Winter School in Algebraic Geometry and Applications

February 17th-20th, 2014 Konkuk University, Seoul, Korea

Exercise Sheet - Day 1

Exercise 1. You entered at the "$\diamondsuit$ $\heartsuit$ Gambling Tensors $\clubsuit$ $\spadesuit$" casino in the worst area of Seoul. In a desk there is a dice game where you can bet on the exits of two dices (not depending on the order of the dice). Before to play you observe a (long) sequence of 96 observations

$$(1,6),(4,2),(3,3),(1,4),(4,1),\ldots$$

which is described by the following matrix:

$$\begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 0 & 0 & 1 & 6 & 5 & 3 \\ 0 & 0 & 1 & 4 & 1 & 3 \\ 1 & 1 & 1 & 3 & 2 & 3 \\ 6 & 4 & 1 & 2 & 2 & 7 \\ 1 & 3 & 0 & 1 & 1 & 4 \\ 6 & 3 & 2 & 6 & 3 & 9 \end{pmatrix} \end{array}$$

i) Using the Algortithm for Statistical inference based on invariants, decide which one of the following models best fits the sequences of observations ?
   (1) the dice are fair,
   (2) one die is fair and one die is loaded,
   (3) the dice are loaded,
   (4) the gambler, after three times with fair dice, switches once, at a pair of loaded dice.

ii) I bet for pair $(6,1)$ while you bet for pair $(4,2)$. Who has more chance to win ?

Exercise 2. Suppose we have three tetrahedral dice with faces A, C, G and T. Two of the three dice are loaded and one is fair. We know the probabilities of rolling the four letters:

	A	C	G	T
first die	0.15	0.33	0.36	0.16
second die	0.27	0.24	0.23	0.24
third die	0.25	0.25	0.25	0.25

We generate DNA sequences according to the following process:

- We first pick one of the three dice at random. where the first die is picked with probability θ_1, the second die is picked with probability θ_2 and the third die is picked with probability $\theta_3 = 1 - \theta_1 - \theta_2$.
- we then make a roll of the selected die and we record the resulting letter

Write the function expressing this model as a parametric statistical model. Suppose we have the sequence

CTCACGTGATGAGAGCATTCTCAGACCGTGACGCGTGTAGCAGCGGCTC

Is the sequence generated by this process, and, if so, which parameters θ_1 and θ_2 does it use?

Exercise 3.

- Let X be the variety of 3×4 matrices of rank one, let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Which of the following matrices are in the tangent space $T_A X$ at A ?

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 6 & 7 & 8 & 9 \end{pmatrix}$$

Which are cartesian equations for $T_A X$ in the $m \times n$ case ?

- Let X be the variety of 3×4 matrices of rank ≤ 2, let

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Compute the two linear equations of T_AX.

note that tangent spaces meet

Exercise 4.

This exercise can be executed with the help of Macaulay2 or other computer algebra package.

Let A, B, C be three vector spaces of dimension two, with basis respectively given by $\{a_0, a_1\}$, $\{b_0, b_1\}$, $\{c_0, c_1\}$.

(1) For any tensor $t = \sum_{i,j,k=0,\ldots 1} t_{ijk} a_i b_j c_k$, write it as a 2×2 with coefficients linear in c_i, so as a *pencil* of 2×2 matrices.

(2) Compute the condition that the previous matrix is singular, it is quadratic in c_i giving a pair of points.

(3) Compute the condition that the pair of points consists of a double point, get a polynomial of degree 4 in t_{ijk}, which is called the hyperdeterminant of t and we denote as $Det(t)$.

(4) Show that the above pair gives the two summands of t, when t has rank two. Conclude that the sign of $Det(t)$ allows to detect if a real $2 \times 2 \times 2$ tensor has rank two or three.

(5) The following $2 \times 2 \times 2$ tensors t_1, t_2, t_3 fill the first column of the following table. Which is which ? Can you decompose them ?

t	$\mathrm{rk}_{\mathbb{R}}(t)$	$\mathrm{rk}_{\mathbb{C}}(t)$
?	2	2
?	3	2
?	3	3

$t_1 = 4a_0b_0c_0 + 2a_1b_0c_0 - a_0b_1c_0 + 2a_0b_0c_1$

$t_2 = a_0b_0c_0 + 2a_1b_1c_0 + 3a_1b_0c_1 + 6a_0b_1c_1$

$t_3 = a_0b_0c_0 - 2a_1b_1c_0 - 3a_1b_0c_1 - 6a_0b_1c_1$