REGULARITY OF THE ETA FUNCTION ON MANIFOLDS WITH CUSPS

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Abstract. On a spin manifold with conformal cusps, we prove under an invertibility condition at infinity that the eta function of the twisted Dirac operator has at most simple poles and is regular at the origin. For hyperbolic manifolds of finite volume, the eta function of the Dirac operator twisted by any homogeneous vector bundle is shown to be entire.

1. Introduction

The eta invariant was first introduced in [1] as a real number associated to certain elliptic first-order differential operators on compact manifolds with boundary, which happened to equal the difference between the Atiyah-Singer integral and the index with respect to the Atiyah-Patodi-Singer spectral boundary condition. During the thirty years since its discovery, this invariant has risen from the status of "error term" to that of a subtle tool, highly efficient in solving otherwise intractable problems from various fields of mathematics. Let us mention in this respect its recent application in finding obstructions for hyperbolic and flat 3-manifolds to bounding hyperbolic 4-manifolds [18].

For a Hermitian vector bundle E over a closed manifold M, consider an elliptic self-adjoint first-order differential operator $D : \mathcal{C}^\infty(M, E) \to \mathcal{C}^\infty(M, E)$. Then the L^2 spectrum of D is purely discrete and distributed according to the classical Weyl law. It follows that the complex function

$$\eta(D, s) := \sum_{\lambda \in \mathrm{Spec}(D)\setminus\{0\}} \lambda|\lambda|^{-s-1}$$

is well-defined (and holomorphic) when $\Re(s) > \dim(M)$. This function admits a meromorphic extension to the complex plane with possible simple poles. If $\dim(M)$ is odd, the possible poles are located at $\dim(M) - 1 - 2\mathbb{N}$ where $\mathbb{N} = \{0, 1, 2, \ldots\}$. If $\dim(M)$ is even and D is a Dirac operator associated to a Clifford connection, the eta function is entire [6].

It is a byproduct of the index theorem of Atiyah, Patodi and Singer that when M is a boundary and D is the tangential part of an elliptic operator as above, the point $s = 0$ is always regular for the eta function. In fact, the eta function is always regular at the origin for general elliptic pseudodifferential operators; this was proved using K-theory in

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the spirit of early index theory by Atiyah, Patodi, and Singer [3] in the odd-dimensional case, and by Gilkey [10] for arbitrary dimensions. The *eta invariant* of D is defined as

$$\eta(D) = \eta(D,0) + \dim\ker(D).$$

Still on a closed manifold another question arises: it is easy to note that the possible residue of the eta function at the origin is the integral on M of a well-defined density determined locally by D, called the local eta residue. Is this density zero? For arbitrary differential operators the answer is negative, but for Dirac operators it was proved by Bismut and Freed [5] that this is indeed the case. The proof uses the properties of the heat coefficients in terms of Clifford filtration, along the lines of Bismut's heat equation proof of the local index formula.

In this note we first revisit the vanishing of the local eta density from the point of view of conformal invariance. We give a self-contained proof, using the APS formula, of the fact that the eta invariant of the spin Dirac operator is insensible to conformal changes. This important fact belongs to the mathematical folklore but we could not find a complete proof in the literature. The existing proofs (e.g. [2, pp. 420-421]) tend to use the index formula of [1] for metrics which are not of product type near the boundary, without explaining why one can do so. In section 2 we show how to apply the APS formula to a true product-type metric in order to prove the conformal invariance of the eta invariant. We deduce the vanishing of the local eta residue from this conformal invariance, by interpreting the variation of the eta invariant in terms of the Wodzicki residue.

Our main results concern the eta function on noncompact spin manifolds with conformal cusps, in particular on complete finite-volume hyperbolic spin manifolds. More precisely, let M be a compact manifold with boundary and $[0,\epsilon)_x \times \partial M$ a collar neighborhood of its boundary. The interior M° of M is called a conformally cusp manifold if it is endowed with a metric g_p which near $x = 0$ takes the form

$$g_p = x^{2p}\left(\frac{dx^2}{x^4} + h\right), \tag{1}$$

for some $p > 0$, where h is a metric on ∂M independent of x. The main examples are complete hyperbolic manifolds of finite volume, for which $p = 1$ and h is flat. Assume now that M is spin. Let E be a bundle with connection over M which is of product-type on the collar, and let D_p denote the associated twisted Dirac operator on $\Sigma \otimes E$ where Σ is the spinor bundle over M. We assume that the spin structure and the connection on E are "nontrivial" (Assumption 1 in Section 5) in the sense that the twisted Dirac operator $D_{(\partial M,h)}$ for the induced spin structure on $(\partial M, h)$ is invertible. Under this assumption, the twisted Dirac operator D_p is essentially self-adjoint with discrete spectrum obeying the Weyl law and the corresponding eta function $\eta(D_p, s)$ has a meromorphic extension to $\mathbb{C}$ with possible double poles [24]. For the untwisted Dirac operator on finite-volume hyperbolic manifolds, it was already noted by Bär [4] that the spectrum of $D = D_1$ is discrete if and only if the spin structure is "nontrivial" on the cusps in the above sense, otherwise the continuous spectrum of D is $\mathbb{R}$. In Appendix A we prove that the same occurs for conformal cusp metrics: If $D_{(\partial M,h)}$ fails to be invertible, then for $p \leq 1$ the twisted Dirac operator D_p has essential spectrum equal to $\mathbb{R}$. In the case $p > 1$, D_p fails to be essentially self-adjoint and although every self-adjoint extension of D_p has discrete spectrum, nothing is known about the meromorphic properties of the corresponding eta functions. These are the reasons the "nontriviality" assumption plays an important rôle in this theory. The main results of this paper are that under Assumption 1 the eta function $\eta(D_p, s)$ in fact has at most simple poles, and is always regular at the origin. Moreover, the poles disappear for hyperbolic manifolds, thus the eta function is entire in that case.

Main Theorem. *Let M° be an odd-dimensional spin manifold with conformal cusps, E a twisting bundle of product type on the cusps, and let D_p be the associated twisted Dirac operator to* (1) *on $\Sigma \otimes E$ satisfying Assumption 1. Then*

(1) *The eta function $\eta(D_p, s)$ of the twisted spin Dirac operator is regular for $\Re(s) > -2$ and has at most simple poles at $s \in \{-2, -4, \dots\}$.*
(2) *When $p = 1$, M° is hyperbolic of finite volume and E is a homogeneous vector bundle, the eta function $\eta(D, s)$ of the twisted Dirac operator is entire.*

In even dimensions the eta function vanishes identically since the spectrum is symmetric, see Section 2.

A related question may be asked on more complicated metrics at infinity, like the fibered-cusp metrics arising on $\mathbb{Q}$-rank one locally symmetric spaces. A similar problem arose from [19], where we could obtain a meromorphic extension of the eta function for cofinite quotients of $\mathrm{PSL}_2(\mathbb{R})$ by using the Selberg trace formula. The methods employed here do not seem to extend easily to such spaces.

We now outline this paper. We begin in Section 2 by proving that on a closed spin manifold the eta invariants are identical for two Dirac operators associated to conformal metrics. In Section 3 we review the Guillemin-Wodzicki residue density and residue trace and derive some of their elementary properties that we need in the sequel. In Section 4 we give a new proof that on any spin manifold, the local eta residue of a twisted Dirac operator vanishes. In Sections 5 and 6 we prove the main theorem based on Melrose's cusp calculus [21], which we review in Appendix B.

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2. Conformal invariance of eta invariants on closed manifolds

The eta invariant of the spin Dirac operator and of the odd signature operator are known to be invariant under conformal changes of the metric. Since on one hand we need to understand this fact in depth, and on the other hand we were unable to find a good reference, we chose to give here a complete proof.

Let (M, g) be a closed spin Riemannian manifold of dimension n, (E, ∇) a Hermitian vector bundle on M with compatible connection ∇, and D the twisted Dirac operator. Note that when n is even, the eta invariant reduces to $\dim \ker(D)$ (which is known to be a conformal invariant, see (2)). Indeed, for n even the operator D is odd with respect to the splitting in positive and negative spinors, more precisely it anti-commutes with the involution defined by Clifford multiplication with the volume form. Thus the spectrum of D is symmetric around 0, so the eta function itself vanishes in even dimensions. For the untwisted spin Dirac operator, the same vanishing occurs in dimensions $4k+1$: for $n=8k+1$ the spinor bundle has a real structure (i.e. a skew-complex map C with $C^2=1$) which anti-commutes with D, while in dimensions $8k+5$ it has a quaternionic structure (i.e. a skew-complex map J with $J^2=-1$) which anti-commutes with D [1, pp. 61, Remark (3)].

Let $f \in \mathcal{C}^\infty(M)$ be a real conformal factor, $g' := e^{-2f}g$ a metric conformal to g and D' the corresponding Dirac operator.

Proposition 1. *The eta invariants of D and D' coincide.*

The proof below fills a gap in the proof of [9, Lemma 3.1], where a certain metric on a cylinder is treated as if it were of product type.

