Contemporary Mathematics

Agranovich-Dynin Formula for the Zeta-determinants of the Neumann and Dirichlet Problems

Jinsung Park and Krzysztof P. Wojciechowski

Abstract. Let Δ denote a Laplace type operator acting on sections of a bundle S over a compact manifold M with boundary Y . Let us assume that the Dirichlet to Neumann operator $\mathcal{N}$ is a positive operator. We offer a detailed proof of the equality

$$\frac{\det_\zeta \Delta_N}{\det_\zeta \Delta_D} = \det_\zeta \mathcal{N} \ , \tag{0.1}$$

where Δ_N, Δ_D denote the Laplacian with Neumann, Dirichlet boundary conditions respectively.

Introduction and statement of the result

Let Δ denote a Laplace type operator acting on sections of a bundle S over a compact manifold M with boundary Y . In the following we assume that there exists a collar neighborhood $N \cong [0,1] \times Y$ of Y in M such that the Riemannian structure on M and the Hermitian structure on S are products when restricted to N . We also assume that the operator Δ restricted to the submanifold N has the following form

$$\Delta = -\partial_u^2 + \Delta_Y \ , \tag{0.2}$$

where u denotes the (inward) normal coordinate in N and Δ_Y is the corresponding Laplacian on Y . The operator $\Delta : C^\infty(M;S) \to C^\infty(M;S)$ itself does not have nice analytical properties. For instance, the kernel of Δ is infinite dimensional. We have to put a boundary condition on Δ in order to obtain a closed Fredholm operator. The two most classical conditions are the Dirichlet condition and the Neumann condition. We introduce the operator

$$\Delta_D = \Delta : \operatorname{dom} \Delta_D \to L^2(M;S) \ ,$$

where

$$\operatorname{dom} \Delta_D = \{ \ s \in H^2(M;S) \ ; \ s|Y = 0 \ \} \ .$$

Similarly we define Δ_N with

$$\operatorname{dom} \Delta_N = \{ \ s \in H^2(M;S) \ ; \ (\partial_u s)|Y = 0 \ \} \ .$$

Both Δ_D and Δ_N enjoy the standard properties of the Laplacian on a closed manifold. In particular they have well-defined ζ-determinant (see Ray and Singer [**10**]).

2000 *Mathematics Subject Classification.* Primary: 58J32, 58J52.

The elliptic boundary value problems Δ_D and Δ_N on M define an elliptic, pseudo-differential, self-adjoint operator $\mathcal{N}$ on Y , the so called *Dirichlet to Neumann operator*. In the following we call $\mathcal{N}$ the *DN operator*. We provide the precise description of the DN operator later on. In this note we assume

$$\text{(A)} \qquad \mathcal{N} \text{ is a positive operator.}$$

We assume (A) in order to prove formula (0.1) without introducing more elaborate technical tools. This gives us an opportunity to present the technique we are going to use in our study of the ζ-determinants in one of the simplest possible set-ups. We will discuss a proof of (0.1) without assuming (A) elsewhere.

Let us explain some consequences of condition (A). First of all, positivity of $\mathcal{N}$ implies that Δ_N is an invertible operator. Then we may use the Dirichlet to Neumann bracketing to show that Δ_D is an invertible operator. We refer to Taylor [**14**] for an exposition of the basic results on Neumann and Dirichlet problems.

The purpose of this note is to study

$$\frac{\det_\zeta \Delta_N}{\det_\zeta \Delta_D} \quad . \tag{0.3}$$

The adiabatic analysis of this quotient was provided by the authors (see [**8**]), and now we briefly describe the main result of their work. Let us introduce a manifold M_R obtained from M by replacing the collar N by $N_R = [0, R] \times Y$. We extend the bundle S and operator Δ to M_R in an obvious way (we use formula (0.2)). We denote this extended operator by Δ_R and the corresponding operators with the Dirichlet, Neumann boundary conditions by $\Delta_{R,D}$, $\Delta_{R,N}$. Then the following equality holds

$$\lim_{R\to\infty} \frac{\det_\zeta \Delta_{R,N}}{\det_\zeta \Delta_{R,D}} = \det_\zeta \sqrt{\Delta_Y} \tag{0.4}$$

under certain conditions (see Remark 0.2). The analysis of the behavior of the small eigenvalues (converging to 0 as $R \to \infty$) allowed us to prove a similar formula in the case of non-invertible Δ_Y (see Park and Wojciechowski [**9**] for the detailed formulation). Still, it is difficult to get excited about this type of results as the adiabatic process ($\lim_{R\to\infty}$) kills interesting geometric and analytical information.

In this paper the adiabatic analysis is replaced by the argument outlined by Forman (see [**6**]). Let us point out, however, that we introduce one important modification to the Forman method. We regularize the determinants and their variations by introducing their counterparts which live on the cylinder. This methods works very well in this note and it simplifies the proofs of the main technical results presented in Section 4. Moreover, this method is now applied in the analysis of more complicated problems related to the cutting and pasting of the ζ-determinant (see Loya and Park [**7**] and future publications).

In this note, we employ our method to prove that (0.1) holds up to a multiplicative constant. Then a separate argument is used to show that this multiplicative constant is equal to 1 . This gives the main result of the paper:

THEOREM 0.1. *The following equality holds*

$$\frac{\det_\zeta \Delta_N}{\det_\zeta \Delta_D} = \det_\zeta \mathcal{N} \quad .$$

REMARK 0.2. In [**9**] we have proved that the DN operator $\mathcal{N}_R$ (defined by Δ_R) converges to $\sqrt{\Delta_Y}$ as $R \to \infty$. In fact we showed that $\mathcal{S}_R := \mathcal{N}_R - \sqrt{\Delta_Y}$ was a smoothing operator whose operator norm is bounded by $\frac{c}{R}$ for a positive constant c . It follows that if we assume that Δ_Y is invertible and that there is no L^2-solutions of the extension of Δ_R on a manifold with cylindrical end, then $\mathcal{N}_R$ is a positive operator for large R . We refer to [**9**] for the details.

In Section 1 we introduce the DN operator.

In Section 2 we digress a little and discuss non-positivity of the DN operator for the Dirac Laplacian.

In Section 3 we show that the difference between the DN operator $\mathcal{N}$ of Δ on M and the corresponding DN operator on the cylinder is a smoothing operator. This result is used later to establish the trace class property for various operators appearing in the proof of Theorem 0.1.

In Section 4 we study the variation of the ζ-determinant and prove (0.1) up to a multiplicative constant.

In Section 5 we show that the aforementioned constant is equal to 1 , which ends the proof of Theorem 0.1.

Acknowledgements We want to express our gratitude to the referee for thoughtful comments, and helpful suggestions which led to the substantial improvements in the exposition.

