Degree - Complexity :

How difficult is it to compute Gröbner Bases ?

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Notations and Definitions

- $\Bbbk$: algebraically closed field of $\mathrm{char}(\Bbbk) = 0$
- $R = \Bbbk[x_0, \ldots, x_n]$: a polynomial ring $/\Bbbk$

Definition [Term ordering on monomials]

For $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_{\geq 0}^{n+1}$,

$$X^\alpha := x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \quad : \text{ a monomial}$$

$\Rightarrow$ A total order τ on monomials is a term order

if τ satisfies

$$X^\alpha <_\tau X^\beta \Rightarrow X^\alpha X^\gamma <_\tau X^\beta X^\gamma$$

In this talk, we will focus on two term orders

- Graded lexicographic order (glex)

$x^\alpha >_{glex} x^\beta$ if and only if

i) $\deg(x^\alpha) > \deg(x^\beta)$ or

ii) $\deg(x^\alpha) = \deg(x^\beta)$ and the leftmost nonzero of $\alpha - \beta$ is positive.

- Graded reverse lexicographic order (grlex)

$x^\alpha >_{grlex} x^\beta$ if and only if

i) $\deg(x^\alpha) > \deg(x^\beta)$ or

ii) $\deg(x^\alpha) = \deg(x^\beta)$ and the rightmost nonzero of $\alpha - \beta$ is negative.

Example

Monomials of degree 3 in $k[x,y,z]$

$xz^2 \longrightarrow \alpha = (1, 0, 2)$

$y^3 \longrightarrow \beta = (0, 3, 0)$

$\Rightarrow \alpha - \beta = (1, 0, -2)$

$\Rightarrow \quad xz^2 > y^3 \text{ (glex)} \qquad xz^2 < y^3 \text{ (grlex)}$

- For a homogeneous polynomial $f(x_0, \cdots, x_n) \in R$,

  $in_\tau(f) :=$ the greatest monomial of $f(x_0 \cdots x_n)$ w.r.t the term order τ.

- For a homogeneous ideal $I \subset R$,

  $in_\tau(I) :=$ the ideal generated by the set $\{ in_\tau(f) \mid f \in I \}$

  We say that $in_\tau(I)$ is the initial ideal of I w.r.t τ.

Let $I = (f_1, \cdots, f_t)$ be a homogeneous ideal of R.

$\Rightarrow$ In general, $(\text{in}_\tau(f_1), \cdots, \text{in}_\tau(f_t)) \subsetneqq \text{in}_\tau(I)$

$\Rightarrow$ We can choose a set of polynomials

$$\{g_1, g_2, \cdots, g_k\} \subset I$$

such that

$$(\text{in}_\tau(f_1), \cdots, \text{in}_\tau(f_t), \text{in}_\tau(g_1), \cdots, \text{in}_\tau(g_k)) = \text{in}_\tau(I)$$

Definition [Gröbner bases]

A set of polynomials $S = \{h_1, h_2, \cdots, h_r\} \subset I$ is a Gröbner basis for I w.r.t τ, if

$$(\text{in}_\tau(h_1), \cdots, \text{in}_\tau(h_r)) = \text{in}_\tau(I)$$

Remark

A reduced Gröbner basis is the unique Gröbner basis $S = \{f_1, \cdots, f_n\}$ for I such that

(i) the leading coefficient of $f_i = 1$ for each $i = 1, \cdots, n$

(ii) no monomial of f_i lies in the ideal $\left(\mathrm{in}_\tau(f_1), \cdots, \overbrace{\mathrm{in}_\tau(f_i)}, \cdots, \mathrm{in}_\tau(f_n)\right)$ for each $i = 1, \cdots, n$

Let I be a homogeneous ideal of $R = k[x_0 \dots x_n]$

- $g = (g_{ij})$: an invertible matrix in $GL_{n+1}(R_1)$
- $f(x_0 \dots x_n) \in R$

We define an action on R by

$$f(x_0, \dots, x_n) \longmapsto f(g(x_0), \cdots, g(x_n))$$

where $g(x_i) = \sum_{k=0}^{n} g_{ik} x_k$.

$\Rightarrow$ We can extend this action to a homogeneous ideal $I \subset R$

$$g(I) = \langle f(g(x_0), \cdots, g(x_n)) \mid f \in I \rangle$$

- $C \subset \mathbb{P}^3$: a rational normal curve
- $I_C = (x_1^2 - x_0x_2,\ x_1x_2 - x_0x_3,\ x_2^2 - x_1x_3)$
- $\mathrm{in}_{\mathrm{glex}}(I_C) = (x_0x_2,\ x_0x_3,\ x_1x_3)$
- For a random $g \in GL(\Bbbk^4)$,

$$\mathrm{in}_{\mathrm{glex}}(g(I_C)) = (x_0^2,\ x_0x_1,\ x_1^3,\ x_0x_2)$$

$\Rightarrow$ The initial ideal of I depends on the choice of coordinates.

(i.e $\mathrm{in}_\tau(g(I))$ varies with the choice of $g = (g_{ij}) \in GL_{n+1}$)

However, by allowing a generic change of coordinates, we may eliminate this dependence.

Theorem [Galligo, Bayer-Stillman]

There is a Zariski open set $U \subset GL_{n+1}(R_1)$, and a monomial ideal J such that

$$J = \mathrm{in}_\tau(g(I)) \text{ for all } g \in U \subset GL_{n+1}$$

Theorem [Galligo, Bayer–Stillman]
There is a Zariski open set $U \subset GL_{n+1}(R_1)$, and a monomial ideal J such that

$$J = in_\tau(g(I)) \text{ for all } g \in U \subset GL_{n+1}$$

Remark

1. We call J the generic initial ideal of I, and denote it by $Gin_\tau(I)$.
2. We say that I is in generic coordinates if $in_\tau(I) = Gin_\tau(I)$.

