JEFFREY-WEITSMAN-WITTEN INVARIANTS OF SEIFERT FIBRED MANIFOLDS

JINSUNG PARK

ABSTRACT. We introduce a result in [4] about Jeffrey-Weitsman-Witten invariants of Seifert fibred manifolds.

1. DEFINITION OF JEFFREY-WEITSMAN-WITTEN INVARIANT

We review the Chern-Simons gauge theory to understand the definition of the Jeffrey-Weitsman-Witten invariant. For the detail of the Chern-Simons gauge theory, see [5], [3].

Let X be a 2-dimensional manifold and P be a principal $SU(2)$-bundle over X. Let $\mathcal{A}, \mathcal{A}_F, \mathcal{G}$ be the affine space of connection one forms of P, the space of flat connections of P and the gauge transformation group of P respectively. Let $\mathcal{M}_2$ be the moduli space of the flat connections of P.

We consider a 3-dimensional manifold Y_1 with a boundary X. Moreover we assume that a neighborhood of X in Y_1 is diffeomorphic to $X \times [0,1)$. For $A \in \mathcal{A}, g \in \mathcal{G}$ we consider a $U(1)$-valued function $\mathcal{S}(A,g)$ defined by

$$\mathcal{S}(A,g) \equiv \exp(2\pi i(CS(\tilde{A}^{\tilde{g}}) - CS(\tilde{A}))) \tag{1}$$

where $\tilde{A}$ and $\tilde{g}$ are the extensions of A and g into Y_1, $\tilde{A}^{\tilde{g}}$ is the gauge transformation of $\tilde{A}$ by $\tilde{g}$ and the Chern-Simons invariant $CS(\tilde{A})$ is given by

$$CS(\tilde{A}) = \frac{1}{8\pi^2} \int_{Y_1} tr(d\tilde{A} \wedge \tilde{A} + \frac{2}{3} \tilde{A} \wedge \tilde{A} \wedge \tilde{A}). \tag{2}$$

Such an extension of $\tilde{g}$ always exists since $\pi_1(SU(2)) = \pi_2(SU(2)) = 0$. We choose the extensions so that $\tilde{A}$ and $\tilde{g}$ are pull-backs of A and g by the projection to X over $X \times [0,1)$ respectively. Then $\mathcal{S}$ is independent on the extension Y_1 and the extension $\tilde{A}$ and $\tilde{g}$. In fact the extensions $(\tilde{A}_1, \tilde{g}_1)$ and $(\tilde{A}_2, \tilde{g}_2)$ into Y_1 and Y_2 give a connection $\tilde{B}$ and a gauge transformation $\tilde{h}$ on $Y = Y_1 \cup Y_2$ so that

$$\begin{aligned} &\exp(2\pi i(CS(\tilde{A}_1^{\tilde{g}}) - CS(\tilde{A}_1))) \exp(2\pi i(CS(\tilde{A}_2^{\tilde{g}}) - CS(\tilde{A}_2)))^{-1} \\ &= \exp(2\pi i(CS(\tilde{B}^{\tilde{h}}) - CS(\tilde{B}))) = 1. \end{aligned} \tag{3}$$

The above function $\mathcal{S}$ over $\mathcal{A} \times \mathcal{G}$ is a cocycle since

$$\mathcal{S}(\tilde{A}, \tilde{g})\mathcal{S}(\tilde{A}^{\tilde{g}}, \tilde{h}) = \mathcal{S}(\tilde{A}, \tilde{g}\tilde{h}). \tag{4}$$

We can define a line bundle $\mathcal{L}$ over $\mathcal{M}_2$ by

$$\mathcal{L} \equiv \mathcal{A}_F \times_{\mathcal{S}} C \tag{5}$$

where the right side is the quotient space given by the equivalence relation

$$(A, z) \sim (A^g, \mathcal{S}(A, g)z) \tag{6}$$

for $A \in \mathcal{A}_F, z \in C$.

We consider a 3-dimensional manifold Y and a principal $SU(2)$-bundle P_Y over Y. We decompose Y into two handle bodies Y_1 and Y_2. Let X be the intersection of Y_1 and Y_2. We apply above construction to 2-dimensional manifold X and $P_Y|_X = P$. We consider the restriction of line bundle $\mathcal{L}$ to Lagrangian submanifolds L_1, L_2 of $\mathcal{M}_2$ where L_1, L_2 are made from the handle bodies Y_1, Y_2 of Y. Let $\mathcal{L}_i$ be the restriction of the line bundle $\mathcal{L}$ to L_i. Then there is section $\mathcal{S}_i(A)$ of $\mathcal{L}_i$ over L_i defined by

$$\mathcal{S}_i(A) = \exp(2\pi i CS(A_{Y_i})) \tag{7}$$

for $[A] \in \mathcal{M}_2$ and A_{Y_i} is an extension of A to Y_i.

