isomorphism types greatly restrict the homeomorphism types of spaces on which the groups can act nicely. See [23] for comprehensive references on this, especially regarding diffeomorphism groups.

The key step in the proof of Rubin's theorem can be rephrased as follows. We emphasize that the formulae below are independent of the choice of the group  $G$  or the space  $X$ .

**Theorem 3.4** (Rubin's Expressibility Theorem, cf. [39]). *There exist first order formulae*

$$\subseteq (\gamma_1, \gamma_2), \quad \text{appl}(\gamma_1, \gamma_2, \gamma_3), \quad \cap(\gamma_1, \gamma_2, \gamma_3), \quad \oplus(\gamma_1, \gamma_2, \gamma_3), \quad \perp(\gamma_1, \gamma_2)$$

*in the language of groups such that if  $G$  be a locally moving group of homeomorphisms of a Hausdorff topological space  $X$ , then the following hold for all  $g_1, g_2, g_3 \in G$ .*

- (1)  $G \models \subseteq (g_1, g_2) \iff \text{supp}^c g_1 \subseteq \text{supp}^c g_2 \iff g_1 \in G[\text{supp}^c g_2]$ .
- (2)  $G \models \text{appl}(g_1, g_2, g_3) \iff \text{appl}(g_1, \text{supp}^c g_2) = \text{supp}^c g_3$ .
- (3)  $G \models \cap(g_1, g_2, g_3) \iff \text{supp}^c g_1 \cap \text{supp}^c g_2 = \text{supp}^c g_3$ .
- (4)  $G \models \oplus(g_1, g_2, g_3) \iff \text{supp}^c g_1 \oplus \text{supp}^c g_2 = \text{supp}^c g_3$ .
- (5)  $G \models \text{ext}(g_1, g_2) \iff \text{supp}^c g_1 = (\text{supp}^c g_2)^\perp$ .

*Proof.* Parts (1) and (2) are given as Theorem 2.5 in [38]; see also [23, Corollary 3.6.9] for a concrete formula. The rests are clear from that the supremum in  $\text{RO}(M)$  is first order expressible in terms of the inclusion relations.  $\square$

Let  $(M, G) \in \mathcal{M}_{\text{vol}}$ . By Proposition 2.7, we have a surjection

$$\rho_0: G \longrightarrow \text{RO}(M)$$

defined as  $g \mapsto \text{supp}^e g$ . Since  $G$  is locally moving on  $M$ , Rubin's expressibility theorem implies that the fiber

$$\{(g, h) \mid \text{supp}^e g = \text{supp}^e h\}$$

of  $\rho_0$  is definable, and that the Boolean symbols and the function symbols  $\text{appl}$  and  $\text{supp}^e$  have group theoretic interpretations; see also parts (1) and (2) of Remark 2.10. As a conclusion, we have the following, which means that Proposition 3.3 holds for the case  $i = 0$ .

**Corollary 3.5.** *Uniformly for  $(M, G) \in \mathcal{M}_{(\text{vol})}$ , the  $L_{G, \text{RO}}$ -structure  $\text{Act}_{G, \text{RO}}(M, G)$  is interpretable in the group structure  $G$ .*

Corollary 3.5 can be summarized as saying that  $G$  interprets the group action structure of  $G$  on the algebra of regular open sets, in a way that preserves the meaning of  $G$ . This interpretation is uniform in the underlying pair  $(M, G)$ , and any formula in the language of  $G$  and  $\text{RO}$  can be expressed entirely in  $G$ , since the formulae in Theorem 3.4 are independent of  $M$ . Henceforth, we will assume that we will work with the expanded language  $L_{\text{Act}}^1 = L_{G, \text{RO}}$ .

**3.3. First order descriptions of basic topological properties.** Recall whenever the expression  $U \sqcup V$  is used it is assumed that  $U$  and  $V$  are disjoint. Let us find first order expressions for some pointset-topological properties.

**Lemma 3.6.** *The following hold for  $(M, G) \in \mathcal{M}_{(\text{vol})}$ .*

- (1) *For  $U, V \in \text{RO}(M)$ , we have that  $G[U] = G[V]$  if and only if  $U = V$ .*
- (2) *For each  $U \in \text{RO}(M)$ , we have that*

$$G[U] = \{g \in G \mid g(V) = V \text{ for all regular open set } V \subseteq U^\perp\}.$$

- (3) *An open subset is path-connected if and only if it is connected.*
- (4) *An arbitrary union of connected components of a regular open set is necessarily regular open. More specifically, if a regular open set  $W$  can be written as  $W = U \sqcup V$  for some disjoint pair of open sets  $U$  and  $V$ , then  $U$  and  $V$  are regular open and  $W = U \oplus V$ . Moreover, we have  $V = W \cap U^\perp$ .*
- (5) *For disjoint pair  $U, V$  of regular open sets, we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).*
  - (i)  *$U$  is connected, and  $U \oplus V = U \sqcup V$ ;*
  - (ii) *Every  $g \in G[U \oplus V]$  satisfies either  $g(U) = U$  or  $g(U) \cap U = \emptyset$ ;*
  - (iii)  *$U \oplus V = U \sqcup V$*
- (6) *Let  $W$  and  $U$  are regular open sets such that  $U$  is connected and such that  $U \subseteq W$ . Then  $U$  is a connected component of  $W$  if and only if  $W = U \oplus V$*

for some regular open  $V$  that is disjoint from  $U$  and every  $g \in G[W]$  satisfies either  $g(U) = U$  or  $g(U) \cap U = \emptyset$ .

(7) The following are all equivalent for a regular open set  $W$ .

(i)  $W$  is disconnected;

(ii)  $W = U \sqcup V$  for some disjoint pair of nonempty regular open sets  $U$  and  $V$  such that  $U$  is connected;

(iii)  $W = U \oplus V$  for some disjoint pair of nonempty regular open sets  $U$  and  $V$ , and every  $g \in G[W]$  satisfies either  $g(U) = U$  or  $g(U) \cap U = \emptyset$ ;

(iv)  $W = U \oplus V = U \sqcup V$  for some disjoint pair of nonempty regular open sets  $U$  and  $V$ .

(8) For two regular open subsets  $U$  and  $V$  satisfying  $U \cap V = \emptyset$ , we have that  $U \sqcup V = U \oplus V$  if and only if each connected component of  $U \oplus V$  is contained either in  $U$  or in  $V$ .

*Proof.* (1) If  $x \in U \setminus V$ , then there exists some  $h \in G[U]$  satisfies  $h(x) \neq x$ ; see [23, Lemma 3.2.3] for instance. In particular, we have that  $h \in G[U] \setminus G[V] \neq \emptyset$ . This proves the nontrivial part of the given implication. We remark that the same statement holds without the assumption that  $U$  and  $V$  are regular open, under the extra hypothesis that  $M \not\cong I$ . Part (2) is similar.

(3) This part is clear from the fact that every manifold is locally path-connected.

(4) Whenever two open sets  $U$  and  $V$  are disjoint we have that  $U^{\perp\perp}$  and  $V^{\perp\perp}$  are also disjoint; see [23, Lemma 3.6.4 (4)], for instance. From

$$W = U \sqcup V \subseteq U^{\perp\perp} \sqcup V^{\perp\perp} \subseteq U^{\perp\perp} \oplus V^{\perp\perp} \subseteq W^{\perp\perp} = W,$$

we see that  $U$  and  $V$  are actually regular open and  $W = U \oplus V = U \sqcup V$ . It is clear that  $V \cap \text{fr } U = \emptyset$ , which implies  $V = W \cap U^\perp$ .

(5) The implication (i) $\Rightarrow$ (ii) is clear from that every setwise stabilizer of  $g \in G[U \oplus V]$  permutes connected components of  $U \oplus V$ .

For the implication (ii) $\Rightarrow$ (iii), assume we have a point

$$p_1 \in (U \oplus V) \setminus (U \cup V).$$

Take a sufficiently small open ball  $B$  around  $p_1$  so that

$$B \subseteq U \oplus V = \text{int cl}(U \cup V) \subseteq \text{cl } U \cup \text{cl } V.$$

Note also from

$$p_1 \notin U = \text{int cl } U$$

that  $B \not\subseteq \text{cl } U$ . Similarly,  $B \not\subseteq \text{cl } V$ . This implies that we can choose distinct points

$$p_2, p_3 \in B \cap U \cap \text{int } M$$

and  $p_4 \in B \cap V \cap \text{int } M$ . Since  $G$  is  $k$ -transitive on  $B \cap \text{int } M$  for all  $k$ , we can find a  $g \in G$  supported in  $B$  satisfying  $g(p_2) = p_3$  and  $g(p_3) = p_4$ ; see also Lemma 2.4. Then  $g(U)$  is neither  $U$  nor disjoint from  $U$ .

Parts (6) and (7) are clear from the previous parts.

(8) The forward direction comes from the observation that  $(U, V)$  is a disconnection of  $U \oplus V$ . The backward direction is trivial since the hypothesis implies that  $U \oplus V \subseteq U \cup V$ .  $\square$

Let us note the following consequences of Lemma 3.6.

**Corollary 3.7.** *There exist first order formulae in the language  $L_{\mathbf{G}, \text{RO}}$  as follows:*

(1) A formula  $\text{contained}(\gamma, u)$ , also denoted as  $\gamma \in \mathbf{G}[u]$  such that

$$\models \text{contained}(g, U) \quad \text{if and only if } \text{supp}^e g \subseteq U.$$

(2) A formula  $\text{conn}(u)$  such that

$$\models \text{conn}(U) \quad \text{if and only if } U \text{ is connected.}$$

(3) A formula  $\text{cc}(u, v)$ , also denoted as  $u \in \pi_0(v)$  such that

$$\models \text{cc}(U, V) \quad \text{if and only if } U \text{ is a connected component of } V.$$

(4) A formula  $\text{ucc}(u, v)$  such that

$$\models \text{ucc}(U, V) \quad \text{if and only if } U \text{ is a union of connected component of } V.$$

(5) For all  $k \in \omega$ , a formula  $\# \text{cc}_k(u)$  such that

$$\models \# \text{cc}_k(U) \quad \text{if and only if } U \text{ has exactly } k \text{ connected components.}$$

(6) A formula  $\text{disj}(u, v)$  such that

$$\models \text{disj}(U, V) \quad \text{if and only if } U \oplus V = U \sqcup V.$$

(7) A formula  $\text{ccpartition}(u, v, w)$  such that

$$\models \text{ccpartition}(U, V, W) \quad \text{if and only if } \text{ucc}(U, W) \wedge \text{ucc}(V, W) \wedge W = U \sqcup V.$$

(8) A formula  $\#_=(u, v)$  such that for all regular open sets  $U$  and  $V$  having finitely many connected components, we have

$$\models \#_=(U, V)$$

*if and only if  $U$  and  $V$  have the same number of connected components.*

*Proof.* The existence of the formula  $\text{contained}(\gamma, u)$  is trivial since  $\text{supp}^e$  and  $\subseteq$  belong to the signature of  $L_{\mathbf{G}, \text{RO}}$ . The formulae  $\text{conn}(u)$  and  $\text{cc}(u, v)$  exist by parts (6) and (7) of Lemma 3.6. We can then set

$$\text{ucc}(u', u) \equiv (\forall w)[\text{cc}(w, u') \rightarrow \text{cc}(w, u)].$$

The construction of the formulae  $\text{cc}_k(u)$  and  $\text{disj}(u, v)$  follow from the same lemma, which also implies that the formula

$$\text{ccpartition}(u, v, w) \equiv [\text{ucc}(u, w) \wedge \text{ucc}(v, w) \wedge w = u \oplus v \wedge u \cap v = \emptyset]$$

has the meaning required in part (7). Finally, we set

$$\#_=(u, v) \equiv (\exists u' \subseteq u, \exists v' \subseteq v) [(\forall \hat{u} \in \pi_0(u)) [\text{conn}(u' \cap \hat{u}) \wedge u' \cap \hat{u} \neq \emptyset] \wedge (\forall \hat{v} \in \pi_0(v)) [\text{conn}(v' \cap \hat{v}) \wedge v' \cap \hat{v} \neq \emptyset] \wedge (\exists \gamma) [\gamma(u') = v']].$$

From the transitivity on good balls (of the same measures, in the measure preserving case) as in Lemma 2.4, we see  $\#_=(u, v)$  has the intended meaning.  $\square$

Using the above formula, we can distinguish the case that  $\dim M = 1$  among all compact connected manifolds.

**Corollary 3.8.** *For each compact connected one–manifold  $M$ , there exist  $L_{G,RO}$ –formulae  $\phi_M$  such that when  $(N, H) \in \mathcal{M}$ , we have that*

$$\text{Act}_{G,RO}(N, H) \models \phi_M$$

*if and only if  $N$  and  $M$  are homeomorphic.*

*Proof.* We let  $\phi_I$  be the  $L_{G,RO}$ –formula expressing that for all pairwise disjoint, proper, nonempty regular open sets  $U_1, U_2$  and  $U_3$  the exterior of  $U_i$  is disconnected for some  $i$ . This formula holds for  $M \cong I$  since at least one of  $\text{cl } U_i$  does not intersect  $\partial M$ , and hence  $U_i^\perp$  separates the two endpoints of  $M$ . It is clear that  $\phi_I$  is never satisfied by any other compact connected manifolds.

We now suppose that  $M$  is a compact connected manifold not homeomorphic to  $I$ . Then for all disjoint, proper, non-empty regular open sets  $U$  and  $V$  satisfying  $U \oplus V = U \sqcup V$ , the set  $M \setminus (U \oplus V)$  is disconnected. From Corollary 3.7, we obtain the formula  $\phi_{S^1}$  expressing that  $M$  and  $S^1$  are homeomorphic.  $\square$

By Proposition 2.8 and Corollary 3.8, we establish Theorem 1.4 for the case when  $M$  is one–dimensional. *So from now on, we modify the definitions of  $\mathcal{M}$  and  $\mathcal{M}_{\text{vol}}$ , replacing these classes by subclasses where all the manifolds in consideration are of dimension at least two.*

**3.4. Further topological properties.** We will need several more general first order formulae to express basic topological properties of regular open sets. One of primary importance will be a formula which implies that a particular regular open set  $U$  is contained in a collared ball inside of another regular open set  $V$ . This is not particularly difficult to state and prove in the class  $\mathcal{M}$ , but is substantially harder in  $\mathcal{M}_{\text{vol}}$ . For the rest of this section, we will make a standing assumption that  $(M, G) \in \mathcal{M}_{(\text{vol})}$ , and that the underlying structure is  $\text{Act}_{G,RO}(M, G)$ .

**3.4.1. Relative-compactness regarding good balls.** Let us use preceding results to find first order formulae that compare measures of regular open sets. For the remainder of this subsection, we assume that  $M$  is a connected, compact  $n$ –manifold with  $n > 1$ , equipped with an Oxtoby–Ulam measure  $\mu$ .