Question How difficult is it to compute GB.?

Since the construction of GB provides the foundation for most computation in algebraic geometry and commutative algebra, it is important to know the complexity of the computation of GB.

Bayer-Mumford introduced

"degree-complexity" and "regularity"

of initial ideals w.r.t a given term order,

as a measure of the complexity of computing GB.

Definition [Regularity]

For a homogeneous ideal I of R, consider the minimal free resolution $\mathbb{F}_\bullet$ of I:

$$\mathbb{F}_\bullet : 0 \to F_r \xrightarrow{\phi_r} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} I \to 0$$

where $F_i = \bigoplus_j R(-i-j)^{\beta_{ij}}$

We say that I is m-regular if $j \leq m$ for all i, j with $\beta_{i,j+i} \neq 0$

We define the regularity of I, denoted by $\operatorname{reg}(I)$, as the smallest m for which I is m-regular.

Definition [Degree-Complexity]

Suppose that I is in generic coordinates. For a term order τ, the degree-complexity $m_\tau(I)$ is the maximum degree in a reduced Gröbner basis of I. In particular,

- $m(I)$:= the degree-complexity w.r.t grlex.
- $M(I)$:= the degree-complexity w.r.t glex.

Remark The degree-complexity of a variety $X \subset \mathbb{P}^n$ is defined for its defining ideal $I_X = \bigoplus_{m \geq 0} H^0(\mathcal{I}_X(m))$

In general, it is so hard to compute the reduced Gröbner basis G in generic coordinates.

We need to use CAS for computation.

eg. CoCoA
Macaulay 2
Singular

```
-- Goal: compute the generic initial ideal of a homogeneous ideal $I$
--        with respect to a $TermOrder$

----------------------code for function "gin" ---------------------------------
-- Input: a homogeneous ideal $I$ and a TermOrder.
-- Output: generic initial ideal w.r.t a TermOrder.
-- Syntax: gin (I: a homogeneous ideal, TermOrder)

gin = (I,TermOrder) -> (
 S := ring I;
 n := # gens S;
 R := (coefficientRing S)[gens S, MonomialOrder=> TermOrder];
 f := map(R,S, gens R);
 J := f(I);
 RtoR := map(R,R,random(R^{0}, R^{numgens R:-1}));
 monomialIdeal RtoR J
 )
```

Example [Macaulay2]

Consider a smooth complete intersection curve $C \subset \mathbb{P}^4$ of type $(2,2,3)$. Then,

- $I = (F_1, F_2, G)$ $\quad$ $(\deg F_i = 2, \deg G = 3)$
- $\mathrm{Gin}_{rlex}(I) = (x_0^2, x_0x_1, x_1^3, x_1^2x_2, x_0x_2^3, x_1x_2^3, x_2^5)$
- $d(I) =$ the maximum degree in $\{F_1, F_2, G\} = 3$
- $m(I) = 5$. (what is $M(I) = ?$)

$$\mathrm{Gin}_{\mathrm{Glex}}(I) =$$

o12 = monomialIdeal $(x_0^2, x_0x_1, x_1^4, x_1^3x_2, x_0x_2^2, x_1^2x_2^4, x_1x_2^7, x_2^{12}, x_0x_2x_3, x_1^2x_2^3x_3, x_1^3x_3^2, x_1^2x_2^2x_3^2, x_1x_2^6x_3^2, x_0x_3^3, x_1^2x_2x_3^4, x_1x_2^5x_3^4, x_1^2x_3^6, x_1x_2^4x_3^6, x_1x_2^3x_3^9,$

$x_1x_2^2x_3^{13}, x_1x_2x_3^{20}, x_1x_3^{42}, x_1^2x_2^3x_4, x_1^3x_3x_4, x_1x_2^6x_3x_4, x_1^2x_2x_3^3x_4, x_1x_2^5x_3^3x_4, x_1^2x_3^5x_4, x_1x_2^4x_3^5x_4, x_1x_2^3x_3^8x_4, x_1x_2^2x_3^{12}x_4, x_0x_2x_4^2, x_1^2x_2^2x_3x_4^2, x_0x_3^2x_4^2,$

$x_1^2x_2x_3^2x_4^2, x_1x_2^5x_3^2x_4^2, x_1^2x_3^4x_4^2, x_1x_2^4x_3^4x_4^2, x_1x_2^3x_3^7x_4^2, x_1x_2^2x_3^{11}x_4^2, x_1x_2x_3^{19}x_4^2, x_1^3x_4^3, x_1^2x_2^2x_4^3, x_1x_2^6x_4^3, x_0x_3x_4^3, x_1x_2^3x_3^6x_4^3, x_1^2x_2x_3x_4^4, x_1x_2^5x_3x_4^4, x_1^2x_3^3x_4^4,$

$x_1x_2^4x_3^3x_4^4, x_1x_2^2x_3^{10}x_4^4, x_1x_2x_3^{18}x_4^4, x_0x_4^5, x_1^2x_2x_4^5, x_1x_2^5x_4^5, x_1^2x_3^2x_4^5, x_1x_2^4x_3^2x_4^5, x_1x_2^3x_3^5x_4^5, x_1x_2^2x_3^9x_4^5, x_1^2x_3x_4^6, x_1x_2^4x_3x_4^6, x_1x_2^3x_3^4x_4^6, x_1x_2x_3^{17}x_4^6, x_1x_2^4x_4^7,$