1. Dirichlet to Neumann operator

We offer here a brief presentation of part of the theory of the elliptic boundary value problems, originated in the works of Calderon [**5**], Agranovich and Dynin [**2**], [**1**], and Seeley [**12**]. The case of a $1st$ order Dirac operator was discussed in Section 2 of Scott and Wojciechowski [**11**] (see also a detailed exposition in Booß-Bavnbek and Wojciechowski [**3**]). We concentrate on the case of a particular operator Δ and we assume for simplicity that all elliptic boundary conditions P considered here define invertible operators $\Delta_P : \operatorname{dom}\Delta_P \to L^2(M;S)$.

We start with the trace map

$$\gamma(s) = (s|Y, (\partial_u s)|Y) : C^\infty(M;S) \to C^\infty(Y;S|Y) \oplus C^\infty(Y;S|Y) \quad ,$$

which extends to a well-defined map

$$\gamma = (\gamma_0, \gamma_1) : H^k(M;S) \to H^{k-\frac{1}{2}}(Y;S|Y) \oplus H^{k-\frac{3}{2}}(Y;S|Y) \quad ,$$

where H^k denotes the k-th Sobolev space for $k > \frac{3}{2}$. The operator Δ determines $\mathcal{H}(\Delta)$, the Cauchy data space on Y ,

$$\{\ (f,g) \in C^\infty(Y;S|Y) \oplus C^\infty(Y;S|Y)\ ;\ \exists s \in C^\infty(M;S) \text{ s.t. } \Delta s = 0 \text{ in } M \setminus Y \text{ and } \gamma(s) = (f,g)\ \} \quad .$$

There exists a pseudo-differential projection onto $\mathcal{H}(\Delta)$, but the construction we use has a choice involved. We have to pick $\tilde{\Delta}$, the extension of Δ to an invertible operator of Laplace type on a closed manifold $\tilde{M}$ which contains M as a submanifold with boundary. The projection $P(\Delta)$ on $\mathcal{H}(\Delta)$ is given by the formula

$$P(\Delta) = \gamma\big(r_M \tilde{\Delta}^{-1} \tilde{\gamma}^* \mathcal{G}\big) \quad . \tag{1.1}$$

Here $\tilde{\gamma}$ denotes the trace on Y of the section defined on $\tilde{M}$, and $\tilde{\gamma}^*$ is the operator adjoint to $\tilde{\gamma}$ (we refer to [**12**] for a detailed exposition of the necessary material), r_M is the restriction operator (operator which maps section s on $\tilde{M}$ to $s|M$). Finally, $\mathcal{G}$ is the Green's form, which in the product case, is the bundle involution $\mathcal{G} : S|Y \oplus S|Y \to S|Y \oplus S|Y$

$$\mathcal{G} = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \quad .$$

As we mentioned before, the pseudo-differential projection $P(\Delta)$ depends on the choice of the operator $\tilde{\Delta}$. This is also the case for the Poisson operator for Δ ,

$$\mathcal{K} = r_M \tilde{\Delta}^{-1} \tilde{\gamma}^* \mathcal{G} \quad ,$$

which maps the boundary data onto the null-solutions of Δ in $M \setminus Y$.

Now we introduce the operators

$$P_D, P_N : C^\infty(Y; S|Y) \oplus C^\infty(Y; S|Y) \to C^\infty(Y; S|Y) \oplus C^\infty(Y; S|Y)$$

determined by the projections of $S|Y \oplus S|Y$ onto first (resp. second) summand

$$P_D = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad P_N = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \quad .$$

These maps provide us with projections onto the 0th and 1st order parts of the Cauchy data of a section of S . It follows from assumption (A) that

$$\begin{aligned} S_D =& P_D P(\Delta) : \mathcal{H}(\Delta) \to \mathrm{range}\, P_D \cong C^\infty(Y; S|Y) \quad , \\ S_N =& P_N P(\Delta) : \mathcal{H}(\Delta) \to \mathrm{range}\, P_N \cong C^\infty(Y; S|Y) \end{aligned}$$

are invertible bounded operators and that the operators $P_D P(\Delta) + P_N(\mathrm{Id} - P(\Delta))$, $P_N P(\Delta) + P_D(\mathrm{Id} - P(\Delta))$ are invertible. Hence we can define

$$\begin{aligned} S_D^{-1} &= P(\Delta)[P_D P(\Delta) + P_N(\mathrm{Id} - P(\Delta))]^{-1} P_D \quad , \\ S_N^{-1} &= P(\Delta)[P_N P(\Delta) + P_D(\mathrm{Id} - P(\Delta))]^{-1} P_N \quad . \end{aligned}$$

The Poisson operator for Dirichlet boundary condition is given by

$$\mathcal{K}_D = \mathcal{K} S_D^{-1} \quad . \tag{1.2}$$

Let us remark that contrary to the operator $\mathcal{K}$, $\mathcal{K}_D$ does not depend on the choice of $\tilde{\Delta}$ since the restriction of $\mathcal{K}$ to $\mathcal{H}(\Delta)$ is independent on the choice of the Calderon projection. This may be explained in the following way. Let $f \in C^\infty(Y; S|Y)$, then $\mathcal{K}_D f$ is a null-solution of Δ in $M \setminus Y$ and

$$P_D \gamma(\mathcal{K}_D f) = P_D \gamma \mathcal{K} S_D^{-1} f = S_D S_D^{-1} f = f \quad .$$

Hence $s = \mathcal{K}_D f$ is the unique solution of Dirichlet problem

$$\Delta s = 0 \;\text{ and }\; s|Y = f \quad .$$

The corresponding DN operator is defined by the formula

$$\mathcal{N} = -\gamma_1 \mathcal{K}_D = -S_N S_D^{-1} \quad . \tag{1.3}$$

The assumption (A) implies that it is an invertible operator and it is not difficult to see that

$$\mathcal{N}^{-1} = -S_D S_N^{-1} = -\gamma_0 \mathcal{K}_N . \tag{1.4}$$

Remark 1.1. Let us comment on the sign convention in the definition (1.3). We follow [**12**] and use the inward normal derivative in the first order component γ_1 . We introduce the minus sign in the definition of $\mathcal{N}$ in order to end-up with positive (in the general case bounded from below) operator.

In the following, we also need a formula for the inverse of the operator Δ_D ,

$$\Delta_D^{-1} = \Delta^{-1} - \mathcal{K} S_D^{-1} P_D \gamma \Delta^{-1} \quad . \tag{1.5}$$

To simplify notation in the formula above, Δ^{-1} denotes the following operator

$$\Delta^{-1} = r_M \tilde{\Delta}^{-1} e_+ \quad , \tag{1.6}$$

where $e_+ : L^2(M;S) \to L^2(\tilde{M};\tilde{S})$ is an extension map (by 0) from M to $\tilde{M}$. The first term on the right side of (1.5) defines an inverse in $M \setminus Y$, the second is a "boundary" correction term, which makes sure that the result is an element of $\operatorname{dom} \Delta_D$. The same formula, with *"N"* replacing *"D"*, gives the inverse of Δ_N .

