HÖLDER CONTINUITY OF A BOUNDED WEAK SOLUTION OF A DEGENERATE PARABOLIC EQUATION

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ABSTRACT. These are some notes taken in a lecture series given by Sukjung Hwang at KIAS during the Spring semester of 2019. All errors and inaccuracies are solely due to the notetaker (Sung-Jin Oh).

The content of this lecture series is as follows:

(1) Equations & References
(2) Hölder continuity & Two alternatives
(3) Energy estimate & Intrinsic scaling
(4) De Giorgi iteration
(5) Expansion of positivities
(6) Proof of the two alternatives

1. Equations & References

Consider the minimization problem

$$\min \int |\nabla u|^2 \, \mathrm{d}x.$$

Since we wish to consider the parabolic problem, we consider its gradient flow, which is $u_t - \Delta u = 0$ in this case. Generalizing the above functional to

$$\min \int |\nabla u|^p \, \mathrm{d}x,$$

we obtain the parabolic equation

$$u_t - \nabla \cdot \left(|\nabla u|^{p-2} \nabla u \right) = 0.$$

while this equation is linear when $p = 2$, it is *degenerate* (resp. *singular*) if $p > 2$ (resp. $1 < p < 2$).

Despite great qualitative differences, in all these cases, bounded solutions are Hölder continuous. The motivating question was: Is there a unified framework for showing this? For this purpose, we may consider

$$\min \int G(|\nabla u|) \, \mathrm{d}x$$

where

$$|\nabla u|^{p_0} \sim G(|\nabla u|) \sim |\nabla u|^{p_1}, \quad 1 < p_0 \leq p_1 < \infty.$$

The corresponding gradient flow is

$$u_t - \nabla \cdot \left(g(|\nabla u|) \frac{\nabla u}{|\nabla u|} \right) = 0,$$

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where $G' = g$. Some references are:

- Di Benedetto '93
- Urbano 2008
- Hwang–Lieberman 2015: I - $2 \le p_0 \le p_1 < \infty$, II - $1 < p_0 \le p_1 \le 2$.

Today, we will consider

$$u_t - \nabla \cdot \left(|\nabla u|^{p-2} \nabla u\right) = 0, \qquad p \ge 2. \tag{1.1}$$

2. Hölder continuity & two alternatives

Our goal is to show the following:

Assume that u is a bounded weak solution to (1.1). Show that

$$|u(x_1,t_1) - u(x_2,t_2)| \le \gamma \|u\|_{L^\infty} \left(\frac{|x_1 - x_2| + \|u\|_{L^\infty}^{1-\frac{2}{p}} |t_1 - t_2|^{1/p}}{\operatorname{dist}_p(\Omega', \partial_p \Omega)} \right)^\alpha$$

for some $\alpha \in (0,1)$; (x_1,t_1), $(x_2,t_2) \in \Omega' \subseteq \Omega$. Here, $\partial_p \Omega$ consists of the lateral boundary and the initial domain.

The main lemma is:

Lemma 2.1. *Assume that*

$$\operatorname{ess\,osc}_{Q_\rho} u \le \omega,$$

for some parabolic cylinder Q_ρ around $(0,0)$. Then there exist $\lambda, \sigma \in (0,1)$ such that

$$\operatorname{ess\,osc}_{Q_{\lambda\rho}} u \le \sigma\omega,$$

The above goal will be achieved by iterating the main lemma; α will be $\log_\sigma \lambda$. Here, λ, σ will only depend on the data (i.e., n (dimension), p and the domain Ω_T). We leave the precise definition of Q_λ for later.

The main lemma will be proved by showing "two alternatives": Consider a nonnegative weak solution u on $Q = K_{2k} \times (-\theta_0 \Delta, 0)$, where K_{2k} is a cube of sidelength $2k$, $\theta_0 > 1$ will be determined later, and $\Delta = \left(\frac{\omega}{2}\right)^{2-p} (2R)^p$, and R is to be chosen according to the data.