**Lemma 3.9.** *There exists a formula  $\text{vol}_{\leq}(u_1, u_2, v)$  in the language  $L_{\mathbf{G}, \text{RO}}$  such that for all  $(M, G) \in \mathcal{M}_{\text{vol}}$  with an Oxtoby–Ulam measure  $\mu$  on  $M$ , and for all any triple  $(U_1, U_2, V)$  with  $U_1$  and  $U_2$  connected and  $U_1, U_2 \subseteq V$ , we have the following.*

- (1) *If  $\mu(U_1) \leq \mu(U_2)$  then  $\text{vol}_{\leq}(U_1, U_2, V)$  holds.*
- (2) *If  $\text{vol}_{\leq}(U_1, U_2, V)$  holds then  $\mu(U_1) \leq \mu((\text{cl } U_2) \cap V)$ .*

*Proof.* Suppose first that  $\mu(U_1) \leq \mu(U_2)$ , and let  $\emptyset \neq W_0 \in \text{RO}(M)$  be arbitrary. By Lemma 2.5, we can find a good ball  $B \subseteq U_1$  such that

$$\mu(U_1) - \mu(W_0) < \mu(B) < \mu(U_1) \leq \mu(U_2),$$

and such that  $U_1 \setminus B$  connected. Lemma 2.4 furnishes  $g \in \text{Homeo}_{0, \mu}(M)[V]$  such that  $g(B) \subseteq U_2$ , but clearly there is no  $\mu$ -preserving  $h$  such that

$$h(W_0) \subseteq U_1 \setminus B = U_1 \cap (\text{int } B)^\perp.$$

We have just established that  $\text{vol}_{\leq}(U_1, U_2, V)$  holds with  $W_1 := \text{int } B$ , where

$$\begin{aligned} \text{vol}_{\leq}(u_1, u_2, v) &\equiv (\forall w_0 \neq \emptyset, \exists w_1) [w_1 \subseteq u_1 \wedge \text{conn}(u_1 \cap w_1^\perp) \wedge \\ &(\forall \gamma \in \mathbf{G}) [\gamma(w_0) \not\subseteq u_1 \cap w_1^\perp] \wedge (\exists \delta \in \mathbf{G}[v]) [\delta(w_1) \subseteq u_2]]. \end{aligned}$$

Let us now suppose for a contradiction that  $\text{vol}_{\leq}(U_1, U_2, V)$  holds but that

$$\mu(U_1) > \mu(V \cap \text{cl } U_2).$$

Let  $W_0$  be the interior of a good ball in  $M$  with measure  $r_0 < \mu(U_1) - \mu(V \cap \text{cl } U_2)$ . It suffices to show that there is no witness  $W_1$  as required by  $\text{vol}_{\leq}$ .

If such a  $W_1$  exists then by the condition on  $\gamma$ , we see again from Lemmas 2.4 and 2.5 that  $\mu(U_1 \cap W_1^\perp) \leq r_0$ . Moreover, there is a group element  $g \in G[V]$  such that  $g(W_1) \subseteq U_2$ , so that in fact

$$g((\text{cl } W_1) \cap V) \subseteq (\text{cl } U_2) \cap V.$$

We then obtain

$$r_0 = \mu(W_0) \geq \mu(U_1 \cap W_1^\perp) \geq \mu(U_1) - \mu(V \cap \text{cl } W_1) \geq \mu(U_1) - \mu(V \cap \text{cl } U_2).$$

This violates the choice of  $r_0$ . □

The foregoing discussion allows us to characterize when a regular open set  $U$  is contained in a collared ball  $B$  inside a regular open set  $V$ . There are separate formulae which apply in the measure-preserving case, and in the general case.

**Lemma 3.10.** *There exists first order formula  $\text{RCB}_{(\text{vol})}(u, v)$  such that for each  $(M, G) \in \mathcal{M}_{(\text{vol})}$ , we have that*

$$\models \text{RCB}_{(\text{vol})}(U, V)$$

*if and only if  $U$  is relatively compact in some good ball contained in  $V$ .*

Recall our convention that this lemma claims to have two formulae, namely  $\text{RCB}(u, v)$  and  $\text{RCB}_{\text{vol}}(u, v)$ .

*Proof of Lemma 3.10.* We first claim that the following formula satisfies the conclusion for  $(M, G) \in \mathcal{M}$ .

$$\text{RCB}(u, v) \equiv (\exists \hat{v} \in \pi_0(v)) [u \subseteq \hat{v} \wedge (\emptyset \neq \forall w \subseteq \hat{v}, \exists \gamma \in \mathbf{G}[\hat{v}]) [\gamma(u) \subseteq w]].$$

Indeed, if  $U$  is relatively compact in a collared ball  $B \subseteq V$ , then there exists a unique  $\hat{V} \in \pi_0(V)$  containing  $B$ , and hence  $U$ . For each nonempty regular open  $W \subseteq \hat{V}$ , we see from Lemma 2.4 that some  $g \in G[\hat{V}]$  satisfies

$$g(U) \subseteq g(B) \subseteq W,$$

as desired. Conversely, suppose  $\models \text{RCB}(U, V)$  holds and let  $\hat{V}$  be the connected component of  $V$  containing  $U$ . Let us fix a collared ball  $B$  in  $\hat{V}$  and set  $W := \text{int } B$ . From the assumption we can find  $g \in G[\hat{V}]$  such that  $g(U) \subseteq W$ . Then  $U$  is relatively compact in the collared ball  $g^{-1}(B)$  in  $V$ .

For the case when  $(M, G) \in \mathcal{M}_{\text{vol}}$ , we set

$$\begin{aligned} \text{RCB}_{\text{vol}}(u, v) \equiv & (\exists u', \hat{v}) [(\hat{v} \in \pi_0(v) \wedge u \subseteq u' \not\subseteq \hat{v} \wedge \text{conn}(u')) \wedge \\ & (\forall w) [(\text{conn}(w) \wedge w \subseteq \hat{v} \wedge \neg \text{vol}_{\leq}(w, u', \hat{v})) \longrightarrow (\exists \gamma \in \mathbf{G}[\hat{v}]) [\gamma(u') \subseteq w]]]. \end{aligned}$$

In order to prove the forward direction, let us assume that  $\text{RCB}_{\text{vol}}(U, V)$  holds for some nonempty  $U, V \in \text{RO}(M)$ . Let  $U'$  and  $\hat{V}$  be witnesses for the existentially quantified variables  $u'$  and  $\hat{v}$ . Since  $U' \not\subseteq \hat{V}$ , the Boolean subtraction  $\hat{V} \cap (U')^\perp$  is nonempty. We now see that

$$\mu(\hat{V}) = \mu(\hat{V} \cap \text{cl } U') + \mu(\hat{V} \cap (U')^\perp) > \mu(\hat{V} \cap \text{cl } U').$$

So, Lemma 2.5 furnishes a good ball  $B \subseteq \hat{V}$  satisfying

$$\mu(B) > \mu(\hat{V} \cap \text{cl } U').$$

By Lemma 3.9, we have that  $\neg \text{vol}_{\leq}(\text{int } B, U', \hat{V})$ , and that some  $g \in G[\hat{V}]$  satisfies that  $g(U') \subseteq \text{int } B$ . It follows that

$$\text{cl } U \subseteq \text{cl } U' \subseteq g^{-1}(B) \subseteq \hat{V} \subseteq V,$$

as desired.

For the backward direction, we pick a good ball  $B$  satisfying  $U \subseteq B \subseteq \hat{V}$  for a suitable  $\hat{V} \in \pi_0(V)$  and set  $U' := \text{int } B$ . Let us consider an arbitrary connected regular open set  $W \subseteq \hat{V}$  satisfying  $\neg \text{vol}_{\leq}(W, U', \hat{V})$ . From Lemma 3.9 again, we see that  $\mu(W) > \mu(U') = \mu(B)$ . We may therefore find some  $g \in G[\hat{V}]$  such that  $g(B) \subseteq W$ . This shows that  $\text{RCB}_{\text{vol}}(U, V)$  holds.  $\square$

When using Lemma 3.10, we will write RCB both in the case of the full homeomorphism group and the measure-preserving homeomorphism group, and suppress the symbol  $\text{vol}$  from the notation. In fact, many of the formulae below will actually have different meanings for  $\mathcal{M}$  and for  $\mathcal{M}_{\text{vol}}$ . Let us also record the fact that  $\text{RCB}_{(\text{vol})}(U, M)$  implies that  $\text{cl } U \subseteq \text{int } M$ .

**3.4.2. Detecting finiteness of components.** From part (5) of Corollary 3.7, we can detect whether or not a given regular open set has exactly  $k$  connected component in the theory of  $G$  for each fixed  $k \in \omega$ . It is not obvious how to express the infinitude of the connected components of  $U \in \text{RO}(M)$ , as such an infinitude would be equivalent to the infinite conjunction

$$\neg \text{cc}_0(U) \wedge \neg \text{cc}_1(U) \wedge \neg \text{cc}_2(U) \wedge \cdots .$$

However, one can express such an infinitude in a single formula.

**Definition 3.11.** Let us set

$$\text{dispersed}(u) \equiv (\forall \hat{u} \in \pi_0(u))[\text{RCB}(\hat{u}, u^\perp \oplus \hat{u})].$$

We say a regular open set  $U$  is *dispersed* if  $\text{dispersed}(U)$  holds.

Note that  $\text{dispersed}(U)$  implies that

$$\text{cl } \hat{U} \cap \text{cl}(U \setminus \hat{U}) = \emptyset$$

for each connected component  $\hat{U}$  of  $U$ . Let us introduce another formula below that will play crucial roles in several places of this paper; the proof is clear.

**Lemma 3.12.** *There exists an  $L_{G, \text{RO}}$ -formula  $\text{seq}(u, v, \gamma)$  such that*

$$\text{Act}_{G, \text{RO}}(M, G) \models \text{seq}(U, V, g)$$

for  $U, V \in \text{RO}(M)$  and  $g \in G$  if and only if the following conditions hold for a unique  $U' \in \pi_0(U)$ .

- (i) *The set  $U$  is dispersed;*
- (ii) *we have that  $V \cup g(V) \subseteq U$ ;*
- (iii) *for all  $\hat{U} \in \pi_0(U)$ , the set  $\hat{U} \cap V$  is nonempty and connected;*
- (iv) *for all  $\hat{U} \in \pi_0(U) \setminus \{U'\}$ , the set  $\hat{U} \cap g(V)$  is nonempty and connected;*
- (v) *we have that  $U' \cap g(V) = \emptyset$ ;*
- (vi) *if a union  $W$  of connected components of  $U$  satisfies that  $U' \subseteq W$  and that  $g(V \cap W) \subseteq W$ , then  $W = U$ .*

In the situation as in Lemma 3.12, we can enumerate the components of  $U$  as

$$\hat{U}_0 = U', \hat{U}_1, \dots$$

such that  $g(V \cap \hat{U}_i) \subseteq \hat{U}_{i+1}$  for each  $i \geq 0$ . Furthermore, we have an injection

$$\sigma = \sigma_{U, V, g}: \pi_0(U) \longrightarrow \pi_0(U) \setminus \{U'\}$$



sending  $\hat{U}_i$  to  $\hat{U}_{i+1}$  for each  $i \in \omega$ . We also note that for each  $i \in \omega$  there exists a uniformly definable function  $\text{seq}_i(u, v, g)$  such that

$$\text{seq}_i(U, V, g) = \hat{U}_i.$$

We can now establish the main result of this subsection.

**Lemma 3.13.** *There exists a formula  $\text{infcomp}(w)$  such that*

$$\models \text{infcomp}(W) \quad \text{if and only if } W \text{ has infinitely many connected components.}$$

*Proof.* Let us define

$$\text{infcomp}(w) \equiv (\exists u, v, \gamma)[\emptyset \neq u \subseteq w \wedge (\forall \hat{w} \in \pi_0(w))[\text{conn}(u \cap \hat{w})] \wedge \text{seq}(u, v, \gamma)].$$

In order to prove the forward direction, suppose we have  $\text{seq}(U, V, g)$  for some nonempty  $U \subseteq W$  such that each connected component  $\hat{W}$  of  $W$  satisfies  $\text{conn}(U \cap \hat{W})$ . In particular, we have  $|\pi_0(U)| \leq |\pi_0(W)|$ . The injection  $\sigma_{U, V, g}$  above certifies that  $\pi_0(U)$  is an infinite set. Hence,  $\pi_0(W)$  is infinite as well.

For the backward direction, suppose that  $W$  has infinitely many components. We will establish  $\models \text{infcomp}(W)$  only in the case of  $(M, G) \in \mathcal{M}_{\text{vol}}$ , since the case  $(M, G) \in \mathcal{M}$  is strictly easier. We use an idea similar to the proof of Proposition 2.7. We first find distinct components  $\{\hat{W}_i\}_{i \in \omega}$  of  $W$  such that some sequence  $\{x_i\}_{i \in \omega}$  satisfying  $x_i \in \hat{W}_i$  converges to some point  $x^* \in M$ . We consider a sufficiently small compact chart neighborhood  $B$  of  $x^*$ , which still intersects infinitely many components of  $W$ . Let  $n = \dim M$ . By the Oxtoby–Ulam theorem, we can simply identify  $B$  with  $B^n(1)$  or  $B^n(1) \cap \mathbb{H}_+^n$  equipped with the Lebesgue measure. The point  $x^*$  is then identified with the origin  $O$ . By shrinking each  $\hat{W}_i$  to  $\hat{U}_i \subseteq \text{int } B$  and passing to a subsequence, we can further require the following for all  $i \geq 0$ .

- The open set  $\hat{U}_i$  is an open Euclidean ball, converging to  $x^* = O$ ;
- We have  $\text{dist}(x^*, \hat{U}_{i+1}) + \text{diam}(\hat{U}_{i+1}) < \text{dist}(x^*, \hat{U}_i)$ .

We set  $U := \bigsqcup_i \hat{U}_i = \oplus_i \hat{U}_i$  and  $U' = \hat{U}_0$ . We can find a disjoint collection of compact topological balls  $\{D_i\}$  such that  $\text{int } D_i$  intersects both  $\hat{U}_i$  and  $\hat{U}_{i+1}$ , and no other  $\hat{U}_j$ 's. Using the path-transitivity as in Lemma 2.4, we can inductively find a  $g_i \in G[\text{int } D_i]$  sending some good ball  $C_i \subseteq \hat{U}_i$  onto another good ball inside  $\hat{U}_{i+1}$ . We will set

$$V = \bigcup_{i \in \omega} \text{int } C_i.$$

By the uniform convergence theorem, the sequence  $\{\prod_{i=0}^k g_i\}_{k \geq 0}$  converges to a homeomorphism  $g \in \text{Homeo}_{0, \mu}(M) \leq G$ , which witnesses the properties  $\text{infcomp}(U)$  requires.  $\square$

We remark that we may also test if a regular open set has finitely many components, and write

$$\text{fincomp}(u) \equiv \neg \text{infcomp}(u).$$

**3.4.3. Touching and containing the boundary.** By a *collar (embedding)* of the boundary in a manifold  $M$ , we mean an embedding  $h: \partial M \times [0, 1) \rightarrow M$  that extends the identity map  $\partial M \times \{0\} \rightarrow \partial M$ ; we sometimes allow  $h$  to be an embedding of  $\partial M \times [0, 1]$ . The image of a collar embedding is called a *collar neighborhood*. A fundamental result due to Brown [6, Theorem 2] says that the boundary of a topological manifold admits a collar. Let us now state several formulae regarding the boundary of a given manifold.