2. Remark on the non-positivity of the DN operator

It is not difficult to show that DN operator $\mathcal{N}$ is a pseudodifferential, elliptic operator of order one, with the principal symbol equal to the principal symbol of the square root of Δ_Y (see for instance [**9**]). This shows that $\mathcal{N}$ is bounded from below. We have assumed in Theorem 0.1 that $\mathcal{N}$ is positive. This is not the case in general and we would like to present an example of the situation in which $\mathcal{N}$ may have not only non-trivial kernel, but also true negative eigenvalues.

We consider a compatible Dirac operator $\mathcal{D} : C^\infty(M;S) \to C^\infty(M;S)$. We keep up the assumptions from the Introduction (with the exception of Condition (A)). Hence in particular we have the product metric structures in the collar N . The operator $\mathcal{D}$ has the following form in N

$$\mathcal{D}|N = G(\partial_u + B) \quad , \tag{2.1}$$

where $B : C^\infty(Y;S|Y) \to C^\infty(Y;S|Y)$ is the corresponding Dirac operator on Y and $G : S|Y \to S|Y$ is a unitary bundle isomorphism, such that

$$G^2 = -\mathrm{Id} \quad \text{and} \quad GB = -BG \quad .$$

The space $\mathcal{H}(\mathcal{D})$, the Cauchy data space of $\mathcal{D}$, has a different structure from the Cauchy data space of Δ . There exists Calderón projector $P(\mathcal{D})$ on $C^\infty(Y;S|Y)$, which is a pseudodifferential operator of order 0 (see [**3**] and [**11**] for details). In particular $\mathcal{H}(\mathcal{D})$ is a Lagrangian subspace of $L^2(Y;S|Y)$ meaning that

$$G(\mathcal{H}(\mathcal{D})) = \mathcal{H}(\mathcal{D})^\perp \quad . \tag{2.2}$$

The following result is a straightforward consequence of Green's formula.

Proposition 2.1. *The DN operator $\mathcal{N}$ of $\mathcal{D}^2$ is equal to B when restricted to $\mathcal{H}(\mathcal{D})$,*

$$\mathcal{N}|\mathcal{H}(\mathcal{D}) = B|\mathcal{H}(\mathcal{D}) \quad .$$

Proof. Let $f \in \mathcal{H}(\mathcal{D})$ and let s denote a section of S , such that

$$\mathcal{D}s = 0 \quad \text{and} \quad s|Y = f \quad .$$

First, we observe that s is the solution of Dirichlet problem. Let us denote by $s_1 \in C^\infty(M;S)$ the solution of

$$\Delta s_1 = 0 \quad \text{and} \quad s_1|Y = f \quad .$$

We put $s_2 = s_1 - s$ so $\Delta s_2 = \mathcal{D}^2 s_2 = 0$ and $s_2|Y = 0$. This gives

$$0 = (\mathcal{D}^2 s_2; s_2) = \|\mathcal{D}s_2\|^2 - \int_Y \langle G((\mathcal{D}s_2)|Y); s_2|Y\rangle \ \mathrm{dvol}(y) = \|\mathcal{D}s_2\|^2 \quad ,$$

so $\mathcal{D}s_2 = 0$ and $s_2|Y = 0$. The vanishing of s_2 is a consequence of the Unique Continuation Property for Dirac operators (see for instance [**3**]). Now, for any

$w \in C^\infty(M;S)$, we have

$$\begin{aligned} 0 =&(\Delta s; w) \\ =&(\mathcal{D}s; \mathcal{D}w) - \int_Y \langle (G(G(\partial_u + B))s)|Y; w|Y\rangle \ \mathrm{dvol}(y) \\ =& \int_Y \langle \partial_u s|_Y + Bf; w|Y\rangle \ \mathrm{dvol}(y) \end{aligned}$$

and $\mathcal{N}f = -(\partial_u s_1)|Y = -(\partial_u s)|Y = Bf$ for $f \in \mathcal{H}(\mathcal{D})$. This completes the proof. □

Now we use formula (2.1) to extend $\mathcal{D}$ to the Dirac operator on a manifold M_∞ , where M_∞ is a manifold with cylindrical end

$$M_\infty = (-\infty, 0] \times Y \cup M \quad .$$

It is well-known that the resulting operator has unique closed self-adjoint extension in the space $L^2(M_\infty; S)$, which we denote by $\mathcal{D}_\infty$. This operator has a finite dimensional kernel (the space of L^2-solutions). More general we may introduce the space of extended L^2-solutions of $\mathcal{D}_\infty$. A section $s \in C^\infty(M_\infty; S)$ is an extended L^2-solution of $\mathcal{D}_\infty$ if $\mathcal{D}_\infty s = 0$ and s has the following form on the cylinder $(-\infty, 0] \times Y$

$$s(u,y) = s_{L^2}(u,y) + s_{lim}(y) \quad , \tag{2.3}$$

where s_{L^2} is in $L^2((-\infty,0] \times Y; S)$ and $s_{lim} \in \ker B$. We have the following Proposition

Proposition 2.2. *Let $f = s|Y$ where s is an extended L^2-solution of $\mathcal{D}_\infty$. Then*

$$(\mathcal{N}f; f) < 0.$$

Proof. By Proposition 2.1,

$$\begin{aligned} \int_Y \langle \mathcal{N}f; f\rangle \ \mathrm{dvol}(y) =& \int_Y \langle Bf; f\rangle \ \mathrm{dvol}(y) \\ =& \int_Y \langle B(s_{L^2}|_{y=0} + s_{lim}); (s_{L^2}|_{y=0} + s_{lim})\rangle \ \mathrm{dvol}(y) \\ =& \int_Y \langle B(s_{L^2}|_{y=0}); s_{L^2}|_{y=0}\rangle \ \mathrm{dvol}(y) \quad . \end{aligned}$$

Now, we only have to recall a formula for $s_{L^2}(u,y)$ (see the discussion around formula (22.69) in [**3**])

$$s_{L^2}(u,y) = \sum_{\lambda<0} a_\lambda e^{-\lambda u} \phi_\lambda(y) \quad ,$$

where $\{\lambda, \phi_\lambda\}$ is the spectral resolution of B. We observe that the summation goes over negative eigenvalues of the tangential operator B. This completes the proof. □

3. Comparison with the cylinder

In this Section, we compare the Calderón projector $P(\Delta)$ with the corresponding object on the cylinder.

We introduce $\{\Delta(t)\}$ a family of operators defined by

$$\{\Delta(t) := \Delta + t\}_{t\geq 0}.$$

In particular the restriction of the operator Δ to the collar N defines a family of Laplace type operators over $N = [0,1] \times Y$,

$$\Delta^c(t) := \Delta^c + t \ = \ -\partial_u^2 + \Delta_Y + t \quad .$$

In the following we assume that $\Delta^c(t)$ is subject to Dirichlet boundary condition at $\{1\} \times Y$, so that if we put a self-adjoint, elliptic boundary condition at $u = 0$, we obtain the operator with a well-defined ζ-determinant. The operator $\Delta^c(t)$ has a well-defined Calderón projector on $Y \cong \{0\} \times Y$ and in this Section we compare $P(\Delta(t))$ with $P(\Delta^c(t))$.

