Results on m–dissimilarity tensors and graphs of genus 1

Joint works with Filip Cools

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Korea Institute for Advanced Study (KIAS), Seoul, Korea

"The Grassmannian is a valuable geometric tool for understanding and designing algorithms for phylogenetic trees"

B. Sturmfels, L. Pachter

Algebraic Statistics for Computational Biology

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From such sequence data, computational biologists infer the distance between any two taxa. There are various algorithms for carrying out this inference.

For our example we may think of the distance between any two strings as a refined version of the Hamming distance (= the proportion of characters where they differ).

$$D = \begin{array}{c} \\ H \\ M \\ R \\ C \end{array} \begin{array}{c} \begin{array}{cccc} H & M & R & C \end{array} \\ \begin{pmatrix} 0 & 1.1 & 1.0 & 1.4 \\ 1.1. & 0 & 0.3 & 1.3 \\ 1.0 & 0.3 & 0 & 1.2 \\ 1.4 & 1.3 & 1.2 & 0 \end{pmatrix} \end{array}$$

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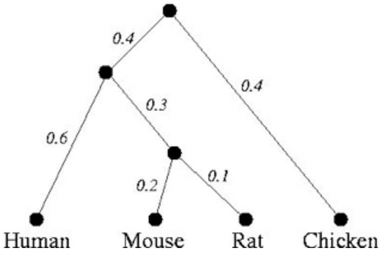
The problem of phylogenetics is to construct a tree with edge lengths which represent this distance matrix, provided such a tree exists.



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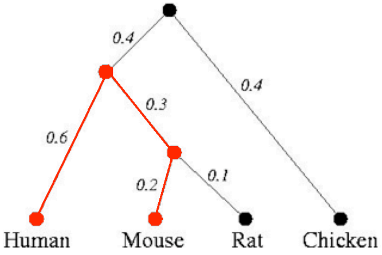

For instance, the distance in this tree between “Human” and “Mouse” equals

0.6+0.3+0.2=1.1

which is the corresponding entry in the inferred distance matrix.

Tree metrics and m–dissimilarity maps

A dissimilarity matrix D is a map $D : X^2 \to \mathbb{R}$, with

- $D(x_i, x_j) = D(x_j, x_i) \geq 0$
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We say that D has a realization if there is a weighted graph G whose node set contains X and the distance $d(u, v)$ between nodes $u, v \in X$ is exactly $D(u, v)$.
In the case the graph is a tree and X corresponds to the set of leaves, D is called a tree metric.

Theorem (Tree-Metric Theorem - the Four-Point Condition)

A metric D is a tree metric if and only if, for every four leaves $i,j,k,l \in X$, the maximum of the three numbers

$$D(i,j) + D(k,l), \quad D(i,k) + D(j,l), \quad D(i,l) + D(j,k).$$

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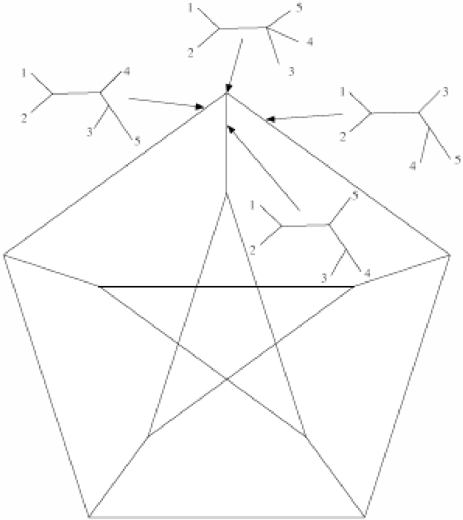
Tree metrics on n leaves are parametrized by a the so-called space of trees $\mathcal{T}_n \subset \mathbb{R}^{\binom{n}{2}}$. $(X=[n])$


1
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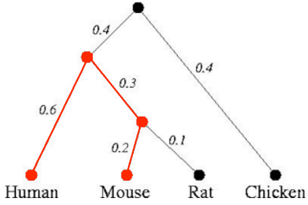
Let T be a n– tree with a positive weight assigned to each edge. Thus, the distance between leaves i and j is the sum of the weights of the path connecting i and j.



