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ON THE ISOSPECTRA AND THE ISOMETRIES OF THE ALOFF-WALLACH SPACES

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Abstract. We use the branching rules on $SU(3)$ to show that if two Aloff-Wallach spaces $M_{k,l}$ and $M_{k',l'}$ are isospectral for the Laplacian acting on smooth functions, they are isometric. We also show that 1 is the non-zero smallest eigenvalue among all Aloff-Wallach spaces and compute the multiplicities.

1. Introduction

The Aloff-Wallach space $M_{k,l}$ is a 7-dimensional homogeneous space obtained from an S^1-action on $SU(3)$ for a pair of coprime integers (k,l). This family of homogeneous spaces gives an example of infinitely many non-homotopic spaces with the positive sectional curvature. In early 1990's, Kreck and Stolz discovered some pairs of the Aloff-Wallach spaces that are homeomorphic but not diffeomorphic (see [7]). To distinguish the homeomorphism types and diffeomorphism types, they used some invariants which are closely related to the eta invariants of some Dirac operators (see [6]). Recently Kruggel classified the homotopy types of this family of spaces (see [8]).

Since the Aloff-Wallach space is obtained from an S^1-action on $SU(3)$, this space admits the metric induced from the Killing form of $SU(3)$. In this case, the spectrum of the Laplacian can be computed from the action of the Casimir operator of $SU(3)$ on the irreducible representation spaces by the branching rules. The branching rule is the method which

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is used commonly on the calculation of the spectrum of the Laplacian on homogeneous spaces (c.f. [2], [4], [9]). Throughout this paper, we assume that each Aloff-Wallach space is equipped with this metric. We also mean by the Laplacian the Hodge Laplacian acting on smooth functions.

The purpose of this paper is to show that there is no pair of Aloff-Wallach spaces which are isospectral but not isometric. In other words, we show that if two Aloff-Wallach spaces are isospectral for the Laplacian acting on smooth functions, they are isometric.

The definition of the Aloff-Wallach space and the elementary facts of the homogeneous vector bundles are given in Section 2. In Section 3, we discuss the multiplicity function $d(k,l;m,n)$ and in Section 4, we prove the following theorem by considering the small eigenvalues of $M_{k,l}$.

THEOREM 1.1. *For two Aloff-Wallach spaces $M_{k,l}$ and $M_{k',l'}$, the following three statements are equivalent to each other.*

(1) *$M_{k,l}$ and $M_{k',l'}$ are isospectral for the Laplacian acting on smooth functions.*
(2) *$M_{k,l}$ is isometric to $M_{k',l'}$.*
(3) *(k',l') is the one of*

$$\begin{aligned}&\pm(k,l),\ \pm(l,k),\ \pm(k,-(k+l)),\ \pm(-(k+l),k),\\&\pm(l,-(k+l)),\ \pm(-(k+l),l).\end{aligned}$$

In the above theorem, the implications of (2) to (1) and of (3) to (2) are trivial. The implication of (1) to (3) is the main part of this theorem.

REMARK. Kreck and Stolz showed in [7] that $M_{-4638661,582656}$ and $M_{-2594149,5052965}$ are diffeomorphic. The above theorem tells that these two spaces are not isometric with respect to the metrics induced from the Killing form.

2. The Aloff-Wallach spaces and the homogeneous vector bundles

In this section we explain the basic facts of the homogeneous vector bundles and introduce the Aloff-Wallach spaces.

Let G be a compact connected Lie group and H be a compact connected subgroup of G. We denoted by $\mathfrak{g},\mathfrak{h}$ the Lie algebras of G,H, respectively. Then $M = G/H$ is a compact homogeneous space and the tangent bundle TM is a homogeneous vector bundle, which is given by

$$TM = G \times_H \mathfrak{g}/\mathfrak{h},$$

where the Lie group H acts on $\mathfrak{g}/\mathfrak{h}$ by the adjoint representation. TM also inherits an inner product induced from the killing form on $\mathfrak{g}$.