From now on, we identify $L^2(N, S)$ with the image of the natural embedding of this into $L^2(M, S)$. Hence any operator over $L^2(N, S)$ can be considered as an operator over $L^2(M, S)$, which is defined to be zero map over the orthogonal complement of $L^2(N, S)$ within $L^2(M, S)$. The main result of this Section is the following

PROPOSITION 3.1. *The difference $P(\Delta(t)) - P(\Delta^c(t))$ is a smoothing operator acting on $C^\infty(Y; S|_Y)$.*

PROOF. We choose the doubled invertible extensions $\tilde{\Delta}(t)$ over $\tilde{M}$, $\tilde{\Delta}^c(t)$ over $\tilde{N}$ of $\Delta(t)$, $\Delta^c(t)$ respectively. The result follows from the fact that, up to a smoothing operator, we can replace $\tilde{\Delta}(t)^{-1}$ in the formula (1.1) by a suitable *parametrix*, which involves the operator $\tilde{\Delta}^c(t)^{-1}$. In the following proof we omit the parameter t .

We introduce a smooth, increasing function $\rho(a, b) : [0, \infty) \to [0, 1]$ equal to 0 for $0 \le u \le a$ and equal to 1 for $b \le u$. We use $\rho(a, b)(u)$ to define

$$\phi_1 = 1 - \rho(\frac{5}{7}, \frac{6}{7}) \quad , \quad \psi_1 = 1 - \psi_2 \quad , \tag{3.1}$$
$$\phi_2 = \rho(\frac{1}{7}, \frac{2}{7}) \quad , \quad \psi_2 = \rho(\frac{3}{7}, \frac{4}{7})$$

and then we extend these functions to be the even functions over $\tilde{M}$ and $\tilde{N}$ in an obvious way. Now, we define Q , a parametrix for the operator $\tilde{\Delta}^{-1}$ by

$$Q(x, z) = \phi_1(x)(\tilde{\Delta}^c)^{-1}(x, z)\psi_1(z) + \phi_2(x)\tilde{\Delta}^{-1}(x, z)\psi_2(z) \ . \tag{3.2}$$

Then we have

$$\begin{aligned} \tilde{\Delta}Q(x, z) \; &= \; \mathrm{Id} \; + \; 2\frac{\partial \phi_1(x)}{\partial u}\frac{\partial (\tilde{\Delta}^c)^{-1}}{\partial u}(x, z)\psi_1(z) \; + \; 2\frac{\partial \phi_2(x)}{\partial u}\frac{\partial \tilde{\Delta}^{-1}}{\partial u}(x, z)\psi_2(z) \\ & \quad + \; \frac{\partial^2 \phi_1}{\partial u^2}(x)(\tilde{\Delta}^c)^{-1}(x, z)\psi_1(z) \; + \; \frac{\partial^2 \phi_2}{\partial u^2}(x)\tilde{\Delta}^{-1}(x, z)\psi_2(z) \quad , \end{aligned}$$

hence

$$\tilde{\Delta}Q \; = \; \mathrm{Id} + \mathcal{S} \quad , \tag{3.3}$$

where $\mathcal{S}$ is the operator with a smooth kernel $\mathcal{S}(x, z)$ equal to 0 if the distance from x to z is smaller than $\frac{1}{7}$. The equality (3.3) allows us to write

$$\tilde{\Delta}^{-1} - Q \; = \; -\tilde{\Delta}^{-1}\mathcal{S}$$

where $-\tilde{\Delta}^{-1}\mathcal{S}$ is a smoothing operator. We reformulate this equality as follows,

$$\tilde{\Delta}^{-1} - (\tilde{\Delta}^c)^{-1} = \mathcal{S}' + \mathcal{T}$$

where $\mathcal{S}'$ is a smoothing operator and the Schwartz kernel of $\mathcal{T}$ has the following form,

$$\begin{aligned} \mathcal{T} \; &= \phi_1(x)(\tilde{\Delta}^c)^{-1}(x, z)\psi_1(z) - (\tilde{\Delta}^c)^{-1}(x, z) + \phi_2(x)\tilde{\Delta}^{-1}(x, z)\psi_2(z) \\ &= \; - (1 - \phi_1(x))(\tilde{\Delta}^c)^{-1}(x, z)(1 - \psi_1(z)) - \phi_1(x)(\tilde{\Delta}^c)^{-1}(x, z)(1 - \psi_1(z)) \\ & \quad - (1 - \phi_1(x))(\tilde{\Delta}^c)^{-1}(x, z)\psi_1(z) + \phi_2(x)\tilde{\Delta}^{-1}(x, z)\psi_2(z) \quad . \end{aligned}$$

Therefore, we can see that

$$P(\Delta) - P(\Delta^c) \; = \; \gamma r_M \tilde{\Delta}^{-1}\tilde{\gamma}^*\mathcal{G} - \gamma r_M (\tilde{\Delta}^c)^{-1}\tilde{\gamma}^*\mathcal{G} \; = \; \gamma r_M \mathcal{S}'\tilde{\gamma}^*\mathcal{G}$$

is a smoothing operator on Y . □

We denote by $\Delta(t)_D$, $\Delta(t)_N$, $\Delta^c(t)_D$, $\Delta^c(t)_N$ the corresponding operators with Dirichlet, Neumann boundary conditions at $\{0\}\times Y\subset M,N$. We also denote by $\mathcal{K}_D(t)$, $\mathcal{K}_N(t)$, $\mathcal{K}^c_D(t)$, $\mathcal{K}^c_N(t)$ the Poisson operators for Dirichlet, Neumann problems of $\Delta(t)$, $\Delta^c(t)$ respectively.

