Letters in Mathematical Physics **52:** 329–337, 2000. 


# Relative $\zeta$-Determinant and Adiabatic Decomposition of the $\zeta$-Determinant of the Dirac Laplacian

JINSUNG PARK[1] and KRZYSZTOF P. WOJCIECHOWSKI[2]
[1] *Korea Institute for Advanced Study, Seoul, Republic of Korea. e-mail: jspark@ns.kias.re.kr*
[2] *Department of Mathematics, IUPUI (Indiana/Purdue), Indianapolis, IN 46202–3216, U.S.A. e-mail: kwojciechowski@math.iupui.edu*

(Received: 4 April 2000)

**Abstract.** In this Letter we announce some results on decomposition formulas for the $\zeta$-determinant of the Dirac Laplacian, which hold in the adiabatic form.

**Mathematics Subject Classifications (2000):** 58J52, 58J32, 58Z05, 81Q70, 35J55.

**Key words:** Dirac operator, $\zeta$-function, $\zeta$-determinant, relative $\zeta$-determinant, self-adjoint elliptic boundary problem, adiabatic decomposition of the $\zeta$-determinant.

## 0. Statement of the Results

Let $\mathcal{D}\colon C^\infty(M;S) \to C^\infty(M;S)$ denote a compatible Dirac operator acting on sections of a bundle of Clifford modules $S$ over a closed manifold $M$. Assume that we have a decomposition of $M$ as $M_1 \cup M_2$, where $M_1$ and $M_2$ are compact manifolds with boundary such that

$$M = M_1 \cup M_2, \qquad M_1 \cap M_2 = Y = \partial M_1 = \partial M_2 \ . \tag{0.1}$$

The $\zeta$-determinant of the operator $\mathcal{D}$ is given by the formula

$$\det{}_\zeta \mathcal{D} = e^{\frac{i\pi}{2}(\zeta_{\mathcal{D}^2}(0) - \eta_{\mathcal{D}}(0))} e^{-\frac{1}{2}\zeta'_{\mathcal{D}^2}(0)} \tag{0.2}$$

(see [17]). The value of the function $\zeta_{\mathcal{D}^2}(s)$ at $s = 0$ is a local invariant in the sense that it is given by a formula $\zeta_{\mathcal{D}^2}(0) = \int_M a(x)\mathrm{d}x$, where $a(x)$ is a density determined at $x$ by the coefficients of the operator $\mathcal{D}$ at the point $x$ (see for instance [6]). This is the reason why the index of an elliptic differential operator, which can be viewed as the difference of the values of two different $\zeta$-functions determined by the operator, has a nice decomposition corresponding to the decomposition of the manifold.

Another contribution to the phase of $\det_\zeta \mathcal{D}$ is the eta-invariant $\eta_{\mathcal{D}}(0)$. This is not a local invariant (see [2]), hence at first sight it is difficult to expect a decomposition formula. It is therefore rather surprising that such a formula actually exists.

In the following, we concentrate on the odd-dimensional case ($\dim M = 2k + 1$). We further assume that $M$ and the operator $\mathcal{D}$ have product structures in a neighborhood of the boundary $Y$. More precisely, we assume that there is a bicollar neighborhood $N = [-1, 1] \times Y$ of $Y$ in $M$ such that both the Riemannian structure on $M$ and the Hermitian structure on $S$ are products when restricted to $N$. This implies that $\mathcal{D}$ has the following form when restricted to the submanifold $N$

$$\mathcal{D} = G(\partial_u + B). \tag{0.3}$$

Here $u$ denotes a normal variable, $G : S|Y \to S|Y$ is a bundle automorphism and $B$ is a corresponding Dirac operator on $Y$. Moreover, $G$ and $B$ do not depend on $u$ and they satisfy

$$G^* = -G, \qquad G^2 = -\mathrm{Id}, \qquad B = B^* \quad \text{and} \quad GB = -BG \tag{0.4}$$