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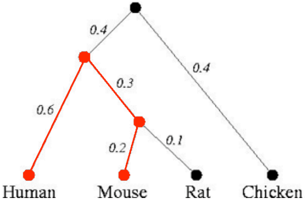

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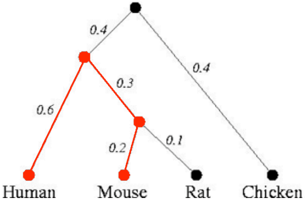

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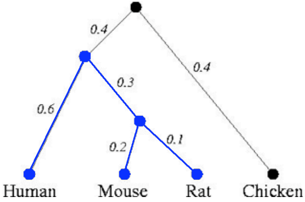
$$\omega([\mathrm{HMR}]) = 0.6 + 0.3 + 0.2 + 0.1 = 1.2$$



An *m–dissimilarity tensor* D is a map $D : X^m \to \mathbb{R}$, with

- $D(x_1, \ldots, x_m) = D(x_{\pi(1)}, \ldots, x_{\pi(m)})$ for all permutations $\pi \in S_m$
- $D(x_1, x_2, \ldots, x_m) = 0$ if $x_i = x_j$ for some i and j.

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We say that a graph G *realizes* D if the node set of G contains X and for every $x_1, \ldots, x_m \in X$, the weight of the smallest subgraph in G containing $x_1, \ldots, x_m$ is $D(x_1, \ldots, x_m)$. An m–dissimilarity tensor which is realizable is called a *m–distance map*.

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Theorem (Pachter–Speyer, 2004)

Let T be a tree with n leaves and no vertices of degree 2. Let $m \geq 3$ be an integer, If $n \geq 2m - 1$, then T is determined by the set of values $\omega([V])$ as V ranges over all m element subset of $[n]$. If $n = 2m - 2$, this is not true.

A motivation

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- All the MLE computation is very difficult, also for a single trees.
- this approach requires examining all exponentially many trees.

One popular way to avoid these problems is the "distance based approach" which is to collapse the data to a dissimilarity map and then to obtain a tree via a projection onto tree space $\mathcal{T}_n$ (*neighbor-joining algorithm*).

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Q1 Is it possible to define the space of m–distance maps D_T arising from trees T as the image of the tree space $\mathcal{T}_n$ under a certain map $\psi^{(m)} : \mathbb{R}^{\binom{n}{2}} \to \mathbb{R}^{\binom{n}{m}}$. ?

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Q2' : Describe $\mathcal{W}_{m,n} := \psi^{(m)}(\mathcal{T}_n)$ as the parameter space of m–distance maps.

Tropical Geometry

We work in the semi-ring

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Consider indeterminates $x_1, \ldots, x_n$

- Tropical monomials $x_1^{a_1} \otimes \cdots \otimes x_n^{a_n}$ represent ordinary linear forms $\sum_{i=i}^{n} a_i x_i$, i.e. linear functions $F : \mathbb{R}^n \to \mathbb{R}$.
- tropical polynomials $\bigoplus_{a \in \mathcal{A}} C_a \otimes x_1^{a_1} \otimes \cdots \otimes x_n^{a_n}$, with $\mathcal{A} \in \mathbb{N}^n$ and $C_a \in \mathbb{R}$, represent piecewise-linear functions $F : \mathbb{R}^n \to \mathbb{R}$. To compute $F(x)$ we take the maximum of the affine-linear forms $C_a + \sum_{i=i}^{n} a_i x_i$ for $a \in \mathcal{A}$.