The space of smooth p-forms $\Gamma(\wedge^p T^*M)$ is an infinite dimensional G-representation space. Let $\hat{G}$ denote the set of all equivalence classes of irreducible representations of G. For each γ in $\hat{G}$, let (π_γ, V_γ) be a representative of γ. Then by the Peter-Weyl theorem we get

$$\Gamma(\wedge^p T^*M) \supset \oplus_{\gamma \in \hat{G}} V_\gamma \otimes Hom_G(V_\gamma, \Gamma(\wedge^p T^*M)), \tag{2.1}$$

where the right hand side is dense in the left hand side.

By the Frobenious law, there is an isomorphism

$$Hom_G(V_\gamma, \Gamma(\wedge^p T^*M)) \cong Hom_H(V_\gamma, \wedge^p(\mathfrak{g}/\mathfrak{h})^*). \tag{2.2}$$

Let Δ_p denote the Bochner Laplacian on $\Gamma(\wedge^p T^*M)$ for $0 \leq p \leq n$. With respect to an orthonormal frame $\{e_i\}$ on $TM\,|_U$ for some open subset U of M

$$\Delta_p = \sum_i (\bigtriangledown^2_{e_i} - \bigtriangledown_{\bigtriangledown_{e_i} e_i}),$$

where $\bigtriangledown$ is the Levi-Civita connection with respect to the metric induced from the Killing form.

Let $\{X_i\}$ be an orthonormal basis of the Lie algebra $\mathfrak{g}$ with respect to the inner product induced from the Killing form. The Casimir operator Cas_G is defined as the element of the universal enveloping algebra of $\mathfrak{g}$ such that

$$Cas_G = -\sum_i X_i \cdot X_i.$$

Similarly Cas_H is defined on H as above.

Then it is well-known (c.f. [3], [4]) that

$$\Delta_p = Cas_G + Cas_H,$$

and that the Casimir operator Cas_G acts on the irreducible representation space V_γ as the scalar multiplication given by

$$Cas_G|_{V_\gamma} = \langle \rho_G + \Lambda_\gamma, \rho_G + \Lambda_\gamma \rangle - \langle \rho_G, \rho_G \rangle,$$

where ρ_G is the half sum of the positive roots of G, Λ_γ is the highest weight of γ and the inner product $\langle , \rangle$ is induced from the Killing form.

The Casimir operator Cas_H acts as an endomorphism of $\wedge^p(\mathfrak{g}/\mathfrak{h})^*$ via the adjoint action of H.

In particular, if $p = 0$, Cas_H acts as a zero endomorphism.

By the Weitzenböck formula we can decompose the Hodge Laplacian $(d + d^*)^2$ by

$$(d + d^*)^2 = \Delta_p + E_p,$$

where E_p is the zero order operator which vanishes for $p = 0$.

Hence if $p = 0$, we get

$$(d + d^*)^2 = Cas_G.$$

Now we introduce the Aloff-Wallach spaces.

Define an embedding $i_{k,l} : S^1 \to SU(3)$ for $k,\ l \in Z$ by

$$\exp(2\pi i\theta) \longrightarrow \begin{pmatrix} \exp(2\pi ik\theta) & 0 & 0 \\ 0 & \exp(2\pi il\theta) & 0 \\ 0 & 0 & \exp(-2\pi i(k+l)\theta) \end{pmatrix}.$$

Then the Aloff-Wallach space $M_{k,l}$ is defined by $SU(3)/i_{k,l}(S^1)$, where $(k,l) = 1$ and $kl(k+l) \neq 0$.

It is shown in [1] that

$$H^4(M_{k,l}, Z) \cong Z/rZ,$$

where $r = |k^2 + kl + l^2|$.

Thus there are infinitely many different homotopy types among $M_{k,l}$'s.

Moreover, there are pairs $(k,l), (k',l')$ such that $M_{k,l}$ is homeomorphic but not diffeomorphic to $M_{k',l'}$.

For example $(-56788, 5227)$ and $(-42652, 61213)$ are such pairs, which were discovered by Kreck and Stolz in [7].