The operator $B$ has a discrete spectrum with infinitely many positive and infinitely many negative eigenvalues. In the following we assume that dim ker $B = \{0\}$. This simplifies the exposition and allows us to avoid a discussion of technicalities. Let $\Pi_>$ denote the spectral projection onto the subspace spanned by the eigensections of $B$ corresponding to the positive eigenvalues. Then $\Pi_>$ is an elliptic boundary condition for $\mathcal{D}_2 = \mathcal{D}|M_2$ (see [1]; see [3] for an exposition of the theory of elliptic boundary problems for Dirac operators). In fact, any orthogonal projection satisfying

$$-GPG = \mathrm{Id} - P \quad \text{and} \quad P - \Pi_> \quad \textit{is a smoothing operator} \tag{0.5}$$

is a self-adjoint elliptic boundary condition for the operator $\mathcal{D}_2$. This means that the associated operator

$$(\mathcal{D}_2)_P : \mathrm{dom}\,(\mathcal{D}_2)_P \to L^2(M_2; S|M_2)$$

with dom $(\mathcal{D}_2)_P = \{s \in H^1(M_2; S|M_2); P(s|Y) = 0\}$ is a self-adjoint Fredholm operator with $\ker((\mathcal{D}_2)_P) \subset C^\infty(M_2; S|M_2)$ and discrete spectrum (see [16]). We denote by $\mathcal{G}r^*_\infty(\mathcal{D}_2)$ the space of P satisfying (5).

The existence of the meromorphic extensions of the functions $\eta_{(\mathcal{D}_2)_P}(s)$, $\zeta_{(\mathcal{D}_2)^2_P}(s)$ to the whole complex plane and their nice behaviour in a neighborhood of $s = 0$ was proved by the second author in late 1994 and announced in the AMS meeting in San Francisco in January 1995. The reader can find the full exposition in [21]. The results of [21] were later generalized by Gerd Grubb [7].

Let us observe that $\mathrm{Id} - P \in \mathcal{G}r^*_\infty(\mathcal{D}_1)$ if $P$ is an element of $\mathcal{G}r^*_\infty(\mathcal{D}_2)$. Let $\eta(G(\partial_u + B); P_1, P_2)(s)$ denote the $\eta$-function of the operator $G(\partial_u + B)$ on $[0, 1] \times Y$ subject to the boundary condition $P_2$ at $u = 0$ and $Id - P_1$ at $u = 1$. We have the following pasting formula proved in [21]

$$\eta_{\mathcal{D}}(0) = \eta_{(\mathcal{D}_1)_{\mathrm{Id}-P_1}}(0) + \eta_{(\mathcal{D}_2)_{P_2}}(0) + \eta(G(\partial_u + B); P_1, P_2)(0) \mod \mathbb{Z}. \tag{0.6}$$

A similar formula for finite-dimensional perturbations of $\Pi_>$ has been discussed by several authors (see [19, 20] and references therein).

The proof of (0.6) offered by the second author goes as follows.

First, we replace the bicollar $N$ by $N_R = [-R, R] \times Y$. Now $\eta_{\mathcal{D}}(0)$, which can be expressed via a suitable heat-kernel formula, splits into contributions coming from each side, plus the cylinder contribution and error terms. The error terms disappear as $R \to \infty$.

Second, though $\eta_{\mathcal{D}}(0)$ is not local, its variation (for instance with respect to the parameter $R$) is local and therefore the value of the contributions do not vary with $R$. This fact and the explicit calculation of the invariant $\eta(G(\partial_u + B); P_1, P_2)(0) \bmod \mathbb{Z}$ proves the formula (0.6).

In this Letter we discuss the corresponding splitting formula for the modulus of the determinant of the Dirac operator $\mathcal{D}$

$$\det{}_\zeta \mathcal{D}^2 = \mathrm{e}^{-\frac{\mathrm{d}}{\mathrm{d}s}(\zeta_{\mathcal{D}^2}(s))|_{s=0}}.$$

We follow the strategy employed above. Step number one works as before. We split

$$\zeta'_{\mathcal{D}_R^2}(0) = \int_0^\infty \frac{1}{t} \mathrm{Tr}\ \mathrm{e}^{-t\mathcal{D}_R^2} \mathrm{d}t \tag{0.7}$$

into contributions coming from different submanifolds plus cylinder contributions and the error terms. Here $\mathcal{D}_R$ denotes the operator $\mathcal{D}$ on the manifold $M_R$ equal to the manifold $M$ with $N$ replaced by $N_R$.