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Given a tropical polynomial $F = \bigoplus_{a \in \mathcal{A}} C_a \otimes x_1^{a_1} \otimes \cdots \otimes x_n^{a_n}$ we define the tropical hypersurface $\mathcal{H}(F)$ as the corner locus of the function F, that is

$$\left\{ (w_1, \ldots, w_n) \in \mathbb{R}^n : \quad \left\{ \sum_{i=1}^{n} a_i w_i + C_a \right\}_{a \in \mathcal{A}} \text{ attain the maximum twice} \right\}$$

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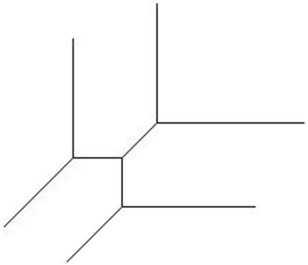

Consider an ordinary polynomial

$$f = \sum_{e \in E} f_{e_1 \cdots e_n} x_1^{e_1} \cdots x_n^{e_n}$$

The tropicalization of f, $Trop(f)$ is defined as

$$Trop(f) = \bigoplus_{e \in E} f_{e_1 \cdots e_n} \otimes x_1^{e_1} \otimes \cdots \otimes x_n^{e^n}$$

Let $I \subset K[x_1, \ldots, x_n]$ be an ideal, we define the tropical variety of I as

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We are mainly interested in the tropical variety defined by

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Definition

The tropical Grassmannian $\mathcal{G}_{m,n}$ is the tropical variety $\mathcal{H}(I_{m,n})$

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Theorem

The ideal $I_{2,n}$ is generated by the quadratic polynomials

$$x_{ik}x_{jl} - x_{ij}x_{kl} - x_{il}x_{jk} \qquad (1 \leq i < j < k < l \leq n) \tag{1}$$

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Theorem (Speyer–Sturmfels, 2004)

The space of trees $\mathcal{T}_n$ is (up to sign) the tropical Grassmannian $\mathcal{G}_{2,n}$.

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For $m \geq 4$ the situation is much harder and more interesting. Here there is no monomial map of which the subtree weight map $\psi^{(m)}$ is the tropicalization.

A necessary condition to be a m–distance map is given by a generalization of the four-point condition, that is the maximum is reach for at least two terms between

$$D(Rij) + D(Rkl), \quad D(Rik) + D(Rjl), \quad D(Ril) + D(Rjk).$$

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Q3 (proposed by R. Yoshida) How much the “generalized four-point condition” is not a sufficient condition for m–distance map ?

The ideal $I_{m,n}$ is generated by quadratic polynomials known as the Plücker relations. Among these are the three-term Plücker relations which are defined as

$$g_{R,ijkl} := x_{Rik}x_{Rjl} - x_{Rij}x_{Rkl} - x_{Ril}x_{Rjk}$$

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where R is any $(m-2)$–element subset of $[n]$ and $i, j, k, l \in [n] \setminus R$.

The three-term Plücker relations are not enough to generate $I_{m,n}$.

$$p_{R,ijkl} := Trop(g_{R,ijkl}) = (x_{Rij} \otimes x_{Rkl}) \oplus (x_{Rik} \otimes x_{Rjl}) \oplus (x_{Ril} \otimes x_{Rjk})$$

Definition

The three-term tropical Grassmannian $\mathcal{T}_{m,n}$ is the intersection

$$\mathcal{T}_{m,n} := \bigcap_{R,i,j,k,l} \mathcal{H}(p_{R,ijkl}) \subset \mathbb{R}^{\binom{n}{m}}$$

$\mathcal{T}_{m,n}$ is also known as the space of m–trees.