Proof. The map of dilation by e^f gives an $\mathrm{SO}(n)$-isomorphism between the orthonormal frame bundles of g and g'. Thus the principal $\mathrm{Spin}(n)$-bundle (for the fixed spin structure) corresponding to g and g' are also isomorphic via the lift of this map. This identifies the spinor bundles for the two metrics; the Dirac operators are linked by the formula

$$g' = e^{-2f}g, \qquad D' = e^{\frac{n+1}{2}f} D e^{-\frac{n-1}{2}f} \tag{2}$$

(see e.g. [26, Proposition 1] for a proof). In particular, the null-spaces of these two operators have the same dimension. Thus, if n is even, the discussion at the beginning of this section finishes the proof. We may therefore assume that n is odd.

Let $\psi : I = [0,1] \to \mathbb{R}$ be a smooth function which is 0 for $t < 1/3$ and which is identically 1 for $t > 2/3$. Set $f_t := \psi(t)f$ and define a metric on $X := I \times M$ by

$$h = dt^2 + e^{-2f_t}g.$$

We denote again by E the pull back of E from the second factor, together with its connection. Therefore, the curvature tensor of E on X satisfies

$$\partial_t \lrcorner R^E = 0. \tag{3}$$

Since n is odd, X is even dimensional so the twisted spinor bundle of X splits into ± 1 eigenspaces of Clifford multiplication by vol_h, denoted $\Sigma^\pm_{X,E}$. The Dirac operator D_X on X is odd with respect to this splitting; let $D^+ : \mathcal{C}^\infty(X, \Sigma^+_{X,E}) \to \mathcal{C}^\infty(\Sigma^-_{X,E})$ denote the restriction of the D_X to positive twisted spinors. The metric h is of product type near ∂X and hence the Atiyah-Patodi-Singer formula can be applied to the (chiral) twisted Dirac operator D^+ on X:

$$\mathrm{index}(D^+) = \int_X \hat{A}(R^h)\mathrm{ch}(R^E) - \frac{1}{2}\eta(D) + \frac{1}{2}\eta(D').$$

On the other hand, this index is also equal to the spectral flow in the space of Riemannian metrics from D to D'; again by (2), there is no spectral flow so the index vanishes. The proof will be concluded by showing that the top component of the integrand in the APS formula vanishes.

From (3) we deduce $\partial_t \lrcorner \exp(R^E) = 0$ so it is enough to show that $\partial_t \lrcorner \hat{A}(R^h) = 0$. Recall that $\hat{A}$ is a polynomial in the Pontrjagin forms $\mathrm{tr}(R^h)^{2k} \in \Omega^{4k}(X)$. Also, recall that the Pontrjagin forms are conformal invariants (they only depend on the Weyl tensor – first proved by Chern and Simons [8]). Let

$$h' = e^{2f_t} h = e^{2f_t} dt^2 + g.$$

We claim that $\partial_t \lrcorner \mathrm{tr}((R^{h'})^{2k}) = 0$ for all k. Indeed, let ∇ be the Levi-Civita connection of h'. For every vector field V on M denote by $\tilde{V}$ its pull-back to X, which is orthogonal to the length-1 vector field $T := e^{-f_t}\partial_t$. Note that

$$[\tilde{V}, T] = -\psi(t)V(f)T, \qquad\qquad [\tilde{V}, \tilde{U}] = \widetilde{[V, U]}.$$

We deduce that

$$\begin{aligned} 2\langle \nabla_{\tilde{V}} T, \tilde{U}\rangle =& \tilde{V}\langle T, \tilde{U}\rangle + T\langle \tilde{V}, \tilde{U}\rangle - \tilde{U}\langle \tilde{V}, T\rangle \\ &+ \langle [\tilde{V}, T], \tilde{U}\rangle + \langle [\tilde{U}, \tilde{V}], T\rangle + \langle [\tilde{U}, T], \tilde{V}\rangle \\ =& 0. \end{aligned}$$

Clearly, since also $\langle \nabla_{\tilde{V}} T, T\rangle = 0$, we infer $\nabla_{\tilde{V}} T = 0$. Directly from the definition of the curvature this implies that $R^{h'}_{\tilde{V}\tilde{U}} T = 0$. If we split TX into $TI \oplus TM$, we see from the symmetry of the curvature tensor that $R^{h'}_{\tilde{V}\tilde{U}}$ is a diagonal linear map, while $R^{h'}_{\tilde{V}T}$ is off-diagonal. It follows that $\partial_T \lrcorner (R^{h'})^{2k}$ is an off-diagonal form-valued endomorphism (since it contains exactly one curvature term involving T). Hence its trace is zero. □

Remark 2. The same proof applies as well to the (twisted) odd signature operator on any orientable manifold, since the Hirzebruch L-form, like the $\hat{A}$-form, is also a polynomial in the Pontrjagin forms.

3. The residue trace and the residue density

We review a refined construction of the residue trace. Let A be a classical pseudodifferential operator A of integer order on a smooth closed manifold M of dimension n, acting on the sections of a vector bundle E. We will later be interested in twisted spinor bundles over spin manifolds, but the description of the residue density does not need these assumptions. Let $\kappa_A(m, m')$ be the Schwartz kernel of A, which is a distributional section in $E \boxtimes (E^* \otimes |\Omega|)$ over $M \times M$ with singular support contained in the diagonal. Choose a diffeomorphism

$$\Phi : U \to V \subset M \times M, \qquad\qquad \Phi(m, v) = (m, \phi_m(v)) \tag{4}$$

from a neighborhood of the zero section in TM to a neighborhood of the diagonal in M, extending the canonical identification of M with the diagonal. Cut-off κ_A away from the diagonal, i.e., multiply it by a function ψ with support in V which is identically 1 near the diagonal. Fix a connection in E, so that we can identify $E^*_{\phi_m(v)}$ with E^*_m using parallel transport along the curve $t \mapsto \phi_m(tv)$. Then $\Phi^*(\psi\kappa_A)$ is a compactly-supported distributional section over TM in the bundle $\mathrm{End}(E)$ pulled back from the base, tensored with the fiberwise density bundle. This distribution is conormal to the zero section, thus by definition there exists a classical symbol $a(m,\xi)$ on T^*M (with values at (m,ξ) in $\mathrm{End}(E_m)$) such that

$$\Phi^*(\psi\kappa_A)(m,v) = \frac{1}{(2\pi)^n}\int_{T^*M/M} e^{i\xi(v)}a(m,\xi)\,\omega^n. \tag{5}$$

Here ω is the canonical symplectic form on T^*M, and $\int_{T^*M/M}$ means integration along the fibers of T^*M. The result on the right-hand side is an $\mathrm{End}(E)$-valued density in the *base* variables; however since the vertical tangent bundle to TM at (m,v) is canonically isomorphic to T_mM, this can be interpreted as a vertical density.

Let $\mathcal{R}$ be the radial (vertical) vector field in the fibers of T^*M. Let $a_{[-n]}$ denote the component of homogeneity $-n$ of the classical symbol a. Fix a Euclidean metric g in the vector bundle T^*M (this amounts to choosing a Riemannian metric on M), thus defining a sphere bundle S^*M inside T^*M.

Definition 3. The residue density of A is the smooth $\mathrm{End}(E)$-valued density

$$\mathrm{res}(A) := \frac{1}{(2\pi)^n}\int_{S^*M/M} a_{[-n]}\,\mathcal{R}\lrcorner\omega^n.$$

At this stage, $\mathrm{res}(A)$ depends on a number of geometric choices: the embedding Φ, the cut-off ψ, the connection in E and the metric g.

One way to show that $\mathrm{res}(A)$ is defined independently of the choices involved is through holomorphic families. Let $(A_s)_{s\in\mathbb{C}}$ be a holomorphic family of pseudodifferential operators on E such that A_s is of order $k-s$, where k is the order of A, and $A_0 - A \in \Psi^{-\infty}(M,E)$. Then for $\Re(s)$ sufficiently large, restricting the Schwartz kernel of A_s to the diagonal Δ gives a well-defined and holomorphic $\mathrm{End}(E)$-valued density

$$F(s) := \kappa_{A_s}|_\Delta.$$

This density extends to $\mathbb{C}$ with possible simple poles at $s\in n+k-\mathbb{N}$, where k is the order of A and $\mathbb{N} = \{0,1,2,3,\ldots\}$. One natural choice of a holomorphic family is $A_s = AQ_s$ or $A_s = Q_sA$ where $(Q_s)_{s\in\mathbb{C}}$ is a holomorphic family of pseudodifferential operators on E such that Q_s is of order $-s$ and $Q_0 - \mathrm{Id}\in\Psi^{-\infty}(M,E)$.