Lemma 3.2. *Each of $\mathcal{K}_D(t)-\mathcal{K}^c_D(t)$, $\mathcal{K}_N(t)-\mathcal{K}^c_N(t)$ has the smoothing Schwartz kernel $K(x,z)$ for $(x,z)\in\big(M-\{1\}\times Y\big)\times Y$ with a jump discontinuity at $\{1\}\times Y$.*

Proof. We have

$$\mathcal{K}_D(t)-\mathcal{K}^c_D(t)=r_M\tilde{\Delta}(t)^{-1}\tilde{\gamma}^*\mathcal{G}S_D(t)^{-1}-r_M\tilde{\Delta}^c(t)^{-1}\tilde{\gamma}^*\mathcal{G}S^c_D(t)^{-1}$$

(see Section 1 for the definition of $S_D(t)^{-1}$ and $S^c_D(t)^{-1}$). We follow the proof of Proposition 3.1 and see that $r_M\big(\tilde{\Delta}(t)^{-1}-\tilde{\Delta}^c(t)^{-1}\big)\tilde{\gamma}^*\mathcal{G}$ has the following Schwartz kernel,

$$r_M\big(-(1-\phi_1(x))(\tilde{\Delta}^c)^{-1}(x,z)\psi_1(z)+\mathcal{S}'(x,z)\big)\tilde{\gamma}^*\mathcal{G}$$

where $x\in M, z\in Y$. We observe that the second term is smoothing for $(x,z)\in M\times Y$ and that the first term is smoothing for $(x,z)\in N\times Y$, which is extended to be zero out of N within M. Hence this Schwartz kernel is smoothing with a jump discontinuity at $\{1\}\times Y$. It is immediate from the Proposition 3.1 that $S_D(t)^{-1}-S^c_D(t)^{-1}$ is also a smoothing operator, and these implies that the Schwartz kernel of $\mathcal{K}_D(t)-\mathcal{K}^c_D(t)$ is smoothing with a jump discontinuity at $\{1\}\times Y$. We work in the same way in the case of $\mathcal{K}_N(t)-\mathcal{K}^c_N(t)$. □

Let $\mathcal{N}(t)$, $\mathcal{N}^c(t)$ denote the DN operators of $\Delta(t)$, $\Delta^c(t)$.

Corollary 3.3. *The difference of the DN operators $\mathcal{N}(t)-\mathcal{N}^c(t)$ is a smoothing operator acting on $C^\infty(Y;S|Y)$.*

Corollary 3.4. *The following operator is of trace class,*

$$(\Delta(t)^{-1}_N-\Delta(t)^{-1}_D)-(\Delta^c(t)^{-1}_N-\Delta^c(t)^{-1}_D):L^2(M;S)\to L^2(M;S).$$

Proof. We apply the equality (1.5) to get

$$\begin{aligned}(3.4)\qquad &(\Delta(t)^{-1}_N-\Delta(t)^{-1}_D)-(\Delta^c(t)^{-1}_N-\Delta^c(t)^{-1}_D)\\ &=-\mathcal{K}_N(t)\gamma_1\Delta(t)^{-1}+\mathcal{K}_D(t)\gamma_0\Delta(t)^{-1}+\mathcal{K}^c_N(t)\gamma_1\Delta^c(t)^{-1}-\mathcal{K}^c_D(t)\gamma_0\Delta^c(t)^{-1}\quad .\end{aligned}$$

Lemma 3.2 implies that $\mathcal{K}_N(t)-\mathcal{K}^c_N(t)$, $\mathcal{K}_D(t)-\mathcal{K}^c_D(t)$ has the smoothing Schwartz kernel with a jump discontinuity at $\{1\}\times Y$. We repeat the proof of Proposition 3.1 to show that $\gamma_1(\Delta(t)^{-1}-\Delta^c(t)^{-1})$ has the smoothing Schwartz kernel with a jump discontinuity at $\{1\}\times Y$. Therefore, the operator on the left side of (3.4) is of trace class. □

4. Variation of the ζ-determinant

In this Section, we study the variation of the ζ-determinant of the family of operators

$$\{\Delta(t)=\Delta+t\}_{t\geq 0}\quad .$$

We begin with the following Lemma:

Lemma 4.1. *The variation of $\mathcal{N}(t)$ with respect to the parameter t is given by*

$$\dot{\mathcal{N}}(t)=\gamma_1\Delta(t)^{-1}_D\mathcal{K}_D(t)\quad .$$

Proof. We have
$$\dot{\mathcal{N}}(t) = -\gamma_1 \dot{\mathcal{K}}_D(t) \quad .$$
Now we differentiate with respect to t the following equalities,
$$\Delta(t)\mathcal{K}_D(t)f = 0 \ , \ \gamma_0\mathcal{K}_D(t)f = f \quad ,$$
where f denote a smooth section of S over Y . The derivative $\dot{\Delta}(t)$ is equal to 1 so
$$\Delta(t)\dot{\mathcal{K}}_D(t)f = -\mathcal{K}_D(t)f \ , \ \gamma_0\dot{\mathcal{K}}_D(t)f = 0 \ . \tag{4.1}$$
The second equality in (4.1) implies that the range of $\dot{\mathcal{K}}_D(t)$ is in the domain of $\Delta(t)_D$ so the first equality gives
$$\dot{\mathcal{K}}_D(t) = -\Delta(t)_D^{-1}\mathcal{K}_D(t) \quad ,$$
which leads to the required identity
$$\dot{\mathcal{N}}(t) = -\gamma_1\dot{\mathcal{K}}_D(t) = \gamma_1\Delta(t)_D^{-1}\mathcal{K}_D(t) \quad .$$
□

Corollary 4.2. *The difference $\dot{\mathcal{N}}(t) - \dot{\mathcal{N}}^c(t)$ is a smoothing operator.*

Here is the first crucial technical result

Lemma 4.3. *The following equality holds,*
$$\frac{d}{dt}\Big(\ \ln\det{}_\zeta\mathcal{N}(t) - \ln\det{}_\zeta\mathcal{N}^c(t) \ \Big) \ = \ \mathrm{Tr}\ \Big(\ \dot{\mathcal{N}}(t)\mathcal{N}(t)^{-1} - \dot{\mathcal{N}}^c(t)\mathcal{N}^c(t)^{-1} \ \Big) \quad . \tag{4.2}$$

Proof. We have
$$\begin{aligned}
&\frac{d}{dt}\Big(\ \ln\det{}_\zeta\mathcal{N}(t) - \ln\det{}_\zeta\mathcal{N}^c(t) \ \Big) \\
= \ & -\frac{d}{dt}(\zeta'_{\mathcal{N}(t)}(0) - \zeta'_{\mathcal{N}^c(t)}(0)) \\
= \ & -\frac{d}{dt}\frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}\mathrm{Tr}\left(e^{-u\mathcal{N}(t)} - e^{-u\mathcal{N}^c(t)}\right)\, du \\
= \ & -\frac{d}{dt}\int_0^\infty u^{-1}\mathrm{Tr}\left(e^{-u\mathcal{N}(t)} - e^{-u\mathcal{N}^c(t)}\right)\, du \qquad \text{by Corollary 3.3} \\
= \ & \int_0^\infty \mathrm{Tr}\left(\dot{\mathcal{N}}(t)e^{-u\mathcal{N}(t)} - \dot{\mathcal{N}}^c(t)e^{-u\mathcal{N}^c(t)}\right)\, du \\
= \ & -\lim_{\varepsilon\to 0}\ \mathrm{Tr}\ \Big(\ \dot{\mathcal{N}}(t)\mathcal{N}(t)^{-1}e^{-u\mathcal{N}(t)} - \dot{\mathcal{N}}^c(t)\mathcal{N}^c(t)^{-1}e^{-u\mathcal{N}^c(t)} \ \Big)\Big]_\epsilon^{\epsilon^{-1}} \\
= \ & \lim_{\varepsilon\to 0}\ \mathrm{Tr}\ \Big(\ \dot{\mathcal{N}}(t)\mathcal{N}(t)^{-1}e^{-\varepsilon\mathcal{N}(t)} - \dot{\mathcal{N}}^c(t)\mathcal{N}^c(t)^{-1}e^{-\varepsilon\mathcal{N}^c(t)} \ \Big).
\end{aligned}$$
To study this limit as $\varepsilon \to 0$, we introduce a family of operators $\{F_t(u)\}_{u\in[0,\infty)}$. The operator $F_t(u)$ is given by the formula
$$F_t(u) = \frac{1}{2\pi i}\int_\Gamma \frac{e^{-u\lambda}}{\lambda}\Big[\frac{\dot{\mathcal{N}}(t)}{\lambda - \mathcal{N}(t)} - \frac{\dot{\mathcal{N}}^c(t)}{\lambda - \mathcal{N}^c(t)}\Big]\ d\lambda \quad ,$$
where Γ is a standard counter-clockwise contour around $\mathbb{R}^+$. Corollary 3.3 and 4.2 immediately imply that the integrand
$$\frac{e^{-u\lambda}}{\lambda}\Big[\frac{\dot{\mathcal{N}}(t)}{\lambda - \mathcal{N}(t)} - \frac{\dot{\mathcal{N}}^c(t)}{\lambda - \mathcal{N}^c(t)}\Big]$$
is a family of smoothing operators with the norm of the size $O(\lambda^{-2})$ as $|\lambda| \to \infty$ for $u \in [0,\infty)$. In particular, this is a family of smoothing operator continuous with respect