**Lemma 3.14.** *There exist  $L_{G,RO}$ -formulae as follows.*

(1) A formula  $\text{touch}_\partial(u)$  such that

$$\models \text{touch}_\partial(U) \quad \text{if and only if the closure of } U \text{ nontrivially intersects } \partial M.$$

(2) A formula  $\text{stab}_\partial(\gamma)$  such that

$$\models \text{stab}_\partial(g) \quad \text{if and only if } g \text{ setwise stabilizes each boundary component of } M.$$

*Proof.* (1) Let us define the formula

$$\text{finint}(u, w) \equiv (\exists u')[\text{fincomp}(u') \wedge (\forall \hat{u} \in \pi_0(u))[\hat{u} \cap w \neq \emptyset \rightarrow \hat{u} \in \pi_0(u')]].$$

It is clear from the formulation that

$$\models \text{finint}(U, W)$$

if and only if  $U$  meets  $W$  in finitely many components on  $U$ . We now set

$$\text{touch}_\partial(u) \equiv (\exists u')[u' \subseteq u \wedge \text{infcomp}(u') \wedge (\forall w)[\text{RCB}(w, M) \rightarrow \text{finint}(u', w)]].$$

Suppose that  $\text{cl } U \cap \partial M \neq \emptyset$ . Choose a sequence of points  $\{p_i\}_{i \in \omega}$  in  $U$  converging to a point in  $\partial M$ , and choose small open balls  $U_i \ni p_i$  in  $U$  with pairwise disjoint closures and with radii tending to zero. Let  $U'$  be the union of these balls. Now, if  $W$  fails to satisfy  $\text{finint}(U', W)$ , then  $W$  must meet infinitely many of the balls  $U_i$ , and so  $\text{cl } W \cap \partial M \neq \emptyset$ . In particular,  $\neg \text{RCB}(W, M)$ .

Conversely, suppose that  $\text{cl } U \cap \partial M = \emptyset$ , and let  $U' \subseteq U$  have infinitely many components  $\{U_i\}_{i \in \omega}$ . As in Lemma 3.13, by shrinking components of  $U'$  and passing to a subsequence, we may assume that each  $U_i$  is an open ball, that the sequence has shrinking radii and converges monotonically to the origin in an open chart in  $\mathbb{R}^n$ . Moreover, the origin in this chart lies in the interior of  $M$ , by assumption.

Thus, we may take  $W$  to be a neighborhood of the origin in this chart, which then satisfies  $\text{RCB}(W, M)$  and meets infinitely many components of  $U'$ . Thus,  $U'$  fails to witness  $\text{touch}_\partial(U)$ , and so  $\text{touch}_\partial(U)$  does not hold.

(2) Setting

$$\text{contain}_\partial(u) \equiv \neg \text{touch}_\partial(u^\perp),$$

we see that  $\text{contain}_\partial(U)$  holds if and only if  $\partial M \subseteq U$ . We now define

$$\text{stab}_\partial(\gamma) \equiv (\forall u, \hat{u})[(\hat{u} \in \pi_0(u) \wedge \text{contain}_\partial(u) \wedge \text{touch}_\partial(\hat{u})) \rightarrow \hat{u} \cap \gamma(\hat{u}) \neq \emptyset].$$

We claim that  $\text{stab}_\partial(g)$  holds for  $g \in G$  if and only if  $g$  setwise stabilizes each component of

$$\partial M = \partial_1 \sqcup \partial_2 \sqcup \cdots \sqcup \partial_k.$$

For the forward direction, suppose we have  $\text{stab}_\partial(g)$ . By the aforementioned result of Brown, we can pick a closure-disjoint collection of collar neighborhoods  $\{V_1, V_2, \dots, V_k\}$ . Setting the set

$$U := \bigsqcup_{i=1}^k V_i,$$

we see from the hypothesis that  $g(V_i) \cap V_i \neq \emptyset$  for each  $i$ , which trivially implies  $g(\partial_i) = \partial_i$ . The backward direction is clear after observing that the hypothesis part of  $\text{stab}_\partial(g)$  simply says that  $\hat{U}$  contains at least one boundary component.  $\square$

#### 4. INTERPRETATION OF SECOND-ORDER ARITHMETIC

The goal of this section is to prove that the group  $G$  interprets second order arithmetic and analysis uniformly for  $(M, G) \in \mathcal{M}_{(\text{vol})}$ , establishing the case of  $i = 1$  in Proposition 3.3.

**4.1. An example of an interpretation of first order arithmetic.** As a warm-up, let us interpret first order arithmetic

$$\text{Arith}_1 = (\mathbb{N}, +, \times, 0, 1)$$

in the structure  $\text{Act}_{G, \text{RO}}(M, G)$ . For this, we consider the surjection

$$\#\pi_0: \{U \in \text{RO}(M) \mid \text{fincomp}(U)\} \longrightarrow \mathbb{N}$$

sending each  $U$  to  $\#\pi_0(U)$ , namely the cardinality of  $\pi_0(U)$ . The domain of this surjection is clearly definable, and so is the fiber by the formula  $\#_=(u, v)$  in Corollary 3.7. To complete an interpretation of  $\text{Arith}_1$ , it now suffices to establish the following.

**Lemma 4.1.** *There exist  $L_{G, \text{RO}}$ -formulae  $\#_+$  and  $\#_\times$  such that the following hold for all  $U, V, W$  having finitely many connected components.*

- (1) We have  $\models \#_+(U, V, W)$  if and only if  $\#\pi_0(W) = \#\pi_0(U) + \#\pi_0(V)$ .
- (2) We have  $\models \#_\times(U, V, W)$  if and only if  $\#\pi_0(W) = \#\pi_0(U) \cdot \#\pi_0(V)$ .

*Proof.* Recall the meaning of the formula  $\text{ccpartition}$  from Corollary 3.7. Let us make the following definitions.

$$\begin{aligned}\#_+(u, v, w) &\equiv (\exists w_0, w_1)[\text{ccpartition}(w_0, w_1, w) \wedge \#_=(w_0, u) \wedge \#_=(w_1, v)], \\ \#_\times(u, v, w) &\equiv (\exists w')[(w' \subseteq u) \wedge \#_=(w, w') \wedge (\forall \hat{u} \in \pi_0(u))[\#_=(\hat{u} \cap w', v)]].\end{aligned}$$

It is straightforward to check that these formulae have the intended meanings.  $\square$

**4.2. Our interpretation of second order arithmetic.** Let us now describe an interpretation of second order arithmetic

$$\text{Arith}_2 = (\mathbb{N}, \mathcal{P}(\mathbb{N}), 0, 1, +, \times, \in),$$

which has two sorts  $\mathbb{N}$  and  $\mathcal{P}(\mathbb{N})$ . In particular, we will have to be able to quantify over subsets of  $\mathbb{N}$ .

In order to achieve this, we will consider more restricted class of regular open sets  $U$ , the components of which admit a linear order as described by the formula  $\text{seq}(U, V, g)$ ; see Section 3.4.2. In this linear order of  $U$ , the  $k$ -th component  $\hat{U}_k$  will interpret the integer  $k \in \omega$ , and a union of the connected components  $W$  will interpret a subset naturally. We will utilize Lemma 4.1, but not the actual interpretation itself from the previous subsection.

To be more concrete, let us first note the following.

**Lemma 4.2.** *There exists a uniformly defined function  $\text{seq}_\uparrow(u, v, \gamma, \hat{u})$  such that if*

$$\models \text{seq}(U, V, g) \wedge \hat{U} \in \pi_0(U),$$

*then for the unique  $k \in \omega$  satisfying  $\hat{U} = \text{seq}_k(U, V, g)$ , we have that*

$$\text{seq}_\uparrow(U, V, g, \hat{U}) = \bigoplus_{0 \leq i \leq k} \text{seq}_i(U, V, g).$$

*Proof.* It is routine to check that the following has the intended meaning:

$$\begin{aligned}(w = \text{seq}_\uparrow(u, v, \gamma, \hat{u})) &\equiv \text{ucc}(w, u) \wedge (\text{seq}_0(u, v, \gamma) \oplus \hat{u}) \subseteq w \wedge \\ &\quad \gamma(v \cap \hat{u}) \cap w = \emptyset \wedge \gamma^{-1}(w \cap \gamma(v)) \subseteq w.\end{aligned}\quad \square$$

Let us consider the set

$$X_1 := \{(U, V, g, \hat{U}) \mid \text{seq}(U, V, g) \wedge \hat{U} \in \pi_0(U)\},$$

which is definable in  $\text{Act}_{\mathbf{G}, \text{RO}}(M, G)$  uniformly for  $(M, G) \in \mathcal{M}_{(\text{vol})}$ . We have a surjection

$$\rho_1 := \# \pi_0 \circ \text{seq}_\uparrow : X_1 \longrightarrow \mathbb{N}.$$

This surjection satisfies that

$$k = \rho_1(U, V, g, \hat{U}) \quad \text{if and only if} \quad \hat{U} = \text{seq}_k(U, V, g).$$

The fiber of  $\rho_1$  is

$$\{(\underline{y}, \underline{z}) \in X_1 \times X_1 \mid \#_=(\text{seq}_\uparrow(\underline{y}), \text{seq}_\uparrow(\underline{z}))\},$$

and hence uniformly definable. It is trivial to check that  $\rho_1$  produces a uniform interpretation of  $\text{Arith}_1$  to  $\text{Act}_{G, \text{RO}}(M, G)$ . For instance, we have

$$\rho_1(\underline{y}) + \rho_1(\underline{y}') = \rho_1(\underline{y}'')$$

if and only if

$$\models \#_+(\text{seq}_\uparrow(\underline{y}), \text{seq}_\uparrow(\underline{y}'), \text{seq}_\uparrow(\underline{y}'')).$$

After this interpretation of  $\text{Arith}_1$ , the symbol  $\#$  has an intended meaning as a function from  $\text{RO}(M)$  to  $\mathbb{N}$ . We have uniformly defined functions  $\text{seqcomp}(u, v, \gamma, \alpha)$  and  $\text{seqcomp}_\uparrow(u, v, \gamma, \alpha)$  satisfying

$$\begin{aligned} \text{seqcomp}(U, V, g, k) &= \text{seq}_k(U, V, g), \\ \text{seqcomp}_\uparrow(U, V, g, k) &= \bigoplus_{i=1}^k \text{seqcomp}(U, V, g, i). \end{aligned}$$

Similarly, we consider another uniformly definable set

$$X'_1 := \{(U, V, g, W) \mid \text{seq}(U, V, g) \wedge \text{ucc}(W, U)\}.$$

We have a surjection

$$\rho'_1: X'_1 \longrightarrow \mathcal{P}(\mathbb{N})$$

defined by the condition

$$\rho'_1(U, V, g, W) := \{\rho_1(U, V, g, \hat{W}) \mid \hat{W} \in \pi_0(W)\}.$$

Since the fiber of  $\rho_1$  is uniformly definable, so is that of  $\rho'_1$ . We will introduce the function symbol  $\mathcal{P}_\#$  in  $L_{\text{Act}}^2$  interpreted as  $\rho'_1$ .

Finally, we have

$$\rho_1(U_1, V_1, g_1, \hat{U}) \in \rho'_1(U_2, V_2, g_2, W_2)$$

if and only if

$$\#\pi_0 \text{seq}_\uparrow(U_1, V_1, g_1, \hat{U}) = \#\pi_0 \text{seq}_\uparrow(U_2, V_2, g_2, \hat{W})$$

for some  $\hat{W} \in \pi_0(W_2)$ . Hence, the pair of surjections  $(\rho_1, \rho'_1)$  produces a desired interpretation of the two-sorted structure  $\text{Arith}_2$ . We note that the order relation symbol  $<$ , the successor symbol  $S$ , and the inclusion symbol  $\subseteq$  are naturally interpreted as a consequence.

**4.3. Analysis.** The interpretation of  $\mathbb{R}$  is now standard. From  $\mathbb{N}$ , we interpret  $\mathbb{Z}$ , together with addition, multiplication, and order, by imposing a suitable definable equivalence relation on a suitable definable subset of  $\mathbb{N}^2$ . We similarly interpret  $\mathbb{Q}$  by imposing a suitable definable equivalence relation on a definable subset of  $\mathbb{Z} \times \mathbb{Z}$ .

We define  $\mathbb{R}$  together with addition, multiplication, and order via Dedekind cuts of  $\mathbb{Q}$ , is interpretable because of our access to  $\mathcal{P}(\mathbb{N})$ . Finally, we have canonical identifications of

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R},$$

wherein we set  $=$  to be the relation identifying natural numbers with their images under this sequence of inclusions. In the sequel, we will simply talk about natural numbers, integers, or rationals as elements of  $\mathbb{R}$  without further comment. We further may assume to have  $\mathbb{R}^k$  in the universe of the structure for all  $k \in \omega$ .

In order to justify the introduction of the sort symbol  $\text{Cont}_{k,\ell}$  in the structure, let us first note that each function in  $C(\mathbb{R}, \mathbb{R})$  is uniquely determined by its restriction on  $\mathbb{Q}$ . Since

$$|\mathbb{R}^{\mathbb{Q}}| = (2^\omega)^\omega = 2^\omega = |\mathbb{R}|,$$

we have an interpretation of  $C(\mathbb{R}, \mathbb{R})$  by  $\mathbb{R}$ , and hence, that of

$$C(\mathbb{R}^k, \mathbb{R}^\ell).$$

This latter set is the domain of  $\text{Cont}_{k,\ell}$ , and the function symbols

$$\text{appl}(\chi, \rho) = \sigma, \quad \text{norm}(\chi) = \rho$$

are interpreted accordingly. In practice, we write

$$f(r) = s, \quad \|f\| = r$$

for the above formulae. The expanded language containing  $\text{Act}_{\mathbf{G}, \text{RO}}$  structure, second order arithmetic, and analysis will be written  $\text{Act}^2 = \text{Act}_{\mathbf{G}, \text{RO}, \mathbb{R}}$ . This establishes the uniform interpretability of  $\text{Act}_{\mathbf{G}, \text{RO}, \mathbb{R}}(M, G)$  to  $\text{Act}_{\mathbf{G}, \text{RO}}(M, G)$ , namely Proposition 3.3 for the case  $i = 1$ .

## 5. INTERPRETATION OF POINTS

We now wish to be able to talk about points of  $M$  more directly, and prove Proposition 3.3 for the case  $i = 2$ . This would complete the proof of Theorem 1.8.

Rubin [38] accesses points in a space with a locally dense action via a certain collection of ultrafilters consisting of regular open sets; in his approach, the intersection of the closures of all the open sets in each ultrafilter corresponds to a single point of the space. We cannot follow this approach directly, as we need to stay within the first order theory of groups and Boolean algebras. Instead, we consider a certain collection of regular open sets such that the components in each of those

open sets converge to a single point of the manifold. We continue to make the standing assumption that  $(M, G) \in \mathcal{M}_{(\text{vol})}$  with  $\dim M > 1$ , unless stated otherwise.