Lemma 4.

$$\mathrm{Res}_{s=0}F(s) = \mathrm{res}(A). \tag{6}$$

Proof. Let $a_s(m,\xi)\in S^s(T^*M)$ be the full symbol of the operator A_s, defined for all s using the fixed geometric choices from the beginning of this section. By (5), we have

$$F(s)(m,m) = \frac{1}{(2\pi)^n}\int_{T^*M/M} a_s(m,\xi)\,\omega^n.$$

Write $a_s = u_s + \sum_{j=0}^k u_s^{j-n}$, where u_s is a holomorphic family of classical symbols of order $-1-n-s$, and $u_s^{j-n}(m,\xi)$ is smooth on T^*M and homogeneous of order $j-n-s$ for $\|\xi\|\geq 1$. On one hand, a straightforward computation shows that $\int_{T^*_mM} u_s^{j-n}(m,\xi)\,d\xi$

extends to $\mathbb{C}$ with a unique simple pole at $s = j$, with residue $\int_{S^*_m M} u_j^{j-n}(m,\xi)\,\mathcal{R}\lrcorner d\xi$, while $\int_{T^*_m M} u_s(m,\xi)\,d\xi$ is regular in s for $\Re(s) > -1$. On the other hand, A_0 coincides with A modulo smoothing operators, hence the homogeneous symbol u_0^{-n} is precisely $a_{[-n]}$. Thus the residue at $s = 0$ of $F(s)$ is $\operatorname{res}(A)$, as claimed. □

Thus, the residue $\operatorname{Res}_{s=0}F(s)$ is well-defined irrespective of the choice of A_s, and also $\operatorname{res}(A)$ is well-defined independently of choices. The residue trace of A is defined by

$$\operatorname{Tr}_R(A) = \operatorname{Res}_{s=0}\operatorname{Tr}(A_s) = \int_M \operatorname{tr}(\operatorname{res}(A)). \tag{7}$$

Armed with this holomorphic family interpretation of $\operatorname{res}(A)$ and $\operatorname{Tr}_R(A)$, one can deduce without effort various properties of res and Tr_R. For example, it follows that Tr_R vanishes on commutators. Indeed, given integer order operators A and B and taking any auxiliary family Q_s as explained above, we can write

$$\operatorname{Tr}([A,B]Q_s) = \operatorname{Tr}(C_s) + \operatorname{Tr}([AQ_s, B]),$$

where $C_s = ABQ_s - AQ_sB$ is a holomorphic family of operators that is smoothing at $s = 0$. One can check that $\operatorname{Tr}[S,T] = 0$ for any pseudodifferential operators S and T with $\operatorname{ord}(S) + \operatorname{ord}(T) < -n$, so for $\Re(s)$ sufficiently large, the trace $\operatorname{Tr}([AQ_s, B])$ vanishes. Therefore, by analytic continuation, $\operatorname{Tr}([AQ_s, B])$ vanishes for all $s \in \mathbb{C}$; in particular, we have $\operatorname{Tr}([A,B]Q_s) = \operatorname{Tr}(C_s)$. Now C_0 is smoothing, and since the residue density of a smoothing operator is zero, we have $\operatorname{Res}_{s=0}\operatorname{Tr}(C_s) = \operatorname{Tr}_R(C_0) = 0$. Thus, $\operatorname{Tr}_R([A,B]) = 0$.

Furthermore, if S is any section in $\operatorname{End}(E)$, using the fact that

$$\kappa_{SA_s} = S\kappa_{A_s} \quad \text{and} \quad \kappa_{A_sS} = \kappa_{A_s}S,$$

where A_s is a holomorphic family as in the formula (6), we have $\operatorname{res}(SA) = S\operatorname{res}(A)$ and $\operatorname{res}(AS) = \operatorname{res}(A)S$. That $\operatorname{res}(SA) = S\operatorname{res}(A)$ also follows directly from Definition 3. In particular, we have

$$\operatorname{res}(uA) = \operatorname{res}(Au) = u\operatorname{res}(A) \tag{8}$$

for every function $u \in \mathcal{C}^\infty(M,\mathbb{C})$. Finally, observing that

$$\kappa_{A_s}|_\Delta = \left(\kappa_{A_s^*}|_\Delta\right)^*,$$

taking the residue at $s = 0$ of both sides we obtain $\operatorname{res}(A) = \operatorname{res}(A^*)^*$; that is, we have $\operatorname{res}(A^*) = \operatorname{res}(A)^*$.

4. Vanishing of the local eta residue

Let $(D_t)_{t\in I}$ be a smooth 1-parameter family of elliptic self-adjoint pseudodifferential operators of order 1 on a closed manifold M. For simplicity assume for a moment that D_0

is invertible, hence D_t is invertible for small enough t. Then

$$\begin{aligned}\partial_t\eta(D_t,s) =&\partial_t \mathrm{Tr}\left(D_t(D_t^2)^{-\frac{s+1}{2}}\right)\\ =&\mathrm{Tr}\left(\dot D_t(D_t^2)^{-\frac{s+1}{2}}\right) - \frac{s+1}{2}\mathrm{Tr}\left(D_t(D_t^2)^{-\frac{s+3}{2}}(\dot D_t D_t + D_t\dot D_t)\right)\\ =&-s\mathrm{Tr}\left(\dot D_t(D_t^2)^{-\frac{s+1}{2}}\right)\\ =&-s\mathrm{Tr}\left(\dot D_t|D_t|^{-1}(D_t^2)^{-\frac{s}{2}}\right).\end{aligned}$$

Now $Q_s := (D_t^2)^{-\frac{s}{2}}$ is an analytic family of operators of order $-s$ and $Q_0 = \mathrm{Id}$, therefore

$$\partial_t\eta(D_t) = \left[-s\mathrm{Tr}\left(\dot D_t|D_t|^{-1}Q_s\right)\right]_{s=0} = -\mathrm{Tr}_R(\dot D_t|D_t|^{-1}). \tag{9}$$

From (7), the Wodzicki residue trace vanishes on smoothing operators, so as a corollary we see that the eta invariant is constant under smoothing perturbations. By this argument, the above expression makes sense even when D_t is not invertible.

In the same spirit, let D be an elliptic self-adjoint invertible pseudodifferential operator of order $k \in (0,\infty)$. Then the residue at $s = 0$ of the eta function $\eta(D,s)$ is

$$\mathrm{Res}_{s=0}\mathrm{Tr}(D|D|^{-1}(D^2)^{-\frac{s}{2}}) = \frac{1}{k}\mathrm{Tr}_R(D|D|^{-1}) = \frac{1}{k}\int_M \mathrm{tr}(\mathrm{res}(D|D|^{-1})).$$

We have been assuming that M is closed, so that the trace on the left is defined. However, notice that by definition, $\mathrm{tr}(\mathrm{res}(D|D|^{-1}))$ is a local quantity in the sense that it depends only on finitely many terms of the local symbol of $D|D|^{-1}$; moreover, each homogeneous term of $D|D|^{-1}$ is given by a universal formula in terms of the local symbol of D in any coordinate patch. Using this universal formula for the $-n$ degree homogeneous term allows us to define the local eta residue on the *interior* of any manifold (with or without boundary, compact or not), even in the case that $|D|^{-1}$ does not exist.

Definition 5. Let M be a possibly non-compact manifold, $E \to M$ a vector bundle and $z \in \mathbb{C}$. For any elliptic pseudodifferential operator $D \in \Psi^z(M,E)$, the density

$$\mathrm{tr}(\mathrm{res}(D|D|^{-1})) \in |\Omega(M)|.$$

is called the *local eta residue* of D.

From the definition of the residue density, the local eta residue is constant under smoothing perturbations, so the definition makes sense when D is not invertible, non symmetric. The local eta residue can be non-vanishing in general (when M is compact, its integral always vanishes for self-adjoint operators of positive order since the eta function is regular at $s = 0$ [10]). However, for Dirac operators we have:

Theorem 6. [5] *Let (M, g) be a spin Riemannian manifold and E a twisting bundle. Then the local eta residue tr$(\mathrm{res}(D|D|^{-1}))$ of the twisted Dirac operator vanishes.*

Proof. We give here a new, easy proof. Assume that M is closed; this theorem is a local question, so this case suffices. Let f be an arbitrary smooth real function on M. Define D_t as the Dirac operator associated to the family of conformal metrics $e^{-2tf}g$. This operator is an unbounded operator in the L^2 space associated to the measure $e^{-ntf}\mu_g$. To work in the fixed Hilbert space $L^2(\mu_g)$, conjugate through the unitary transformation

$$L^2(M, \Sigma \otimes E, e^{-ntf}\mu_g) \to L^2(M, \Sigma \otimes E, \mu_g) \qquad\qquad \phi \mapsto e^{-\frac{ntf}{2}}\phi$$

where Σ denotes the spinor bundle over M. Using (2), D_t conjugates to

$$\tilde{D}_t = e^{\frac{tf}{2}} D e^{\frac{tf}{2}} \text{ acting in } L^2(M, \Sigma \otimes E, \mu_g)$$

and we compute

$$\partial_t \tilde{D}_t = \frac{1}{2}(f\tilde{D}_t + \tilde{D}_t f). \tag{10}$$

Using Proposition 1 we have on one hand

$$\partial_t \eta(\tilde{D}_t) = \partial_t \eta(D_t) = 0.$$

On the other hand, plugging (10) at $t = 0$ into (9) we write:

$$\begin{aligned} -\partial_t \eta(\tilde{D}_t)_{|t=0} &= \mathrm{Tr}_R \left[\frac{1}{2}(fD + Df)|D|^{-1} \right] && \text{since } \tilde{D}_0 = D \\ &= \frac{1}{2}\mathrm{Tr}_R(fD|D|^{-1} + f|D|^{-1}D) && \text{since } \mathrm{Tr}_R \text{ is a trace} \\ &= \mathrm{Tr}_R(fD|D|^{-1}) && \text{since } D \text{ commutes with } |D| \\ &= \int_M \mathrm{tr}(\mathrm{res}(fD|D|^{-1})). \end{aligned}$$

From the definition (see also (8)), $\mathrm{res}(fD|D|^{-1}) = f\mathrm{res}(D|D|^{-1})$. Since f was arbitrary, we deduce $\mathrm{tr}(\mathrm{res}(D|D|^{-1})) = 0$ as claimed.