Step two, however, cannot be repeated in this situation. The variation of the right side of (0.7) is not local and in fact it is not difficult to see that $\det_\zeta \mathcal{D}_R^2$ blows up as $R \to \infty$. We therefore study the ratio of the determinants

$$\frac{\det_\zeta \mathcal{D}_R^2}{\det_\zeta (\mathcal{D}_{1,R})^2_{\mathrm{Id}-\Pi_>} \det_\zeta (\mathcal{D}_{2,R})^2_{\Pi_>}}. \tag{0.8}$$

We use *Duhamel's Principle* as in [5] (see also [8]) in order to show that the expression in (8) is convergent as $R \to \infty$. The main result of this announcement is the following Theorem.

THEOREM 0.1. *Assume that* $\ker B = \{0\}$ *and that the operator* $\mathcal{D}_R$ *is an invertible operator for* $R$ *sufficiently large. Then the following formula holds:*

$$\lim_{R\to\infty} \frac{\det_\zeta \mathcal{D}_R^2}{(\det_\zeta (\mathcal{D}_{1,R})^2_{\mathrm{Id}-\Pi_>})(\det_\zeta (\mathcal{D}_{2,R})^2_{\Pi_>})} = 2^{-\zeta_{B^2}(0)}. \tag{0.9}$$

*Remark*. Related results were discussed by Piazza (see [10, 11]) in the context of the *b-calculus* introduced by Melrose. Piazza proved a formula that relates the numerator of the left side in (0.9) to the product of the *regularized* determinants on two manifolds with asymptotically cylindrical ends and the determinant of the Dirac operator on the hypersurface (see (4.11) in [10]). The authors hope to reprove Piazza's result on the splitting of determinant line bundles for the families of Dirac operators and on the additivity of the curvature of the determinant line bundle without using *b-calculus*. We also hope to study the relation with the splitting formulas presented in [12].

## 1. Heat Kernels, Relative $\zeta$-Determinant and Decomposition Formulas

The *Relative $\zeta$-determinant*, as discussed for instance in [15], is an important ingredient in our approach to the pasting formulas. The operator $\mathcal{D}$ on $M$ is in this Section understood as a pair $\{\mathcal{D}_1, \mathcal{D}_2\}$, with $\mathcal{D}_i = \mathcal{D}|M_i$ and acting on (the pair of) sections $(s_1, s_2)$, where $s_i \in C^\infty(M_i; S)$ satisfy the transmission condition $s_1|Y = s_2|Y$. The corresponding Atiyah–Patodi–Singer problem is the couple $\{\mathcal{D}_1, \mathcal{D}_2\}$ acting on $(s_1, s_2)$ , which now satisfies

$$(\mathrm{Id} - \Pi_<)(s_1|Y) = 0, \qquad \Pi_>(s_2|Y) = 0.$$

It was explained in [4] that there is a natural interpolation between the Atiyah–Patodi–Singer condition and the transmission condition given by the formula

$$\sin r\,\Pi_>(s_1|Y) = \cos r\,\Pi_>(s_2|Y), \quad \cos r\,\Pi_<(s_1|Y) = \sin r\,\Pi_<(s_2|Y), \tag{1.1}$$

where $0 \leqslant r \leqslant \pi/4$.

We follow [4] and combine (1.1) into a single boundary condition. To do that we consider the manifold $X = M \setminus Y$. This is a manifold with boundary $Z = Y \sqcup Y$ . The Dirac operator $\mathcal{D}$ in the collar $W = [0, 1] \times Z$ can be represented as

$$\begin{aligned}\tilde{\mathcal{D}} &= \begin{pmatrix} \Gamma(\partial_u + B) & 0 \\ 0 & \Gamma(-\partial_u + B) \end{pmatrix} \\ &= \begin{pmatrix} \Gamma & 0 \\ 0 & -\Gamma \end{pmatrix}\left(\partial_u + \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}\right) = \tilde{\Gamma}(\partial_u + \tilde{B}).\end{aligned}$$