to u, even when $u = 0$. Hence $F_t(u)$ is a family of trace class operators, which is continuous for $u \in [0,\infty)$ with respect to the trace norm. Therefore, we obtain

$$\lim_{\varepsilon\to 0} \mathrm{Tr}\,\big(F_t(\varepsilon)\big) = \mathrm{Tr}\,\big(F_t(0)\big)$$

and by the Cauchy integral formula,

$$\mathrm{Tr}\,\big(F_t(0)\big) = \mathrm{Tr}\,\big(\ \dot{\mathcal{N}}(t)\mathcal{N}(t)^{-1} - \dot{\mathcal{N}}^c(t)\mathcal{N}^c(t)^{-1}\ \big).$$

Finally we conclude

$$\begin{aligned}
&\frac{d}{dt}\big(\ \ln\det{}_\zeta\mathcal{N}(t) - \ln\det{}_\zeta\mathcal{N}^c(t)\ \big)\\
&= \lim_{\varepsilon\to 0}\ \mathrm{Tr}\,\big(\ \dot{\mathcal{N}}(t)\mathcal{N}(t)^{-1}e^{-\varepsilon\mathcal{N}^c(t)} - \dot{\mathcal{N}}^c(t)\mathcal{N}^c(t)^{-1}e^{-\varepsilon\mathcal{N}^c(t)}\ \big)\\
&= \mathrm{Tr}\,\big(\ \dot{\mathcal{N}}(t)\mathcal{N}(t)^{-1} - \dot{\mathcal{N}}^c(t)\mathcal{N}^c(t)^{-1}\ \big)\ \ .
\end{aligned}$$

□

PROPOSITION 4.4. *The following equality holds,*

$$\frac{d}{dt}\big(\ \ln\det{}_\zeta\mathcal{N}(t) - \ln\det{}_\zeta\mathcal{N}^c(t)\ \big) = \ \mathrm{Tr}\,\big(\ \Delta(t)_N^{-1} - \Delta(t)_D^{-1} - \Delta^c(t)_N^{-1} + \Delta^c(t)_D^{-1}\ \big)\ \ .$$

PROOF. Formula (1.4) and Lemma 4.1 lead to the expression

$$\dot{\mathcal{N}}(t)\mathcal{N}(t)^{-1} = -\gamma_1\Delta(t)_D^{-1}\mathcal{K}_D(t)\gamma_0\mathcal{K}_N(t)\ \ .$$

The similar formula holds also for $\dot{\mathcal{N}}^c(t)\mathcal{N}^c(t)^{-1}$. Therefore,

$$\begin{aligned}
&\mathrm{Tr}\,\big(\dot{\mathcal{N}}(t)\mathcal{N}(t)^{-1} - \dot{\mathcal{N}}^c(t)\mathcal{N}^c(t)^{-1}\big)\\
&= \mathrm{Tr}\,\big(-\gamma_1\Delta(t)_D^{-1}\mathcal{K}_D(t)\gamma_0\mathcal{K}_N(t) + \gamma_1\Delta^c(t)_D^{-1}\mathcal{K}^c_D(t)\gamma_0\mathcal{K}^c_N(t)\big).
\end{aligned}$$

Now we observe that

$$\mathcal{K}_D(t)\gamma_0\mathcal{K}_N(t) = \mathcal{K}_N(t)\quad,\quad \mathcal{K}_N(t)\gamma_1\mathcal{K}_D(t) = \mathcal{K}_D(t).$$

To prove the first equality, we note that $\mathcal{K}_N(t)f$ is the unique solution of $\Delta(t)u = 0$ with the Dirichlet data $\gamma_0\mathcal{K}_N(t)f$ where $f \in C^\infty(Y, S|Y)$. On the other hand, $\mathcal{K}_D(t)\gamma_0\mathcal{K}_N(t)f$ satisfies same conditions, so that $\mathcal{K}_D(t)\gamma_0\mathcal{K}_N(t) = \mathcal{K}_N(t)$. The second equality can be proved in a similar way. Now we have

$$\begin{aligned}
&\mathrm{Tr}\,\big(-\gamma_1\Delta(t)_D^{-1}\mathcal{K}_D(t)\gamma_0\mathcal{K}_N(t) + \gamma_1\Delta^c(t)_D^{-1}\mathcal{K}^c_D(t)\gamma_0\mathcal{K}^c_N(t)\big)\\
&= \mathrm{Tr}\,\big(-\gamma_1\Delta(t)_D^{-1}\mathcal{K}_N(t) + \gamma_1\Delta^c(t)_D^{-1}\mathcal{K}^c_N(t)\big)\\
&= \mathrm{Tr}\,\big(-\mathcal{K}_N(t)\gamma_1\Delta(t)_D^{-1} + \mathcal{K}^c_N(t)\gamma_1\Delta^c(t)_D^{-1}\big)\\
&= \mathrm{Tr}\,\big(-\mathcal{K}_N(t)\gamma_1(\Delta(t)^{-1} - \mathcal{K}_D(t)\gamma_0\Delta(t)^{-1}) + \mathcal{K}^c_N(t)\gamma_1(\Delta^c(t)^{-1} - \mathcal{K}^c_D(t)\gamma_0\Delta^c(t)^{-1})\big)\\
&= \mathrm{Tr}\,\big(-\mathcal{K}_N(t)\gamma_1\Delta(t)^{-1} + \mathcal{K}_D(t)\gamma_0\Delta(t)^{-1} + \mathcal{K}^c_N(t)\gamma_1\Delta^c(t)^{-1} - \mathcal{K}^c_D(t)\gamma_0\Delta^c(t)^{-1}\big)\\
&= \mathrm{Tr}\,\big(\Delta(t)_N^{-1} - \Delta(t)_D^{-1} - (\Delta^c(t)_N^{-1} - \Delta^c(t)_D^{-1})\big)
\end{aligned}$$

where we used (1.5) and the corresponding equalities for the other operators. Now the application of Lemma 4.3 completes the proof. □

We argue as above to obtain the corresponding statement for the ζ-determinants of Dirichlet and Neumann problems.