□

We will need such a vanishing result for a larger class of first-order symmetric differential operators:

Corollary 7. *Let D be a twisted Dirac operator on a spin manifold (M, g). For any $u \in \mathcal{C}^\infty(M, \mathbb{R})$, the operator $D_u := e^{-u}De^u$ is symmetric on M with respect to the measure $\mu_u := e^{2u}\mu_g$, and the local eta residue of D_u vanishes.*

Proof. It is clear that D_u is formally self-adjoint with respect to the measure μ_u. As in the proof of Theorem 6, to prove that the local eta residue of D_u vanishes we can assume that M is compact. Then $|D_u| = e^{-u}|D|e^u$, hence $D_u|D_u|^{-1} = e^{-u}D|D|^{-1}e^u$. By (8), $\mathrm{res}(D_u|D_u|^{-1}) = \mathrm{res}(e^{-u}D|D|^{-1}e^u) = \mathrm{res}(D|D|^{-1})$. The trace of this last endomorphism-valued density vanishes by Theorem 6. □

5. Eta function on conformally cusp manifolds

We turn now to our main object of study.

Let M° be the interior of a compact manifold with boundary M of dimension n. We assume that M is spin with a fixed spin structure, and that the metric is of conformally cusp type as in [24]. To explain this notion, let $x : M \to [0, \infty)$ be a boundary-defining function for the smooth structure of M, namely

(1) $x \in \mathcal{C}^\infty(M)$;
(2) $\{x = 0\} = \partial M$;
(3) The 1-form dx is non-vanishing on ∂M.

There exists a neighbourhood $U \subset M$ of ∂M and a diffeomorphism $\Phi_U : U \to [0, \epsilon) \times \partial M$ such that $x_{|U}$ is the composition of Φ_U with the projection on the first factor. In the sequel we fix such a product decomposition near the boundary.

The metric g on M° is said to be of conformally cusp type if on $U \cap M^\circ$ it is of the form

$$g_p = x^{2p} \left(\frac{dx^2}{x^4} + h \right) \tag{11}$$

where $p \in (0, \infty)$ and h is a metric on ∂M which does not depend on x. For $p = 1$ this metric is a particular case of the d-metrics of Vaillant [29]. Thus $g_p = x^{2p} g_c$, where g_c is a particular case of an exact cusp metric as in [21, 24]. Geometrically, g_c, which takes the form $g_c = \frac{dx^2}{x^4} + h$ on $U \cap M^\circ$, is simply a metric with infinite cylindrical ends, as one can see by switching to the variable $v = 1/x$. Recall that x is a global function, thus g_p is defined on M°. The motivating example is given by complete hyperbolic manifolds of finite volume. Outside a compact set, such a hyperbolic manifold is isometric to an infinite cylinder $(1, \infty) \times T$ where (T, h) is a (possibly disconnected) flat manifold; the metric takes the form

$$dt^2 + e^{-2t} h$$

which is easily seen to be of the form (11) with $p = 1$ if we set $x := e^{-t}$.

Let E be a twisting bundle on M, with a connection which is flat in the direction of ∂_x. This implies that near the boundary, E together with its connection are pull-backs of their restrictions to the boundary $E_{|\partial M}$. Finally, let D_p (where $2p$ is the power in the conformal metric g_p) denote the twisted Dirac operator associated to the aforementioned data.

The main assumption under which we work is the invertibility of the boundary Dirac operator. More precisely,

Assumption 1. *For each connected component N of ∂M, we assume that the Dirac operator $D_{(\partial M, h)}$ on N with respect to the metric h and twisted by E, is invertible.*

Under this assumption, the results of [24] imply that the L^2 spectrum of the essentially self adjoint operator D_p is discrete and obeys a Weyl-type law; moreover the eta function $\eta(D_p, s)$ is holomorphic for $\Re(s) > n$ and extends to a meromorphic function with possible *double* poles at certain points. In particular, for n odd, $s = 0$ is such a possible double pole.

Theorem 8. *Under Assumption 1, the eta function of the twisted Dirac operator on M° is regular at $s=0$.*

It follows that we can define a "honest" eta invariant, depending on the eigenvalues of D_p, as the regular value at $s=0$ of $\eta(D_p,s)$.

Proof. We need to revisit the construction giving the meromorphic extension and the structure of the poles of the eta function. The main tool is the calculus of cusp pseudodifferential operators, first introduced in [21, 20], whose definition we review in Appendix B.

The spinor bundles for conformal metrics are canonically identified together with their metrics. It follows that D_c, the Dirac operator for g_c, is linked to D_p by formula (2). However these two operators act on different L^2 spaces because the measures μ_p and μ_c induced by the metrics g_p, g_c are not the same. We view D_p as acting in $L^2(M^\circ, \Sigma\otimes E, \mu_p)$ and we conjugate it through the Hilbert space isometry

$$L^2(M^\circ, \Sigma\otimes E, \mu_p) \to L^2(M^\circ, \Sigma\otimes E, \mu_c), \qquad\qquad \sigma \mapsto x^{np/2}\sigma.$$

It follows that D_p is unitarily equivalent to the operator

$$A = x^{\frac{np}{2}} D_p x^{-\frac{np}{2}} \text{ acting in } L^2(M^\circ, \Sigma\otimes E, \mu_c).$$

Using the formula (2) with $e^{-f} = x^p$, we see that

$$D_p = x^{-p\frac{n+1}{2}} D_c x^{p\frac{n-1}{2}}.$$

In the sequel we thus replace D_p by the unitarily equivalent operator

$$A = x^{-\frac{p}{2}} D_c x^{-\frac{p}{2}} \text{ acting in } L^2(M^\circ, \Sigma\otimes E, \mu_c).$$

This operator is an elliptic operator in the weighted cusp calculus $x^{-p}\mathrm{Diff}^1_c(M, \Sigma\otimes E)$. The *normal operator* of a cusp operator P in $\mathrm{Diff}^1_c(M,\Sigma\otimes E)$ is the 1-parameter family of operators on ∂M defined by

$$\mathcal{N}(P)(\xi)\phi = [e^{i\frac{\xi}{x}} P(e^{-i\frac{\xi}{x}}\tilde{\phi})]_{|x=0}$$

for $\xi\in\mathbb{R}$, where $\tilde{\phi}$ is any extension of the spinor ϕ from ∂M to M.

Since g_c and the twisting bundle E and its connection are products near infinity, we have

$$D_c = c(\nu)x^2\partial_x + D_{(\partial M,h)}, \qquad\qquad A = x^{-p}\left[\left(x^2\partial_x - \frac{px}{2}\right)c(\nu) + D_{(\partial M,h)}\right] \tag{12}$$

where $D_{(\partial M,h)}$ is the Dirac operator on ∂M, $\nu = \frac{dx}{x^2}$, and $c(\nu)$ is Clifford multiplication by ν. It follows from the definition that

$$\mathcal{N}(D_c)(\xi) = c(\nu)i\xi + D_{(\partial M,h)}. \tag{13}$$

The boundary operator $D_{(\partial M,h)}$ anti-commutes with $c(\nu)$ for algebraic reasons and is invertible by Assumption 1. Since

$$\mathcal{N}(D_c)(\xi)^2 = \xi^2 + D^2_{(\partial M,h)}$$

is strictly positive, it follows that $\mathcal{N}(D_c)(\xi)$ is invertible for all ξ. Such an operator is called *fully elliptic*. From [24], we know that A has essentially the same properties as a symmetric elliptic operator on a closed manifold: it admits a compact parametrix, it is essentially self-adjoint, Fredholm, and has compact resolvent and pure-point spectrum. The eigenvalues are distributed according to a suitable Weyl-type law; in particular, the eta function $\eta(A,s)$ is well-defined for large real parts of s. Moreover, $A(A^2)^{-\frac{s+1}{2}}$ is a holomorphic family of cusp operators in $x^{ps}\Psi_c^{-s}(M,\Sigma\otimes E)$ if we define it to be 0 on the finite-dimensional null-space of A, see [24, Proposition 15]. It follows from [24, Proposition 14] that the trace of this family (i.e., the eta function) extends meromorphically to $\mathbb{C}$ with possible poles when $s\in\{n,n-1,n-2,\dots\}$ and when $ps\in\{1,0,-1,-2\dots\}$; the poles are at most double at points in the intersections of these two sets, otherwise they are at most simple. The content of the theorem is that $s=0$ is in fact a regular point. We will see later that some of the above singularities do not occur in our setting.