By the definition of the Aloff-Wallach spaces, we can see easily that $M_{k,l}$ is isometric to $M_{l,k}, M_{k,-(k+l)}, M_{-(k+l),l}$ by the orientation preserving isometry which are given from the permutation of the diagonal elements.

Similarly $M_{k,l}$ is isometric to $M_{-k,-l}$ by the orientation reversing isometry induced from the complex conjugation.

Thus it is enough to consider $M_{k,l}$ for $1 \leq k \leq l$.

Now we consider the homogeneous vector bundle structure on $M_{k,l}$.

Let α, β, ρ denote the positive roots of the adjoint representation of the Cartan subalgebra $\mathfrak{t}_{SU(3)}$.

Every irreducible representation of $SU(3)$ is determined by the highest weight $m\sigma + n\tau$, where m, n are nonnegative integers and $\sigma = \frac{1}{3}(2\alpha + \beta), \tau = \frac{1}{3}(\alpha + 2\beta)$ (*c.f.* [3]). We denote by $V_{m\sigma+n\tau}$ the representation space determined by the highest weight $m\sigma + n\tau$.

Then $Cas_{SU(3)}$ acts on $V_{m\sigma+n\tau}$ as the multiplication of $\frac{1}{9}(m^2 + n^2 + mn + 3m + 3n)$ and the dimension of $V_{m\sigma+n\tau}$ is $\frac{1}{2}(m+1)(n+1)(m+n+2)$. We denote by $d(k, l; m, n)$ the dimension of

$$Hom_{i_{k,l}(S^1)}(V_{m\sigma+n\tau}, \wedge^0(\mathfrak{g}/\mathfrak{h})^*) = Hom_{i_{k,l}(S^1)}(V_{m\sigma+n\tau}, \mathbb{C}).$$

Then the dimension of $V_{m\sigma+n\tau} \otimes Hom_G(V_{m\sigma+n\tau}, \Gamma(\wedge^0 T^* M))$ is $\frac{1}{2}(m+1)(n+1)(m+n+2)d(k, l; m, n)$ and $(d + d^*)^2$ acts on this space as the multiplication of $\frac{1}{9}(m^2 + n^2 + mn + 3m + 3n)$.

Note that $d(k, l; m, n)$ may be zero. In this case $V_{m\sigma+n\tau}$ does not appear in the right hand side of (2.1).

PROPOSITION 2.1. *Define* $\phi : (\mathbb{Z}^+ \cup \{0\}) \times (\mathbb{Z}^+ \cup \{0\}) \to \mathbb{R}$ *by* $\phi(m, n) = \frac{1}{9}(m^2 + n^2 + mn + 3m + 3n)$. *For* $\lambda \in Im\phi$, *let* $\phi^{-1}(\lambda) = \{(m_1, n_1), \cdots, (m_r, n_r)\}$. *Then* λ *is an eigenvalue of the Laplacian on* $M_{k,l}$ *with the multiplicity* $\sum_{i=1}^r d(k, l; m_i, n_i) \cdot \frac{1}{2}(m_i + 1)(n_i + 1)(m_i + n_i + 2)$ *and all the eigenvalues are of this type.*

REMARK.

(1) ϕ may have the same value on different pairs of coprime integers. For example, $\phi(9, 11) = \phi(1, 17) = \frac{361}{9}$.

(2) $d(k, l; m, n)$ may be 0 for all i. In this case, λ is, in fact, not an eigenvalue of $M_{k,l}$. For instance, if $k = l = 1$ and $m = 1$, $n = 2$, then $\lambda = \frac{16}{9}$ is not an eigenvalue of $M_{1,1}$, (i.e. see (3.2)).

3. The multiplication function $d(k, l; m, n)$

In this section we analyze the multiplication function $d(k, l; m, n)$ for $1 \le k \le l$, $m, n \ge 0$ and $\phi(m, n) \le \phi(k, l)$.

