Expressing a polynomial of degree *kd* as a sum of *k*-th powers
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The Waring decomposition

Let f be a complex homogeneous polynomial of degree d, in $(n+1)$ variables, an expression of f as sum of d-th powers

$$f = \sum_{i=1}^{r} g_i^d,$$

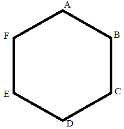
where $\deg g_i = 1$, is called a *Waring decomposition* of f.

The relevance of this decomposition is increased in recent years, due to applications, in the setting of more general *tensor decomposition*.

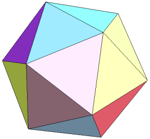
The first basic question is what is the minimal number of summands, which is called the *rank*. From a naive dimensional count, one expects $r(n,d) = \lceil \frac{1}{n+1}\binom{n+d}{d} \rceil$ summands.

Examples from regular polytopes

- $(x^2+y^2)^s = \sum_{k=1}^{s+1} (a_k x + b_k y)^{2s}$ where $(\pm a_k, \pm b_k)$ are the vertices of a regular polygon with $2s+2$ sides, centered at the origin.

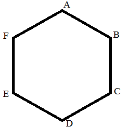


- $(x^2+y^2+z^2)^2 = \sum_{i=1}^{6} (a_k x + b_k y + c_k z)^4$, where $(\pm a_k, \pm b_k, \pm c_k)$ are the vertices of a icosahedron centered at the origin.

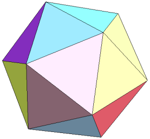


The naive dimensional count may produce wrong results

- The $(p+1) = r(1, 2p)$ summands of the polygons agree the naive dimensional count,



- the 6 summands of the icosahedron are one more than the expected value $5 = r(2, 4)$ from the naive dimensional count. Even for general polynomials of degree 4 in 3 variables, 6 summands are needed [Clebsch, 1861].



The Alexander-Hirschowitz Theorem

Theorem (Alexander-Hirschowitz)

The general homogeneous polynomial f *in* $(n+1)$ *variables, of degree* d *over* $\mathbb{C}$*, can be written as*

$$f = \sum_{i=1}^{r} g_i^d,$$

with suitable g_i *such that* $\deg g_i = 1$*, where* $r = r(n,d) = \lceil \frac{1}{n+1}\binom{n+d}{d} \rceil$*, unless*

- $d = 2$, $2 \le n$, *where we need* $r = n+1$,
- $d = 4$, $2 \le n \le 4$, *where we need* $r = r(n,4)+1$,
- $d = 3$, $n = 4$, *where we need* $r = 8 = r(4,3)+1$.

Link with secant varieties

The theorem has a reformulation in terms of secant varieties of Veronese varieties, since the powers g^d form the d-Veronese variety $v_d(\mathbb{P}^n)$.

Theorem (AH Theorem, reformulated)

The s-th secant variety $\sigma_s(v_d(\mathbb{P}^n))$ has the expected dimension $\min\left(s(n+1)-1, \binom{n+d}{d}-1\right)$, unless the exceptions listed before.

Link with interpolation

Theorem (3rd version of AH Theorem)

A polynomial of degree d in $(n+1)$ variables, such that its first partial derivatives vanish at $r(d,n)$ general points of $\mathbb{P}^n$ is zero, unless the exceptions listed before.

The interpolation point of view makes the problem more elementary.
It is enough to find a *specific* set $\{P_1, \ldots, P_{r(d,n)}\}$ of points of $\mathbb{P}^n$ such that $\{2P_1, \ldots, 2P_{r(d,n)}\}$ impose independent conditions on polynomials of degree d. Then the AH Theorem follows by semicontinuity.

Nobody has never found, up to now, such a collection of points.
For small d, n, they can be found as random points, but the proof becomes computational and cannot be generalized to higher d, n.

The proof of AH Theorem and its history

- The case $n = 1$ goes back to Newton and Leibniz.
- Campbell proved the case $n = 2$ in 1892.
- Terracini proved the case $n = 3$ in 1915.
- Alexander and Hirschowitz proved the general case in 1995. Their original proof is quite difficult and inductive, is divided in 4 papers and it is 150 pages long.
- Today the proof has been simplified and it can be covered in less than 20 pages.
- Still a elegant explanation of why this list of exceptions appears is missing.

Partial results on real version of AH Theorem

Let f be a *real* homogeneous polynomial of degree d, a Waring decomposition has the form

$$f = \sum_{i=1}^{r} \lambda_i g_i^d,$$

where $\deg g_i = 1$, $\lambda_i = 1$ if d is odd, $\lambda_i = \pm 1$ if d is even.