To start the proof, consider the holomorphic family in two complex variables

$$(s,w)\mapsto x^w A(s), \qquad A(s)=x^{-ps}A(A^2)^{-\frac{s+1}{2}}\in\Psi_c^{-s}(M,\Sigma\otimes E) \tag{14}$$

and the function

$$F(s,w):=\operatorname{Tr}(x^w A(s)).$$

Clearly $F(s,ps)=\eta(D_p,s)$.

Lemma 9. *The operator $x^wA(s)$ is of trace-class for $\Re(s)>n,\Re(w)>1$. Moreover, $F(s,w)$ is holomorphic for $\Re(s)>n,\Re(w)>1$ and extends to $\mathbb{C}\times\mathbb{C}$ as a meromorphic function with possibly simple poles in s at $s\in\{n,n-1,n-2,\dots\}$ and in w at $w\in\{1,0,-1,-2,\dots\}$.*

Proof. The operator kernel of $x^wA(s)$ is smooth outside the diagonal and continuous at the diagonal for $\Re(s)>n$. Its restriction to the diagonal is a smooth multiple of the cusp volume density for such s, and has an asymptotic expansion in powers of x as $x\to 0$, starting from x^w. This is due to the fact that $A(s)$ is a conormal distribution on the cusp double space M_c^2, with Taylor expansion at the front face.

The trace of $x^wA(s)$ equals the integral on the lifted diagonal of the above density. The normal bundle to Δ_c in M_c^2 is canonically identified with ${}^cTM$. By the Fourier inversion formula, this is equal to

$$\int_{{}^cT^*M} x^w a_s(p,\xi)\,\omega^n \tag{15}$$

where $a_s(p,\xi)$ is a holomorphic family of classical symbols of order $-s$ on ${}^cT^*M$ (smooth down to $x=0$) and ω is the canonical symplectic form on ${}^cT^*M$. The volume form ω^n is singular at M, however x^2 times it extends smoothly to the boundary of ${}^cT^*M$. It follows that the integral is absolutely convergent (hence holomorphic in s,w) for $\Re(s)>n,\Re(w)>1$.

It is now easy to construct the analytic extension of (15) in w by expanding $a_s(p,\xi)$ in Taylor series at $x=0$, using that for any $k\in\mathbb{N}$, we have

$$\int_0^1 x^{w+k}\frac{dx}{x^2}=\frac{1}{w+k-1}.$$

To get the analytic extension of (15) in s we expand $a_s(p,\xi)$ in homogeneous components in ξ of order $-s-k$ where $k\in\mathbb{N}$, then switching to polar coordinates and using that

$$\int_1^\infty r^{-s-k+n-1}dr = \frac{1}{s-n+k}.$$

□

We first note that there is no pole at $s=0$. From here on, we assume that the dimension of the manifold M is odd, otherwise the eta function is 0 so there is nothing to prove.

Proposition 10. *The function $F(s,w)$ is regular in s at $s=0$.*

Proof. From the construction of the analytic extension of F it follows that for every w with $\Re(w)>1$,

$$\mathrm{Res}_{s=0}F(s,w) = \int_M x^w \mathrm{tr}(\mathrm{res}(A|A|^{-1})).$$

As A is unitarily conjugated to D_p by a real function, we see from Corollary 7 that the density $\mathrm{tr}(\mathrm{res}(A|A|^{-1}))$ vanishes identically. In other words, the holomorphic function $w\mapsto \mathrm{Res}_{s=0}F(s,w)$ is identically 0 on a half-plane, and by unique continuation it is identically zero for all w. □

It remains to show that there is no pole in w at $w=0$ either. This will imply Theorem 8 since $\eta(D_p,s)=F(s,ps)$.

For this, we fix s with $\Re(s)>n$ and we examine $F(s,w)$ as a meromorphic function in the complex variable w.

For any cusp operator $B\in x^z\Psi_c^{-s}(M)$ (where we suppress the bundles for brevity) consider the power series expansion of its Schwartz kernel $x^{-z}\kappa_B$ at the front face of the cusp double space. Although x is not everywhere a defining function for the front face, we do get such an expansion since κ_B vanishes in Taylor series at faces other than the front face. Taking the inverse Fourier transform of the coefficients we can regard the coefficients as lying in the suspended calculus $\Psi_{\mathrm{sus}}^{-s}(\partial M)$. This gives a short exact sequence of spaces of operators

$$0\mapsto x^\infty\Psi_c^{-s}(M)\hookrightarrow x^z\Psi_c^{-s}(M)\xrightarrow{q} x^z\Psi_{\mathrm{sus}}^{-s}(\partial M)[[x]]\to 0. \tag{16}$$

For a weighted cusp operator $B\in x^z\Psi_c^{-s}(M)$, we write

$$q(B)=x^z\left(q_0(B)+xq_1(B)+x^2q_2(B)+\dots\right)$$

It is easy to see that we have $q_0(B)=\mathcal{N}(x^{-z}B)$, see [21].

We use a result from [15]. Let $P\in\Psi_c^{-s}(M)$ with $\Re(s)>n$. Then x^wP is trace-class for $w>1$, and for $k\in\mathbb{N}$,

$$\mathrm{Res}_{w=1-k}\mathrm{Tr}(x^wP)=\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{Tr}(q_k(P)(\xi))d\xi. \tag{17}$$

Proposition 11. *The function $F(s,w)$ does not have any poles in w.*

Proof. Let $R \in \mathrm{Diff}^1_c(M, \Sigma \otimes E)$ be any cusp differential operator which equals $D_\partial := c(\nu)D_{(\partial M,h)}$ near the boundary. This makes sense since we have a fixed product decomposition near the boundary. By definition we have

$$q(R) = D_\partial.$$

We notice that near ∂M, D_∂ anticommutes with the cusp differential operator A from (12). Therefore, if we denote by $\mathcal{I}^s := \ker(q) \subset \Psi^s_c(M, \Sigma \otimes E)$ the subspace of operators which vanish to every order at the front face, we have

$$RA + AR \in \mathcal{I}^2.$$

This implies

$$[R, A^2] \in \mathcal{I}^3, \qquad [R, (A^2)^{-\frac{s+1}{2}}] \in \mathcal{I}^{-s};$$

the latter, by the construction of the complex powers [7]. Together with the obvious commutation $[R, x^{-ps}] \in \mathcal{I}^0$, we get for the operator $A(s)$ defined in (14)

$$RA(s) + A(s)R \in \mathcal{I}^{-s}.$$

Now for every cusp operator Q we have

$$q(RQ) = D_\partial q(Q), \qquad q(QR) = q(Q)D_\partial$$

because R is constant in x near the boundary. Therefore $D_\partial q(A(s)) = -q(A(s))D_\partial$. Using conjugation with the invertible operator D_∂ on ∂M, we see that for every ξ,

$$\mathrm{Tr}\left(q(A(s))(\xi)\right) = \mathrm{Tr}\left(D_\partial q(A(s))(\xi)D_\partial^{-1}\right) = -\mathrm{Tr}\left(q(A(s))(\xi)D_\partial D_\partial^{-1}\right)$$

so $\mathrm{Tr}\left(q(A(s))(\xi)\right) = 0$. Thus for all $k \in \mathbb{N}$ the integrand in (17) for $P = A(s)$ vanishes. □

Together with Proposition 10 this finishes the proof of Theorem 8 since $\eta(D_p, s) = F(s, ps)$. □

In fact, by invoking the regularity results of Bismut and Freed [5], we can restrict further the possible poles of the eta function. By a different argument, it turns out that if M° is a hyperbolic manifold, then there are no poles at all (see Theorem 13)! Of course, we assume that n is odd since otherwise the eta function is 0.