Results on $\psi^{(m)}$ and $\mathcal{W}_{m,n}$

What about

$$\psi^{(m)}: \begin{array}{ccc} \mathcal{G}_{2,n} & \rightarrow & \mathbb{R}^{\binom{n}{m}} \\ (\ldots, D(i,j), \ldots) & \mapsto & (\ldots, D(i_1, i_2, \ldots, i_m), \ldots) \end{array} \quad ?$$

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Theorem (–, Cools, 2010)

Suppose $I = \{i_1, ..., i_m\}$. Then

$$D(I) = \frac{1}{2}\left(\min_{s \in r(S_m)} \left\{D(i_1, i_{s(1)}) + D(i_{s(1)}, i_{s^2(1)}) + ... + D(i_{s^{m-1}(1)}, i_1)\right\}\right)$$

where $r(S_m)$ is the subset of S_m of "real" permutations, i.e. permutations with only one term in the disjoint cycle notation.

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Corollary

$$D(I) = \left(\bigoplus_{s \in r(S_m)} \frac{1}{D(i_1, i_{s(1)}) \otimes D(i_{s(1)}, i_{s^2(1)}) \otimes \cdots \otimes D(i_{s^{m-1}(1)}, i_1)} \right)^{-\frac{1}{2}}$$

$$= \left(\bigoplus_{s \in r(S_m)} \left(D(i_1, i_{s(1)}) \otimes D(i_{s(1)}, i_{s^2(1)}) \otimes \cdots \otimes D(i_{s^{m-1}(1)}, i_1) \right)^{-1} \right)^{-\frac{1}{2}}$$

For $m = 3$ one has $r(S_3) = \{s_1, s_2\}$ with

$$s_1 = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix} \text{ and } s_1 = s_2^{-1}$$

Thus

$$\begin{aligned} D(i_1 i_2 i_3) &= \left(\bigoplus_{s \in r(S_3)} \left(D(i_1, i_{s(1)}) \otimes D(i_{s(1)}, i_{s^2(1)}) \otimes D(i_{s^2(1)}, i_{s^3(1)}) \right)^{-1} \right)^{-\frac{1}{2}} \\ &= \left((D(i_1, i_2) \otimes D(i_2, i_3) \otimes D(i_1, i_3))^{-1} \right)^{-\frac{1}{2}} = \\ &= -\frac{1}{2} \left(- \left((D(i_1, i_2) + D(i_2, i_3) + D(i_1, i_3) \right) \right) = \\ &= \frac{1}{2} \left((D(i_1, i_2) + D(i_2, i_3) + D(i_1, i_3) \right) \end{aligned}$$

Example: $n = 7, m = 4,$ $\qquad \mathcal{G}_{2,7} \subset \mathbb{R}^{21},$ $\qquad \mathcal{W}_{4,7} \subset \mathbb{R}^{35}$

$$\begin{pmatrix} i_2 & i_3 & i_4 & i_1 \end{pmatrix} \quad \begin{pmatrix} i_3 & i_4 & i_2 & i_1 \end{pmatrix} \quad \begin{pmatrix} i_2 & i_4 & i_1 & i_3 \end{pmatrix}$$

$$D(i_1 i_2 i_3 i_4) = \frac{1}{2} \cdot \min \begin{pmatrix} D(i_1, i_2) + D(i_2, i_3) + D(i_3, i_4) + D(i_1, i_4) \\ D(i_1, i_3) + D(i_2, i_3) + D(i_2, i_4) + D(i_1, i_4) \\ D(i_1, i_2) + D(i_2, i_4) + D(i_3, i_4) + D(i_1, i_3) \end{pmatrix}$$

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$$D := (1, 1, 1, 1, 1, \ldots, 1, 1, 1, 1, 2) \notin \mathcal{G}_{2,7}$$

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$Q \in \mathcal{W}_{4,7}$ since $Q = \psi^{(4)}(D')$ where

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$\psi^{(m)}$ is not injective but $\psi^{(m)}_{|\mathcal{G}_{2,n}}$ is injective

Theorem (–, Cools, 2010)

One has

$$\mathcal{W}_{m,n} \subseteq \mathcal{T}_{m,n} \cap \psi^{(m)}(\mathbb{R}^{\binom{n}{2}})$$

Moreover, for $m = 3$ the equality holds

$$\mathcal{W}_{3,n} = \mathcal{T}_{3,n} \cap \psi^{(3)}(\mathbb{R}^{\binom{n}{2}})$$

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If $n \geq 5$, one has $\phi^{(3)}(\mathcal{G}_{2,n}) = \phi^{(3)}(\mathbb{R}^{\binom{n}{2}}) \cap \mathcal{G}_{3,n}$.