To study the cylinder contribution we consider the operator $\tilde{\mathcal{D}}$ acting on $C^\infty([0, \infty) \times Z; \tilde{S})$ , where $\tilde{S}$ denotes a corresponding bundle of Clifford modules over $[0, 1] \times Z$ extended to the whole unbounded cylinder (we refer to [4] for more details). The bundle $\tilde{S}$ is a direct sum $S \oplus S$ . The condition (1.1) now corresponds to the boundary condition

$$\cos r\,\Pi_>(\tilde{B})(s|Y) = \sin r\,\tilde{I}\Pi_<(\tilde{B})(s|Y), \tag{1.2}$$

where $\tilde{I} : \tilde{S} \to \tilde{S}$ is the involution switching the first and second summand in $S \oplus S$. Next let us observe that the condition (1.2) is determined by the projection

$$P_r = \cos^2 r\,\Pi_>(\tilde{B}) + \sin^2 r\,\Pi_<(\tilde{B}) - \frac{1}{2}\sin(2r)\,\tilde{I}. \tag{1.3}$$

This led Brüning and Lesch to study a special class of elliptic self-adjoint conditions in order to obtain a general pasting law for the $\eta$-invariant. We refer to the beautiful paper [4] for a full exposition. Here we only present the calculations made in the special simple case of a Dirac operator on an odd-dimensional manifold, which we use in order to study adiabatic pasting of the $\zeta$-determinants. The key for us (and for Brüning and Lesch as well) is that we can explicitly construct the heat kernel of the corresponding operator $\exp(-t\tilde{\mathcal{D}}^2_{P_r})$. More generally we consider the boundary condition $P$ which satisfies the following conditions (fulfilled by $P_r$ in (1.3))

$$-\tilde{G}P\tilde{G} = \mathrm{Id} - P, \qquad [P; \tilde{B}^2] = 0. \tag{1.4}$$

We do not require that $P$ commutes with $\tilde{B}$, however we assume that

$$P\tilde{B}P = b|\tilde{B}|P, \tag{1.5}$$

for some constant $b > -1$. For instance, in the case of $P_r$ defined in (1.3) we have

$$P_r\tilde{B}P_r = \cos(2r)|\tilde{B}|P_r. \tag{1.6}$$

Now assume that we have given a boundary condition $P$ which satisfies (1.4) and (1.5). Let us consider the operator $\tilde{\mathcal{D}}$ on the cylinder $[0,\infty) \times Z$. Then $\mathcal{E}_P(t; u, v)$, the kernel of the operator $\exp(-t\tilde{\mathcal{D}}^2_P)$, is given by the formula (see Section 4 of [4])

$$\begin{aligned}\mathcal{E}_P(t; u, v) = {} & \frac{1}{\sqrt{4\pi t}}\left(\mathrm{e}^{-\frac{(u-v)^2}{4t}} + (\mathrm{Id} - 2P)\mathrm{e}^{-\frac{(u+v)^2}{4t}}\right)\mathrm{e}^{-t\tilde{B}^2} + \\ & + \frac{1}{\sqrt{\pi t}}(\mathrm{Id} - P)\int_0^\infty \mathrm{e}^{-\frac{(u+v+z)^2}{4t}}\tilde{B}^\perp(P)\mathrm{e}^{z\tilde{B}^\perp(P) - t\tilde{B}^2}\mathrm{d}z,\end{aligned}$$

where

$$\tilde{B}^\perp(P) = (\mathrm{Id} - P)\tilde{B}(\mathrm{Id} - P).$$

It follows from (1.7) that the $\zeta$-function of the operator $\tilde{\mathcal{D}}^2_P$ is not well-defined because the operator $\mathrm{e}^{-t\tilde{\mathcal{D}}^2_P}$ is not of trace class. However, the difference $\mathrm{e}^{-t\tilde{\mathcal{D}}^2_{P_1}} - \mathrm{e}^{-t\tilde{\mathcal{D}}^2_{P_2}}$, where $P_1$ and $P_2$ are Brüning–Lesch conditions, has a well-defined trace and we can study the relative $\zeta$-function

$$\zeta(\tilde{\mathcal{D}}^2; P_1, P_2)(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\mathrm{Tr}(\mathrm{e}^{-t\tilde{\mathcal{D}}^2_{P_1}} - \mathrm{e}^{-t\tilde{\mathcal{D}}^2_{P_2}})\mathrm{d}t. \tag{1.8}$$