$x_1x_2^3x_3^3x_4^7, x_1x_2^2x_3^8x_4^7, x_1^2x_4^8, x_1x_2^2x_3^7x_4^8, x_1x_2x_3^{16}x_4^8, x_1x_2^3x_3^2x_4^9, x_1x_2^3x_3x_4^{10}, x_1x_2^2x_3^6x_4^{10}, x_1x_2x_3^{15}x_4^{10}, x_1x_2^3x_4^{11}, x_1x_2^2x_3^5x_4^{11}, x_1x_2x_3^{14}x_4^{12}, x_1x_2^2x_3^4x_4^{13},$

$x_1x_2^2x_3^3x_4^{14}, x_1x_2x_3^{13}x_4^{14}, x_1x_2^2x_3^2x_4^{16}, x_1x_2x_3^{12}x_4^{16}, x_1x_2^2x_3x_4^{17}, x_1x_2x_3^{11}x_4^{18}, x_1x_2^2x_4^{19}, x_1x_2x_3^{10}x_4^{20}, x_1x_2x_3^9x_4^{22}, x_1x_2x_3^8x_4^{24}, x_1x_2x_3^7x_4^{26}, x_1x_2x_3^6x_4^{28},$

$x_1x_2x_3^5x_4^{30}, x_1x_2x_3^4x_4^{32}, x_1x_2x_3^3x_4^{34}, x_1x_2x_3^2x_4^{36}, x_1x_2x_3x_4^{38}, x_1x_2x_4^{40})$

Example Consider a smooth rational curve $C \subseteq \mathbb{P}^3$ defined by $[s.t] \to [s^5, s^4t, st^4, t^5]$

$$\Rightarrow I_C = (\underbrace{x_1x_2 - x_0x_3}_{F_1}, \underbrace{x_2^4 - x_1x_3^3}_{F_2}, \underbrace{x_0x_2^3 - x_1^2x_3^2}_{F_3}, \underbrace{x_0^2x_2^2 - x_1^3x_3}_{F_4}, \underbrace{x_1^4 - x_0^3x_2}_{F_5})$$

$$\Rightarrow \mathrm{Gin}_{glex}(I) = (x_0^2, x_0x_1^3, x_1^5, x_0x_1x_2^2, x_0x_2^6, x_0x_1^2x_3, x_0x_1x_2x_3^2, x_0x_1x_3^4)$$

- $d(I_C) = 4$
- $M(I_C) = 7$

One of the important problems is to bound the regularity of $\mathrm{Gin}_\tau(I)$ for a term order τ.

The following problem was given by Bayer-Mumford

Problem

1. How much bigger $m(I)$ $(M(I))$ can be than $d(I)$?

2. Bound $m(I)$ $(M(I))$ with respect to $d(I)$ and the number of variables n.

Review of Known Results.

- For a monomial term order τ, $\mathrm{Gin}_\tau(I)$ has Borel fixed property (i.e $x_i X^\alpha \in \mathrm{Gin}_\tau(I) \Rightarrow x_j X^\alpha \in \mathrm{Gin}_\tau(I)$ for all $j < i$.

- $\mathrm{reg}(\mathrm{Gin}_\tau(I)) = m_\tau(I)$ [Bayer-Stillman]

In particular,

$$m(I) = \mathrm{reg}(\mathrm{Gin}_{grlex}(I)) \qquad M(I) = \mathrm{reg}(\mathrm{Gin}_{glex}(I))$$

Theorem Let $I \subset R$ be a homogeneous ideal.

$$\operatorname{reg}(I) = \operatorname{reg}(\operatorname{Gin}_{grlex}(I)) \overset{\text{Hence}}{=} m(I).$$

↑ Bayer-Stillman

Remark We notice that knowing an upper bound for $m(I)$ is equivalent to knowing the regularity bound $\operatorname{reg}(I)$.

A great deal of research has been conducted with the aim of answering this question.

Theorem In general,

$$m(I) \leq (2d(I))^{n!}$$

The following example shows that this bound is the best possible, or nearly so.

[E. Mayr and A. Meyer]

$$R = k[\, S^{(m)}, F^{(m)}, C_i^{(m)}, B_i^{(m)} \mid 1 \leq i \leq 4,\ 1 \leq m \leq n \,]$$

Let $I \subset R$ be the ideal defined by the $10n-6$ generators

$$2 \le m \le n \quad \begin{cases} S^{(m)} - S^{(m-1)} C_1^{(m-1)} \\ F^{(m)} - S^{(m-1)} C_4^{(m-1)} \\ C_i^{(m)} F^{(m-1)} B_2^{(m-1)} - C_i^{(m)} B_i^{(m)} F^{(m-1)} B_3^{(m-1)} \end{cases}$$

$$(1 \le i \le 4)$$

$$1 \le m \le n-1 \quad \begin{cases} F^{(m)} C_1^{(m)} B_1^{(m)} - S^{(m)} C_2^{(m)} \\ F^{(m)} C_2^{(m)} - F^{(m)} C_3^{(m)} \\ S^{(m)} C_3^{(m)} B_1^{(m)} - S^{(m)} C_2^{(m)} B_4^{(m)} \\ S^{(m)} C_3^{(m)} - F^{(m)} C_4^{(m)} B_4^{(m)} \end{cases}$$

$$C_i^{(1)} S^{(1)} - C_i^{(1)} F^{(1)} (B_i^{(1)})^2, \quad 1 \le i \le 4$$

Theorem [Mayer - Meyer]

Let I^H be the homogenization of the ideal I. Then

$$d(I^H) = 4 \Rightarrow m(I^H) \geq 2^{2^n} + 1$$

This is only one known family of ideals where $reg(I)$ is doubly exponential in the number of variables.