Theorem (Blekherman, 2012)

Let
$A_r = \{\textit{real homog. polynomials } f(x_0, x_1) \textit{ of deg } d \textit{ and rank } r\}.$

$$\textit{Then } \mathring{A}_r \neq \emptyset \iff \mathrm{vol} A_r > 0 \iff r \in [\lceil \frac{d+1}{2} \rceil, \ldots, d]$$

The above values of the rank are called typical ranks.

Very few results are known for real polynomials in more variables, unless the classical results by Sylvester in case $d = 2$.

The general Waring problem for polynomials.

Given a homogeneous polynomial f of degree kd, express it as

$$f = \sum_{i=1}^{r} g_i^k \text{ over } \mathbb{C}, \qquad f = \sum_{i=1}^{r} \lambda_i g_i^k \text{ over } \mathbb{R},$$

where $\deg g_i = d$.

Question: what is the minimal number r of summands ?

The case $k = 2$ has a particolar relevance, since sums of squares detect positivity in the real case.

Sum of squares of binary forms

Lemma

Let $f = f(x_0, x_1)$ be a complex polynomial in two variables of degree $2d$. Then there are polynomials g_1, g_2 of degree d such that $f = g_1^2 + g_2^2$.

Proof.

Split $f = f_1 f_2 = (\frac{f_1+f_2}{2})^2 + (\sqrt{-1}\frac{f_1-f_2}{2})^2$, where $\deg f_1 = \deg f_2 = d$. $\square$

Note there are $\frac{1}{2}\binom{2d}{d} = \binom{2d-1}{d}$ ways to split the polynomial f.

Newton solution for nonnegative real polynomials

Newton modified the above proof to work with the nonnegative real case.

Theorem (Newton)

Let $f = f(x_0, x_1)$ be a real nonnegative polynomial in two variables of degree $2d$. Then there are real polynomials g_1, g_2 of degree d such that $f = g_1^2 + g_2^2$.

Proof.

We may split $f = f_1\overline{f_1}$ and we get
$f = (\frac{f_1+\overline{f_1}}{2})^2 - (\frac{f_1-\overline{f_1}}{2})^2 = (\mathrm{Re} f_1)^2 + (\mathrm{Im} f_1)^2$. □

Theorem (Naldi, 2012)

f_1 may be chosen in such a way that both $(\mathrm{Re} f_1)$, $(\mathrm{Im} f_1)$ have only real roots.

Hilbert theorem

Theorem (Hilbert)

Any nonnegative real polynomials of degree d in $(n+1)$ variables can be written as sum of squares only if $n=1$ (binary forms) or $(n,d)=(2,4)$ (ternary quartics).

Sum of squares for plane curves

There are some sporadic results about representations of (complex) plane curves as sum of squares.

Theorem (see Dolgachev book on Classical Algebraic Geometry)

Every smooth plane quartic C can be written as sum of three squares in 63 different ways, they correspond to the 63 nontrivial 2-torsion element in $\mathrm{Jac}(C)$.

Theorem (Blekherman, Hauenstein, Ottem, Ranestad, Sturmfels, 2011)

Every plane sextic can be written as sum of four squares. The closure of the set of plane sextics which are a sums of three squares forms a hypersurface of degree 83200 in the space of all sextics.

Asymptotic value in general case

Denote by S_n^d the vector space of all forms of degree d in $n+1$ variables. Recall that $\dim S_n^d = \binom{d+n}{n}$.

The naive dimensional count gives the ratio $\lceil \frac{\dim S_n^{kd}}{\dim S_n^d} \rceil$ as the expected number of k-th powers neded to express a general polynomial of degree kd.

Simple calculations show that

$$\frac{\dim S_n^{kd}}{\dim S_n^d} < k^n \quad \text{and} \quad \lim_{d\to\infty} \frac{\dim S_n^{kd}}{\dim S_n^d} = k^n.$$

Our main result

Theorem (Fröberg-O-Shapiro, 2012)

Let $d \geq 2$. The general complex homogeneous polynomial f of degree kd in $(n+1)$ variables can be expressed as

$$f = \sum_{i=1}^{r} g_i^k$$

with $\deg g_i = d$, where

$$r = k^n.$$

The number of summands is sharp when $d \gg 0$.

Reformulation with ideals generated by sum of $(k-1)$-th powers

Theorem

Given a linear space V, a general polynomial in $S^{kd}V$ is a sum of r k-th powers $g_1^k, \dots g_r^k$ where $g_i \in S^d V$ if and only if for r general forms $g_i \in S^d V$, $i = 1, \dots r$, the ideal generated by $g_1^{k-1}, \dots g_r^{k-1}$ contains $S^{kd}V$. (We shall call such an ideal kd-regular.)