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$$\mathcal{W}_{3,n} = \mathcal{H}(f_1^t) \cap \cdots \cap \mathcal{H}(f_c^t) \cap \mathcal{G}_{3,n}$$

and $f_1^t, \ldots, f_c^t$ have an explicit combinatorial description.

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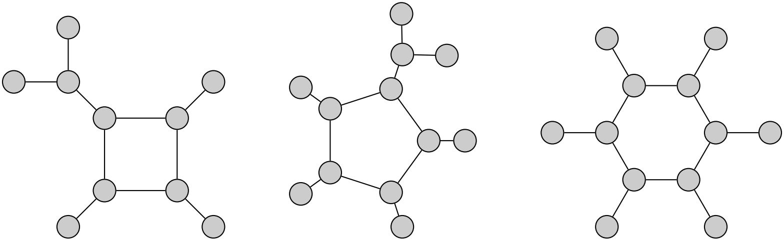

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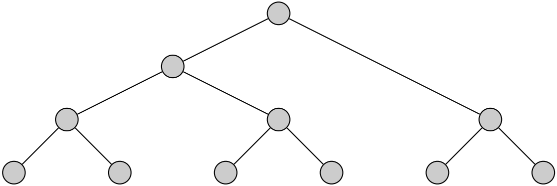

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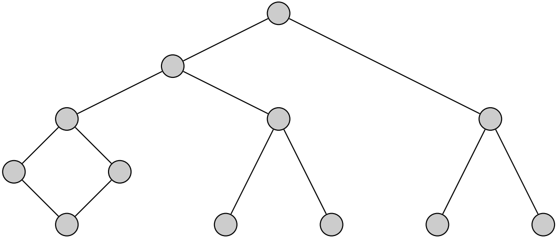

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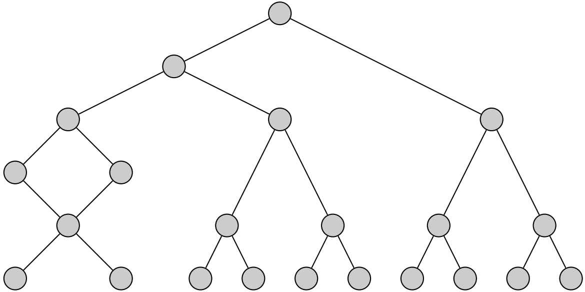

Theorem (–,Cools, 2014)

A metric $D = (D_{ij}) \in \mathbb{R}^{15}$ *on* 6 *vertices arises from a metric graph of genus* ≤ 1 *if and only if there exist 3 quarters* $I_1, I_2, I_3 \subset [6] = \{1, \ldots, 6\}$ *such that:*

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iii) *if* $|I_i \cap I_j| = 2$*, with* $I_i = \{v_1, v_2, v_3, v_4\}$ *and* $I_j = \{v_1, v_2, v_5, v_6\}$ *then the minimum in*
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E_i be the $n \times n$–matrix where

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$$D_i(\alpha) = D - \alpha \cdot E_i$$

Theorem

$D_i(\alpha)$ is a distance matrix if and only if

$$\alpha \leq \frac{1}{2} \cdot (d_{pi} + d_{ir} - d_{pr}), \text{ for all } p, r \neq i.$$

The new metric $D_i(\alpha)$, obtained from D, is called a compaction.

The compaction of an index i of D leads to a new matrix with a possible pair of equal rows (and by symmetry of equal columns).