**5.1. Encoding points of a manifold.** Using the  $L_{G, \text{RO}, \mathbb{R}}$ -formulae introduced in the preceding sections, we define the following new formulae:

$$\begin{aligned} \text{cof}(w, u) &\equiv \text{infcomp}(w) \wedge w \subseteq u \wedge (\forall \hat{u} \in \pi_0(u))[\text{conn}(w \cap \hat{u})], \\ \text{cofcontain}(w, u) &\equiv (\exists w')[w' \subseteq w \wedge \text{cof}(w', u)], \\ \text{cofmove}(\gamma, u_0, u_1) &\equiv (\exists w)[\text{cof}(w, u_0) \wedge \text{cof}(\gamma(w), u_1) \wedge (\forall \hat{w} \in \pi_0(w))[\text{RCB}(\hat{w}, u_0)]] \end{aligned}$$

Note that when  $\models \text{cofmove}(g, U_0, U_1)$ , we can find some  $W$  whose connected components can be written as

$$W = \bigsqcup_{i \in \omega} W_i,$$

with the property that each  $W_i$  is contained in some relatively compact ball inside  $U_0$ ; moreover, no two components of  $W$  belong to the same component of  $U_0$ , and similarly for  $g(W)$  and  $U_1$ .

We consider the definable subgroup

$$\text{St}_G^{\text{lim}}(U) := \bigcup \{G[W^\perp] \mid W \in \text{RO}(M) \text{ and } \text{cof}(W, U)\},$$

which we call as the *limit stabilizer* of  $U$  in  $G$ . Intuitively, each element of this group fixes some open sets that are arbitrarily close to a certain limit point of the components of  $U$ . We will write  $\gamma \in \text{St}_G^{\text{lim}}(u)$  for the formula corresponding to  $g \in \text{St}_G^{\text{lim}}(U)$ .

*Remark 5.1.* One can rephrase Rubin's interpretation of points in second order logic [38] as follows, as summarized in [23, Theorem 3.6.17]. Rubin allowed certain collections (called, *good ultrafilters*) of regular open sets to interpret a single point in the space, by taking the intersection of the closures of those open sets. He then proved that two good ultrafilters  $P$  and  $Q$  interpret different points  $p$  and  $q$  if and only if the group

$$G\{Q^\perp\} := \bigcup \{G[W^\perp] \mid W \in Q\}$$

acts *sufficiently transitively*, in the sense that for some  $U \in P$ , every  $V \in \text{RO}(M)$  satisfying  $\emptyset \neq V \subseteq U$  is an element of the set

$$G\{Q^\perp\}(P).$$

In our approach, we will utilize the sufficient transitivity of the limit stabilizer group characterized in terms of the formula  $\text{cofmove}(\gamma, w_0, w_1)$ .

Let us consider the set  $X_2 := \text{Act}^2(M, G)(\phi_{\text{dom}}^2)$ , defined by the following formula:

$$\begin{aligned} \phi_{\text{dom}}^2(u, v, \gamma) \equiv & \text{seq}(u, v, \gamma) \wedge (\forall w_0, w_1)[\text{cof}(w_0, u) \wedge \text{cof}(w_1, u) \longrightarrow \\ & (\exists \delta \in \text{St}^{\text{lim}}(w_0))[\text{cofmove}(\delta, w_0, w_1)]]]. \end{aligned}$$

The following lemma furnishes an interpretation of the points.

**Lemma 5.2.** *For each  $(U, V, g) \in X_2$  and for an arbitrary sequence  $\{x_i\}_{i \in \omega}$  satisfying  $x_i \in \text{seqcomp}(U, V, g, i)$  for all  $i \in \omega$ , the limit*

$$\rho_2(U, V, g) := \lim_{i \rightarrow \infty} x_i$$

*exists in  $M$ , and is independent of the choice of  $\{x_i\}$ . Moreover, the following conclusions hold.*

- (1) *The map  $\rho_2: X_2 \longrightarrow M$  is surjective.*
- (2) *We have*

$$\rho_2(U_0, V_0, g_0) = \rho_2(U_1, V_1, g_1)$$

*if and only if some  $g \in \text{St}_G^{\text{lim}}(U_0)$  satisfies*

$$\text{cofmove}(g, U_0, U_1).$$

- (3) *We have*

$$h(\rho_2(U, V, g)) = \rho_2(U', V', g')$$

*if and only if*

$$\rho_2(h(U), h(V), hgh^{-1}) = \rho_2(U', V', g').$$

- (4) *We have  $\rho_2(U, V, g) \notin W$  if and only if some  $(U', V', g') \in X_2$  satisfies*

$$U' \cap W = \emptyset \wedge (\rho_2(U, V, g) = \rho_2(U', V', g')).$$

- (5) *We have  $\rho_2(U, V, g) \in \text{int } M$  if and only if there exists some  $W$  such that  $\text{RCB}(W, M)$  and such that  $\text{cofcontain}(W, U)$ .*

*Proof.* Let  $(U, V, g) \in X_2$ , and let

$$\{x_i \in \text{seqcomp}(U, V, g, i)\}_{i \in \omega}$$

be a sequence. In particular, we have  $x_i \in \text{int } M$ . Suppose two subsequences

$$\{y_{0,j}\}_{j \in \omega}, \{y_{1,j}\}_{j \in \omega} \subseteq \{x_i\}_{i \in \omega}$$

converge to two distinct points  $y_0$  and  $y_1$ . For  $i = 0$  and  $i = 1$ , we let  $W_i$  be the union of sufficiently small good open balls  $W_{i,j}$  centered at  $y_{i,j}$ . In particular, we may assume that  $\text{cof}(W_i, U)$ , and that

$$\lim_j W_{i,j} = \{y_i\}$$



in the Hausdorff sense. By hypothesis, we have some  $h \in \text{St}^{\text{lim}}(W_0)$  such that

$$\models \text{cofmove}(h, W_0, W_1).$$

Since  $h$  fixes arbitrarily close points to  $y_0$ , we have  $h(y_0) = y_0$ . It follows that

$$y_0 = h(y_0) = \lim h(y_{0,j}) = y_1.$$

This proves the existence of the claimed limit. The same argument also implies the independence of the limit from the choice of  $\{x_i\}_{i \in \omega}$ , and also the backward direction of part (2). The surjectivity of  $\rho_2$  in part (1) is clear, after choosing  $U$  to be a suitable sequence of good open balls converging to a given point in the Hausdorff sense.

Let us verify the forward direction of part (2). Assuming the hypothesis, we can find two sequences  $\{x_{0,j}\}_{j \in \omega}$  and  $\{x_{1,j}\}_{j \in \omega}$  such that

$$x_{i,j} \in \text{seqcomp}(U_i, V_i, g_i, j).$$

As in the proof of Lemma 3.13, we can find a disjoint collection of good balls  $\{D_i\}$  of decreasing sizes such that each  $D_i$  contains  $x_{0,j}$  and  $x_{1,j}$ , after passing to a subsequence if necessary. By the uniform convergence theorem, we have some  $h \in G$  such that  $h(x_{0,j}) = x_{1,j}$  for all  $j$ , and such that  $h$  pointwise fixes some nonempty open set inside

$$\text{seqcomp}(U_0, V_0, g_0, j) \cap D_j^\perp.$$

In particular, we have that  $h \in \text{St}_G^{\text{lim}}(U_0)$  and that  $\text{cofmove}(h, U_0, U_1)$ , as claimed. The remaining parts of the lemma are straightforward to check.  $\square$

In part (2) of the lemma, we see that the relation

$$\rho_2(U, V, g) = \rho_2(U', V', g')$$

is first order expressible; hence, we deduce that the functional relation  $g(p) = q$  and the membership relation  $p \in W$  in parts (3) and (4) are interpretable for  $p, q \in M$ ,  $g \in M$  and  $W \in \text{RO}(M)$ . Part (5) of the lemma separates out the interior points.

Direct access to points allows us to make direct reference to set theoretic operations. For instance, we can define  $\text{union}(u, v, w)$  by

$$\text{union}(u, v, w) \equiv (\forall \pi)[(\pi \in u \vee \pi \in v) \leftrightarrow \pi \in w].$$

Clearly,  $\text{union}(U, V, W)$  for regular open sets  $\{U, V, W\}$  if and only if  $W = U \cup V$ . Henceforth, we will include the usual set-theoretic union symbol in the language such as  $\cup$ ,  $\cap$  and  $\setminus$ . We are also able now to talk directly about the closure  $\text{cl } U$  of a regular open set  $U$ , both in  $M$  and in  $V$  for arbitrary  $U \subseteq V$ ; for this, it suffices to note that  $p \in \text{cl } U$  if and only if  $p \notin U^\perp$ .

**5.2. Encoding discrete sets of points in a manifold.** Let us now interpret the set

$$\mathcal{P}^{\text{disc}}(\text{int } M) := \{A \subseteq \text{int } M \mid A \text{ is discrete}\}.$$

In particular, every finite subset of  $\text{int } M$  belongs to  $\mathcal{P}^{\text{disc}}(\text{int } M)$ .

We recall from Lemma 3.14 the formula  $\text{finint}(u, w)$ . We first let  $X'_2$  be the set of quadruples  $(U, V, g, W)$  defined by the following formula:

$$\begin{aligned} \psi_{\text{dom}}^2(u, v, \gamma, w) \equiv & \text{dispersed}(w) \wedge (u \oplus \text{supp}^e g \subseteq w) \wedge \\ & \forall \hat{w} \in \pi_0(w) [\phi_{\text{dom}}^2(u \cap \hat{w}, v \cap \hat{w}, \gamma) \wedge \text{RCB}((u \oplus \text{supp}^e \gamma) \cap \hat{w}, \hat{w})]. \end{aligned}$$

For such a quadruple, we set

$$\rho'_2(U, V, g, W) := \{\rho_2(U \cap \hat{W}, V \cap \hat{W}, g) \mid \hat{W} \in \pi_0(W)\}.$$

It is routine to check that this map defines a surjection

$$\rho'_2: X'_2 \longrightarrow \mathcal{P}^{\text{disc}}(\text{int } M)$$

with a definable fiber. Namely, we have

$$\rho'_2(U_0, V_0, g_0, W_0) \neq \rho'_2(U_1, V_1, g_1, W_1)$$

if and only if there exists some regular open sets  $W', W''$  satisfying that

$$\text{RCB}(W', W'')$$

and that

$$\neg \text{finint}(U_i, W') \wedge \text{finint}(U_{1-i}, W'')$$

for some  $i \in \{0, 1\}$ .

We interpret the membership between a point and a set; namely, we have

$$\rho_2(U, V, g) \in \rho'_2(U', V', g', W')$$

if and only if there exists some  $W''$  satisfying that  $\text{RCB}(W'', W')$  and that

$$\text{cofcontain}(W'', U).$$

We also interpret the group action

$$h(\rho'_2(U, V, g, W)) = \rho'_2(U', V', g', W')$$

as

$$\rho'_2(h(U), h(V), hgh^{-1}, h(W)) = \rho'_2(U', V', g', W')$$

Finally, the set  $\rho'_2(U, V, g, W) \in \mathcal{P}^{\text{disc}}(\text{int } M)$  has finite cardinality if and only if  $W$  has finitely many connected components. In this case, the cardinality function  $\#$  for  $T \in \mathcal{P}^{\text{disc}}(\text{int } M)$  is clearly definable by

$$\#(\rho'_2(U, V, g, W)) = \#\pi_0(W).$$

We omit the details, which are very similar to those in Section 5.1. We denote by  $\mathbf{M}^{\text{disc-int}}$  the sort symbol for sets belong to  $\mathcal{P}^{\text{disc}}(\text{int } M)$ .

**5.3. Interpreting exponentiation.** We now interpret the map

$$G \times \mathbb{Z} \times M \longrightarrow M, \quad (g, k, p) \mapsto g^k \cdot p,$$

so that the exponentiation map

$$\exp : G \times \mathbb{Z} \longrightarrow G$$

is definable. Note that  $g^k(p) = p'$  holds with  $k \in \omega$  if and only if we can write  $k = mq + r$  for some integers  $0 \leq r < m$  and  $q$  such that we have a period- $m$  orbit

$$p, g(p), \dots, g^m(p) = p,$$

and a sequence of distinct points

$$p, g(p), \dots, g^r(p) = p'.$$

Let us now define formulae  $\exp_{\text{cyc}}$  and  $\exp_{\text{lin}}$ , which will express the existences of a periodic orbit and of a sequence without repetitions, respectively. More precisely, we set

$$\begin{aligned} \exp_{\text{cyc}}(\gamma, \alpha, \pi) &\equiv (\alpha = 0) \vee (\exists \tau \in \mathbf{M}^{\text{disc-int}})[\# \tau = \alpha \wedge (\pi \in \tau = \gamma(\tau)) \\ &\quad \wedge \neg(\exists \tau' \in \mathbf{M}^{\text{disc-int}})[\emptyset \neq \tau' \subsetneq \tau \wedge \gamma \cdot \tau' = \tau']], \\ \exp_{\text{lin}}(\gamma, \alpha, \pi, \pi') &\equiv (\exists \tau \in \mathbf{M}^{\text{disc-int}})[\# \tau = \alpha + 1 \wedge \{\pi, \pi'\} \subseteq \tau \wedge \gamma(\tau \setminus \{\pi'\}) = \tau \setminus \{\pi\} \\ &\quad \wedge \neg(\exists \tau' \in \mathbf{M}^{\text{disc-int}})[\emptyset \neq \tau' \subsetneq \tau \wedge \gamma \cdot \tau' = \tau']]. \end{aligned}$$

We see that  $\exp(g, k) \cdot p = p'$  with  $k \geq 0$  if and only if the tuple  $(g, k, p, p')$  satisfies the formula

$$\exp(\gamma, \alpha, \pi, \pi') \equiv (\exists \alpha', \beta_1, \beta_2)[\alpha = \beta_2 \alpha' + \beta_1 \wedge \exp_{\text{cyc}}(\gamma, \beta_2, \pi) \wedge \exp_{\text{lin}}(\gamma, \beta_1, \pi, \pi')].$$

It is then trivial to extend the definition for the case  $k < 0$ , establishing the definability of the exponentiation function.

**5.4. The AGAPE structure.** We now define our ultimate structure

$$\text{Act}^3(M, G) = \text{Act}_{\mathbf{G}, \text{RO}, \mathbb{R}, \mathbf{M}}(M, G) = \text{AGAPE}(M, G)$$

as the extension of  $\text{Act}^2(M, G) = \text{Act}_{\mathbf{G}, \text{RO}, \mathbb{R}}(M, G)$  by including the points in  $M$  and adding the relations

$$g(p) = q, \quad p \in W$$

for  $g \in G$ ,  $p, q \in M$  and  $W \in \text{RO}(M)$ . We are then justified to use the expressions such as

$$p \in \text{int } M, \quad p \in \partial M, \quad p \in \text{cl } U, \quad g^n = h, \quad \text{fix } g = \text{cl } U, \quad U \cup V = W$$

for points  $p$ , regular open sets  $U, V, W$ , group elements  $g, h \in G$  and integer  $n \in \mathbb{Z}$  in  $\text{AGAPE}(M, G)$ .