PROPOSITION 4.5. *The following equality holds,*

$$\begin{aligned}(4.3)\qquad &\frac{d}{dt}\Big(\ \ln\big(\det_\zeta\Delta(t)_N(\det_\zeta\Delta(t)_D)^{-1}\big)-\ln\big(\det_\zeta\Delta^c(t)_N(\det_\zeta\Delta^c(t)_D)^{-1}\big)\ \Big)\\ &= \mathrm{Tr}\left(\Delta(t)_N^{-1}-\Delta(t)_D^{-1}-\Delta^c(t)_N^{-1}+\Delta^c(t)_D^{-1}\right)\ .\end{aligned}$$

PROOF. We follow the proof of the Lemma 4.3 and obtain

$$\begin{aligned}&\frac{d}{dt}\Big(\ \ln\big(\det_\zeta\Delta(t)_N(\det_\zeta\Delta(t)_D)^{-1}\big)-\ln\big(\det_\zeta\Delta^c(t)_N(\det_\zeta\Delta^c(t)_D)^{-1}\big)\ \Big)\\ &=-\frac{d}{dt}\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}\mathrm{Tr}\left(e^{-u\Delta(t)_N}-e^{-u\Delta(t)_D}-e^{-u\Delta^c(t)_N}+e^{-u\Delta^c(t)_D}\right)du\\ &=\lim_{\varepsilon\to0}\mathrm{Tr}\left(\Delta(t)_N^{-1}e^{-\varepsilon\Delta(t)_N}-\Delta(t)_D^{-1}e^{-\varepsilon\Delta(t)_D}-\Delta^c(t)_N^{-1}e^{-\varepsilon\Delta^c(t)_N}+\Delta^c(t)_D^{-1}e^{-\varepsilon\Delta^c(t)_D}\right).\end{aligned}$$

To evaluate the limit as $\varepsilon\to0$ in the last line, we consider $\{G_t(u)\}$ a family of operators for $u\in[0,\infty)$ defined by

$$G_t(u):=\frac{1}{2\pi i}\int_\Gamma\frac{e^{-u\lambda}}{\lambda}\Big[\frac{1}{\lambda-\Delta(t)_N}-\frac{1}{\lambda-\Delta(t)_D}-\frac{1}{\lambda-\Delta^c(t)_N}+\frac{1}{\lambda-\Delta^c(t)_D}\Big]\,d\lambda$$

where Γ is a contour we use in the proof of Lemma 4.3. Corollary 3.4 implies that the integrand

$$\frac{e^{-u\lambda}}{\lambda}\Big[\frac{1}{\lambda-\Delta(t)_N}-\frac{1}{\lambda-\Delta(t)_D}-\frac{1}{\lambda-\Delta^c(t)_N}+\frac{1}{\lambda-\Delta^c(t)_D}\Big]$$

is a family of trace class operators with the norm of the size $O(\lambda^{-2})$ as $|\lambda|\to\infty$ for $u\in[0,\infty)$. By definition, this family is continuous for $u\in[0,\infty)$, so that $G_t(u)$ is a family of trace class operators continuous with respect to the trace class norm for $u\in[0,\infty)$. Therefore we have

$$\lim_{\varepsilon\to0}\mathrm{Tr}\left(G_t(\varepsilon)\right)=\mathrm{Tr}\left(G_t(0)\right)=\mathrm{Tr}\left(\Delta(t)_N^{-1}-\Delta(t)_D^{-1}-\Delta^c(t)_N^{-1}+\Delta^c(t)_D^{-1}\right)$$

where we used the Cauchy integral formula in the second equality. Thus, we conclude

$$\begin{aligned}&\frac{d}{dt}\Big(\ \ln\big(\det_\zeta\Delta(t)_N(\det_\zeta\Delta(t)_D)^{-1}\big)-\ln\big(\det_\zeta\Delta^c(t)_N(\det_\zeta\Delta^c(t)_D)^{-1}\big)\ \Big)\\ &=\mathrm{Tr}\left(\Delta(t)_N^{-1}-\Delta(t)_D^{-1}-\Delta^c(t)_N^{-1}+\Delta^c(t)_D^{-1}\right)\ .\end{aligned}$$

□

Now, Proposition 4.4 and Proposition 4.5 give the main result of Section 4

THEOREM 4.6. *If $\mathcal{N}$ is positive, we have*

$$(4.4)\qquad \frac{d}{dt}\ \ln\left(\frac{\det_\zeta\Delta(t)_N}{\det_\zeta\Delta(t)_D}\cdot\frac{\det_\zeta\Delta^c(t)_D}{\det_\zeta\Delta^c(t)_N}\right)=\frac{d}{dt}\ \ln\left(\frac{\det_\zeta\mathcal{N}(t)}{\det_\zeta\mathcal{N}^c(t)}\right)$$

for any $t\in\mathbb{R}^+$.

5. The quotient of the ζ-determinants

Identity (4.4) implies that

$$(5.1)\qquad \frac{\det_\zeta\Delta(t)_N}{\det_\zeta\Delta(t)_D}\cdot(\det_\zeta\mathcal{N}(t))^{-1}\ =\ C\ \frac{\det_\zeta\Delta^c(t)_N}{\det_\zeta\Delta^c(t)_D}\cdot(\det_\zeta\mathcal{N}^c(t))^{-1}\ ,$$

where C is a constant independent of t .