This function has a nice meromorphic extension to the whole complex plane, regular in a neighborhood of $s = 0$, and we define a relative $\zeta$-determinant in the usual way

$$-\ln\det{}_\zeta(\tilde{\mathcal{D}}^2; P_1, P_2) = \frac{\mathrm{d}}{\mathrm{d}s}(\zeta(\tilde{\mathcal{D}}^2; P_1, P_2)(s))|_{s=0}. \tag{1.9}$$

Now we introduce the functions

$$\mathrm{er\,f\,c}(q) = \frac{2}{\sqrt{\pi}}\int_q^\infty \mathrm{e}^{-x^2}\mathrm{d}x,$$

and

$$\mathcal{F}_b(w) = \int_0^{\infty} \operatorname{erfc}(v) w \mathrm{e}^{-2bvw-w^2} dv.$$

Let $M\mathcal{F}_b(z)$ be the Mellin transform of $\mathcal{F}_b(w)$. We have

$$M\mathcal{F}_b(z) = \frac{1}{4b}\left[(1-(1-b^2)^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right) + \frac{2}{\sqrt{\pi}}(1-b^2)^{-\frac{z}{2}}\Gamma\left(\frac{z+1}{2}\right)\int_0^b (1-t^2)^{\frac{z}{2}-1}\mathrm{d}t\right],$$

for $0 < |b| < 1$ and

$$M\mathcal{F}_0(z) = \frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{z+1}{2}\right), \qquad M\mathcal{F}_1(z) = \frac{1}{4}\left[\Gamma\left(\frac{z}{2}\right) - \frac{2}{z\sqrt{\pi}}\Gamma\left(\frac{z+1}{2}\right)\right].$$

The properties of $M\mathcal{F}_b(z)$ are summarized in Lemma 3.3 in [4]. The most important point for us is the fact that $M\mathcal{F}_b(z)$ is meromorphic in $\mathbb{C}$ with simple poles at the points $z_k = -k$ for $k = 1, 2, 3, \ldots$ . It follows that those functions are holomorphic in a neighborhood of $s = 0$ . The straightforward computation shows that the relative $\zeta$-function is given by

$$\zeta(\tilde{\mathcal{D}}^2; P_1, P_2)(s) = \frac{2}{\Gamma(s)}\zeta_{B^2}(s)[b_2 M\mathcal{F}_{b_2}(2s) - b_1 M\mathcal{F}_{b_1}(2s)], \tag{1.10}$$

where $b_i$ is a constant $b$ from (1.5) corresponding to the projection $P_i$. The relative $\zeta$-function is a well-defined holomorphic function in a neighborhood of $s = 0$ . We now differentiate (1.10) at $s = 0$ in order to obtain the logarithm of the relative determinant. Let us recall that

$$\frac{1}{\Gamma(s)} = \frac{s}{1 + s\gamma + s^2 h(s)}$$

in a neighborhood of $s = 0$ , where denotes the Euler constant and $h(s)$ is holomorphic near $s = 0$ . We obtain the following Proposition giving an explicit formula for the relative $\zeta$-determinant of the Brüning–Lesch elliptic boundary problems

PROPOSITION 1.1. *The logarithm of the relative $\zeta$-determinant is equal to*

$$-\frac{\mathrm{d}}{\mathrm{d}s}(\zeta(\tilde{\mathcal{D}}^2; P_1, P_2)(s))|_{s=0} = 2\zeta_{B^2}(0)[b_1 M\mathcal{F}_{b_1}(0) - b_2 M\mathcal{F}_{b_2}(0)]. \tag{1.11}$$

It is not difficult to find the values of $M\mathcal{F}_b(0)$ we are interested in

$$M\mathcal{F}_0(0) = \frac{1}{2}, \quad M\mathcal{F}_1(0) = \frac{\ln 2}{2}$$

and

$$M\mathcal{F}_b(0) = \frac{1}{2b}\ln(1+b) \quad \text{for} \quad 0 < |b| < 1.$$

The smooth transmission condition corresponds to $b_1 = \cos\pi/2 = 0$ and the Atiyah–Patodi–Singer condition corresponds to $b_2 = \cos 0 = 1$ due to (1.6), which gives the main technical result of the Note.