- *First alternative:* If there exists $T_0 \in [-\theta_0 \Delta, -\Delta]$ such that

  $$|K_{2k} \times (T_0, T_0 + \Delta) \cap \{u \le \frac{\omega}{2}\}| \le \nu_0 |K_{2k}| \Delta,$$

  then there exists $\delta_1 = \delta_1(\text{data}) \in (0,1)$ such that

  $$\operatorname*{ess\,inf}_Q u \ge \delta_1 \omega$$

  with $Q = Q_{\omega/4, k/2} = K_{k/2} \times \left[-\left(\frac{\omega}{4}\right)^{2-p} \left(\frac{R}{2}\right)^p, 0 \right]$.
- *Second alternative:* If

  $$|K_{2k} \times (T_0, T_0 + \Delta) \cap \{u > \frac{\omega}{2}\}| \le (1 - \nu_0) |K_{2k}| \Delta$$

  for all $T_0 \in [-\theta_0 \Delta, \Delta]$, then there exists $\delta_2 \in (0,1)$ such that

  $$\operatorname*{ess\,sup}_Q u \le \mu_+ - \delta_2 \omega.$$

3. Energy estimate & intrinsic scaling

For a test function $u\zeta^p$,

$$\begin{aligned} 0 &= \iint_{Q_\rho} \left[u_t - \nabla\left(|\nabla u|^{p-2}\nabla u\right)\right] u\zeta^p \,\mathrm{d}x\mathrm{d}t \\ &= I + II, \end{aligned}$$

where

$$\begin{aligned} I &= \iint \partial_t \left(\frac{1}{2}u^2\right) \zeta^p \,\mathrm{d}x\mathrm{d}t \\ &= \iint \partial_t \left(\frac{1}{2}u^2\zeta^p\right) - \frac{p}{2}u^2\zeta^{p-1}\zeta_t \,\mathrm{d}x\mathrm{d}t \\ &= \int_{\{K_\rho\times\{t_1\}\}} \frac{1}{2}u^2\zeta^p + \int_{\{K_\rho\times\{t_0\}\}} \frac{1}{2}u^2\zeta^p - \iint \frac{p}{2}u^2\zeta^{p-1}\zeta_t \,\mathrm{d}x\mathrm{d}t. \end{aligned}$$

We will set ζ to vanish at the initial time t_0, so that the second term vanishes.

$$\begin{aligned} II &= \iint_{Q_\rho} |\nabla u|^{p-2}\nabla u \cdot \nabla(u\zeta^p) \,\mathrm{d}x\mathrm{d}t \\ &= \underbrace{\iint_{Q_\rho} |\nabla u|^p\zeta^p \,\mathrm{d}x\mathrm{d}t}_{II_1} + \underbrace{\iint_{Q_\rho} |\nabla u|^{p-2}\nabla u \cdot up\zeta^{p-1}\nabla\zeta \,\mathrm{d}x\mathrm{d}t}_{II_2}. \end{aligned}$$

We estimate

$$\begin{aligned} |II_2| &\le \iint |\nabla u|^{p-2}|\nabla u|up\zeta^{p-1}|\nabla\zeta| \,\mathrm{d}x\mathrm{d}t \\ &\le \epsilon II_1 + p^p\epsilon^{-1} \iint u^p|\nabla\zeta|^p \,\mathrm{d}x\mathrm{d}t. \end{aligned}$$

The first term may be absorbed into II_1. In conclusion,

$$\operatorname*{ess\,sup}_t \int u^2\zeta^p \,\mathrm{d}x + \iint |\nabla u|^p\zeta^p \,\mathrm{d}x\mathrm{d}t \le C\iint u^2\zeta^{p-1}\zeta_t + C\iint u^p|\nabla\zeta|^p.$$

When $p = 2$ and $Q_\rho = K_\rho \times [-\rho^2, 0]$, then a cutoff adapted to Q_ρ obeys

$$|\zeta_t| \le \frac{1}{\rho^2}, \quad |\nabla\zeta| \le \frac{1}{\rho},$$

and the two terms on the RHS have the same strength.