First of all, suppose that (m, n) is a pair of nonnegative integers and denote by $\chi_{m\sigma+n\tau}$ the character of the irreducible representation determined by the highest weight $m\sigma + n\tau$. Then we have

$$
\begin{aligned}
d(k,l;m,n) &= dim\ Hom_{i_{k,l}(S^1)}(V_{m\sigma+n\tau}, \mathbb{C}) \\
&= \int_{i_{k,l}(S^1)} \overline{\chi_{m\sigma+n\tau}}.
\end{aligned} \tag{3.1}
$$

By the Weyl character formula, we can express the character $\chi_{m\sigma+n\tau}$ $(i_{k,l}(\exp(2\pi i\theta))$ explicitly. Setting $e(t) = \exp((2\pi i t)\theta)$, we have (*c.f.* [3])

$$
\chi_{m\sigma+n\tau}\,(i_{k,l}(\exp(2\pi i\theta))) = \frac{j_{m\sigma+n\tau}(\theta)}{j(\theta)},
$$

where

$$
\begin{aligned}
j_{m\sigma+n\tau}(\theta) = e\big(&(m+n+2)k + (m+1)l\big) \\
&- e\big((m+1)k + (m+n+2)l\big) \\
&- e\big(-(m+n+2)k - (n+1)l\big) \\
&+ e\big(-(n+1)k - (m+n+2)l\big) \\
&- e\big((n+1)k - (m+1)l\big) \\
&+ e\big(-(m+1)k + (n+1)l\big),
\end{aligned}
$$

$$
\begin{aligned}
j(\theta) = e(2k+l) &- e(k+2l) - e(-2k-l) \\
&+ e(-k-2l) - e(k-l) + e(-k+l).
\end{aligned}
$$

Setting $z = \exp(2\pi i\theta)$,

$$
j(\theta) = z^{-(2k+l)}(z^{2k+l}-1)(z^{k-l}-1)(z^{k+2l}-1)
$$

and

$$
\begin{aligned}
j_{m\sigma+n\tau}(\theta) = z^{-(m+n+2)k-(n+1)l} \\
\times \Big\{ &z^{(m+n+2)k-(m-2n-1)l} \\
&(z^{(n+1)(k-l)} - 1)(z^{(m+1)(k+2l)} - 1) \\
&- (z^{(m+1)(k-l)} - 1)(z^{(n+1)(k+2l)} - 1)\Big\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\chi_{m\sigma+n\tau}(i_{k,l}(\exp(2\pi i\theta)))\\
=&\frac{j_{m\sigma+n\tau}(\theta)}{j(\theta)}\\
=&\frac{z^{-(n+m)k-nl}}{(z^{2k+l}-1)(z^{k-l}-1)(z^{k+2l}-1)}\\
&\times\Big\{z^{(m+n+2)k-(m-2n-1)l}\\
&\quad\times(z^{(n+1)(k-l)}-1)(z^{(m+1)(k+2l)}-1)\\
&\quad-(z^{(m+1)(k-l)}-1)(z^{(n+1)(k+2l)}-1)\Big\}\\
=&\frac{z^{-(m+n)k-nl}}{(z^{2k+l}-1)}\Big\{z^{(m+1)(k-l)+(n+1)(k+2l)}\\
&\qquad\times(z^{n(k-l)}+\cdots+z^{k-l}+1)\\
&\qquad\times(z^{m(k+2l)}+\cdots+z^{k+2l}+1)\\
&\qquad-(z^{m(k-l)}+\cdots+z^{k-l}+1)\\
&\qquad\times(z^{n(k+2l)}+\cdots+z^{k+2l}+1)\Big\}.
\end{aligned}
$$

Setting $x=z^{k+2l}$ and $y=z^{-(k-l)}$, the above equation is

$$
\begin{aligned}
&\frac{z^{-(m+n+1)k-(n-1)l}}{(x-y)}\Big\{x^{n+1}y^{-(m+1)}(x^m+\cdots+x+1)\\
&\qquad\times(y^{-n}+\cdots+y^{-1}+1)\\
&\qquad-(x^n+\cdots+x+1)(y^{-m}+\cdots+y^{-1}+1)\Big\}\\
=&\frac{z^{-(m+n+1)k-(n-1)l}y^{-(m+n+1)}}{x-y}\sum_{i=0}^{m}\sum_{j=0}^{n}(x^{n+1+i}y^j-x^jy^{n+1+i}).
\end{aligned}
$$