## 6. BALLS WITH DEFINABLE PARAMETRIZATIONS

From this point on, we work in the AGAPE language  $L_{\text{AGAPE}} = L_{\mathbf{G}, \text{RO}, \mathbb{R}, \mathbf{M}}$ , containing second order arithmetic and points. The underlying structure will be  $\text{AGAPE}(M, G)$ ; recall our further standing assumption that  $\dim M > 1$ . We will use the notation  $I^n = [0, 1]^n$  and  $Q^n(r) := [-r, r]^n$ . The main objective of this section is to interpret the dimension and collared balls inside of a manifold, as described in the following two theorems.

**Theorem 6.1.** *For each  $n \geq 2$ , there exists a formula  $\text{dim}_n$  such that we have  $\models \text{dim}_n$  if and only if  $M$  is an  $n$ -manifold.*

**Theorem 6.2.** *For each  $n \geq 2$ , there exist formulae*

$$\text{flows}_n(u, \gamma, \pi), \text{Param}_n(u, \gamma, \pi, \rho, \pi')$$

*such that the following hold for all  $(M, G) \in \mathcal{M}_{(\text{vol})}$  with  $n = \dim M$ .*

(1) *Let  $U \in \text{RO}(M)$ ,  $\underline{g} \in G^n$  and  $p \in M$ . If*

$$\models \text{flows}_n(U, \underline{g}, p)$$

*then we have a unique homeomorphism*

$$\Psi = \Psi[U, \underline{g}, p]: I^n \longrightarrow \text{cl } U$$

*the graph  $\Gamma$  of which satisfies*

$$\Gamma = \{(r, q) \in I^n \times M: \text{AGAPE}(M, G) \models \text{Param}_n(U, \underline{g}, p, r, q)\},$$

*and also  $(0, p) \in \Gamma$ .*

(2) *Let  $U$  and  $V$  be good open balls inside  $\text{int } M$  such that  $\text{cl } U \subseteq V$ ; if  $(M, G) \in \mathcal{M}_{\text{vol}}$ , we further assume that  $\text{vol}(U)/\text{vol}(V)$  is sufficiently small compared to some positive number determined by  $n$ . Then we have*

$$\models (\exists \underline{\gamma} \exists \pi) \text{flows}_n(U, \underline{\gamma}, \pi).$$

In Section 8, we will modify the definition of  $\Psi[U, \underline{g}, p]$  so that the domain is  $Q^n(2)$ , instead of  $I^n$ . We emphasize again that the above formulae for  $\mathcal{M}$  and  $\mathcal{M}_{\text{vol}}$  may differ; for instance, the sentence  $\text{dim}_n$  may be more precisely denoted as  $\text{dim}_n$  and  $\text{dim}_n^{\text{vol}}$  separately depending on the context.

**6.1. Detecting the dimension of a manifold.** We prove Theorem 6.1 by interpreting a sufficient amount of dimension theory. For a topological space  $X$ , the *order* of a finite open cover  $\mathcal{U}$  is defined as the number

$$\sup_{x \in X} |\{U \in \mathcal{U} \mid x \in U\}|.$$

Though in classical literature one considers general open covers, it is sufficient (especially in our situation) to consider finite covers only; cf. [12, 15].

We say the *topological dimension* of  $X$  is at most  $n$ , and write  $\dim X \leq n$ , if every finite open cover of  $X$  is refined by an open cover with order at most  $n + 1$ . The topological dimension  $\dim X$  is defined to be  $n$ , if  $\dim X \leq n$  holds but  $\dim X \leq n - 1$  does not. A topological  $n$ -manifold has the topological dimension  $n$ .

A collection of open sets  $\mathcal{V} = \{V_i\}_{i \in \mathcal{I}}$  is said to *shrink* to another collection  $\mathcal{W} = \{W_i\}_{i \in \mathcal{I}}$  if  $W_i \subseteq V_i$  holds for each  $i$  in the index set  $\mathcal{I}$ . Let us note the following well-known facts.

- Lemma 6.3.** (1) (*Lebesgue's Covering Theorem* [21, Theorem IV.2]) *If  $\mathcal{U}$  is a finite open cover of  $I^n$  such that no element of  $\mathcal{U}$  intersects an opposite pair of codimension one faces, then  $\mathcal{U}$  cannot be refined by an open cover of order at most  $n$ .*
- (2) (*Čech* [8]) *If  $X$  is a metrizable space and if  $Y \subseteq X$ , then  $\dim Y \leq \dim X$ .*
- (3) (*Ostrand's Theorem* [30, Theorem 3]) *If  $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$  is a locally finite open cover of a normal space  $X$  satisfying  $\dim X \leq n$ , then for each  $j = 0, \dots, n$ , the cover  $\mathcal{U}$  shrinks to some pairwise disjoint collection  $\mathcal{V}^j = \{V_i^j\}_{i \in \mathcal{I}}$  of open sets such that the collection  $\bigcup_j \mathcal{V}^j$  is a cover.*

We can now give a characterization of manifold dimension.

**Lemma 6.4.** *For each positive integer  $n$  and for each compact manifold  $M$ , the following two conditions are equivalent.*

- (A) *The dimension of  $M$  is at most  $n$ ;*  
 (B) *Let  $W$  be a regular open set in  $M$ . If*

$$\mathcal{U} = \{U_i : i = 1, 2, \dots, 2^{n+1}\}$$

*is a regular open cover of  $\text{cl } W$ , then there exists a pairwise disjoint collection*

$$\mathcal{V}^j = \{V_i^j : i = 1, 2, \dots, 2^{n+1}\}$$

*of regular open sets for each  $j = 0, 1, \dots, n$  such that  $\mathcal{U}$  shrinks to each  $\mathcal{V}^j$ , and such that  $\bigcup_j \mathcal{V}^j$  is a cover of  $\text{cl } W$ .*

*Proof.* Suppose we have  $\dim M \leq n$ , and assume the hypothesis of part (B). We see from Lemma 6.3 (2) that  $\dim \text{cl } W \leq n$ . Part (3) of the same lemma implies that  $\mathcal{U}$  shrinks to a pairwise disjoint collection of (not necessarily regular) open sets  $\mathcal{W}^j = \{W_i^j\}_{i=1, \dots, 2^{n+1}}$  for each  $j = 0, \dots, n$  with the property that  $\bigcup_j \mathcal{W}^j$  is a cover of the normal space  $\text{cl } W$ . By Lemma 2.6, there exists a regular open cover

$$\mathcal{V} := \{V_i^j\}_{i,j}$$

of  $\text{cl } W$  satisfying

$$\text{cl } V_i^j \subseteq W_i^j \subseteq U_i$$

for all  $i$  and  $j$ . This implies the conclusion of (B).

Conversely, suppose we have condition (B) and assume for contradiction that  $m := \dim M > n$ . We first note the following.

**Claim.** *The unit  $m$ -cube  $[0, 1]^m$  admits a finite regular open cover of cardinality  $2^{n+1}$  that cannot be refined by another open cover with order at most  $n + 1$ .*

Let  $C$  denote the unit cube  $[0, 1]^{n+1}$  in  $\mathbb{R}^{n+1}$ , which is embedded in  $\mathbb{R}^m$  as the subset with the last  $m - n - 1$  coordinates being zero. For each vertex  $v \in C^{(0)}$ , let us consider the translated open cube

$$U_v := v + (-1, 1)^{n+1} \subseteq \mathbb{R}^{n+1}.$$

We then have a regular open cover

$$\mathcal{U} := \{U_v : v \in C^{(0)}\}$$

of  $C$  with cardinality  $2^{n+1}$ . Note that each open cube  $U_v$  does not intersect an opposite pair of codimension one faces of  $C$ . By taking the Cartesian product  $U'_v$  of each  $U_v$  with  $(-1, 2)^{m-n-1}$ , we obtain a finite regular open cover

$$\mathcal{U}' = \{U'_v \mid v \in C^{(0)}\}$$

of  $[0, 1]^m$ . If  $\mathcal{U}'$  is refined by another finite open cover  $\mathcal{V}$  of  $[0, 1]^m$  with order at most  $n + 1$ , then the intersection of the elements in  $\mathcal{V}$  with  $\mathbb{R}^{n+1} \subseteq \mathbb{R}^m$  is a finite open cover of  $C = [0, 1]^{n+1}$  with order at most  $n + 1$ . This violates Lebesgue's Lemma (Lemma 6.3), and the claim is thus proved.

Let us now consider a good ball  $Q$  in  $M$ , which comes with an embedding

$$\phi: \mathbb{R}^m \longrightarrow M$$

satisfying  $\phi[0, 1]^m = \text{cl } Q$ . By applying the above claim, we obtain a finite regular open cover of  $\text{cl } Q$  that cannot be refined by a finite open cover with order at most  $n + 1$ . This contradicts condition (B), which we have assumed.  $\square$

Note that the cardinalities of covers  $\mathcal{U}$  and  $\bigcup_j \mathcal{V}^j$  in condition (B) of the above lemma are explicitly bounded above by  $2^{n+1}$  and  $(n + 1)2^{n+1}$ , respectively. Note also that conditions such as

$$\text{cl } W \subseteq U_1 \cup \cdots \cup U_{2^{n+1}}$$

are expressible in the AGAPE language. It is therefore clear that condition (B) is expressible in this language, for each fixed positive integer  $n$ . As a consequence, we obtain Theorem 6.1.

**6.2. Parametrizing balls in  $M$  in dimension two and higher.** For the proof of Theorem 6.2, let us consider the quotient map  $\text{pr} : \mathbb{R} \rightarrow \mathbb{R}/\sqrt{2}\mathbb{Z}$  defined as

$$x \mapsto [x] := x + \sqrt{2}\mathbb{Z}.$$

The image of  $\mathbb{Z}$  is dense in the circle  $\mathbb{R}/\sqrt{2}\mathbb{Z}$ , equipped with the natural cyclic order. The expression  $\sqrt{2}$  will be regarded as a constant symbol in  $L_{\text{AGAPE}}$ . We have chosen this value for concreteness, but for our purpose we may use an arbitrary irrational number that is definable without parameters in arithmetic. There exists a definable function  $\text{ang}(\rho_1, \rho_2)$  satisfying

$$r = \text{ang}(r_1, r_2)$$

if and only if the (unsigned) angular metric between  $[r_1]$  and  $[r_2]$  is  $r \in [0, \sqrt{2})$ .

Let us also define an  $L_{\text{AGAPE}}$  formula

$$\text{fcov}(u, v_0, \dots, v_n) \equiv (\text{cl } u) \subseteq \bigcup_{i=0}^n v_i \wedge \bigwedge_{i=0}^n \text{fincomp}(v_i).$$

We also use the formula

$$\text{clshrink}(v_0, \dots, v_n, v'_0, \dots, v'_n) \equiv \bigwedge_{i=0}^n \text{cl } v'_i \subseteq v_i.$$

We will equip  $M$  with a compatible metric  $d$ , and denote by  $d_\infty$  the induced uniform metric on the homeomorphism group. We have the following characterization of uniform convergence.

**Lemma 6.5.** *Let  $U$  be a regular open set in  $M$  such that  $\text{cl } U \subseteq \text{int } M$ , and let*

$$F_1 \supseteq F_2 \supseteq \dots$$

*be a sequence of subsets of  $\text{Homeo}(M)$  such that each  $f \in F_1$  setwise stabilizes  $U$ . Then the following two conditions are equivalent.*

(A) *We have that*

$$\limsup_{i \rightarrow \infty} \{d_\infty(f \upharpoonright_U, \text{Id} \upharpoonright_U) \mid f \in F_i\} = 0.$$

(B) *Suppose we have two tuples of regular open sets*

$$\underline{V} = (V_0, \dots, V_n), \quad \underline{V}' = (V'_0, \dots, V'_n)$$

*such that*

$$\text{fcov}(U, \underline{V}) \wedge \text{fcov}(U, \underline{V}') \wedge \text{clshrink}(\underline{V}, \underline{V}').$$

*Then there exist some  $i \in \omega$  such that whenever a pair  $(\hat{V}', \hat{V})$  belongs to*

$$A := \left\{ (\hat{V}', \hat{V}) \in \bigcup_{j=0}^n (\pi_0(V'_j) \times \pi_0(V_j)) \mid \hat{V}' \subseteq \hat{V} \right\},$$

each  $f \in F_i$  satisfies

$$f(\hat{V}' \cap \text{cl } U) \subseteq \hat{V}.$$

*Proof.* Let us assume part (A), and also the hypotheses of (B). We set

$$\epsilon_0 := \inf \{d(\text{cl } \hat{V}', M \setminus \hat{V}) \mid (\hat{V}', \hat{V}) \in A\},$$

which is positive since  $A$  is finite. Choosing  $i$  so that

$$d_\infty(f \upharpoonright_U, \text{Id} \upharpoonright_U) < \epsilon_0$$

for all  $f \in F_i$ , we obtain the conclusion.

Conversely, we assume the condition (B) and pick an arbitrary  $\epsilon > 0$ . Let  $\mathcal{U}$  be a finite cover of  $\text{cl } U$  by regular open sets with radius less than  $\epsilon$ . Applying Lemma 6.4 (after replacing the number  $2^{n+1}$  in the lemma by the size of  $\mathcal{U}$ ), we obtain a tuple of regular open sets

$$\underline{V} = (V_0, \dots, V_n)$$

such that every connected component of each  $V_j$  has diameter at most  $2\epsilon$ , and such that  $\text{fcov}(U, \underline{V})$  holds. By Lemma 2.6 and by compactness of  $\text{cl } U$ , we obtain

$$\underline{V}' = (V'_0, \dots, V'_n)$$

such that

$$\text{fcov}(U, \underline{V}') \wedge \text{clshrink}(\underline{V}, \underline{V}').$$

Pick  $i \in \omega$  as given by the condition (B), and let  $f \in F_i$  and  $x \in \text{cl } U$  be arbitrary. Since we have some  $(\hat{V}', \hat{V}) \in A$  such that  $x \in \hat{V}'$ , we see that

$$d(x, f(x)) \leq \text{diam } \hat{V} \leq 2\epsilon.$$

This implies that  $d_\infty(f \upharpoonright_U, \text{Id} \upharpoonright_U) \leq 2\epsilon$  and that the condition (A) holds.  $\square$

We now interpret non-integral powers of group elements, in the following sense.

**Lemma 6.6.** *There exist formulae*

$$\text{conv}(u, \gamma, \rho, \delta), \quad \text{flow}(u, \gamma)$$

such that the following hold for each  $(M, G) \in \mathcal{M}_{(\text{vol})}$ .