However, if X is a smooth variety of $\dim(X)=r$ then $m(I_X)$ is much smaller, like the $d(I_X)^n$ or better

Theorem $X^r \subset \mathbb{P}^n$: a smooth variety

(a) $m(I_X) \leq (r+1)(\deg(X)-2)+2$

(b) $0 \leq r \leq 2 \Rightarrow m(I_X) \leq \deg(X) - \operatorname{codim}(X)+1$

(c) $I_X = (f_1, \dots, f_t)$, $a_i = \deg(f_i)$

$\Rightarrow m(I_X) \leq a_1 + a_2 + \cdots + a_t - \operatorname{codim}(X)+1$

Remark It is well known that $\deg(X) \leq d(I_X)^{n-r}$

Hence, these results shows that $m(I_X) \leq C \cdot d(I_X)^n$

The bound of $m(I_X)$ is still open for $\dim(X) \geqslant 4$ [Eisenbud-Goto]

Eisenbud - Goto Conjecture

(Regularity Conjecture)

If $X \subset \mathbb{P}^n$ is a non-degenerate variety of codimension e,

then $m(I_X) \leq \deg(X) - e + 1$.

Main Results (The case of $M(I)$)

Question What can we say about the corresponding problem for $M(I)$?

Remark It is known from experience that $\mathrm{Gin}_{glex}(I)$ tends to be extremely complicated.

So. $M(I)$ can climb to much higher degrees than $m(I)$.

Our resuts show that

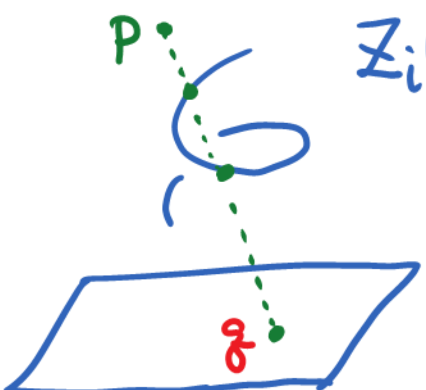
$M(I_X)$ is deeply related to generic projections of X.

Partial Elimination Ideals

- $X \subset \mathbb{P}^n$: a reduced closed subscheme.
- $I_X \subset R$: the defining ideal of X.
- $\pi_p : X \to \mathbb{P}^{n-1}$: a projection from a point $p \in \mathbb{P}^n \setminus X$

We may assume that $p = [1,0,0 \cdots 0]$

$\Rightarrow$ What is the defining ideal of the set



$$Z_i(X) = \{ q \in \pi_p(X) \mid \mathrm{mult}_q(\pi_p(X)) > i \} ?$$

$i = 0, 1, 2, \ldots$

$q \in Z_1(X)$

Definition

Let $I \subset R = k[x_0 \cdots x_n]$.

$$\Rightarrow K_i(I) = \left\{ \frac{\partial^i}{\partial x_0{}^i}(f) \,\middle|\, \frac{\partial^{i+1}}{\partial x_0{}^{i+1}}(f) = 0,\ f \in I \right\}$$

(i-th partial elimination ideal)

Theorem [M. Green]

① $K_i(I_X) \subset k[x_1 \cdots x_n]$: homogeneous ideal

② $Z(K_i(I_X)) = \{ z \in \pi_p(X) \mid \ell(\pi_p^{-1}(z)) > i \}$.

where $\pi_p : X \to \mathbb{P}^{n-1}$, $p = [1, 0, \cdots, 0] \in \mathbb{P}^n \setminus X$

$\Rightarrow$ the radical ideal $\sqrt{K_i(I_X)}$ defines the algebraic set $Z_i(X)$.

How to compute $K_i(I_X)$?

Proposition Let $I \subset R$ be a homogeneous ideal in generic coordinates

If G is a Gröbner basis for I w.r.t glex order.

then

$$G_i = \left\{ \left(\frac{\partial}{\partial x_0}\right)^i (f) \;\middle|\; \left(\frac{\partial}{\partial x_0}\right)^{i+1} (f) = 0, \; f \in G \right\} \subset G$$

is a Gröbner basis for $K_i(I)$ for each $i \geq 0$

```
---------------------------code for function "partial"--------------------------------
--Input: ideal I, integer k
--Output: A matrix of (non-minimal) generators of the k-th partial elimination ideal of I.
--Syntax:  partial(I,2).

--5/5/03
--This is a version of the partial elimination ideal function that uses the product order for speed.  (Thanks to R. Thomas for
the suggestion.)
partial = (I,k) -> (

--This part creates a new ring with the product order.
  R := ring I;
  n := # gens R;
  S := (coefficientRing R)[gens R, MonomialOrder=>ProductOrder{1,n-1}];
  f := map(S,R, gens S);
  J := f(I);
--Now proceed with the computation of the {1,n} Groebner basis.
--find the lead variable in the ring S
  a := (flatten entries vars S)_0;
--compute a lex groebner basis of J (which is really I)
  gblist := flatten entries gens gb J;

  L := #gblist;
  partialideal := {};
  for i from 0 to L-1 do(
    --get the next element in the Groebner basis
    temp := gblist_i;
    --if the a-degree of the initial term is less than k+1, put it in the ideal
    if (initialexp temp) < k+1 then (
    --take partial derivatives until we get rid of the highest variable
    while (initialexp temp )> 0 do (
        temp = diff(a, temp);
         );
    partialideal = append(partialideal, temp);
      );
    );
   H = map (ring I, S, flatten entries vars ring I);
   H   matrix {partialideal}
)
```

Example 6.10. *A complete intersection curve of type (3,3) in* $\mathbf{P}^3$.