Finally, the above equation is as follows.

$$\begin{aligned}
&\frac{z^{-(2n+m)l}}{(x-y)}\sum_{i=0}^{m}\sum_{j=0}^{n}(x^{n+1+i}y^j - x^j y^{n+1+i}) \\
&= z^{-(2n+m)l}\sum_{i=0}^{m}\sum_{j=0}^{n} x^j y^j \\
&\quad \times (x^{n+i-j} + x^{n+i-j-1}y + \cdots + xy^{n+i-j-1} + y^{n+i-j}) \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n}\Big(z^{(n+i-j)k+(2i+j-m)l} + z^{(n+i-j-2)k+(2i+j-m-1)l} \\
&\qquad\qquad + \cdots + z^{(j-i-n)k+(i+2j-n-m)l}\Big) \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{s=0}^{i+n-j} z^{(n+i-j-2s)k+(2i+j-m-s)l}.
\end{aligned} \tag{3.2}$$

From (3.1), $d(k, l; m, n)$ is the number of triple pairs of (i, j, s)'s satisfying $(n + i - j - 2s)k + (2i + j - m - s)l = 0$ for $0 \le i \le m$, $0 \le j \le n$ and $0 \le s \le i + n - j$. Since $(k, l) = 1$, there exists an integer $u \in \mathbb{Z}$ such that

$$\begin{cases} m + s - 2i - j = uk \\ n + i - j - 2s = ul. \end{cases}$$

Adding these two equations, we get

$$m + n - (i + 2j + s) = u(k + l).$$

From the ranges of i, j, s, one can check easily that

$$\begin{aligned}
-(m+n) &\le m + n - (i + 2j + s) \\
&= u(k+l) \\
&\le m + n.
\end{aligned} \tag{3.3}$$

Now we assume that for a given pair (k, l) $(1 \le k \le l)$, (m, n) is a pair of nonnegative integers satisfying $m^2 + n^2 + mn + 3m + 3n \le k^2 + l^2 + kl + 3k + 3l$. If $m + n < k + l$, the only possible integer for u is 0. If $m + n \ge k + l$, $(m + n)^2 \ge u^2(k + l)^2$ and hence

$$\begin{aligned}
0 &\ge \{(m+n)^2 + 3(m+n) - mn\} - \{(k+l)^2 + 3(k+l) - kl\} \\
&\ge (\frac{1}{2}(m+n)^2 + 3(m+n)) - (\frac{3}{2}(k+l)^2 + 3(k+l)) \\
&\ge \frac{1}{2}(u^2 - 3)(k+l)^2 + 3(u-1)(k+l)
\end{aligned}$$

From (3.3) and the fact that $k \geq 1$, $l \geq 1$, the only possible integers for u are 0 and ± 1.

Now we want to find the number of solutions of

$$\text{(I)} \quad \begin{cases} m+s-2i-j=0 \\ n+i-j-2s=0 \end{cases}$$

$$\text{(II)} \quad \begin{cases} m+s-2i-j=k \\ n+i-j-2s=l \end{cases}$$

$$\text{(III)} \quad \begin{cases} m+s-2i-j=-k \\ n+i-j-2s=-l, \end{cases}$$

where $0 \leq i \leq m,\ \ 0 \leq j \leq n,\ \ 0 \leq s \leq i+n-j$.

For given k,l,m,n, denote by $f(m,n)$, $g(k,l;m,n)$ and $h(k,l;m,n)$ the numbers of solutions of (I), (II), and (III), respectively. Then

$$d(k,l;m,n) = f(m,n) + g(k,l;m,n) + h(k,l;m,n).$$

Here $f(m,n)$ does not depend on the pair (k,l). One can also check that for $1 \leq k \leq l$, $(k,l)=1$

$$\begin{aligned} g(k,l;k,l) &= 1, \quad g(k,l;l,k) = 0, \\ h(k,l;k,l) &= 0, \quad h(k,l;l,k) = 1. \end{aligned}$$

The following theorem can be shown by Proposition 2.1 and considering the equations (I), (II), and (III).