- (1) For  $\{g, h\} \subseteq G$ ,  $U \in \text{RO}(M)$  and  $r \in \mathbb{R}$  satisfying  $\text{cl } U \subseteq \text{int } M$  and  $g(U) = U = h(U)$ , we have

$$\models \text{conv}(U, g, r, h)$$

if and only if

$$\lim_{\delta \rightarrow +0} \sup \{d_\infty(g^s \upharpoonright_U, h \upharpoonright_U) \mid s \in \mathbb{Z} \text{ and } \text{ang}(s, r) < \delta\} = 0.$$



(2) For  $g \in G$  and  $U \in \text{RO}(M)$  satisfying  $\text{cl } U \subseteq \text{int } M$  and  $g(U) = U$ , we have

$$\models \text{flow}(U, g)$$

if and only if there exists a unique topological flow

$$\Phi = \Phi_{U,g}: \mathbb{R}/\sqrt{2}\mathbb{Z} \times U \longrightarrow U$$

such that, writing  $\Phi([t], p) = \Phi^t(p)$ , we have the conditions below.

- for each  $m \in \mathbb{Z}$ , we have  $\Phi^m = g^m \upharpoonright_U$ ;
- the map  $[t] \mapsto \Phi^t$  is a topological embedding of  $\mathbb{R}/\sqrt{2}\mathbb{Z}$  into the group

$$G \upharpoonright_U := \{h \upharpoonright_U \mid h \in G \text{ and } h(U) = U\} \leq \text{Homeo}(U);$$

- for each  $[t] \neq [0]$ , we have  $\text{fix } \Phi^t \cap U = \emptyset$ .

In this case, for  $r \in \mathbb{R}$  and  $p \in U$ , the map

$$(U, g, r, p) \mapsto \Phi_{U,g}^r(p)$$

is definable.

(3) If  $\models \text{flow}(U, g) \wedge \text{flow}(V, g)$ , then for  $p \in U \cap V$  and  $r \in \mathbb{R}$ , we have

$$\Phi_{U,g}^r(p) = \Phi_{V,g}^r(p).$$

*Proof.* Applying Lemma 6.5 for the definable set

$$F_i := \{h^{-1}g^s \mid s \in \mathbb{N} \text{ and } \text{ang}(s, r) < 1/i\} \subseteq G,$$

we immediately obtain a desired formula  $\text{conv}(\gamma, \delta, u, \rho)$ .

It is straightforward to check

$$\text{flow}(u, \gamma) \equiv (\forall \rho \exists \delta)[\text{conv}(u, \gamma, \rho, \delta) \wedge (\rho \in \sqrt{2}\mathbb{Z} \vee \text{fix } \delta \cap u = \emptyset)].$$

satisfies the desired conditions in (2). In particular, the uniqueness is a consequence of the fact that the formula  $\text{conv}(U, g, r, h)$  uniquely determines the restriction of  $h$  on  $U$ , as an approximation of the form

$$\{g^{k_n} \upharpoonright_U\}$$

satisfying  $k_n \longrightarrow r$  in  $\mathbb{R}/\sqrt{2}\mathbb{Z}$ . The definability of the flow in (2) and the independence on the choice of  $U$  in part (3) also follow by the same reason, completing the proof.  $\square$

In the situation of Lemma 6.6, we will say that  $g$  defines a *circular flow* on the open set  $U$ . When we have  $\text{conv}(U, g, r, h)$ , the element  $g$  is viewed as an irrational rotation through a specified angle, and  $h$  is the rotation of the  $r$ -multiple of this angle. By the definability of  $\Phi_{U,g}^r(p)$  for  $p \in U$ , we are justified to use the expression such as

$$\Phi_{u,\gamma}^\rho(\pi) = \pi'$$

in an  $L_{\text{AGAPE}}$  formula with the hypothesis that  $\pi \in u$ . When the meaning is clear, we also use the more succinct notation

$$g^r := \Phi_{U,g}^r.$$

We are now ready to complete the proof of Theorem 6.2.

*Proof of Theorem 6.2.* By Lemma 6.6, we have an  $L_{\text{AGAPE}}$  formula  $\text{flows}_n(U, \underline{g}, p)$  that expresses the following:

- there exists some  $\underline{V} = \{V_i\}$  such that

$$\models \text{cl } V_i \subseteq \text{int } M \wedge g_i(V_i) = V_i \wedge \text{flow}(V_i, g_i)$$

for each  $i$ , and such that

$$p \in U \in \text{cl } U \subseteq \cap_i V_i;$$

- there exists a continuous bijection  $[0, 1]^n \longrightarrow \text{cl } U$  defined by

$$(r_1, \dots, r_n) \mapsto \prod_i g_i^{r_i}(p);$$

- For all  $r_i \in [0, 1]$  and for all permutation  $\sigma$  of  $\{1, \dots, n\}$ , we have

$$\prod_i g_i^{r_i}(p) = \prod_i g_{\sigma(i)}^{r_{\sigma(i)}}(p).$$

Here, it is implicitly required that

$$\prod_{i=1}^{j-1} g_i^{r_i}(p) \in \text{cl } U$$

for all  $j \leq n$ , so that

$$\prod_{i=1}^j g_i^{r_i}(p) = g_j^{r_j} \circ \prod_{i=1}^{j-1} g_i^{r_i}(p)$$

is well-defined. The formula  $\text{Param}_n$  is simply obtained from the map  $(U, g_i, r_i, p) \mapsto g_i^{r_i}(p)$ . This proves part (1).

For part (2), we may identify  $\text{cl } U = Q^n(1)$  and  $V = \text{int } Q^n(R)$  for some sufficiently large  $R$ . We can then choose  $n$  independent circular flows such that each flow rotates  $U$  in some compact solid torus  $B^{n-1}(1) \times S^1$  with the rotation number  $1/\sqrt{2}$ , and such that on the outside of  $V$  the restrictions of the flows are the identity maps; see Figure 1 (a), where a suitable homeomorphism is applied to  $U$  for an illustrational purpose. Such choices of flows will yield the desired conclusion.  $\square$

We remark that in the measure preserving case, if  $\text{vol}(U)/\text{vol}(V)$  is not sufficiently small, then there may not be enough room for a solid torus inside  $V$  such that  $\text{cl } U$  occupies  $(1/\sqrt{2})$ -fraction of the torus. For instance, one may consider an annulus that is homeomorphic to  $S^1 \times I$ , but which is equipped with a measure that



(a) The ball  $U$  is exactly  $1/\sqrt{2}$  fraction of the domain of a flow.

(b) There may not be enough room for a desired measure-preserving flow.

FIGURE 1. The proof of Theorem 6.2 (2) and a potential issue when  $U$  is not “spaciously collared”.

is not the product of the Lebesgue measures on the two factors. Thus, the annulus may be “throttled” in some interval as in Figure 1 (b), and thus there may be no measure preserving flow that globally rotates the annulus.

## 7. PARAMETRIZATION OF COLLAR NEIGHBORHOODS

Let us fix an integer  $n > 1$ . We now describe a definable parametrization of collar neighborhoods of the boundary of a compact  $n$ -manifold. More specifically, we will establish the following.

**Theorem 7.1.** *Then there exist formulae*

$$\text{collar}(\underline{\kappa}), \text{collar-embed}(\underline{\kappa}, \pi, \rho, \pi')$$

for some tuple  $\underline{\kappa}$  of variables in the AGAPE language such that each pair  $(M, G) \in \mathcal{M}_{(\text{vol})}$  with  $\dim M = n$  satisfies the following.

- (1) We have that  $\models (\exists \underline{\kappa})[\text{collar}(\underline{\kappa})]$ .
- (2) Let  $\underline{K}$  be a tuple of elements in  $\text{AGAPE}(M, G)$  satisfying

$$\text{collar}(\underline{K}).$$

Then there exists a unique a collar embedding

$$u = u[\underline{K}]: \partial M \times [0, 1) \longrightarrow M$$

of  $\partial M$  such that for all  $p \in \partial M, q \in M$  and  $r \in [0, 1)$  we have

$$u(p, r) = q \iff (\text{AGAPE}(M, G) \models \text{collar-embed}(\underline{K}, p, r, q)).$$

**7.1. Decomposition of a unit cube.** Let us fix  $n > 1$ . We will use a certain partition of a cube to parametrize a collar neighborhood of  $\partial M$ . We set

$$\begin{aligned} \Lambda &:= \{0, 1\}^{n-1} \subseteq I^{n-1}, \\ \mathbf{0} &:= (0, \dots, 0), \quad \mathbf{1} := (1, \dots, 1) \in \Lambda, \\ \mathbf{0}^k, \mathbf{1}^k &\in \Lambda^k \subseteq (I^{n-1})^k && \text{for } k > 0, \\ \text{len}(w) &:= k && \text{for } w \in \Lambda^k, \\ \text{par}(m) &:= m - 2\lfloor m/2 \rfloor && \text{for } m \in \omega. \end{aligned}$$

For convention, we also let

$$\Lambda^0 = \{\mathbf{0}^0\} = \{\emptyset\}.$$

By abuse of notation, we move or remove parantheses rather freely and often write

$$X^{(v_1, \dots, v_k)} = X^{(v_1, \dots, v_{k-1}), v_k} = X^{v_1, \dots, v_k}$$

when the vector  $(v_1, \dots, v_k)$  is used to index certain objects  $X^*$ . For each

$$w = (v_1, \dots, v_k) \in \Lambda^k$$

with  $k \in \omega$ , we let  $\bar{S}^w$  be the dyadic cube of side length  $1/2^k$  that contains the following two points as opposite vertices:

$$\sum_{i=1}^k v_i/2^i, \quad \sum_{i=1}^k v_i/2^i + \mathbf{1}/2^k.$$

For instance, we have  $\bar{S}^\emptyset = I^{n-1}$ ,  $\bar{S}^{\mathbf{0}} = [0, 1/2]^{n-1}$ ,  $\bar{S}^{(\mathbf{0}, \mathbf{1})} = [1/4, 1/2]^{n-1}$  and so forth. We have partitions (with disjoint interiors):

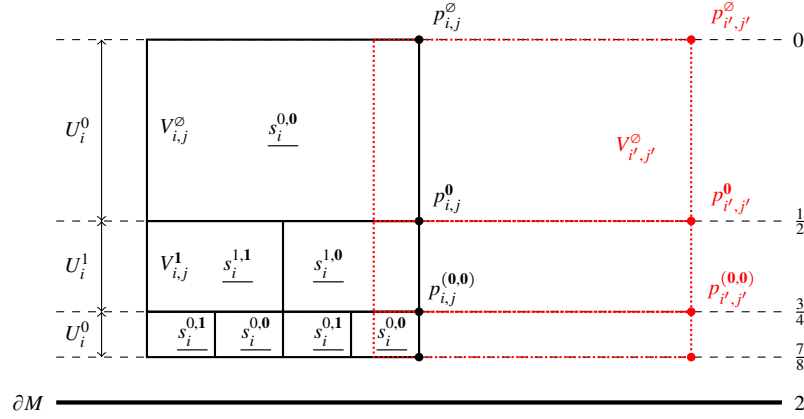
$$\begin{aligned} I^{n-1} &= \bigcup \{ \bar{S}^w \mid w \in \Lambda^k \} && \text{for each } k \in \omega, \\ I^{n-1} \times [0, 2) &= \bigcup \left\{ S^w := \bar{S}^w \times \left[ 2 - \frac{1}{2^{\text{len}(w)-1}}, 2 - \frac{1}{2^{\text{len}(w)}} \right] \mid w \in \bigcup_{k \in \omega} \Lambda^k \right\}. \end{aligned}$$

We have a unique parametrization  $\sigma^w: I^n \rightarrow S^w$  of the regular cube  $S^w$  obtained by positive homothety and translation.

**7.2. The condition for a collar neighborhood.** Let us first consider the case that  $(M, G) \in \mathcal{M}$ . For a tuple

$$\underline{K} = (U_i, U_i^0, U_i^1, U_i^{0,v}, U_i^{1,v}, T_i^0, T_i^1, p_{i,0}^\emptyset, \text{hor}_i, \text{vert}_i, s_i^{0,v}, s_i^{1,v} \mid 1 \leq i \leq n \text{ and } v \in \Lambda)$$

in the universe of  $\text{AGAPE}(M, G)$ , we consider the collection of the conditions and notations itemized as (a) through (i) below; see Figure 2 for an illustration when  $n = 2$ .


 FIGURE 2. The condition  $\text{COL}(M, G; \underline{K})$ .

**Condition**  $\text{COL}(M, G; \underline{K})$ .

(a) We have regular open sets  $U^*$  and  $U_1, \dots, U_n$  such that

$$\partial M \subseteq U^* = \bigcup_{1 \leq i \leq n} U_i,$$

and such that every regular open neighborhood of  $\partial M$  contains  $g(U^*)$  for some  $g \in G$ ; moreover, each  $U_i$  has finitely many components, and the closures of distinct components are disjoint.

(b) We have dispersed (Definition 3.11) regular open sets

$$U_i^0, U_i^1, U_i^{0,v}, U_i^{1,v}$$

for each  $i \leq n$  and  $v \in \Lambda$ ; moreover, we have for each  $\epsilon \in \{0, 1\}$  that

$$U_i = U_i^0 \oplus U_i^1, \quad U_i^\epsilon = \bigoplus_{v \in \Lambda} U_i^{\epsilon,v}.$$

(c) For each  $i \leq n$ , we have  $\text{hor}_i, \text{vert}_i \in G$  and

$$p_{i,0}^\emptyset \in T_i^0 \subseteq T_i^1 \in \mathcal{P}^{\text{disc}}(\text{int } M)$$

such that  $T_i^0$  is a nonempty, finite, minimal  $\text{hor}_i$ -invariant set; moreover, the map

$$(j, k) \mapsto p_{i,j}^{0^k} := \text{vert}_i^k \circ \text{hor}_i^j (p_{i,0}^\emptyset)$$

is a bijection from  $\{0, \dots, \#T_i^0 - 1\} \times \omega$  to  $T_i^1$ .

(d) For each  $i \leq n$  and  $j < \#T_i^0$ , there exists a unique  $U_{i,j} \in \pi_0 U_i$  satisfying

$$p_{i,j}^\emptyset \in \text{fr } U_{i,j}.$$

For each  $k \in \omega$ , there also exists a unique  $U_{i,j}^k \in \pi_0 U_i^{\text{par}(k)}$  such that

$$p_{i,j}^{\mathbf{0}^k} \in \text{fr } U_{i,j}^k.$$

We further have closure–disjoint unions

$$U_i = \bigsqcup_j U_{i,j}, \quad U_i^0 = \bigsqcup_{j,k} U_{i,j}^{2k}, \quad U_i^1 = \bigsqcup_{j,k} U_{i,j}^{2k+1}.$$

(e) For each  $i \leq n$ , we have  $\underline{s}_i^\varnothing \in G^n$ . Setting  $V_{i,j}^\varnothing := U_{i,j}^0$ , we also have

$$\begin{aligned} & \models \text{flows}_n (V_{i,j}^\varnothing, \underline{s}_i^\varnothing, p_{i,j}^\varnothing), \\ \Psi_{i,j}^\varnothing & := \Psi [V_{i,j}^\varnothing, \underline{s}_i^\varnothing, p_{i,j}^\varnothing] : I^n \longrightarrow \text{cl } V_{i,j}^\varnothing. \end{aligned}$$

For all  $k > 0$  and  $(v_1, \dots, v_k) \in \Lambda^k$  we have that

$$p_{i,j}^{(v_1, \dots, v_k)} := \text{vert}_i^k \circ \Psi_{i,j}^\varnothing \left( \sum_{i=1}^k \frac{v_i}{2^i}, 0 \right) \in \text{fr } U_{i,j}^{k-1} \cap \text{fr } U_{i,j}^k.$$

(f) For each  $(i, j, w = (v_1, \dots, v_k))$  in the index set

$$\mathcal{I} := \left\{ (i, j, w) \mid 1 \leq i \leq n, 0 \leq j < \#T_i^0, w \in \bigcup_{k \in \omega} \Lambda^k \right\},$$

there exists a unique

$$V_{i,j}^w \in \pi_0 U_i^{\text{par}(k), v_k}$$

the closure of which contains  $p_{i,j}^w$ .