The lex gin of a general complete intersection of two cubics in 4 variables is x_1^3, $x_1^2(x_2, x_3^2, x_3x_4^2, x_4^4)$, $x_1(x_2^4, x_2^3x_3^2, x_2^3x_3x_4^2, x_2^3x_4^3, x_2^2x_3^5, x_2^2x_3^4x_4, x_2^2x_3^3x_4^2, x_2^2x_3^2x_4^4, x_2^2x_3^5, x_2^2x_4^7, x_2x_3^8, x_2x_3^7x_4^2, x_2x_3^6x_4^4, x_2x_3^5x_4^6, x_2x_3^4x_4^8, x_2x_3^3x_4^{10}, x_2x_3^2x_4^{12}, x_2x_3x_2^{14}, x_2x_4^{16}, x_3^{18})$, x_2^9. The lex gins of $K_2(I)$ and $K_1(I)$ occur in parentheses.

Remark $K_3(I) = (1) = k[x_2, x_3, x_4] = \overline{R}$

Theorem For a homogeneous ideal of $I \subset R = k[x_0 \ldots x_n]$,

$$\mathrm{Gin}_{\mathrm{glex}}(I) = \bigoplus_{i \geq 0} x_0^i \, \mathrm{Gin}_{\mathrm{glex}}(K_i(I))$$

$\Rightarrow$ This implies the following two facts:

(a) $$M(I) = \max_{i \geq 0} \{ i + M(K_i(I)) \mid 0 \leq i \leq \alpha \}$$

where α is the least degree satisfying $I_\alpha \neq 0$.

(b) $$H(R/I, k) = \sum_{i=0}^{\alpha} H(\overline{R}/K_i(I), k-i)$$

where $H(R/I, -) : \mathbb{N} \to \mathbb{N}$: Hilbert fuction of R/I.

Theorem Let X be a reduced subscheme in $\mathbb{P}^n$ and let $\pi : X \to \mathbb{P}^n$ be a generic projection of X to $\mathbb{P}^{n-1}$.
If π is a isomorphism then

$$M(I_X) = reg(Gin_{glex}(I_X))$$

$$= reg(Gin_{glex}(I_{\pi(X)})) = M(I_{\pi(X)}).$$

Corollary Let X be a reduced points in $\mathbb{P}^n$

$$\Rightarrow M(I_X) = \deg(X)$$

pf) If we take a generic projection $\pi_\Lambda : X \to Y = \pi_\Lambda(X) \subset \mathbb{P}^1$

then π_Λ is an isomorphism.

$\Rightarrow$ From Theorem, $M(I_X) = M(I_Y)$

Since $I_Y = (F_1)$ for a homogeneous polynomial F_1 of degree $= \deg(X)$,

we have $Gin(I_Y) = (x_{n-1}^{\deg(X)})$.

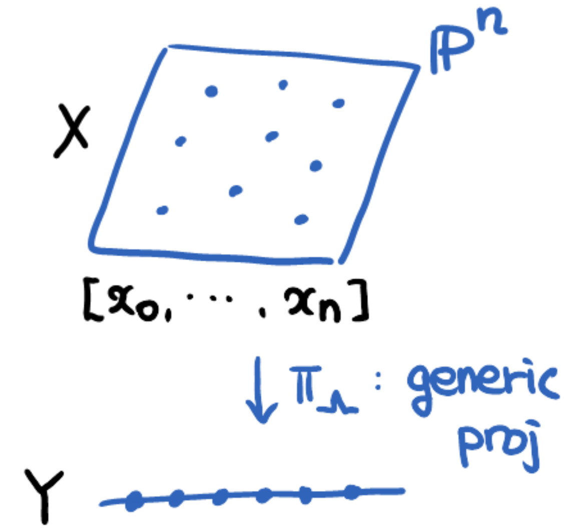
$$\Rightarrow M(I_Y) = \deg(X)$$

Example Let $I=(F_1, F_2) \in k[a,b,c]$ where F_1, F_2 are general forms of degree 3.

$\Rightarrow$ $X = V(I) \subset \mathbb{P}^2$ is a reduced set of points of $\deg(X) = 9$.

```
+ M2 --no-readline --print-width 116
Macaulay 2, version 1.2
with packages: Elimination, IntegralClosure, LLLBases, PrimaryDecomposition,

i2 : R=QQ[a..c,MonomialOrder=>GLex];

i3 : F1=random(3,R);F2=random(3,R);

i5 : I=ideal(F1, F2);

o5 : Ideal of R

i6 : RtoR := map(R,R,random(R^{0}, R^{numgens R:-1}));

o6 : RingMap R <--- R

i8 : GinI= monomialIdeal RtoR I

                        3    2       4    9      3    2 2     2 3       5     7
o8 = monomialIdeal (a , a b, a*b , b , a*b c, a c , a*b c , a*b*c , a*c )

o8 : MonomialIdeal of R
```

$\mathrm{Gin}(I)$



$M(I_X) = M(I_Y)$

$\Rightarrow$ $I_Y = (f)$ where f is a poly of $\deg(X)$.

$\Rightarrow$ $\mathrm{Gin}(I_Y) = (x_{n-1}^{\deg(X)})$.

$\Rightarrow$ $M(I_Y) = \deg(X)$

Conca and Sidman have shown that if I is the defining ideal of a smooth irreducible complete intersection of type $(a.b)$ in $\mathbb{P}^3$, then $M(I)$ is as follows.