THEOREM 1.2. *Let $\{P_r\}_{0\leqslant 1}$ denote a 1-parameter family of boundary conditions defined by (1.3). The following formula gives the relative ζ-determinant of $P_1$ and $P_0$*

$$\frac{\mathrm{d}}{\mathrm{d}s}(\zeta(\tilde{\mathcal{D}}^2; P_1, P_2)(s))|_{s=0} = \ln 2 \cdot \zeta_{B^2}(0)\ . \tag{1.12}$$

We employ *Duhamel's Principle* in order to deduce Theorem 0.1 from Theorem 1.2. The method, based on ideas of Singer [18], was worked out in [5] (see also [8]) and is now fairly standard. The key here is that the kernel $\mathcal{E}_P(t;(u,w),(v,z))$ satisfies the estimate

$$\|\mathcal{E}_P(t;(u,w),(v,z))\| \leqslant c_1 t^{-N} \mathrm{e}^{-c_2 t} \mathrm{e}^{-c_3\frac{(u-v)^2}{t}} \tag{1.13}$$

for some natural $N$ and positive constants $c_1, c_2, c_3$ . Once the estimate (1.13) is established there is no problem with following [5]. Details can be found in [9].

## 2. Concluding Remarks

We want to point out that Atiyah–Patodi–Singer conditions can be replaced by elliptic boundary conditions from the Grassmannian $\mathcal{G}r^*_\infty(\mathcal{D})$. Let us discuss the operator $\mathcal{D}$ on the cylinder $[0,\infty)\times Y$ and the boundary condition $P \in \mathcal{G}r^*_\infty(\mathcal{D})$. Once again the ζ-function of the operator $\mathcal{D}^2_P$ is not well-defined. However, if we consider two conditions $P_1, P_2 \in \mathcal{G}r^*_\infty(\mathcal{D})$, then the relative ζ-function is well-defined and we may consider the relative ζ-determinant

$$-\ln\det{}_\zeta(\mathcal{D}^2; P_1, P_2) = \frac{\mathrm{d}}{\mathrm{d}s}(\zeta(\mathcal{D}^2; P_1, P_2)(s))|_{s=0}.$$

Moreover, it was observed in [15], that we can use the results of [14] (see also [13]) to express $\det_\zeta(\mathcal{D}^2; P_1, P_2)$ as the Fredholm determinant of an elliptic pseudo-differential operator of the form Id *plus smoothing operator* on $Y$. We denote this determinant by $\det_{\mathbb{C}c}(\mathcal{D}; P_1, P_2)$. The corresponding pasting formula which involves the boundary condition $\mathrm{Id} - P_1$ on $\partial M_1$ and $P_2$ on $\partial M_2$ contains an additional factor $\det_{\mathbb{C}c}(\mathcal{D}; P_1, P_2)$.

To end this Letter let us discuss one more variant of the adiabatic pasting formula. The manifold $M$ is odd-dimensional, hence the spinor bundle splits on the boundary as $S|Y = S^{+} \oplus S^{-}$. Let $R_{\pm}$ denote a boundary condition $R_{\pm}(s|Y) = 0$. To be more precise, let us point out here that spinors of positive chirality (sections of $S^{+}$) on $Y = \partial M_2$ are spinors of negative chirality on $Y = \partial M_1$. We also denote by $\mathcal{D}_{\pm}$ the operator $\mathcal{D}$ subject to the boundary condition $R_{\pm}$. The operators $\mathcal{D}_{\pm}$ are not self-adjoint and in fact we have $\mathcal{D}_{\pm}^{*} = \mathcal{D}_{\mp}$. The following results follow easily from [8], where the corresponding formula for the splitting of the *analytic torsion* was discussed.

THEOREM 2.1.

$$\lim_{R\to\infty} \frac{\det_{\zeta} \mathcal{D}_R{}^2}{\det_{\zeta}(\mathcal{D}_{1,R_-}\mathcal{D}_{1,R_+})\cdot \det_{\zeta}(\mathcal{D}_{2,R_-}\mathcal{D}_{2,R_+})} = 1. \tag{2.1}$$

Once again we refer to [9] for the proof.