Trying to balancing the two terms also in the case $p > 2$ motivate the notion of *intrinsic scaling*. Writing ω for $\|u\|_{L^\infty}$, the two terms are roughly of size, respectively,

$$\frac{1}{T}\omega^2, \quad \omega^p\frac{1}{\rho^p}.$$

Making these two terms equal, we are motivated to choose $T \simeq \omega^{2-p}\rho^p$.

4. De Giorgi iteration

Let $Q_{k,\rho}(\theta) = K_\rho \times [-\theta k^{2-p}\rho^p, 0]$

Proposition 4.1. *For a given θ, k, ρ, there exists $\nu_0 = \nu_0(\theta, \text{data}) \in (0,1)$ such that, if*

$$|Q_{k,2\rho}(\theta) \cap \{u \le k\}| \le \nu_0 |Q_{k,2\rho}(\theta)|$$

then

$$\operatorname*{ess\,inf}_{Q_{k,\rho}(\theta)} u(x,t) \ge k/2$$

Proof. Construct the sequences

$$\rho_n = \rho + \frac{\rho}{2^n}$$

so that $\rho_0 = 2\rho$ and $\rho_\infty = \rho$;

$$k_n = \frac{k}{2} + \frac{k}{2^{n+1}}$$

so that $k_0 = k$ and $k_\infty = k/2$; let

$$Q_n = K_{\rho_n} \times [-\theta k^{2-p}\rho_n^p, 0], \quad \zeta_n = \begin{cases} 1 & \text{in } Q_{n+1} \\ 0 & \text{on } \partial_p Q_n \end{cases}.$$

We use the energy estimate with $(u-k_n)_- = \max\{0, k_n - u\}$ instead of u, and with a cutoff ζ adapted to Q_{n+1}. Then

$$\begin{aligned} &\operatorname*{ess\,sup}_{-\theta k^{2-p}\rho_n^p < t < 0} \int_{K_{\rho_n}} (u-k_n)^2 \zeta_n^p + \iint_{Q_n} |\nabla (u-k_n)_- \zeta_n|^p \\ &\le C \iint_{Q_n} (u-k_n)_- \zeta_n^{p-1} (\zeta_n)_t + \iint_{Q_n} (u-k_n)_-^p |\nabla \zeta_n|^p \end{aligned}$$

Note that

$$|\nabla \zeta_n| \le \frac{2^n}{\rho}, \quad (\zeta_n)_t \le \frac{2^{np}}{\theta k^{2-p}\rho^p}.$$

Since u is nonnegative, $(u-k_n)_- \le k_n \le k$. Thus, the RHS is bounded by

$$\left(k^p \frac{2^{np}}{\rho^p} + k^2 \frac{2^{np}}{\theta k^{2-p}\rho^p} \right) A_n$$

where

$$A_n := |Q_n \cap \{u < k_n\}|.$$

Observe that

$$(u-k_n)_-^2 = (u-k_n)_-^p (u-k_n)_-^{2-p} \ge k^{2-p} (u-k_n)_-^p$$

so

$$\operatorname*{ess\,sup}_{-\theta k^{2-p}\rho_n^p < t < 0} \int_{K_{\rho_n}} (u-k_n)^2 \zeta_n^p \ge \operatorname*{ess\,sup}_{-\theta k^{2-p}\rho_n^p < t < 0} k^{2-p} \int_{K_{\rho_n}} (u-k_n)^p \zeta_n^p.$$