Theorem (A. Conca and J. Sidman, 2005)

Let I be the definig ideal of a smooth irreducible complete intersection curve C of type (a, b) in $\mathbb{P}^3$. Then

$$M(I) = \begin{cases} 4 & \text{if } a = b = 2. \\ 1 + \frac{ab(a-1)(b-1)}{2} & \text{otherwise} \end{cases}$$

Remark For a generic projection $\pi : C \subset \mathbb{P}^3 \to \mathbb{P}^2$, the image $C' = \pi(C)$ has only nodes as singularities.

One may check that $\frac{ab(a-1)(b-1)}{2} = \binom{ab-1}{2} - g(C)$, which is the number of nodes in the plane curve $C' \subset \mathbb{P}^2$.

For a non-degenerate smooth curve $C \subset \mathbb{P}^r$, if $r > 3$ then a generic projection $\pi : C \longrightarrow \mathbb{P}^3$ is an isomorphism and

$$M(I_C) = M(I_{\pi(C)}).$$

To give the exact formula for $M(I_C)$ w.r.t $\deg(C)$ and $g(C)$, we only need to consider the case $n=3$.

In this case, we proved the followings

(a) $M(I_C) = \max\{M(K_0(I_C)),\ M(K_1(I_C))+1\}$

(b) $M(K_0(I_C)) = \deg(C)$

(c) $M(K_1(I_C)) = \binom{\deg(C)-1}{2} - g(C)$

(d) Note that $M(I_C) = \max\{M(K_0(I_C)), M(K_1(I_C))+1\}$

$\Rightarrow M(K_0(I_C)) \geq M(K_1(I_C))+1$

if and only if either

C is a rational normal curve

or

C is a elliptic normal curve

Theorem (J. Ahn, 2008)

Let C be a non-degenerate smooth integral curve of degree ℓ and genus $g(C)$ in $\mathbb{P}^r$. Then, $M(I_C)$ is given as follows:

$$\begin{cases} 3 & \text{if } C \text{ is a rational normal curve in } \mathbb{P}^3. \\ 4 & \text{if } C \subset \mathbb{P}^3 \text{ is an elliptic curve of degree 4.} \\ 1+\binom{\ell-1}{2}-g(C) & \text{otherwise} \end{cases}$$

Example 2.6 (*Macaulay 2*). Consider a smooth rational curve $C \subset \mathbb{P}^3$ of degree 5 defined by the map

$$[s, t] \to [s^5, s^4 t, st^4, t^5].$$

If I_C is the defining ideal of C, then

- $I_C = (x_2x_3 - x_1x_4, x_3^4 - x_2x_4^3, x_1x_3^3 - x_2^2x_4^2, x_1^2x_3^2 - x_2^3x_4, x_2^4 - x_1^3x_3)$

- $\mathrm{Gin}_{\mathrm{DegRevLex}}(I_C) = (x_1^2, x_1x_2^3, x_2^4, x_1x_2^2x_3, x_2^3x_3)$.

$\Rightarrow$ By the result of Bayer-Stillman, $m(I_C) = \mathrm{reg}(I_C) = 4$.

On the other hand, $M(I_C) = 1 + \binom{5-1}{2} - g = 7$.

- $\mathrm{Gin}_{\mathrm{DegLex}}(I_C) = (x_1^2, x_1x_2^3, x_2^5, x_1x_2^2x_3, x_1x_2x_3^2, x_1x_3^6, x_1x_2^2x_4, x_1x_2x_3x_4^2, x_1x_2x_4^4)$

$x_1\, \mathrm{Gin}(K_1(I_C))$

$= x_1(x_2^3, x_2^2x_3, x_2x_3^2, x_3^6, x_2^2x_4, x_2x_3x_4^2, x_2x_4^4)$

$6 = \binom{\deg(C)-1}{2} - g$

Example Let X be a smooth c.i curve of type $(4.4\ 4)$ in $\mathbb{P}^4$

$\Rightarrow$ $m(I_X) = \mathrm{reg}(I_X)$ by the result of Bayer-Stillman.

and $0 \to R(-12) \to R(-8)^3 \to R(-4)^3 \to R \to R/I_X \to 0$

$\Rightarrow$ $\mathrm{reg}(I_X) = 10$. In fact, $\mathrm{Gin}_{\mathrm{drlex}}(I_X)$ is the following,

$R = k[a.b.c.d.e]$

```
i6 : gin (I,{})

              4   3    2 2    4   5    3 2   4 2   3 4   2   4    2 4   3 4   2 6      6   2 6    8
o6 = monomialIdeal (a , a b, a b , a*b , b , a*b c , b c , a c , a b*c , a*b c , b c , a c , a*b*c , b c , a*c ,
     ------------------------------------------------------------------------------------------------------
        8   10
     b*c , c  )
```

What is $M(I_X)$? If we try to compute $M(I_X)$. we will find that it is almost impossible

However, we know that $\deg(I_X) = 4^3$ and $g(C) = 225$

$\Rightarrow M(I_X) = \binom{\deg(X)-1}{2} - g(C) + 1 = 1729$

Example $\quad I_C = (x^3 - yz^2,\ y^3 - z^2 t)$

$\Rightarrow$ C has one singular point $z = [0.0.0.1]$

In this case, $\dim(\mathrm{Tan}_z(C)) = 3$. $\quad p_a(C) = 10$

$$(y^4, y^3z^2, y^2z^5, yz^8, \mathbf{z}^{15}, y^2z^4t, y^3zt^2, y^2z^3t^2, yz^7t^2,$$
$$y^3t^3, y^2z^2t^4, yz^6t^4, y^2zt^5, yz^5t^6, y^2t^7, yz^4t^8, yz^3t^{10}).$$

$$\Rightarrow \quad M(I_C) = 16 \not\leq 1 + \binom{\deg C - 1}{2} - p_a(C) = 19$$