## References


1. Atiyah, M. F., Patodi, V. K. and Singer, I. M.: Spectral asymmetry and Riemannian geometry. I, *Math. Proc. Cambridge Philos. Soc.* **77** (1975), 43–69.
2. Atiyah, M. F., Patodi, V. K., and Singer, I. M.: Spectral asymmetry and Riemannian geometry. II, *Math. Proc. Cambridge Philos. Soc.* **78** (1975), 405–432.
3. Booß–Bavnbek, B. and Wojciechowski, K. P.: *Elliptic Boundary Problems for Dirac Operators*, Birkhäuser, Boston, 1993.
4. Brüning, J. and Lesch, M.: On the $\eta$-invariant of certain nonlocal boundary value problems, *Duke Math. J.* **96** (1999), 425–468.
5. Douglas, R. G. and Wojciechowski, K. P.: Adiabatic limits of the $\eta$–invariants. The odd–dimensional Atiyah–Patodi–Singer problem, *Comm. Math. Phys.* **142** (1991), 139–168.
6. Gilkey, P. B.: *Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theory*, CRC Press, Boca Raton, Florida, 1995.
7. Grubb, G.: Trace expansions for pseudodifferential boundary problems for Dirac-type operators and more general systems, *Ark. Mat.* **37** (1999), 45–86.
8. Klimek, S. and Wojciechowski, K. P.: Adiabatic cobordism theorems for analytic torsion and $\eta$–invariant, *J. Funct. Anal.* **136** (1996), 269–293.
9. Park, J. and Wojciechowski, K. P.: Relative $\zeta$-determinant and adiabatic pasting formula for the $\zeta$-determinat of the Dirac Laplacian. Preprint, 2000.
10. Piazza, P.: Determinant bundles, manifolds with boundary and surgery, I, *Comm. Math. Phys.* **178** (1996), 597–626.
11. Piazza, P.: Determinant bundles, manifolds with boundary and surgery, II, *Comm. Math. Phys.* **193** (1998), 105–124.
12. Scott, S. G.: Splitting the curvature of determinant line bundle, to appear in *Proc. Amer. Math. Soc.*
13. Scott, S. G. and Wojciechowski, K. P.: $\zeta$-determinant and the Quillen determinant on the Grassmannian of elliptic self-adjoint boundary conditions, *C.R. Acad. Sci. Paris Sér. I Math.*, **328** (1999), 139–144.
14. Scott, S. G. and Wojciechowski, K. P.: The $\zeta$-determinant and Quillen determinant for a Dirac operator on a manifold with boundary, *Geom. Funct. Anal.*, 1999, to appear.

15. Scott, S. G. and Wojciechowski, K. P.: Heat kernels, determinants and elliptic boundary problems, to appear in: *Noncommutative Differential Geometry and Its Applications to Physics,* Proc. Shonan Kokusaimura June 1999 Workshop.
16. Seeley, R. T.: Topics in pseudodifferential operators, In: Cremonese (ed.), *CIME Conference on Pseudo-differential Operators (Stresa 1968)*, Rome, 1969, pp. 167–305.
17. Singer, I. M.: Families of Dirac operators with applications to physics, *Asterisque, hors série*, (1985), 323–340.
18. Singer, I. M.: The $\eta$-invariant and the index, In: S.-T. Yau (ed.), *Mathematical Aspects of String Theory*, World Scientific, Singapore, 1988, pp. 239–258.
19. Wojciechowski, K. P.: The additivity of the $\eta$-invariant: The case of an invertible tangential operator, *Houston J. Math.* **20** (1994), 603–621.
20. Wojciechowski, K. P.: The additivity of the $\eta$-invariant. The case of a singular tangential operator, *Comm. Math. Phys.* **169** (1995), 315–327.
21. Wojciechowski, K. P.: The $\zeta$-determinant and the additivity of the $\eta$-invariant on the smooth, self-adjoint Grassmannian, *Comm. Math. Phys.* **201** (1999), 423–444.