THEOREM 3.1. *Suppose that $1 \leq k \leq l$ and $(k,l)=1$. Then the smallest non-zero eigenvalue of the Laplacian on $M_{k,l}$ is 1, which is common for all Aloff-Wallach spaces. If $k=l=1$, the multiplicity of 1 is 32 and if $1 \leq k < l$, the multiplicity of 1 is 16. Hence, if $1 \leq k < l$, $M_{k,l}$ is not isospectral to $M_{1,1}$.*

4. Proof of Theorem 1.1

In this section, we discuss the eigenvalues with their multiplicities on $M_{k,l}$ which are less than or equal to $\phi(k,l) = \frac{1}{9}(k^2+l^2+kl+3k+3l)$. And then, we are going to prove Theorem 1.1.

LEMMA 4.1. *Suppose that for coprime integers k, l with $1 \leq k \leq l$, m, n are non-negative integers satisfying $(m,n) \neq (k,l)$, $(m,n) \neq (l,k)$ and $(m^2+n^2+mn+3m+3n) \leq (k^2+l^2+kl+3k+3l)$. Then $g(k,l;m,n)=0$ and $h(k,l;m,n)=0$.*

Proof. Adding two equations in (II) and in (III), we get $m+n-(i+2j+s) = \pm(k+l)$.

(*Case 1*) If $m+n < k+l$, obviously there is no (i,j,s) satisfying the equations (II) and (III). Thus $g(k,l;m,n) = h(k,l;m,n) = 0$.

(*Case 2*) Suppose that $m+n \geq k+l$. Then there are four possibilities for (m,n) as follows.

$$\text{(i)}\quad m<k<l<n, \qquad \text{(ii)}\quad k<m<n<l,$$
$$\text{(iii)}\quad n<k<l<m, \qquad \text{(iv)}\quad k<n<m<l.$$

We are going to prove the assertion for the case (i) and we can use the similar argument for other cases.

From the equation (II), we get $2m+n-3(i+j) = 2k+l$, and hence $2m+n \geq 2k+l$. Then,

$$\begin{aligned}
0 &\leq (2m+n)^2 - (2k+l)^2 \\
&= 4(m^2+n^2+mn+3m+3n) - 3n^2 - 12m - 12n \\
&\quad - 4(k^2+l^2+kl+3k+3l) + 3l^2 + 12k + 12l \\
&\leq 3(l^2-n^2) + 6(2k+l-2m-n) + 6(l-n) < 0.
\end{aligned}$$

It is a contradiction and there is no (i,j,s) satisfying the equation (II). Thus we get $g(k,l;m,n)=0$.

Similarly, from the equation (III), we get $m+2n-3(j+s) = -2l-k$. Note that

$$m+2n-3(j+s) \geq m+2n-3(j+i+n-j) \geq -2m-n.$$

Hence, $2m+n \geq 2l+k$. Similarly,

$$\begin{aligned}
0 &\leq (2m+n)^2 - (2l+k)^2 \\
&\leq 3(k^2-n^2) + 6(2l+k-2m-n) + 6(k-n) \\
&< 0.
\end{aligned}$$

It is a contradiction and hence, $h(k,l;m,n)=0$. $\square$

Corollary 4.2. *Suppose that k, l are coprime integers with $1 \leq k < l$. Recall that $\phi : (\mathbb{Z}^+ \cup \{0\}) \times (\mathbb{Z}^+ \cup \{0\}) \to \mathbb{R}$ by $\phi(m,n) = \frac{1}{9}(m^2+n^2+mn+3m+3n)$.*