(g) For each  $i \leq n$  and  $v \in \Lambda$ , we have

$$\underline{s}_i^{0,v}, \underline{s}_i^{1,v} \in G^n.$$

We further have that

$$\begin{aligned} & \models \text{flows}_n (V_{i,j}^w, \underline{s}_i^{\text{par}(k), v_k}, p_{i,j}^w), \\ \Psi_{i,j}^w & := \Psi [V_{i,j}^w, \underline{s}_i^{\text{par}(k), v_k}, p_{i,j}^w] : I^n \longrightarrow \text{cl } V_{i,j}^w. \end{aligned}$$

(h) For each  $(i, j, \varnothing) \in \mathcal{I}$ , there exists a homeomorphism

$$\Psi_{i,j} : I^{n-1} \times [0, 2] \longrightarrow \text{cl } U_{i,j}$$

such that for each  $w \in \Lambda^k$  we have

$$\Psi_{i,j} \upharpoonright_{S^w} = \Psi_{i,j}^w \circ (\sigma^w)^{-1},$$

and such that  $\Psi_{i,j}(I^{n-1} \times \{2\}) \subseteq \partial M$ .

(i) If  $x \in \text{cl } U_{i,j} \cap \text{cl } U_{i',j'}$ , then some  $v, v' \in I^{n-1}$  and  $t \in [0, 2]$  satisfy that

$$x = \Psi_{i,j}(v, t) = \Psi_{i',j'}(v', t).$$

Moreover, in this case we have for each  $t' \in [0, 2]$  that

$$\Psi_{i,j}(v, t') = \Psi_{i',j'}(v', t').$$

We now claim three points. First, these conditions are first order expressible. Second, these conditions produce a definable collar embedding; for this, we will actually need only the conditions (h) and (i). Third, every pair  $(M, G) \in \mathcal{M}$  satisfies these conditions with a suitable choice of  $\underline{K}$ .

The first point is trivial to check from the preceding results, possibly except for the continuity condition in (h) at the level-2 subset of  $I^{n-1} \times [0, 2]$ . At such a point  $x_0$ , we then can simply require the convergence of the values of the form

$$\Psi_{i,j}^w \circ (\sigma^w)^{-1}(x)$$

whenever  $x \in S^w$  gets arbitrarily close to  $x_0$ ; we also require the bijectivity of the resulting map onto  $\text{cl } U_{i,j}$ . We can now let  $\text{collar}(\underline{K})$  be the formula expressing the condition  $\text{COL}(M, G; \underline{K})$ .

Regarding the second point, we note the following.

**Claim.** *Under the hypothesis  $\text{COL}(M, G; \underline{K})$ , we have a collar embedding*

$$u = u[\underline{K}] : \partial M \times [0, 2] \longrightarrow M$$

which is unambiguously defined by

$$u(\Psi_{i,j}(v, 2), r) = \Psi_{i,j}(v, r)$$

for all  $(i, j, \emptyset) \in \mathcal{I}$ ,  $v \in I^{n-1}$  and  $r \in [0, 2]$ . In particular, the image of the level-2 set under the map  $u$  coincides with  $\partial M$ .

*Proof.* The well-definedness and the injectivity follow from the condition (i) above. This map  $u$  is continuous since so is  $\Psi_{i,j}$  for all  $i$  and  $j$ . The condition (h) further implies that this map  $u$  is a collar embedding of the boundary.  $\square$

From the above claim and from the definability of  $\Psi_{i,j}^w$ , we obtain the desired formula  $\text{collar-emb}(\underline{\kappa}, \pi, \rho, \pi')$  expressing the map  $u$ . We complete the proof of part (2) in Theorem 7.1 by simply reparametrizing  $u$  so that the level-0 set corresponds to the boundary.

For the third claim, and hence part (1) of the theorem, we note that the condition (a) is equivalent to that  $\text{cl } U^*$  is contained in a collar neighborhood. Hence, we may simply start with a homeomorphism

$$u : \partial M \times [0, 2] \longrightarrow \text{cl } U^*$$

that satisfies  $u(x, 2) = x \in \partial M$ . Using Ostrand's theorem (Lemma 6.3 (3)), we can write

$$\partial M = \bigcup_{1 \leq i \leq n} \text{cl } \bar{W}_i = \bigcup_{1 \leq i \leq n} \bar{W}_i$$

for some  $\text{cl } \bar{W}_i \subseteq \partial M$  each of whose components  $\text{cl } W_{i,j}$  is homeomorphic to  $I^{n-1}$ . We have a natural homeomorphism

$$u_{i,j}: I^{n-1} \times [0, 2] \longrightarrow U_{i,j} := u(W_{i,j} \times [0, 2]).$$

Denote by  $p_{i,j}^{0^k}$  the image of  $(0, 0, \dots, 0, 2 - 1/2^{k-1})$  under this homeomorphism. We can find  $\text{hor}_i$  that permutes the components  $\text{cl } U_{i,j}$  of  $\text{cl } U_i$  as in condition (d). We let  $T_i^0 := \{p_{i,j}^{\emptyset}\}_j$  and  $T_i^1 := \{p_{i,j}^{0^k}\}_{j,k}$ . We further define

$$V_{i,j}^w := u_{i,j}(S^w),$$

and set

$$U_{i,j}^k := \bigoplus_{w \in \Lambda^k} V_{i,j}^w, \quad U_i^0 := \bigsqcup_{j,k} U_{i,j}^{2k}.$$

The regular open sets  $U_i^1, U_i^{0,v}, U_i^{1,v}$  are similar and straightforward to define. The homeomorphism  $\text{vert}_i$  is clearly defined, so that  $\text{vert}_i(p_{i,j}^w) = p_{i,j}^{w, \mathbf{0}}$ . After decomposing  $U_{i,j}^k$  modeled on  $\{S^w\}$ , we find  $\underline{s}_i^{\text{par}(k),v}$  for the current setup using the uniform convergence theorem. Here, it is crucially used that the diameters of the cubes  $V_{i,j}^w$  converge to zero as they approach the boundary. This completes the proof of the case  $(M, G) \in \mathcal{M}$ .

Slightly more care is needed in the measure preserving case  $(M, G) \in \mathcal{M}_{\text{vol}}$ . To guarantee the existence of a measure preserving flow avoiding issues as described in Figure 1, we need that the components of the supports of flow-generating homeomorphisms  $\underline{s}_i^v$  to be sufficiently far from each other. More precisely, we will pick a sufficiently large  $n_0 > 0$  depending on  $M$ , and replace condition (g) by the following two conditions; we also change the definition of the tuple  $\underline{\kappa}$ , which is now required to contain the group tuple variables  $\underline{s}_{i,j}^w$  as below.

(f)' For each  $k \in \{1, \dots, n_0\}$ ,  $\epsilon \in \{0, 1\}$  and  $w = (v_1, \dots, v_k) \in \Lambda^k$ , we have

$$\underline{s}^w, \underline{s}^{\epsilon, w} \in G^n.$$

We further have that

$$\begin{aligned} & \models \text{flows}_n (V_{i,j}^w, \underline{s}_i^w, p_{i,j}^w), \\ \Psi_{i,j}^w & := \Psi [V_{i,j}^w, \underline{s}_i^w, p_{i,j}^w]: I^n \longrightarrow \text{cl } V_{i,j}^w. \end{aligned}$$



(f)'' For each  $k > n_0$  and  $w = (v_1, \dots, v_k)$ , after setting  $w' := (v_{k-n_0+1}, \dots, v_k)$  we have that

$$\begin{aligned} & \models \text{flows}_n \left( V_{i,j}^w, \underline{s}_i^{\text{par}(k), w'}, p_{i,j}^w \right), \\ \Psi_{i,j}^w & := \Psi \left[ V_{i,j}^w, \underline{s}_i^{\text{par}(k), w'}, p_{i,j}^w \right] : I^n \longrightarrow \text{cl } V_{i,j}^w. \end{aligned}$$

Part (2) of Theorem 7.1 is still proved in the same way, even independent of the choice of  $n_0 \geq 1$ . For part (1), we choose  $n_0$  sufficiently large that, under a fixed metric and a measure on some chart neighborhood of  $M$ . We will require that for each fixed  $w' := (v_1, \dots, v_{n_0}) \in \Lambda^{n_0}$ , each open set in the collection

$$\left\{ V_{i,j}^w \mid w = (\dots, v_1, \dots, v_{n_0}) \in \bigcup_{k > n_0} \Lambda^k \right\}$$

is contained in some closure-disjoint collection of open balls

$$\left\{ W_{i,j}^w \mid w = (\dots, v_1, \dots, v_{n_0}) \in \bigcup_{k > n_0} \Lambda^k \right\}$$

with the additional requirement that  $\text{vol}(V_{i,j}^w) / \text{vol}(W_{i,j}^w)$  is sufficiently small in the sense of Theorem 6.2. This guarantees the existence and the convergence of each measure preserving homeomorphism of the required form  $\underline{s}^{\epsilon, w'}$ , completing the proof.

## 8. COMPLETING THE PROOF

Cheeger and Kister [9] proved that there exist only countably many homeomorphism types of compact manifolds. A key step in their proof was that the topological type of a manifold is invariant under ‘‘small’’ perturbations, in some quantitatively precise sense. As is more concretely described below, this step will be crucial for the construction of the sentences  $\phi_M$  and  $\phi_M^{\text{vol}}$ .

For positive integers  $n, k$  and  $\ell$ , we denote by  $\mathcal{E}(n, k, \ell)$  the set of all tuples of embeddings

$$\underline{f} = (f_{1,1}, \dots, f_{1,k}, f_{2,1}, \dots, f_{2,\ell})$$

from  $Q^n(2)$  to  $\mathbb{R}^{2n+1}$  such that the following conditions hold.

(i) The following set is a compact connected  $n$ -manifold:

$$M = C(\underline{f}) := \bigcup_{i,j} \text{im } f_{i,j} \subseteq \mathbb{R}^{2n+1}.$$

(ii) There exists a collar  $u: \partial M \times [-2, 2] \longrightarrow M$  such that  $u(x, -2) = x$  for all  $x$ .

(iii) We have that

$$M \setminus u(\partial M \times [-2, 0]) \subseteq \bigcup_i f_{1,j}(\text{int } Q^n(1)) \subseteq \bigcup_i f_{1,j}(Q^n(2)) \subseteq M \setminus u(\partial M \times [-2, -1]).$$

(iv) For each  $i = 1, \dots, \ell$ , the restriction

$$f_{2,j} \upharpoonright_{Q^{n-1}(2) \times \{-2\}}$$

is an embedding of  $Q^{n-1}(2)$  into  $\partial M$  such that

$$f_{2,j}(x, t) = u(f_{2,j}(x, -2), t)$$

where  $x \in Q^{n-1}(2)$  and  $t \in [-2, 2]$ , and such that

$$\partial M = \bigcup_i f_{2,j}(\text{int } Q^{n-1}(1) \times \{-2\}).$$

Every compact  $n$ -manifold  $M$  is homeomorphic to  $C(\underline{f})$  for some tuple  $\underline{f} = (f_{i,j}) \in \mathcal{E}(n, k, \ell)$  as above, which we call as a *parametrized cover* of  $C(\underline{f})$ . The space  $\mathcal{E}(n, k, \ell)$  inherits the uniform separable metric from  $C^0(Q^n(2), \mathbb{R}^{(2n+1)(k+\ell)})$ . The proof of Cheeger and Kister essentially boils down to the following rigidity result, along with a deep result of Edwards and Kirby on deformation of embeddings in manifolds [16].

**Lemma 8.1.** [9] *For each  $\underline{f} \in \mathcal{E}(n, k, \ell)$  and for each  $\epsilon > 0$ , there exists  $\delta > 0$  such that every  $\underline{g} \in \mathcal{E}(n, k, \ell)$  that is at most  $\delta$ -far from  $\underline{f}$  admits a homeomorphism*

$$C(\underline{f}) \longrightarrow C(\underline{g})$$

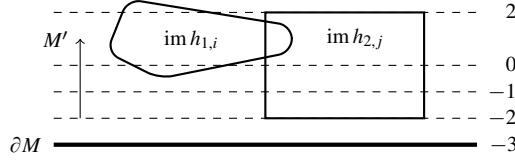
*that is at most  $\epsilon$ -far from the identity map.*

We choose a sufficiently small  $\delta > 0$  for which the conclusion of Lemma 8.1 holds, and call it as a *Cheeger–Kister number* of  $\underline{f} \in \mathcal{E}(n, k)$ ; for our purpose we will further require  $\delta$  to be rational. Our strategy of proving Theorem 1.4 is providing a sentence in  $L_{\text{AGAPE}}$  which is modeled by an input manifold  $M \subseteq \mathbb{R}^{2n+1}$  such that the sentence holds for the structure  $\text{AGAPE}(N, H)$  if and only if  $N$  admits an embedding into Euclidean space that is within the Cheeger–Kister number of a fixed parametrized cover of  $M$ .

In order to execute this strategy, let us fix a pair  $(M, G) \in \mathcal{M}_{(\text{vol})}$  with  $\dim M = n$ . We will slightly modify the definition in Theorem 6.2 by affine transformations, so that  $\Psi[U, \underline{g}, p]$  is a map from  $Q^n(2)$  into  $M$ , sending  $(-2, \dots, -2)$  to  $p$ . We let  $k$  and  $\ell$  be positive integers, and consider a tuple

$$\underline{f} = (f_{1,1}, \dots, f_{1,k}, f_{2,1}, \dots, f_{2,\ell})$$

of functions in  $C^0(\mathbb{R}^n, \mathbb{R}^{2n+1})$ . Let us denote by  $\text{EMB}(M, G; \underline{f})$  the collection of all the conditions below from (a) through (e); see also Figure 3.

FIGURE 3. Parts (c) and (d) of the condition  $\text{EMB}(M, G; f)$ .