The statement is a direct consequence of Terracini's lemma. Indeed, consider the subvariety X in the ambient space $\mathbb{P}S^{kd}V$ consisting of the k-th powers of all forms from $S^d V$. The tangent space to X at $g_i^k \in X$ is of the form $\{g_i^{k-1} f | f \in S^d V\}$.

To prove our main result, it suffices to find k^n specific polynomials $\{g_1, \dots g_{k^n}\}$ of degree d such that the ideal generated by the powers g_i^{k-1} is kd-regular. We will choose as g_i's the powers of certain linear forms.

For powers of linear forms one can use classical apolarity, as follows.
The space $T_{g^k}X^\perp$ orthogonal to $T_{g^k}X = \{g^{k-1}f | f \in S^d V\}$ is given by $T_{g^k}X^\perp = \{h \in S^{kd}V^\vee | h \cdot g^{k-1} = 0 \in S^d V^\vee\}$, i.e., it is the space of polynomials in $V^\vee$ apolar to g^{k-1}.

Lasker Lemma

A form $f \in S^m V^\vee$ is apolar to l^{m-k}, i.e., $l^{m-k} f = 0$ if and only if all the derivatives of f of order $\leq k$ vanish at $l \in V$.

We need to show the following

Theorem

Let $k \geq 2$. A form of degree kd in $(n+1)$ variables, which has all derivatives of order $\leq d$ vanishing at k^n general points, vanishes identically.

In order to prove this Theorem, by semicontinuity, it is enough to find k^n special points in $\mathbb{P}V \simeq \mathbb{P}^n$ such that a polynomial of degree kd in $\mathbb{P}^n$ which has all derivatives of order $\leq d$ vanishing at these points must necessarily vanish identically.

The lattice with k^n points works in general

Let $\xi_i = e^{2\pi i\sqrt{-1}/k}$ for $i = 0, \ldots, k-1$ be the k-th roots of unity.

As set of k^n points, we choose the points $(1, \xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_n})$ where $0 \leq i_j \leq k-1$, $1 \leq j \leq n$.
Then we prove the following more general interpolation result.

Theorem

Let $k \geq 2$. A form of degree $kd + k - 1$ in $(n+1)$ variables which has all derivatives of order $\leq d$ vanishing at k^n general points vanishes identically.

Inductive proof with the lattice

Denote by $x_0, \ldots x_n$ a basis of V.

We prove first the case $n = 1$ directly.

For $n \geq 2$ consider the arrangement of $\binom{n}{2}k$ hyperplanes given by $x_i = \xi_s x_j$ where $1 \leq i < j \leq n$, $0 \leq s \leq k-1$. One can easily check that this arrangement has the property that each hyperplane contains exactly k^{n-1} points and, furthermore, each point is contained in exactly $\binom{n}{2}$ hyperplanes.

The proof works by double induction on n and d.

A corollary of independent interest

Corollary

Any form of degree kd in $(n+1)$ variables can be expressed as a linear combination of the polynomials $(x_0 + \xi_{i_1}x_1 + \xi_{i_2}x_2 + \ldots + \xi_{i_n}x_n)^{(k-1)d}$, with coefficients being polynomials of degree d.

The real case, revisited

We have shown that any form of degree $2d$ in $(n+1)$ variables can be expressed as a linear combination of the polynomials $(x_0 \pm x_1 \pm x_2 + \ldots + \pm x_n)^d$ with coefficients being polynomials of degree d. Note that the former polynomials have real coefficients.

Our main result does not hold over the reals. It only implies that there is, in the usual topology, an open set of real polynomials of degree $2d$ which can be expressed as real linear combinations of 2^n squares of real polynomials of degree d.

In other words, 2^n is a typical rank. Notice that other typical ranks might also appear on other open subsets of polynomials.

Two natural open problems

- Can the typical ranks be determined in the real case ?
- Apart from the asymptotic value, are there exceptions for small values, like in AH Theorem ?

The Fröberg Conjecture

Let I be the ideal generated by s general forms of degree d_i in $(n+1)$ variables.

Fröberg Conjecture

The quotient ring R/I has the expected Hilbert function

$$\frac{\prod_{i=1}^{s}(1-t^{m_i})}{(1-t)^{n+1}}$$

where the series is truncated when it gets ≤ 0 values.
The Fröberg Conjecture has been proved for $n \leq 2$ [Anick], for $s \leq n+2$ [Stanley].
In this conjecture there are exceptions if we replace the general forms with some powers of linear forms.

Thanks for your attention !!