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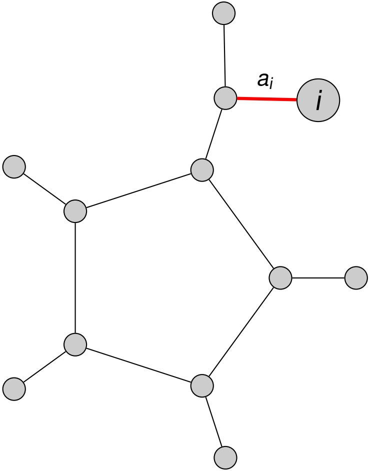
Definition

Given a distance matrix D of order $n \times n$ define the vector $\mathbf{a} = (a_1, \ldots, a_n)$ where

$$a_i = \frac{1}{2} \cdot \min_{p \neq i, r \neq i} \{d_{pi} + d_{ir} - d_{pr}\}$$

The vector $\mathbf{a}$ is called the compaction vector of D.

Notice that, in the case i is a leaf, then the entry a_i is the length of the edge connecting the i to its internal node.



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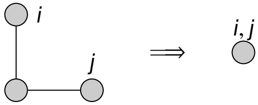

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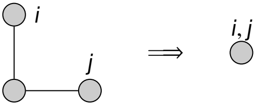


Theorem

A distance matrix D, of order n, as an n−gon as a realization if and only if there exists a real permutation π such that

$$d_{i\pi^s(i)} = \min\{d_{i\pi(i)} + d_{\pi(i)\pi^2(i)} + \cdots + d_{\pi^{s-1}(i)\pi^s(i)}, d_{\pi^s(i)\pi^{s+1}(i)} + \cdots + d_{\pi^{n-1}(i)i}\}$$

for all $i = 1, \ldots, n$ and for all $s = 1, \ldots, n$.

Algorithm[–,Cools, 2014]

INPUT: a distance matrix D

Step 1 Compute the compaction vector $\mathbf{a}$ of D. If $\mathbf{a}$ is the null vector then go to step 4

Step 2 compute the compaction matrix $D(\mathbf{a})$ of D with respect to $\mathbf{a}$. Keep track of $\mathbf{a}$ and of equal rows in $D(\mathbf{a})$.

step 3 Remove equal rows (and columns) in $D(\mathbf{a})$ obtaining a matrix D'. Pose $D = D'$ and go to step 1

step 4 Check if D is realizable by a n–gon. If so, going forward on the procedure, we can construct a graph of genus 1 which is a realization of the initial matrix D.

$$D_1 = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 2 & 5 & 6 & 7 & 4 \\ 2 & 2 & 0 & 5 & 6 & 7 & 4 \\ 3 & 5 & 5 & 0 & 4 & 5 & 3 \\ 4 & 6 & 6 & 4 & 0 & 3 & 2 \\ 5 & 7 & 7 & 5 & 3 & 0 & 3 \\ 6 & 4 & 4 & 3 & 2 & 3 & 0 \end{array}$$

$$D_1 = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 2 & 5 & 6 & 7 & 4 \\ 2 & 2 & 0 & 5 & 6 & 7 & 4 \\ 3 & 5 & 5 & 0 & 4 & 5 & 3 \\ 4 & 6 & 6 & 4 & 0 & 3 & 2 \\ 5 & 7 & 7 & 5 & 3 & 0 & 3 \\ 6 & 4 & 4 & 3 & 2 & 3 & 0 \end{array}$$

$$\mathbf{a}_1 = \left(1, 1, \frac{3}{2}, 1, 2, 0\right)$$

$$D_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 2 & 5 & 6 & 7 & 4 \\ 2 & 0 & 5 & 6 & 7 & 4 \\ 5 & 5 & 0 & 4 & 5 & 3 \\ 6 & 6 & 4 & 0 & 3 & 2 \\ 7 & 7 & 5 & 3 & 0 & 3 \\ 4 & 4 & 3 & 2 & 3 & 0 \end{pmatrix} \end{array}$$