We have to show that C is 1 . The equality follows from the asymptotic expansion of $\ln \det_{\zeta} L(t)$ as the parameter $t \to \infty$, where $L(t)$ is a pseudo-differential operator with the parameter of weight k

$$\ln \det_{\zeta} L(t) \;\sim\; \sum_{j=-\dim N}^{\infty} a_j t^{-\frac{j}{k}} + \sum_{j=0}^{\dim N} b_j t^{\frac{j}{k}} \ln t \quad \text{as} \quad t \to \infty \ . \tag{5.2}$$

In formula (5.2) the coefficients a_j and b_j are determined by the local formulas in terms of the symbol of the operator $L(1)$ (see Burghelea, Friedlander and Kappeler [**4**]). The same formula holds for the elliptic, self-adjoint differential operator with the local elliptic boundary condition. We apply (5.2) to the families of operators $\Delta(t)_D,\ \Delta(t)_N,\ \mathcal{N}(t)$ and $\Delta^c(t)_D,\ \Delta^c(t)_N,\ \mathcal{N}^c(t)$. We denote by $a_D,\ a_N,\ a_{\mathcal{N}}$ and $a^c_D,\ a^c_N,\ a^c_{\mathcal{N}}$ the constant terms of these expansions. Then the following equality holds

$$\ln C \;=\; (a_D - a_N - a_{\mathcal{N}}) - (a^c_D - a^c_N - a^c_{\mathcal{N}}) \ \ .$$

The constant terms a_D, a_N, a^c_D, a^c_N are locally computable from the symbols of $\Delta(1)_D$, $\Delta(1)_N$, $\Delta^c(1)_D$, $\Delta^c(1)_N$ so

$$(a_D - a_N) - (a^c_D - a^c_N) \;=\; (a_D - a^c_D) - (a_N - a^c_N) \;=\; 0 \ \ .$$

Let us remark that the boundary contributions near $u = 0$ of $a_D,\ a^c_D$ $(a_N,\ a^c_N)$ cancel each other and the contributions out of $u = 0$ of $a_D,\ a_N$ $(a^c_D,\ a^c_N)$ cancel each other. It also follows from Corollary 3.3 that the following equality holds

$$a_{\mathcal{N}} - a^c_{\mathcal{N}} \;=\; 0 \ \ .$$

We conclude that C is equal to 1 . In particular, if we put $t = 0$ in (5.1) then

$$\frac{\det_{\zeta}\Delta_N}{\det_{\zeta}\Delta_D}\cdot(\det_{\zeta}\mathcal{N})^{-1} = \frac{\det_{\zeta}\Delta^c_N}{\det_{\zeta}\Delta^c_D}\cdot(\det_{\zeta}\mathcal{N}^c)^{-1} \ \ . \tag{5.3}$$

Now the following Proposition completes the proof of Theorem 0.1.

Proposition 5.1. *The following equality holds,*

$$\frac{\det_{\zeta}\Delta^c_N}{\det_{\zeta}\Delta^c_D} = \det_{\zeta}\mathcal{N}^c \ \ .$$

Proof. We can make explicit computations of the ζ-determinants of the Laplace type operators over the finite cylinder to obtain

$$\det_{\zeta}\Delta^c_N = 2^{h_Y}\cdot \exp(-(2\sqrt{\pi})^{-1}(\Gamma(s)^{-1}\Gamma(s-1/2)\zeta_{\Delta_Y}(s-1/2))'(0))\cdot \prod_{l=h_Y+1}^{\infty}(1+e^{-2\mu_l}) \ ,$$

$$\det_{\zeta}\Delta^c_D = 2^{h_Y}\left(\det_{\zeta}\sqrt{\Delta_Y}\right)^{-1}$$
$$\cdot \exp(-(2\sqrt{\pi})^{-1}(\Gamma(s)^{-1}\Gamma(s-1/2)\zeta_{\Delta_Y}(s-1/2))'(0))\cdot \prod_{l=h_Y+1}^{\infty}(1-e^{-2\mu_l}) \ ,$$

where $h_Y = \dim\ker(\Delta_Y)$ and $\{\mu_l^2\}$ is the spectrum of Δ_Y. The computations of $\det_{\zeta}\Delta^c_D$ are performed in section 5 of [**7**] and $\det_{\zeta}\Delta^c_N$ can be computed in a similar way. We have

$$\frac{\det_{\zeta}\Delta^c_N}{\det_{\zeta}\Delta^c_D} = \det_{\zeta}\sqrt{\Delta_Y}\cdot \prod_{l=h_Y+1}^{\infty}\frac{1+e^{-2\mu_l}}{1-e^{-2\mu_l}}.$$

We follow computations from section 5 of [**7**] to obtain the following formula

$$\det_\zeta \mathcal{N}^c = \det_\zeta \sqrt{\Delta_Y} \cdot \prod_{l=h_Y+1}^{\infty} \frac{1+e^{-2\mu_l}}{1-e^{-2\mu_l}}.$$

Now the proof is complete. □

References

[1] Agranovich, M. S.: 1965, 'Elliptic singular integro-differential operators', *Uspehi Mat. Nauk.* **20**, 1–121.

[2] Agranovich, M. S. and Dynin, A. S.: 1962, 'General boundary value problems for elliptic systems in an n-dimensional domain', *Dokl. Akad. Nauk. SSR* **146**, 511–514.

[3] Booß–Bavnbek, B. and Wojciechowski, K.P.: 1993, *Elliptic Boundary Problems for Dirac Operators*, Birkhäuser, Boston.

[4] Burghelea, D., Friedlander, L. and Kappeler, T.: 1992, 'Mayer-Vietoris type formula for determinants of differential operators', *J. Funct. Anal.* **107**, 34-65.

[5] Calderón, A. P.: 1963, 'Boundary value problems for elliptic equations', *Outlines of Joint Soviet-American Symposium on Partial Differential Equations*, 303–304.

[6] Forman, R.: 1987, 'Functional determinants and geometry', *Invent. Math.* **88**, 447 –493.

[7] Loya, P. and Park, J.: 2003, 'Decomposition of the ζ-determinant for the Laplacian on manifolds with cylindircal end', *Preprint.*

[8] Park, J. and Wojciechowski, K. P.: 2002, 'Analytic surgery of the ζ-determinant of the Dirac operator', *Nuclear Physics B. Proc. Supp.* **104** , 89–115.

[9] Park, J. and Wojciechowski, K. P.: 2003, 'Adiabatic decomposition of the ζ-determinant and Dirichlet to Neumann operator'. *Preprint, DG/0301170.*

[10] Ray, D. and Singer, I.M.: 1971, 'R–torsion and the Laplacian on Riemannian manifolds', *Adv. Math.* **7**, 145–210.

[11] Scott, S. G. and Wojciechowski, K. P.: 1999, 'The ζ-determinant and Quillen determinant for a Dirac operator on a manifold with boundary', *Geom. Funct. Anal.* **10**, 1202–1206.

[12] Seeley, R. T.: 1966, 'Singular integrals and boundary value problems', *Amer. J. Math.* **88**, 781–809.

[13] Seeley, R. T.: 1969, 'Topics in pseudo-differential operators' in: CIME *Conference on Pseudo-Differential Operators*, Ed. Cremonese, Rome, pp. 167–305.

[14] Taylor, M.: 1996, *Partial Differential Equations. Basic Theory*, Springer Verlag, New York.

Mathematisches Institut, Universität Bonn, Beringstrasse 1, D-53115 Bonn, Germany
E-mail address: jpark@math.uni-bonn.de

Department of Mathematics, IUPUI (Indiana/Purdue), Indianapolis IN 46202–3216, U.S.A.
E-mail address: kwojciechowski@math.iupui.edu

Received by the editors January 3, 2004; revised April 26, 2004.