Now we consider the intersection of the two Lagrangian submanifolds L_1, L_2 in $\mathcal{M}_2$. Then we can see that this intersection is the moduli space of flat connections of P_Y over Y which we denote by $\mathcal{M}_3$. By the correspondence between the flat connection A and the $SU(2)$ -representation ρ_Y of $\pi_1(Y)$, $\mathcal{M}_3$ can be identified with $R(Y, SU(2))$. There may occur singularities of $\mathcal{M}_2$ within $\mathcal{M}_3$. But the set of singularities is a measure zero set in $\mathcal{M}_3$. So we may not care these singularities in the following construction since we shall integrate over the dense subset $R(Y, SU(2))^-$ of $\mathcal{M}_3 = R(Y, SU(2))$.

We consider the k-tensor power of $\mathcal{L}$, $\mathcal{L}^{\otimes^k}$ over $\mathcal{M}_2$ and their restrictions to two Lagrangian submanifolds L_1, L_2. We denote these by $\mathcal{L}_1^{\otimes^k}, \mathcal{L}_2^{\otimes^k}$. Then we can pair two sections $\mathcal{S}_i^k = \mathcal{S}_i^{\otimes^k}$ of $\mathcal{L}_i^{\otimes^k}$ in $\mathcal{M}_3$ by the hermitian product of the complex line C. We denote this pairing by $\langle \mathcal{S}_1^k, \mathcal{S}_2^k \rangle$. Then this can be considered as an $U(1)$ valued function on $\mathcal{M}_3 = R(Y, SU(2))$. We can see easily that $\langle \mathcal{S}_1^k, \mathcal{S}_2^k \rangle$ at a connection A is the exponential of the Chern-Simons invariant of Y, that is, $\exp(2k\pi i CS(A))$.

We recall that the half density derived from the Reidemeister torsion (R-torsion) can be considered as a measure of $R(Y, SU(2))^-$. The Jeffrey-Weitsman-Witten invariant is defined by integrating the pairing $\langle \mathcal{S}_1^k, \mathcal{S}_2^k \rangle$

with respect to half density $\tau(Y, Ad(\rho_Y))^{\frac{1}{2}}$ over $R(Y, SU(2))^-$ using the correspondence between the flat connection A of P and the $SU(2)$-representation ρ_Y of $\pi_1(Y)$. We formulate this construction as the following definition.

Definition 1.1. For integer k, the Jeffrey-Weitsman-Witten invariant $Z(Y, k)$ is defined by

$$Z(Y,k) = \int_{R(Y,SU(2))^-} \langle \mathcal{S}_1^k, \mathcal{S}_2^k \rangle \tau(Y, Ad(\rho_Y))^{\frac{1}{2}}. \tag{8}$$

This definition is given in [3]. This definition is motivated from the asymptotic expansion of the Witten invariant $Z_Y(k)$ of 3-dimensional manifold Y in [6]. The asymptotic expansion of $Z_Y(k)$ is given by

$$\begin{aligned} Z_Y(k) \simeq \tfrac{1}{2} \textstyle\sum_i (\tau(Y, A_i))^{\frac{1}{2}} \exp(-\tfrac{3\pi i + 2\pi i SF(A_i)}{4}) \\ \exp(2(k+2)\pi i CS(A_i)) \end{aligned} \tag{9}$$

where the sum is taken over the finite set of flat connections A_i, $\tau(Y, A_i)$ is the R-torsion for A_i of Y and $SF(A_i)$ is the spectral flow from trivial connection to the flat connection A_i. The above formula is given in [2]. We can see that if the moduli space of flat connections is a discrete set, then definition 5.1 is almost same as the leading term of the above asymptotic expansion since the square root of the R-torsion becomes a point mass in this case.

2. Jeffrey-Weitsman-Witten invariants for Seifert fibred manifolds

Now we introduce our result in [4] for Jeffrey-Weitsman-Witten invariant $Z(M, k)$ of Seifert fibred manifold M with the non-discrete irreducible $SU(2)$-representation space $R(M, SU(2))^-$.

We denote the Seifert fibred manifold with the Seifert invariant $\{g; (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\}$ by $M = M(g; (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m))$. We have that $\pi_1(M)$ is

$$\{a_i, b_i, q_j, h : [h, a_i] = [h, b_i] = [h, q_j] = 1, q_j^{\alpha_j} h^{\beta_j} = 1, \prod_{j=1}^{m} q_j \prod_{i=1}^{g} [a_i, b_i] = 1\}.$$

We assume that g is larger than 2 or equal to 2 from now on. The irreducible $SU(2)$-representation of $\pi_1(M)$ is well known. We review some facts that we need.

Since h is central in $\pi_1(M)$, an irreducible representation ρ takes h to ± 1 in $SU(2)$. So the trace of $\rho(q_j)$ is $2\cos(\frac{\pi n_j}{\alpha_j})$. The set of numbers $\{n_1, \ldots, n_m\}$ which are called as 'rotation numbers' determines a connected component of the irreducible $SU(2)$-representation space $R(M, SU(2))^-$. The rotation number n_j is even(odd) if β_j is even(odd).