We now make a change of variable

$$\bar{t} = k^{p-2} t \in [-\theta \rho_n^p, 0].$$

Then $\mathrm{d}\bar{t} = k^{p-2}\,\mathrm{d}t$. Denoting all variables in the new coordinates by putting a bar,

$$\operatorname*{ess\,sup}_{-\theta k^{2-p}\rho_n^p < t < 0} \int_{K_{\rho_n}} (\bar{u}-k_n)^p \bar{\zeta}_n^p + \iint_{\bar{Q}_n} |\nabla (\bar{u}-k_n)_- \bar{\zeta}_n|^p$$

$$\leq C\left(k^p\frac{2^{np}}{\rho^p}+k^2\frac{2^{np}}{\theta k^{2-p}\rho^p}\right)|\bar{A}_n|$$

We now apply the (parabolic) Sobolev embedding

$$\iint_{\bar{Q}_n}|(\bar{u}-k_n)_-\bar{\zeta}_n|^{p\cdot N+pN}\leq\left(\iint_{\bar{Q}_n}|\nabla(\bar{u}-k_n)_-\bar{\zeta}_n|^p\right)\left(\operatorname*{ess\,sup}_t\int\left((\bar{u}-k_n)_n\bar{\zeta}_n\right)^p\right)^{\frac{p}{N}}.$$

Then

$$\iint_{\bar{Q}_{n+1}}(\bar{u}-k_n)_-^p\lesssim\iint_{\bar{Q}_n}(\bar{u}-k_n)_-^p\zeta^p\leq|\bar{A}_n|^{\frac{p}{N+p}+1}\left(\frac{2^{np}k_p}{\rho_p}+\frac{k^p2^{np}}{\theta\rho^p}\right).$$

In the set $\bar{u}>k_{n+1}$,

$$(\bar{u}-k_n)_-=k_n-\bar{u}\geq k_n-k_{n+1}=\frac{k}{2^{n+1}}.$$

Thus, by Chebyshev,

$$\left(\frac{k}{2^{n+1}}\right)^p|\bar{Q}_{n+1}\cap\{\bar{u}\leq k_{n+1}\}|\leq\frac{k^p2^{np}}{\theta\rho^p}|\bar{A}_n|^{1+\frac{p}{N+p}}.$$

Thus

$$|\bar{A}_{n+1}|\leq\frac{2^{2np}}{\theta\rho^p}|\bar{A}_n|^{1+\frac{p}{N+p}}.$$

Dividing by $|\bar{Q}_n|$, and recalling that $\rho^p\simeq|\bar{Q}_n|^{\frac{p}{N+p}}$ etc.,

$$\frac{|\bar{A}_{n+1}|}{|\bar{Q}_{n+1}|}\leq C2^{2np}\left(\frac{|\bar{A}_n|}{|\bar{Q}_n|}\right)^{1+\frac{p}{N+p}}.$$

Scaling back to t,

$$\frac{|A_{n+1}|}{|Q_{n+1}|}\leq C2^{2np}\left(\frac{|A_n|}{|Q_n|}\right)^{1+\frac{p}{N+p}}.$$

We now use the following convergence lemma: If $\{Y_n\}$ satisfies $Y_{n+1}\leq Cb^nY_n^{1+\alpha}$ and $Y_0\leq C^{-1/\alpha}b^{-1/\alpha^2}$, then $Y_n\to 0$ as $n\to\infty$. Now the proposition follows by taking

$$\nu_0=C^{-\frac{N+p}{p}}(2^{2p})^{-\left(\frac{N+p}{p}\right)^2},$$

making the iteration, and noting that at the end $|A_\infty|=0$, where $A_\infty=Q_\infty\cap\{u<k_\infty\}=Q_{k,\rho}\cap\{u<\frac{k}{2}\}$. □

For a later step, we need the following variant of Proposition 4.1: If

$$|\{(x,t)\in Q_{k,2\rho}(\theta):u(x,t)<k\}|<\frac{\nu_0}{\theta}|Q_{k,2\rho}(\theta)|$$

and

$$u(x,-T_{k,2\rho}(\theta))\geq k$$

then

$$\operatorname*{ess\,inf}_{Q_{k,\rho}(\theta)}u\geq k/2.$$

The idea is now to choose ζ that does not depend on time.

TO BE CONTINUED.