(1) *Let $\lambda_0 = \phi(k,l)$ and $\phi^{-1}(\lambda_0) = \{(k,l),(l,k),(m_1,n_1),\cdots,(m_r,n_r)\}$, where $(m_i,n_i) \neq (k,l)$ and $(m_i,n_i) \neq (l,k)$ for $1 \leq i \leq r$. Then λ_0 is an eigenvalue of the Laplacian on $M_{k,l}$ with multiplicity*

$$\frac{1}{2}(k+1)(l+1)(k+l+2)(f(k,l)+f(l,k)+2) \\ + \frac{1}{2}\sum_{i=1}^{r}(m_i+1)(n_i+1)(m_i+n_i+2)f(m_i,n_i).$$

(2) *Let $\lambda \in Im\phi$ with $\lambda < \lambda_0$. If $\phi^{-1}(\lambda) = \{(m_1,n_1),\cdots,(m_p,n_p)\}$, λ is an eigenvalue of the Laplacian on $M_{k,l}$ with multiplicity $\frac{1}{2}\sum_{i=1}^{p}(m_i+1)(n_i+1)(m_i+n_i+2)f(m_i,n_i)$.*

Corollary 4.3. *Suppose that $M_{k,l}$ and $M_{k',l'}$ are two Aloff-Wallach spaces with $1 \leq k < l$, $1 \leq k' < l'$ and $\phi(k,l) \neq \phi(k',l')$. Then $M_{k,l}$ and $M_{k',l'}$ are not isospectral.*

Proof. Without loss of generality, we may assume that $\phi(k,l) < \phi(k',l')$. Set $\lambda_0 = \phi(k,l)$ and denote by q, q' the multiplicities of λ_0 on $M_{k,l}$ and on $M_{k',l'}$, respectively. Then by Corollary 4.2,

$$q - q' = (k+1)(l+1)(k+l+2) > 0$$

and $M_{k,l}$ and $M_{k',l'}$ are not isospectral. □

Now we consider a pair of the Aloff-Wallach spaces $M_{k,l}$ and $M_{k',l'}$ with $\phi(k,l) = \phi(k',l')$. The following lemma can be shown easily by direct calculation.

Lemma 4.4. *Suppose that k,l,k',l' are positive integers satisfying $\phi(k,l) = \phi(k',l')$, $(k,l) \neq (k',l')$ and $(k,l) \neq (l',k')$. Then $k+l \neq k'+l'$. Furthermore, if $k+l < k'+l'$, then $kl < k'l'$.*

Corollary 4.5. *Suppose that $M_{k,l}$ and $M_{k',l'}$ are two Aloff-Wallach spaces with $1 \leq k < l$, $1 \leq k' < l'$, $(k,l) \neq (k',l')$, $(k,l) \neq (l',k')$ and $\phi(k,l) = \phi(k',l')$. Then $M_{k,l}$ and $M_{k',l'}$ are not isospectral.*

Proof. Let us set $\lambda_0 = \phi(k,l) = \phi(k',l')$ and denote by q, q' the multiplicities of λ_0 on $M_{k,l}$ and on $M_{k',l'}$, respectively.. Then by Corollary 4.2,

$$q - q' = (k+1)(l+1)(k+l+2) - (k'+1)(l'+1)(k'+l'+2).$$

From Lemma 4.4, we may assume that $k+l < k'+l'$ and hence $kl < k'l'$. Then $q < q'$ and $M_{k,l}$ and $M_{k',l'}$ are not isospectral. □

Now we are ready to prove Theorem 1.1.

Theorem 1.1. *Suppose that $M_{k,l}$ and $M_{k',l'}$ are two Aloff-Wallach spaces which are isospectral. Then (k',l') is one of $\pm(k,l)$, $\pm(l,k)$, $\pm(k,-(k+l))$, $\pm(-(k+l),k)$, $\pm(l,-(k+l))$, $\pm(-(k+l),l)$.*

Proof. From Theorem 3.1, we may assume that $(k,l) \neq \pm(1,1)$ and $(k',l') \neq \pm(1,1)$. We may also assume that $1 \leq k < l$ and $1 \leq k' < l'$. Then from Corollary 4.3 and Corollary 4.5 we get $k = k'$ and $l = l'$. This completes the proof. □

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