$$\mathbf{a}_1 = (1, 1, \frac{3}{2}, 1, 2, 0)$$

$$D_1(\mathbf{a}_1) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & \frac{5}{2} & 4 & 4 & 3 \\ 0 & 0 & \frac{5}{2} & 4 & 4 & 3 \\ \frac{5}{2} & \frac{5}{2} & 0 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ 4 & 4 & \frac{3}{2} & 0 & 0 & 1 \\ 4 & 4 & \frac{3}{2} & 0 & 0 & 1 \\ 3 & 3 & \frac{3}{2} & 1 & 1 & 0 \end{pmatrix} \end{array}$$

$$D_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 2 & 5 & 6 & 7 & 4 \\ 2 & 0 & 5 & 6 & 7 & 4 \\ 5 & 5 & 0 & 4 & 5 & 3 \\ 6 & 6 & 4 & 0 & 3 & 2 \\ 7 & 7 & 5 & 3 & 0 & 3 \\ 4 & 4 & 3 & 2 & 3 & 0 \end{pmatrix} \end{array}$$

$$\mathbf{a}_1 = (1, 1, \frac{3}{2}, 1, 2, 0)$$

$$D_1(\mathbf{a}_1) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & \frac{5}{2} & 4 & 4 & 3 \\ 0 & 0 & \frac{5}{2} & 4 & 4 & 3 \\ \frac{5}{2} & \frac{5}{2} & 0 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ 4 & 4 & \frac{3}{2} & 0 & 0 & 1 \\ 4 & 4 & \frac{3}{2} & 0 & 0 & 1 \\ 3 & 3 & \frac{3}{2} & 1 & 1 & 0 \end{pmatrix} \end{array}$$

$$D_2 = \begin{array}{c} \\ 1,2 \\ 3 \\ 4,5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 1,2 & 3 & 4,5 & 6 \end{array} \\ \begin{pmatrix} 0 & \frac{5}{2} & 4 & 3 \\ \frac{5}{2} & 0 & \frac{3}{2} & \frac{3}{2} \\ 4 & \frac{3}{2} & 0 & 1 \\ 3 & \frac{3}{2} & 1 & 0 \end{pmatrix} \end{array}$$

$$D_2 = \begin{array}{c} \\ 1,2 \\ 3 \\ 4,5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 1,2 & 3 & 4,5 & 6 \end{array} \\ \begin{pmatrix} 0 & \frac{5}{2} & 4 & 3 \\ \frac{5}{2} & 0 & \frac{3}{2} & \frac{3}{2} \\ 4 & \frac{3}{2} & 0 & 1 \\ 3 & \frac{3}{2} & 1 & 0 \end{pmatrix} \end{array}$$

$$D_2 = \begin{array}{c} \\ 1,2 \\ 3 \\ 4,5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 1,2 & 3 & 4,5 & 6 \end{array} \\ \begin{pmatrix} 0 & \frac{5}{2} & 4 & 3 \\ \frac{5}{2} & 0 & \frac{3}{2} & \frac{3}{2} \\ 4 & \frac{3}{2} & 0 & 1 \\ 3 & \frac{3}{2} & 1 & 0 \end{pmatrix} \end{array}$$

$$\mathbf{a}_2 = (2, 0, \frac{1}{2}, 0)$$

$$D_2 = \begin{array}{c} \\ 1,2 \\ 3 \\ 4,5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 1,2 & 3 & 4,5 & 6 \end{array} \\ \begin{pmatrix} 0 & \frac{5}{2} & 4 & 3 \\ \frac{5}{2} & 0 & \frac{3}{2} & \frac{3}{2} \\ 4 & \frac{3}{2} & 0 & 1 \\ 3 & \frac{3}{2} & 1 & 0 \end{pmatrix} \end{array}$$