Theorem 12. *Under Assumption 1, the eta function $\eta(D_p, s)$ is regular for $\Re(s) > -2$ and has at most simple poles at $s \in \{-2, -4, \dots\}$.*

Proof. We have written $\eta(D_p, s) = F(s, ps)$ for an analytic function $F(s, w)$ in $w \in \mathbb{C}, s \in \mathbb{C} \setminus \{n, n-1, \dots, 1, -1, -2, -3, \dots\}$ by Theorem 8, Lemma 9, and Proposition 11. We claim that $F(s, w)$ is in fact regular at $s \in \mathbb{C} \setminus \{-2, -4, \dots\}$. Indeed, for $\Re(w) > 1$ the residue in s at $s = n - k$ is given by

$$\int_M x^{w-p(n-k)} \mathrm{tr}(\mathrm{res}(A|A|^{k-n-1})).$$

Since $A = x^{\frac{np}{2}} D_p x^{-\frac{np}{2}}$, we have $A|A|^{k-n-1} = x^{\frac{np}{2}} D_p |D_p|^{k-n-1} x^{-\frac{np}{2}}$, so by (8),

$$\mathrm{Res}_{s=n-k} F(s,w) = \int_M x^{w-p(n-k)} \mathrm{tr}(\mathrm{res}(D_p|D_p|^{k-n-1})).$$

Consider the well-known odd heat kernel small-time expansion given in Lemma 1.9.1 of [11]

$$D_p e^{-tD_p^2}(y,y) \sim \sum_{k=0}^{\infty} t^{\frac{-n+k-1}{2}} b_k(y) \tag{18}$$

valid on the interior of any manifold, by locality of the coefficients; moreover, for k even, we have $b_k \equiv 0$. From the relationship

$$D_p|D_p|^{-s-1} = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^{\infty} t^{\frac{s-1}{2}} D_p e^{-tD_p^2} dt \tag{19}$$

we deduce that

$$\mathrm{res}(D_p|D_p|^{k-n-1}) = \frac{2}{\Gamma(\frac{n-k+1}{2})} b_k. \tag{20}$$

In particular, when k is even, this residue vanishes. By the regularity results of Bismut-Freed [5], the pointwise traces of the local coefficients $b_0, b_1, \ldots, b_n$ vanish identically, i.e. $\mathrm{tr}(b_k(y)) \equiv 0$ for $k = 0, \ldots, n$ so for these k the density $\mathrm{tr}(\mathrm{res}(D_p|D_p|^{k-n-1}))$ also vanishes. (The vanishing of the term with $k = n$, corresponding to the residue of the eta function at the origin, has already been proved in Proposition 10 by using conformal invariance, although here we could have also deduced this fact from the regularity results of Bismut-Freed [5].) This proves that $F(s,w)$ is regular at $s \in \mathbb{C} \setminus \{-2, -4, \ldots\}$. □

In the hyperbolic case we have a stronger result. The necessary local vanishing of the heat trace is proved in the next section.

Theorem 13. *If M° is an odd dimensional hyperbolic manifold of finite volume, the eta function of the Dirac operator twisted by a homogeneous vector bundle is entire.*

Proof. By Proposition 14 in Section 6, we have that for $m \in \mathbb{H}^n(\mathbb{R})$ with $n = 2d+1$,

$$\mathrm{tr}(De^{-tD^2}(m,m)) = 0.$$

This implies that $\mathrm{tr}(b_k) = 0$ for all the coefficients b_k on $\mathbb{H}^n(\mathbb{R})$, and thus also on the locally symmetric space M°. In view of (20), the possible poles in s of the function $F(s,w)$ actually do not occur. Together with Proposition 11, this shows that the eta function is entire since $\eta(D,s) = F(s,s)$. □

6. Odd heat kernels for homogeneous vector bundles over hyperbolic space

The real hyperbolic space $\mathbb{H}^n(\mathbb{R})$ is given as the symmetric space $\mathrm{SO}(n,1)/\mathrm{SO}(n)$. But, for our purpose, we use the realization of $\mathbb{H}^n(\mathbb{R}) = G/K$ where $G = \mathrm{Spin}(n,1)$, $K = \mathrm{Spin}(n)$, which are the double covering groups of $\mathrm{SO}(n,1)$, $\mathrm{SO}(n)$. We denote the Lie algebras of G and K by $\mathfrak{g}$ and $\mathfrak{k}$, respectively. The Cartan involution θ on $\mathfrak{g}$ gives the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ and $\mathfrak{p}$ are, respectively, the $+1$ and -1 eigenspaces of θ. The subspace $\mathfrak{p}$ can be identified with the tangent space $T_o(G/K) \cong \mathfrak{g}/\mathfrak{k}$ at $o = eK \in G/K$. Let $\mathfrak{a}$ be a fixed maximal abelian subspace of $\mathfrak{p}$. Then the dimension of $\mathfrak{a}$ is one. Let $M = \mathrm{Spin}(n-1)$ be the centralizer of $A = \exp(\mathfrak{a})$ in K with Lie algebra $\mathfrak{m}$. We put β to be the positive restricted root of $(\mathfrak{g}, \mathfrak{a})$. Note that $A \cong \mathbb{R}$ via $a_r = \exp(rH)$ with $H \in \mathfrak{a}$, $\beta(H) = 1$.

From now on we assume that n is odd, that is, $n = 2d + 1$.

The spinor bundle Σ over $\mathbb{H}^n(\mathbb{R}) = G/K = \mathrm{Spin}(n,1)/\mathrm{Spin}(n)$ is defined by

$$\Sigma = \mathrm{Spin}(n,1) \times_{\tau_s} V_{\tau_s} \longrightarrow \mathbb{H}^n(\mathbb{R}) = \mathrm{Spin}(n,1)/\mathrm{Spin}(n) \tag{21}$$

where (τ_s, V_{τ_s}) denotes the spin representation of $\mathrm{Spin}(n)$. Here points of $\mathrm{Spin}(n,1) \times_{\tau_s} V_{\tau_s}$ are given by equivalence classes $[g, v]$ of pairs (g, v) under $(gk, v) \sim (g, \tau_s(k)v)$. In general, any G-homogeneous Clifford module bundle over $\mathbb{H}^n(\mathbb{R})$ is associated to $(\tau_s \otimes \tau, V_{\tau_s} \otimes V_\tau)$ for a unitary representation (τ, V_τ) of $\mathrm{Spin}(n)$ as in (21), which we denote by $\Sigma \otimes E$. For instance, the representation $\tau_s \otimes \tau_s$ of $\mathrm{Spin}(n)$ determines a homogeneous vector bundle $\Sigma \otimes \Sigma$ over $\mathbb{H}^n(\mathbb{R})$ whose fiber is $V_{\tau_s} \otimes V_{\tau_s} \cong \oplus_{k=0}^d \wedge^k (\mathfrak{p} \otimes \mathbb{C})$.

The space of smooth sections from $\mathbb{H}^n(\mathbb{R})$ to $\Sigma \otimes E$ is denoted by $\mathcal{C}^\infty(\mathbb{H}^n(\mathbb{R}), \Sigma \otimes E)$ and can be identified with $[\mathcal{C}^\infty(G) \otimes V_{\tau_s} \otimes V_\tau]^K$ where K acts on $\mathcal{C}^\infty(G)$ by the right regular representation R. Now a natural connection $\nabla : \mathcal{C}^\infty(\mathbb{H}^n(\mathbb{R}), \Sigma \otimes E) \to \mathcal{C}^\infty(\mathbb{H}^n(\mathbb{R}), \Sigma \otimes E \otimes T^*(G/K))$ is given by

$$\nabla f = \sum_{i=1}^n (R(X_i) \otimes \mathrm{Id}) f \otimes X_i^* \tag{22}$$

where $\{X_i\}$ is an orthonormal basis of $\mathfrak{p}$ and $\{X_i^*\}$ is its dual basis. This connection is the unique connection on $\mathcal{C}^\infty(\Sigma \otimes E)$ which is G-homogeneous and anti-commutes with the Cartan involution θ (see Lemma 3.2 of [25]). Now the Dirac operator D on $\Sigma \otimes E$ associated to the connection ∇ is defined by

$$D = \sum_{i=1}^n R(X_i) \otimes c(X_i)$$

where $c(X_i)$ denotes the Clifford multiplication.

Proposition 14. *For $m \in \mathbb{H}^n(\mathbb{R})$, we have $\mathrm{tr}(De^{-tD^2}(m,m)) = 0$.*

Proof. Recalling $\mathcal{C}^\infty(\mathbb{H}^n(\mathbb{R}), \Sigma \otimes E) \cong [\mathcal{C}^\infty(G) \otimes V_{\tau_s} \otimes V_\tau]^K$, the Schwartz kernel of De^{-tD^2} is given by a section H_t in $[\mathcal{C}^\infty(G) \otimes \mathrm{End}(V_{\tau_s} \otimes V_\tau)]^{K \times K}$ satisfying

$$H_t(k_1 g k_2) = (\tau_s \otimes \tau)^{-1}(k_2) H_t(g) (\tau_s \otimes \tau)(k_1)^{-1} \tag{23}$$

for $k_1, k_2 \in K, g \in G$, which acts on $[\mathcal{C}^\infty(G) \otimes V_{\tau_s} \otimes V_\tau]^K$ by convolution. For each $t > 0$, H_t lies in $[\mathcal{S}(G) \otimes \mathrm{End}(V_{\tau_s} \otimes V_\tau)]^K$ where $\mathcal{S}(G) = \cap_{p>0} \mathcal{S}^p(G)$ with $\mathcal{S}^p(G)$ the Harish-Chandra L^p-Schwartz space. For more details, we refer to Section 3 of [25]. Taking the local trace of H_t, we have that $h_t := \mathrm{tr}(H_t) \in \mathcal{S}(G)$. From (23) and recalling that a point in the homogeneous vector bundle $\Sigma \otimes E$ is given by an equivalence class through the relation

$(gk, v) \sim (g, (\tau_s \otimes \tau)(k)v)$, we can see that the local trace of $De^{-tD^2}(m, m)$ is given by $h_t(e)$ for the identity element $e \in G$.