Corollary Let $C \subset \mathbb{P}^n$ be nondegenerate smooth curve

a) $m(I_C) = \operatorname{reg}(\operatorname{Gin}_{Hlex}(I_C)) \leq \deg(C) \leq d(I_C)^{n-1}$

b) $M(I_C) = \operatorname{reg}(\operatorname{Gin}_{lex}(I_C)) \leq \binom{\deg(C)-1}{2} \leq \frac{1}{2} d(I_C)^{2(n-1)}$

Next Question [B. Sturmfels, 2009]

B. Sturmfels asked us to compute $M(I)$ during KIAS summer school on Tropical Geometry

"Generic forms f_1 and f_2 of degree 4 in $k[x_0, \dots, x_4]$ define a smooth surface in $\mathbb{P}^4$. What is $M(I)$?"

We don't know it yet, but this motivates our study to compute $M(I)$ for surfaces in $\mathbb{P}^4$

Theorem $S \subset \mathbb{P}^4$: a smooth surface. Then

(a) $K_1(I_S)$ is a saturated ideal

(b) If S is contained in a quadric hypersurfaces then $K_1(I_S)$ is a reduced ideal.

This theorem is proved by the following facts:

1) M. Green's spectral sequence on Koszul complex

2) 1st Fitting ideal $\mathcal{F}itt_1^{\mathbb{P}^3}(\pi_* \mathcal{O}_S)$ defines the reduced scheme structure on $V(K_1(I_S))$

$$\Rightarrow \quad \widetilde{K_1}(I_S) = \mathcal{F}itt_1^{\mathbb{P}^3}(\pi_* \mathcal{O}_S).$$

Remark Let S be a locally Cohen-Macaulay surface lying on a quadric hypersurface Q in $\mathbb{P}^4$. Then,

a) If $\deg(S) = 2\alpha$ then $I_S = (Q, F)$ where $\deg(F) = \alpha$

(S is a complete intersection of type $(2, \alpha)$)

$\Rightarrow m(I_S) = \mathrm{reg}(I_S) = \alpha + 1$.

b) If $\deg(S) = 2\alpha - 1$ then $I_S = (Q, G_1, G_2)$ where $\deg(G_i) = \alpha$

$\Rightarrow$ By Hilbert-Burch Theorem,

$$0 \to R(-\alpha-1)^2 \xrightarrow{\quad A \quad} R(-2) \oplus R(-\alpha)^2 \longrightarrow I_S \to 0$$

$$A = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \\ F_5 & F_6 \end{pmatrix}$$

I_S = determinantal ideal of A.

$$\Rightarrow I_S = (\underbrace{L_1L_4 - L_2L_3}_{=Q},\ \underbrace{L_1F_5 - L_2F_6}_{=G_1},\ \underbrace{L_3F_5 - L_4F_6}_{=G_2})$$

Theorem [___ , Kwak , Song , 2012]

Let S be a smooth surface in $\mathbb{P}^4$ with $h^0(\mathcal{I}_S(2)) \neq 0$.

$$M(I_S) = \begin{cases} 3 & \text{if } S \text{ is a rational normal scroll with } d = 3 \\ 4 & \text{if } S \text{ is a complete intersection of (2,2)-type} \\ 5 & \text{if } S \text{ is a Castelnuovo surface with } d = 5 \\ 2 + \binom{\deg Y_1(S)-1}{2} - g(Y_1(S)) & \text{for } d \geq 6 \end{cases}$$

where $Y_1(S) \subset \mathbb{P}^3$ is double curve of degree $\binom{d-1}{2} - g(S \cap H)$.

under a generic projection of S to $\mathbb{P}^3$.

Moreover, $\deg(S) = 2\alpha$

a) $M(I_S) = \frac{1}{2}(\alpha^4 - 4\alpha^3 + 5\alpha^2 - 2\alpha + 4)$

b) $M(I_S) = \frac{1}{2}(\alpha^4 - 6\alpha^3 + 13\alpha^2 - 12\alpha + 8)$.

$\deg(S) = 2\alpha - 1$

Example $\deg(S) = 7$, $h^0(\mathcal{I}_S(2)) \neq 0$

In this case, $2\alpha - 1 = 7 \Rightarrow \alpha = 4$

$$M(I_S) = \frac{1}{2}(\alpha^4 - 6\alpha^3 + 13\alpha^2 - 12\alpha + 8) = 20$$

In fact,

$$\begin{aligned}\mathrm{Gin}(I_S) = (&x_0^2,\ x_0x_1^3,\ x_1^7,\ x_0x_1^2x_2,\ x_0x_1x_2^4,\ x_0x_2^9,\ x_0x_1^2x_3^2,\ x_0x_1x_2^3x_3^2,\ x_0x_1x_2^2x_3^5,\\ &x_0x_1x_2x_3^8,\ \mathbf{x_0x_1x_3^{18}},\ x_0x_1x_2^2x_3^4x_4,\ x_0x_1^2x_3x_4^2,\ x_0x_1x_2^3x_3x_4^2,\ x_0x_1x_2^2x_3^3x_4^2,\\ &x_0x_1x_2x_3^7x_4^2,\ x_0x_1x_2^3x_4^3,\ x_0x_1^2x_4^4,\ x_0x_1x_2^2x_3^2x_4^4,\ x_0x_1x_2x_3^6x_4^4,\ x_0x_1x_2^2x_3x_4^5,\\ &x_0x_1x_2x_3^5x_4^6,\ x_0x_1x_2^2x_4^7,\ x_0x_1x_2x_3^4x_4^8,\ x_0x_1x_2x_3^3x_4^{10},\ x_0x_1x_2x_3^2x_4^{12},\\ &x_0x_1x_2x_3x_4^{14},\ x_0x_1x_2x_4^{16}).\end{aligned}$$

Example $\deg(S) = 8$, $h^0(\mathcal{J}_S(2)) \neq 0$.