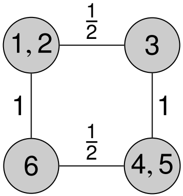
$$\mathbf{a}_2 = (2, 0, \frac{1}{2}, 0)$$

$$D_2(a_2) = \begin{array}{c} \\ 1,2 \\ 3 \\ 4,5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 1,2 & 3 & 4,5 & 6 \end{array} \\ \begin{pmatrix} 0 & \frac{1}{2} & \frac{3}{2} & 1 \\ \frac{1}{2} & 0 & 1 & \frac{3}{2} \\ \frac{3}{2} & 1 & 0 & \frac{1}{2} \\ 1 & \frac{3}{2} & \frac{1}{2} & 0 \end{pmatrix} \end{array}$$

The compaction vector of $D_2(\mathbf{a}_2)$is the null vector.

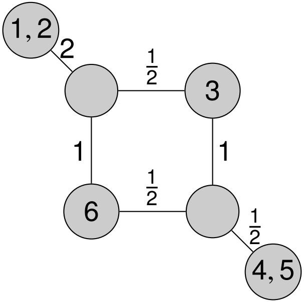
However, $D_2(\mathbf{a}_2)$ has a realization by an 4–gon.

$$
\begin{array}{c} \\ 1,2 \\ 3 \\ 4,5 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{cccc} 1,2 & 3 & 4,5 & 6 \end{array} \\
\begin{pmatrix}
0 & \frac{1}{2} & \frac{3}{2} & 1 \\
\frac{1}{2} & 0 & 1 & \frac{3}{2} \\
\frac{3}{2} & 1 & 0 & \frac{1}{2} \\
1 & \frac{3}{2} & \frac{1}{2} & 0
\end{pmatrix}
\end{array}
$$



$$\begin{array}{c|cccc} & 1,2 & 3 & 4,5 & 6 \\ \hline 1,2 & 0 & \frac{1}{2} & \frac{3}{2} & 1 \\ 3 & \frac{1}{2} & 0 & 1 & \frac{3}{2} \\ 4,5 & \frac{3}{2} & 1 & 0 & \frac{1}{2} \\ 6 & 1 & \frac{3}{2} & \frac{1}{2} & 0 \end{array}$$

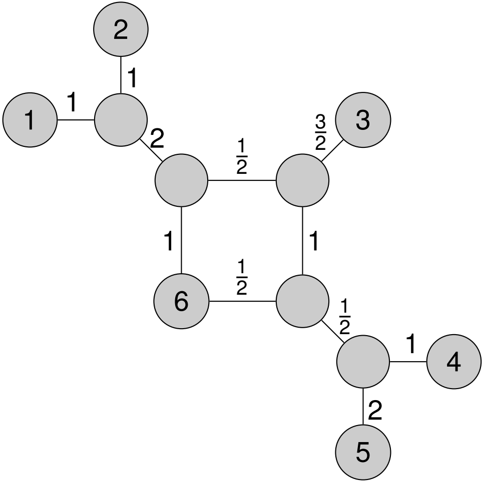
$$\mathbf{a}_2 = (2, 0, \frac{1}{2}, 0)$$



$$\begin{array}{c|cccc} & 1,2 & 3 & 4,5 & 6 \\ \hline 1,2 & 0 & \frac{1}{2} & \frac{3}{2} & 1 \\ 3 & \frac{1}{2} & 0 & 1 & \frac{3}{2} \\ 4,5 & \frac{3}{2} & 1 & 0 & \frac{1}{2} \\ 6 & 1 & \frac{3}{2} & \frac{1}{2} & 0 \end{array}$$

$$\mathbf{a}_2 = (2, 0, \frac{1}{2}, 0)$$

$$\mathbf{a}_1 = (1, 1, \frac{3}{2}, 1, 2, 0)$$



Thank you for your attention