By the Plancherel theorem (see Theorem 4.1 in [23]), we have the following expression for h_t at $e \in G$ (up to a constant depending on a normalization),

$$h_t(e) = \sum_{\sigma \in \hat{M}} \int_{-\infty}^{\infty} \Theta_{\sigma,i\lambda}(h_t)\, p(\sigma, i\lambda) d\lambda, \tag{24}$$

where $\hat{M}$ denotes the set of equivalence classes of irreducible unitary representations of M,

$$\Theta_{\sigma,i\lambda}(h_t) = \mathrm{Tr} \int_G h_t(g) \pi_{\sigma,i\lambda}(g)\, dg,$$

and $p(\sigma, i\lambda)$ denotes the Plancherel measure associated to the unitary principal representation $\pi_{\sigma,i\lambda}$. Here the unitary principal representation $\pi_{\sigma,i\lambda} = \mathrm{Ind}_{MAN}^G(\sigma \otimes e^{i\lambda} \otimes \mathrm{Id})$ acts by the left regular representation on

$$H_{\sigma,i\lambda} = \{\ f : G \to V_\sigma \mid f(gma_r n) = e^{-(i\lambda+d)rH} \sigma(m)^{-1} f(g)\ \}$$

where $n = 2d + 1$. By Proposition 3.6 in [25], it follows that

$$\Theta_{\sigma,i\lambda}(h_t) = [(\tau_s \otimes \tau)|_M : \sigma]([\sigma : \sigma_+] - [\sigma : \sigma_-])\, \lambda e^{-t\lambda^2} \tag{25}$$

where $\sigma_\pm$ denotes the half spin representation of M such that $\tau_s|_M = \sigma_+ \oplus \sigma_-$. By the branching rule from $K = \mathrm{Spin}(2d+1)$ to $M = \mathrm{Spin}(2d)$ given in Theorem 8.1.3 of [14], we have that for any $\tau \in \hat{K}$, $\sigma \in \hat{M}$, $[\tau|_M : \sigma] \le 1$ and $[\tau|_M : \sigma] = 1$ if and only if

$$a_i - b_i \in \mathbb{Z} \quad (i, j = 1, 2, \ldots, d), \quad a_1 \ge b_1 \ge \ldots a_{d-1} \ge b_{d-1} \ge a_d \ge |b_d|$$

where $\tau = \sum_{i=1}^d a_i e_i$, $\sigma = \sum_{i=1}^d b_i e_i$. Here we denote the highest weights of the representations τ, σ with respect to the standard basis. This implies that

$$\begin{aligned} [(\tau_s \otimes \tau)|_M : \sigma][\sigma : \sigma_+] = [(\tau_s \otimes \tau)|_M : \sigma_+] &= [(\tau_s \otimes \tau)|_M : \sigma_-] \\ &= [(\tau_s \otimes \tau)|_M : \sigma][\sigma : \sigma_-] \end{aligned}$$

since $\sigma_\pm = \frac{1}{2}(e_1 + e_2 + \ldots + e_{d-1} \pm e_d)$. Now, by Theorem 3.1 of [22], we also have $p(\sigma_+, i\lambda) = p(\sigma_-, i\lambda)$. This implies $h_t(e) = 0$ by (24) and (25), which completes the proof. □

Remark 15. It can be proved, using the Selberg trace formula, that the eta function vanishes at negative odd integers under the condition $\Gamma \cap P = \Gamma \cap N$ for the fundamental group Γ of the given hyperbolic manifold where $P = MAN$ (Langlands decomposition) denotes a parabolic subgroup of G fixing the infinity point of a cusp. Recall that this fact is true for arbitrary operators of Dirac type over closed manifolds, as follows immediately from (19) and from the odd heat trace expansion (18). We believe that this vanishing holds also in the context of manifolds with conformal cusps without this technical condition but the necessary work, which surpasses the scope of this paper, is left for a future publication.

Appendix A. The spectrum of the Dirac operator

Recall that under Assumption 1, the Dirac operator D_p is always essentially self-adjoint with discrete spectrum [24]. One may ask what happens with the eta invariant when Assumption 1 does not hold. Like in [4], for $p \leq 1$ when the manifold is complete with respect to g_p, the answer is that the continuous spectrum of the twisted Dirac operator (which is essentially self adjoint by [30], [28]) becomes the full real line, hence the usual definition of the eta invariant breaks down. We will not attempt here to extend the definition in that case, note however that for finite-volume hyperbolic manifolds this has been done in [27]. For $p = 1$, the spectrum of the Dirac operator was computed in [29] for a larger class of metrics than ours. The proof of the following result is very similar to the corresponding statements from [12, 13] concerning magnetic and Hodge Laplacians.

Theorem 16. *Let M be a spin manifold with conformal cusps and E a twisting bundle of product type near the cusps. Let D_p denote the Dirac operator associated to the metric* (1) *on M, twisted by E. If Assumption 1 does not hold, then*

- *if $0 < p \leq 1$, the essential spectrum of D_p is $\mathbb{R}$.*
- *if $p > 1$, then D_p is not essentially self-adjoint, and every self-adjoint extension of D_p in L^2 has purely discrete spectrum.*

Proof. The idea is to reduce the problem to a 1-dimensional problem, essentially to the computation of the spectrum of $i\partial_t$ on an interval.

When $p > 1$, the metric is of metric horn type, and self-adjoint extensions of D_p on M (given by boundary conditions at $x = 0$) are in 1-to-1 correspondence with Lagrangian subspace in $\ker(D_{(\partial M,h)})$ with respect to the symplectic form

$$\omega(u, v) := \langle c(\nu)u, v\rangle_{L^2(\partial M,\Sigma\otimes E)},$$

see [17]. Such subspaces exist by the cobordism invariance of the index (note that ∂M may be disconnected). Moreover, since Assumption 1 does not hold, there exist infinitely many Lagrangian subspaces in $\ker(D_{(\partial M,h)})$ thus D_p is not essentially self-adjoint.

We work with the operator A from (12), which is unitarily conjugated to D_p hence has the same spectrum as D_p. When $p > 1$, for each Lagrangian subspace $W \subset \ker(D_{(\partial M,h)})$, A is essentially self-adjoint on the initial domain

$$\mathcal{D}_W(A) = \mathcal{C}_c^\infty(M, \Sigma \otimes E) \oplus \{\phi(x)x^{p/2}w; w \in W\}$$

for some fixed cut-off function ϕ supported in the cusps which equals 1 near infinity. When $0 < p \leq 1$, A is essentially self-adjoint on $\mathcal{D}(A) = \mathcal{C}_c^\infty(M, \Sigma \otimes E)$.

The essential spectrum of A (with the above boundary condition when $p > 1$) can be computed on the complement of any compact set in M, i.e., on the union of the cusps, by imposing self-adjoint boundary conditions. More precisely, consider the non-compact manifold with boundary $M_\epsilon := \{x \leq \epsilon\}$. We need to specify a self-adjoint boundary condition for A at $x = \epsilon$, which is obtained by the APS condition and by choosing yet another Lagrangian subspace in $\ker(D_{(\partial M,h)})$. With these self-adjoint boundary conditions, the decomposition principle (see [4, Prop. 1]) states that the essential spectrum of A on M_ϵ coincides with the essential spectrum of A on M.

We decompose the space of L^2 spinors on ∂M twisted by E into the space of zero-modes (i.e., the kernel of $D_{(\partial M,h)}$) and its orthogonal complement consisting of "high energy modes". Accordingly we get an orthogonal decomposition of $L^2(M_\epsilon, \Sigma \otimes E)$ into zero-modes and high energy modes, the main point being that A preserves this decomposition. As in [13, Prop. 5.1] the high energy modes do not contribute to the essential spectrum. The reason is that there exists a cusp pseudodifferential operator $R \in x^{-2p}\Psi_c^{-\infty}(M, \Sigma \otimes E)$, localized on the cusps and acting as 0 on high energy modes, such that $A^2 + R$ is fully elliptic. Therefore this operator has discrete spectrum so in particular, on high energy modes A^2 has discrete spectrum as claimed. We are left with the formally self-adjoint operator

$$A_0 := x^{1-p}c(\nu)\left(x\partial_x - \tfrac{p}{2}\right)$$

acting in $L^2(0,\epsilon) \otimes \ker(D_{(\partial M,h)})$ with respect to the volume form $\frac{dx}{x^2}$ (with a certain boundary condition at $x = \epsilon$).

We claim that A_0 is unitarily equivalent to $c(\nu)t\partial_t$ over a certain interval depending on ϵ and p, with respect to the measure $\frac{dt}{t}$. We start by conjugating with $x^{1/2}$, so the volume form becomes $\frac{dx}{x}$ and

$$x^{-\frac{1}{2}}A_0 x^{\frac{1}{2}} = x^{2-p}c(\nu)\partial_x + \frac{c(\nu)}{2}(1-p)x^{1-p}.$$

For $p = 1$, we already obtain the desired expression for our operator by setting $t := x$. For $p \neq 1$, we write

$$x^{2-p}\partial_x = y^2\partial_y, \qquad\qquad y := (1-p)x^{1-p}.$$

Then, after conjugating with $y^{-1/2}$, we obtain the operator $c(\nu)y^2\partial_y$ acting in L^2 with respect to the measure $\frac{dy}{y^2}$. With the change of variable *cusp-to-b*

$$t := e^{-1/y}$$

we get the desired operator.