In this case, $2\alpha = 8 \Rightarrow \alpha = 4$

$$M(I_S) = \frac{1}{2}(\alpha^4 - 4\alpha^3 + 5\alpha^2 - 2\alpha + 4) = 38.$$

: Computed Gin of I_S

$$\begin{aligned}
\mathrm{Gin}(I_S) = (&x_0^2,\ x_0x_1^3,\ x_1^8,\ x_0x_1^2x_2^2,\ x_0x_1x_2^6,\ x_0x_2^{12},\ x_0x_1^2x_2x_3^2,\ x_0x_1x_2^5x_3^2,\\
&x_0x_1^2x_3^5,\ x_0x_1x_2^4x_3^5,\ x_0x_1x_2^3x_3^7,\ x_0x_1x_2^2x_3^{11},\ x_0x_1x_2x_3^{17},\ \mathbf{x_0x_1x_3^{36}},\\
&x_0x_1^2x_3^4x_4,\ x_0x_1x_2^4x_3^4x_4,\ x_0x_1x_2^3x_3^6x_4,\ x_0x_1x_2^2x_3^{10}x_4,\ x_0x_1^2x_2x_3x_4^2,\\
&x_0x_1x_2^5x_3x_4^2,\ x_0x_1^2x_3^3x_4^2,\ x_0x_1x_2^4x_3^3x_4^2,\ x_0x_1x_2^2x_3^9x_4^2,\ x_0x_1x_2x_3^{16}x_4^2,\\
&x_0x_1^2x_2x_4^3,\ x_0x_1x_2^5x_4^3,\ x_0x_1x_2^4x_3^2x_4^3,\ x_0x_1x_2^3x_3^5x_4^3,\ x_0x_1^2x_3^2x_4^4,\\
&x_0x_1x_2^3x_3^4x_4^4,\ x_0x_1x_2^2x_3^8x_4^4,\ x_0x_1x_2x_3^{15}x_4^4,\ x_0x_1^2x_3x_4^5,\ x_0x_1x_2^4x_3x_4^5,\\
&x_0x_1x_2^3x_3^3x_4^5,\ x_0x_1x_2^2x_3^7x_4^5,\ x_0x_1x_2^4x_4^6,\ x_0x_1x_2x_3^{14}x_4^6,\ x_0x_1^2x_4^7,\\
&x_0x_1x_2^3x_3^2x_4^7,\ x_0x_1x_2^2x_3^6x_4^7,\ x_0x_1x_2^3x_3x_4^8,\ x_0x_1x_2^2x_3^5x_4^8,\ x_0x_1x_2x_3^{13}x_4^8,\\
&x_0x_1x_2^3x_4^9,\ x_0x_1x_2^2x_3^4x_4^{10},\ x_0x_1x_2x_3^{12}x_4^{10},\ x_0x_1x_2^2x_3^3x_4^{11},\ x_0x_1x_2x_3^{11}x_4^{12},\\
&x_0x_1x_2^2x_3^2x_4^{13},\ x_0x_1x_2^2x_3x_4^{14},\ x_0x_1x_2x_3^{10}x_4^{14},\ x_0x_1x_2^2x_4^{16},\ x_0x_1x_2x_3^9x_4^{16},\\
&x_0x_1x_2x_3^8x_4^{18},\ x_0x_1x_2x_3^7x_4^{20},\ x_0x_1x_2x_3^6x_4^{22},\ x_0x_1x_2x_3^5x_4^{24},\ x_0x_1x_2x_3^4x_4^{26},\\
&x_0x_1x_2x_3^3x_4^{28},\ x_0x_1x_2x_3^2x_4^{30},\ x_0x_1x_2x_3x_4^{32},\ x_0x_1x_2x_4^{34}). \quad \square
\end{aligned}$$

Table 1
The complete intersection S of $(2, \alpha)$-type in $\mathbb{P}^4$.

α	5	6	7	8	9	10	20	50	100
$M(I_S)$	122	302	632	1178	2018	3242	64982	2881202	48024902
$m(I_S)$	6	7	8	9	10	11	21	51	101

Table 2
The smooth surface $S \subset \mathbb{P}^4$ of degree $(2\alpha - 1)$ lying on a quadric.

α	5	6	7	8	9	10	20	50	100
$M(I_S)$	74	202	452	884	1570	2594	58484	2765954	47064404
$m(I_S)$	5	6	7	8	9	10	20	50	100

Problems

X: a smooth projective variety of degree d, dimension n and codimension 2. Then, it is expected that

- $m(I_X) = \mathrm{reg}(I_X) \leq d-1$ (Castelnuovo-Mumford-Eisenbud-Goto conjecture),
- $M(I_X) \sim 2(\frac{d}{2})^{2^n}$ (Asymptotic expectation).

Remark

- *It is true for $m(I_X)$ when $n = 0, 1, 2, 3$ and still open for $n \geq 4$,*
- *For $M(I_X)$, it is still open for $n \geq 3$.*

Thank you !!

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