To compute $Z(Y,k)$, we must integrate the pairing $\langle \mathcal{S}_1^k, \mathcal{S}_2^k\rangle$ with respect to the half density of the R-torsion over $R(M,SU(2))^-$. We know that the value $\langle \mathcal{S}_1^k, \mathcal{S}_2^k\rangle$ at the flat connection A is simply the Chern-Simons invariant of A. This invariant is constant in each connected component $R(M,(n_j))$. By the result of [1], the value for the fixed connected component $R(M,(n_j))$ is given by

$$\exp\left[2k\pi i\left(-\sum_{j=1}^{m}\left(\frac{\beta_j^* n_j^2}{\alpha_j}+\frac{2\epsilon n_j}{\alpha_j}\right)+\epsilon^2\sum_{j=1}^{m}\frac{\beta_j}{\alpha_j}\right)\right] \tag{11}$$

where $\epsilon = \frac{1}{2}, 1$ if $\rho_M(h) = -1, 1$ and $\beta_j\beta_j^* = 1$ (mod) α_j as above.

From the result in [4] for the half density volume of $R(M,SU(2))^-$ the value of the Jeffrey-Weitsman-Witten invariant $Z(Y,k)$ over the connected component $R(M,(n_i))$ which is determined by $\{n_1,\ldots,n_m\}$ is given by

$$\begin{aligned}&\int_{R(M,(n_j))}\langle \mathcal{S}_1^k, \mathcal{S}_2^k\rangle(\tau(M,Ad(\rho_M)))^{\frac{1}{2}}\\ &= 2^m\langle \mathcal{S}_1^k, \mathcal{S}_2^k\rangle\prod_{j=1}^{m}\frac{|\sin(\frac{\pi n_j\beta_j^*}{\alpha_j})|}{|\alpha_j|^{\frac{1}{2}}}\int_{R(M,(n_j))}D_{H_1(M,su(2)_{\rho_M})}.\end{aligned} \tag{12}$$

From (10), (11) we have the following theorem.

Theorem 2.1. *For the integer k, the Jeffrey-Weitsman-Witten invariant $Z(M,k)$ of the Seifert fibred manifold M with the Seifert invariant $(g,(\alpha_1,\beta_1,\cdots,\alpha_m,\beta_m))$ is given by*

$$\begin{aligned}c_M\sum_{\{(n_j)\}}\ &\exp\left[2k\pi i\left(-\sum_{j=1}^{m}\left(\frac{\beta_j^* n_j^2}{\alpha_j}+\frac{2\epsilon n_j}{\alpha_j}\right)+\epsilon^2\sum_{j=1}^{m}\frac{\beta_j}{\alpha_j}\right)\right]\\ &\times\prod_{j=1}^{m}\left[\frac{|\sin(\frac{\pi n_j\beta_j^*}{\alpha_j})|}{|\alpha_j|^{\frac{1}{2}}}\sum_{n=1}^{\infty}\left(\frac{1}{n^{2(g-1)+m}}\prod_{j=1}^{m}\frac{|\sin(\frac{\pi n n_j}{\alpha_j})|}{|\sin(\frac{\pi n_j}{\alpha_j})|}\right)\right]\end{aligned} \tag{13}$$

where the above sum is taken over the finite set of the rotation numbers $\{(n_j)\}$, $c_M = 2^{m-1}Vol(S^2)^m Vol(SU(2))^{g+m-1}$ and $\epsilon = \frac{1}{2}$ or 1 if $\rho_M(h) = -1$ or 1.

We can see that $Z(M,k)$ depends only on the manifold M via the Seifert invariant $(g;(\alpha_i,\beta_i))$ since the set of all the rotation numbers (n_j) is determined by $\pi_1(M)$.

References

[1] D. R. Auckly, *Topological methods to compute Chern-Simons invariants,* Math. Proc. Camb. Phil. Soc. **115** (1994), 229-251.

[2] D. S. Freed, R. E. Gompf, *Computer calculation of Witten's 3-manifold invariant,* Commun. Math. Phys. **141** (1991), 79-117.
[3] L. C. Jeffrey, J. Weitsman, *Half density Quantization of the moduli space of flat connections and Witten's semiclassical manifold invariants,* Topology **32** (1993), 509-529.
[4] J. Park, *Half-density volumes of representation spaces of some 3-manifolds and their application,* Duke Math. Jour. **86** (1997), 493-515.
[5] T. R. Ramadas, I. M. Singer and J. Weitsman, *Some Comments on Chern-Simons Gauge theory,* Commun. Math. Phys. **126** (1989), 409-420.
[6] E. Witten, *Quantum Field Theory and the Jones Polynomial,* Commun. Math. Phys. **121** (1989), 351-399.

School of Mathematics, Korea Institute for Advanced Study, 207-43 Cheongryangri-dong, Dongdaemun-gu, Seoul 130-012, Republic of Korea
E-mail address: `jspark@math.snu.ac.kr`