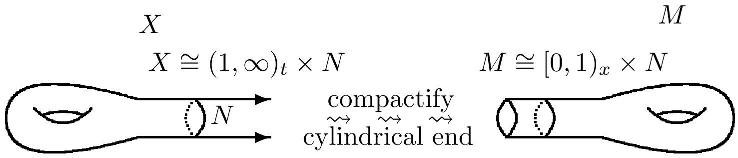
Now for $p \leq 1$ the operator $c(\nu)t\partial_t$ acts on an interval of the form $(0, \beta)$, while for $p > 1$ it acts on $(1, \beta)$ for some strictly positive β. Here $c(\nu)$ is a diagonalizable automorphism with $\pm i$ eigenvalues. For every self-adjoint extension, in the first case the spectrum is $\mathbb{R}$ while in the second case it is discrete. □

Alternatively, one could prove the first part of the theorem similarly to [4] by constructing Weyl sequences for each real number.

Although for $p > 1$ the spectrum of any self-adjoint extension of D_p is discrete even when Assumption 1 does not hold, the methods of this paper do not show the meromorphic extension of the eta function in that case.

Appendix B. Elements of the cusp calculus

In this appendix we give a short introduction to the cusp calculus (first defined by Melrose and Nistor [21]). Consider a Riemannian manifold X with a cylindrical end as shown in the left-hand side of Figure 1. The metric takes the form $dt^2 + h$ on the cylinder



$$x = 1/t \Longleftrightarrow t = 1/x$$

FIGURE 1. Compactifying X into M.

where h is a metric on the cross section N. Changing coordinates to $x = 1/t$ and noting that for $t \in (1, \infty)$ we have $x \in (0, 1)$, and $t \to \infty$ implies $x \to 0$, it follows that we can view X as the interior of the *compact* manifold M obtained from X by replacing the infinite cylinder $(1, \infty)_t \times N$ with the finite cylinder $[0, 1)_x \times N$, see Figure 1.

A cusp pseudodifferential operator is just a "usual" pseudodifferential operator on X that "behaves nicely" near $t = \infty$. To make this precise, recall that in local coordinates on the cylinder, the Schwartz kernel of an m-th order ($m \in \mathbb{C}$) pseudodifferential operator A on X takes the form

$$\kappa_A = \int e^{i(t-t')\tau + i(y-y')\cdot\eta}\, a(t, y, \tau, \eta)\, đ\tau đ\eta, \tag{26}$$

where $(t, t', y, y') \in (1, \infty)^2 \times \mathcal{U}^2$ with $\mathcal{U}$ local coordinates on N and a is a classical symbol of order m in τ and η. We say that the operator A is, by definition, an m-th order *cusp pseudodifferential operator* if in the compactified coordinates,

$$\tilde{a}(x, y, \tau, \eta) := a(1/x, y, \tau, \eta)$$

is smooth at $x = 0$ (and is still classical in τ and η). This definition only works on the cylinder, so to be more precise, A is a *cusp operator* if it can be written as $A_1 + A_2 + A_3$, where A_1 is of the form (26) such that $\tilde{a}(x, y, \tau, \eta) := a(1/x, y, \tau, \eta)$ is smooth at $x = 0$, A_2 is a usual pseudodifferential operator on the compact part of X, and finally, where A_3 is a smoothing operator on X that vanishes, with all derivatives, at ∞ on the cylinder (or equivalently, the Schwartz kernel of A_3 is a smooth function on M^2 vanishing to infinite order at $\partial(M^2)$). The space of cusp pseudodifferential operators of order $m \in \mathbb{C}$ is denoted Ψ_c^m. If the symbol a is polynomial in τ, η, the resulting operator is differential, and can be written near $x = 0$ as sums of compositions of partial differentials on N and of $\partial_t = -x^2\partial_x$ with smooth coefficients on the compactification M. The space of cusp differential operators of order $m \in \mathbb{Z}$ is denoted Diff_c^m.

Cusp operators are usually presented geometrically in relation to blown-up spaces, which might obscure their straightforward definition, so we shall explain this relationship. Setting $z = (t - t', y - y')$, which is a *normal* coordinate to the set $\{z = 0\} = \{t = t', y = y'\} =$ diagonal in X^2, we see from (26) that κ_A is written as the inverse Fourier transform of a symbol using the normal coordinate z. Hence κ_A is a distribution on X^2 that is, by definition, *conormal* to the diagonal in X^2. Expressing the kernel (26) in the compactified

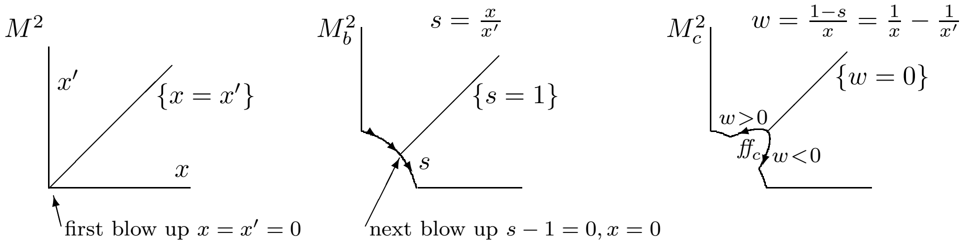


FIGURE 2. The blown-up manifold M_c^2. (We omit the N^2 factor.)

coordinates, we obtain

$$\kappa_A = \int e^{iz\cdot(\tau,\eta)}\, \tilde{a}(x,y,\tau,\eta)\, đ\tau đ\eta\ , \quad \text{where } z = \Big(\frac{1}{x} - \frac{1}{x'}, y - y'\Big). \tag{27}$$

Note that z is not a normal coordinate to the diagonal (given by $\{x = x', y = y'\}$) in M^2 because the coordinate $\frac{1}{x} - \frac{1}{x'}$ fails to be smooth at the corner $x = x' = 0$. Thus, it seems like switching to compactified coordinates destroys the conormal distribution portrayal of pseudodifferential operators. However, we now show how to interpret this kernel as being conormal, not in M^2, but on a related blown-up manifold. The idea is to blow-up the singular point $x = x' = 0$ until the kernel (27) can be interpreted as conormal. To begin this program, we first write

$$M^2 \cong [0,1)_x \times [0,1)_{x'} \times N^2$$

near the corner $\{x = x' = 0\}$ as shown pictorially on the left in Figure 2. Next, we introduce polar coordinates (r,θ) in the x, x' variables, where $r = \sqrt{x^2 + (x')^2}$ and $\theta = \arctan(x'/x)$. Geometrically we can think of introducing polar coordinates as "blowing up" the set $\{x = x' = 0\}$ by replacing it with a quarter-circle (the angular θ coordinate). The resulting manifold is called the *b-double space* M_b^2, see the middle picture in Figure 2. Actually, instead of using the standard polar coordinates (r,θ), in the sequel it is helpful to use the projective coordinates (x, s) where $s = x/x'$, which can be used as coordinates instead of (r,θ) for θ away from 0 and $\pi/2$. Here, x represents the radial coordinate and s represents the angular coordinate along the quarter circle.

Now we form the *cusp double space* M_c^2. To do so, we introduce polar coordinates where $s = 1$ and $x = 0$, in M_b^2, as shown in Figure 2. This blow-up geometrically replaces the set $s = 1, x = 0$ in M_b^2 with a half circle, which is called the *cusp front face* and which we denote by ff_c. Since the set $s = 1, x = 0$ is the set of points where $1 - s = 0$ and $x = 0$, we can use the projective coordinate

$$w = \frac{1-s}{x} = \frac{1}{x} - \frac{1}{x'}$$

as an angular coordinate along ff_c and we can use x as the radial variable, at least if we stay away from the extremities of ff_c. Thus, (x, w) can be used as coordinates near the blown-up face ff_c. Note that the set $\{x = x'\}$ corresponds to the set $\{w = 0\}$ in M_c^2, therefore w is a normal coordinate to $\{x = x'\}$. Moreover, in view of the formula (27), we see that κ_A is, by definition, a distribution conormal to the set $\{w = 0, y = y'\}$ in M_c^2, which is called the *cusp diagonal.* In fact, one can prove the following theorem.

Theorem 17. *The Schwartz kernels of cusp pseudodifferential operators are in one-to-one correspondence with distributions on M_c^2 that are conormal to the cusp diagonal and vanish to infinite order at all boundary hypersurfaces of M_c^2 except the cusp front face where they are smooth.*

Our original definition of a cusp pseudodifferential operator as presented after (26) is usually disregarded in favor of the more geometric definition presented in the above theorem. It is evident how to extend the definition from the scalar case to cusp operators acting on sections of vector bundles which are of product type in t near $t = \infty$.

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