- (a) each  $f_{i,j}$  restricts to an embedding of  $Q^n(2)$  into  $\mathbb{R}^{2n+1}$ ;  
 (b) for the indices as above, we have some

$$U_{i,j} \in \text{RO}(M), p_{i,j} \in \text{int } M, \underline{g}_{i,j} \in G^n$$

satisfying flows  $s_n(U_{i,j}, \underline{g}_{i,j}, p_{i,j})$ , corresponding to the homeomorphism

$$h_{i,j} := \Psi \left[ U_{i,j}, \underline{g}_{i,j}, p_{i,j} \right] : Q^n(2) \longrightarrow \text{cl } U_{i,j} \subseteq \text{int } M;$$

- (c) there exists a collar  $u : \partial M \times [-3, 2] \longrightarrow M$  such that  $u \upharpoonright_{\partial M \times \{-3\}} = 1 \upharpoonright_{\partial M}$ , and such that

$$M \setminus u(\partial M \times [-3, 0]) \subseteq \bigcup_j h_{1,j}(\text{int } Q^n(1)) \subseteq \bigcup_j h_{1,j}(Q^n(2)) \subseteq M \setminus u(\partial M \times [-3, -1]);$$

- (d) for each  $j = 1, \dots, \ell$ , the restriction

$$h_{2,j} \upharpoonright_{Q^{n-1}(2) \times \{-2\}}$$

is an embedding of  $Q^{n-1}(2)$  into  $u(\partial M \times \{-2\})$  such that

$$h_{2,j}(x, t) = u(h_{2,j}(x, -2), t)$$

where  $x \in Q^{n-1}(2)$  and  $t \in [-2, 2]$ , and such that

$$u(\partial M \times \{-2\}) = \bigcup_j h_{2,j}(\text{int } Q^{n-1}(1) \times \{-2\}).$$

- (e) whenever  $x \in \text{cl } U_{a,b} \cap \text{cl } U_{c,d}$  for some  $a, b, c, d$ , we have

$$f_{a,b} \circ h_{a,b}^{-1}(x) = f_{c,d} \circ h_{c,d}^{-1}(x).$$

The condition  $\text{EMB}(M, G; \underline{f})$  implies that  $\underline{f} \circ \underline{h}^{-1} := (f_{i,j} \circ h_{i,j}^{-1})$  defines an embedding

$$M' := M \setminus u(\partial M \times [-3, -2]) \hookrightarrow \mathbb{R}^{2n+1},$$

and that the tuple  $\underline{f}$  is a parametrized cover of the image.

Recall the domain of the sort symbol  $\text{Cont}_{n,2n+1}$  is  $C^0(\mathbb{R}^n, \mathbb{R}^{2n+1})$ . By the preceding results, there exists a formula

$$\text{Embed}_{n,k,\ell}(\underline{\chi})$$

expressing  $\text{EMB}(M, G; \underline{f})$  in  $\text{AGAPE}(M, G)$ . We emphasize that although the maps  $h_{i,j}$  do not belong to the universe of  $\text{AGAPE}(M, G)$ , Theorem 7.1 together with our access to the real numbers enables us to have such expressions. Let us record this fact.

**Lemma 8.2.** *Let  $(M, G) \in \mathcal{M}_{(\text{vol})}$  satisfy  $\dim M = n$ . For positive integers  $n, k$  and  $\ell$ , there exists a formula  $\text{Embed}_{n,k,\ell}(\underline{\chi})$  with a  $(k + \ell)$ -tuple of  $\text{Cont}_{n,2n+1}$  variables*

$$\underline{\chi} = (\chi_{1,1}, \dots, \chi_{1,k}, \chi_{2,1}, \dots, \chi_{2,\ell})$$

in  $\text{AGAPE}$  language such that

$$\models \text{Embed}_{n,k,\ell}(\underline{f})$$

if and only if the condition  $\text{EMB}(M, G; \underline{f})$  is satisfied.

We can now establish the main result of this paper.

*Proof of Theorem 1.4.* We may assume that  $n := \dim M > 1$  and that  $M \subseteq \mathbb{R}^{2n+1}$ . Consider a parametrized cover

$$\underline{f} \in \mathcal{E}(n, k, \ell) \subseteq C^0(Q^n(2), \mathbb{R}^{(2n+1)(k+\ell)})$$

of  $M = C(\underline{f})$ . We have a corresponding Cheeger–Kister rational number  $\delta = \delta(M, \underline{f}) > 0$ . Let us pick  $\delta_0 > 0$  such that

$$\sup_{\|x-y\| \leq \delta_0} \|\underline{f}(x) - \underline{f}(y)\| < \delta/3.$$

We can find a partition  $\{C_1, \dots, C_s\}$  of  $Q^n(2)$  having diameters less than  $\delta_0$  such that each  $C_i$  is the intersection of  $Q^n(2)$  with a cube with rational corners. Each  $C_i$  is definable in  $L_{\text{AGAPE}}$ , since so is every rational number. We arbitrarily pick  $x_i$  in  $C_i$ , and choose  $q_i \in \mathbb{Q}^{(2n+1)(k+\ell)}$  such that

$$\|\underline{f}(x_i) - q_i\| < \delta/3.$$

Let us now consider the following conditions for an arbitrary  $(N, H) \in \mathcal{M}_{(\text{vol})}$ , which are first order expressible in  $L_{\text{AGAPE}}$  by preceding results:

- $\dim N = n$ ;
- some tuple  $\underline{g} \in C^0(\mathbb{R}^n, \mathbb{R}^{(2n+1)(k+\ell)})$  satisfies that

$$\text{AGAPE}(N, H) \models \text{Embed}_n(\underline{g}),$$

and that

$$\sup_{x \in C_i} \|\underline{g}(x) - q_i\| < \delta/3.$$

The above conditions are obviously met in the case when  $(N, H) = (M, G)$ . We also note for each  $x \in C_i$  that

$$\|\underline{g}(x) - \underline{f}(x)\| \leq \|\underline{g}(x) - q_i\| + \|q_i - \underline{f}(x_i)\| + \|\underline{f}(x_i) - \underline{f}(x)\| < \delta.$$

By Lemma 8.1, we see that  $N$  is homeomorphic to  $M$ .  $\square$

## 9. FURTHER QUESTIONS

A large number of interesting open questions remain. We already mentioned Question 1.5. Part of the motivation for this question is the theory of critical regularity of groups, which seeks to distinguish between diffeomorphism groups of various regularities of a given manifold by the isomorphism types of finitely generated subgroups; cf. [22, 27]. Along this line, one may ask whether or not the  $C^k$ -analogue of Theorem 1.4 holds.

**Question 9.1.** *Let  $M$  be a compact, connected, smooth manifold, and let  $N$  be an arbitrary smooth manifold. Is there a sentence  $\phi_{k,M}$  in the language of groups such that if  $\text{Diff}^k(N)$  satisfies  $\phi_{k,M}$  then  $N$  is  $C^k$ -diffeomorphic to  $M$ ?*

Relatedly, leaving the framework of first order rigidity, we have the following.

**Question 9.2.** *Let  $M$  be a compact, connected, smooth manifold. Is there a finitely generated (or countable) group  $G_M$  such that  $G_M$  acts faithfully by  $C^k$  diffeomorphisms of a compact, connected, smooth manifold  $N$  of the same dimension if and only if  $N$  is  $C^k$  diffeomorphic to  $M$ ?*

The discussion in the present article depended heavily on the compactness of the comparison manifold.

**Question 9.3.** *Let  $M$  be an arbitrary manifold. Under what conditions is there a sentence  $\phi_M$  in the language of groups such that if  $N$  is an arbitrary manifold then  $\text{Homeo}(N)$  satisfies  $\phi_M$  if and only if  $N$  is homeomorphic to  $M$ ? More generally, under what conditions does  $\text{Homeo}(M) \equiv \text{Homeo}(N)$  imply  $M \cong N$ ?*

We conclude by asking what the weakest hypotheses on  $G$  can be.

**Question 9.4.** *For what classes of subgroups of  $\text{Homeo}(M)$  do the conclusions of Theorem 1.4 hold?*

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#### REFERENCES

1. M. Ajtai, *Isomorphism and higher order equivalence*, Ann. Math. Logic **16** (1979), no. 3, 181–203. MR548473
2. Nir Avni, Alexander Lubotzky, and Chen Meiri, *First order rigidity of non-uniform higher rank arithmetic groups*, Invent. Math. **217** (2019), no. 1, 219–240. MR3958794
3. Augustin Banyaga, *The structure of classical diffeomorphism groups*, Mathematics and its Applications, vol. 400, Kluwer Academic Publishers Group, Dordrecht, 1997. MR1445290 (98h:22024)
4. Thomas William Barrett and Hans Halvorson, *Quine’s conjecture on many-sorted logic*, Synthese **194** (2017), no. 9, 3563–3582.
5. Salomon Bochner and Deane Montgomery, *Groups of differentiable and real or complex analytic transformations*, Ann. of Math. (2) **46** (1945), 685–694. MR14102
6. Morton Brown, *Locally flat imbeddings of topological manifolds*, Ann. of Math. (2) **75** (1962), 331–341. MR133812
7. ———, *A mapping theorem for untriangulated manifolds*, Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961), Prentice-Hall, Englewood Cliffs, N.J., 1962, pp. 92–94. MR0158374
8. Eduard Čech, *Příspěvek k teorii dimenze [contribution to the theory of dimension]*, Časopis Pěst. Mat. **62** (1933), 277–291.
9. J. Cheeger and J. M. Kister, *Counting topological manifolds*, Topology **9** (1970), 149–151. MR256399
10. Lei Chen and Kathryn Mann, *Structure theorems for actions of homeomorphism groups*, Duke Mathematical Journal **172** (2023), no. 5, 915 – 962.
11. Charles O. Christenson and William L. Voxman, *Aspects of topology*, Pure and Applied Mathematics, Vol. 39, Marcel Dekker, Inc., New York-Basel, 1977. MR0487938
12. Michel Coornaert, *Topological dimension and dynamical systems*, Universitext, Springer, Cham, 2015, Translated and revised from the 2005 French original. MR3242807
13. Jean-Louis Duret, *Sur la théorie élémentaire des corps de fonctions*, J. Symbolic Logic **51** (1986), no. 4, 948–956. MR865921
14. ———, *Équivalence élémentaire et isomorphisme des corps de courbe sur un corps algébriquement clos*, J. Symbolic Logic **57** (1992), no. 3, 808–823. MR1187449
15. Gerald Edgar, *Measure, topology, and fractal geometry*, second ed., Undergraduate Texts in Mathematics, Springer, New York, 2008. MR2356043
16. Robert D. Edwards and Robion C. Kirby, *Deformations of spaces of imbeddings*, Ann. of Math. (2) **93** (1971), 63–88. MR283802
17. A. Fathi, *Structure of the group of homeomorphisms preserving a good measure on a compact manifold*, Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 1, 45–93. MR584082
18. R. P. Filipkiewicz, *Isomorphisms between diffeomorphism groups*, Ergodic Theory Dynam. Systems **2** (1982), no. 2, 159–171 (1983). MR693972
19. Casper Goffman and George Pedrick, *A proof of the homeomorphism of Lebesgue-Stieltjes measure with Lebesgue measure*, Proc. Amer. Math. Soc. **52** (1975), 196–198. MR376995
20. Be’eri Greenfeld, *First-order rigidity of rings satisfying polynomial identities*, Ann. Pure Appl. Logic **173** (2022), no. 6, Paper No. 103109, 8. MR4393207

21. Witold Hurewicz and Henry Wallman, *Dimension Theory*, Princeton Mathematical Series, vol. 4, Princeton University Press, Princeton, N. J., 1941. MR0006493
22. S.-h. Kim and T. Koberda, *Diffeomorphism groups of critical regularity*, *Invent. Math.* **221** (2020), no. 2, 421–501. MR4121156
23. Sang-hyun Kim and Thomas Koberda, *Structure and regularity of group actions on one-manifolds*, Springer Monographs in Mathematics, Springer, Cham, [2021] ©2021. MR4381312
24. Robion C. Kirby, *Stable homeomorphisms and the annulus conjecture*, *Ann. of Math. (2)* **89** (1969), 575–582. MR242165
25. Frédéric Le Roux, *On closed subgroups of the group of homeomorphisms of a manifold*, *J. Éc. polytech. Math.* **1** (2014), 147–159. MR3322786
26. Gérard LeLoup, *Équivalence élémentaire entre anneaux de monoïdes*, Séminaire Général de Logique 1983–1984 (Paris, 1983–1984), *Publ. Math. Univ. Paris VII*, vol. 27, Univ. Paris VII, Paris, 1986, pp. 149–157. MR938759
27. Kathryn Mann and Maxime Wolff, *Reconstructing maps out of groups*, arXiv preprint (2019), 1–16, Preprint, arXiv:1907.03024.
28. David Marker, *Model theory*, Graduate Texts in Mathematics, vol. 217, Springer-Verlag, New York, 2002, An introduction. MR1924282
29. Francis Oger, *Équivalence élémentaire entre groupes finis-par-abéliens de type fini*, *Comment. Math. Helv.* **57** (1982), no. 3, 469–480. MR689074
30. Phillip A. Ostrand, *Covering dimension in general spaces*, *General Topology and Appl.* **1** (1971), no. 3, 209–221. MR288741
31. J. C. Oxtoby and S. M. Ulam, *Measure-preserving homeomorphisms and metrical transitivity*, *Ann. of Math. (2)* **42** (1941), 874–920. MR5803
32. Eugene Plotkin, *On first order rigidity for linear groups*, *Toyama Math. J.* **41** (2020), 45–60. MR4353090
33. Florian Pop, *Elementary equivalence versus isomorphism*, *Invent. Math.* **150** (2002), no. 2, 385–408. MR1933588
34. MathOverflow Post, *Does a \*topological\* manifold have an exhaustion by compact submanifolds with boundary?*, <https://mathoverflow.net/questions/129321>.
35. ———, *On a result by rubin on elementary equivalence of homeomorphism groups and homeomorphisms of the underlying spaces*, <https://mathoverflow.net/questions/400602>.
36. Frank Quinn, *Ends of maps. III. Dimensions 4 and 5*, *J. Differential Geometry* **17** (1982), no. 3, 503–521. MR679069
37. ———, *Topological transversality holds in all dimensions*, *Bull. Amer. Math. Soc. (N.S.)* **18** (1988), no. 2, 145–148. MR929089
38. Matatyahu Rubin, *On the reconstruction of topological spaces from their groups of homeomorphisms*, *Trans. Amer. Math. Soc.* **312** (1989), no. 2, 487–538. MR988881
39. ———, *Locally moving groups and reconstruction problems*, *Ordered groups and infinite permutation groups*, *Math. Appl.*, vol. 354, Kluwer Acad. Publ., Dordrecht, 1996, pp. 121–157. MR1486199
40. Floris Takens, *Characterization of a differentiable structure by its group of diffeomorphisms*, *Bol. Soc. Brasil. Mat.* **10** (1979), no. 1, 17–25. MR552032
41. Katrin Tent and Martin Ziegler, *A course in model theory*, *Lecture Notes in Logic*, vol. 40, Association for Symbolic Logic, La Jolla, CA; Cambridge University Press, Cambridge, 2012. MR2908005
42. Xavier Vidaux, *Équivalence élémentaire de corps elliptiques*, *C. R. Acad. Sci. Paris Sér. I Math.* **330** (2000), no. 1, 1–4. MR1741158

43. John von Neumann, *Collected works. Vol. II: Operators, ergodic theory and almost periodic functions in a group*, Pergamon Press, New York-Oxford-London-Paris, 1961, General editor: A. H. Taub. MR0157872
44. James V. Whittaker, *On isomorphic groups and homeomorphic spaces*, *Ann. of Math. (2)* **78** (1963), 74–91. MR150750

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