 

# Ruelle zeta function and Prime geodesic theorem for hyperbolic manifolds with cusps

Jinsung Park

**Abstract.** For a $d$-dimensional real hyperbolic manifold with cusps, we obtain more refined error terms in the prime geodesic theorem (PGT) using the Ruelle zeta function instead of the Selberg zeta function. To do this, we prove that the Ruelle zeta function over this type manifold is a meromorphic function of order $d$ over $\mathbb{C}$.

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## 1 Introduction

Let $X_\Gamma$ be a $d$-dimensional locally symmetric space given by $X_\Gamma = \Gamma\backslash G/K$ where $G$ is a semi-simple Lie group of rank 1 and $K$ is a maximal compact subgroup of $G$, and $\Gamma$ is a discrete torsion free subgroup of $G$. We also assume that the Riemannian metric over $X_\Gamma$ induced from the Killing form is normalized so that the sectional curvature of $X_\Gamma$ is $-1$. Now let us recall that a prime geodesic $C_\gamma$ over $X_\Gamma$ corresponds to a conjugacy class of a primitive hyperbolic element $\gamma \in \Gamma$. Let $\pi_\Gamma(x)$ denote a function counting the prime geodesic $C_\gamma$ of length $l(C_\gamma)$ whose norm $N(\gamma) = e^{l(C_\gamma)}$ is not larger than $x$. Then the prime geodesic theorem (PGT) states

$$\pi_\Gamma(x) \ \sim \ \frac{x^{d-1}}{(d-1)\log x} \tag{1.1}$$

where $f(x) \sim g(x)$ means $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. This was proved by Gangolli [9] and DeGeorge [5] independently when $X_\Gamma$ is compact, and by Gangolli–Warner [10] when $X_\Gamma$ has a finite volume.

In [13], [14], Hejhal extensively studied the Selberg zeta function over a hyperbolic Riemann surface $X_\Gamma$, that is, when $\Gamma$ is a co-finite discrete subgroup of $G = \mathrm{PSL}(2,\mathbb{R})$. Applying these results, Hejhal proved PGT with error terms (see also [15], [19], [24]),

$$\pi_\Gamma(x) = \sum_{\frac{3}{4}<s_n\le 1} \mathrm{li}(x^{s_n}) + \mathrm{O}(x^{\frac{3}{4}}(\log x)^{-\frac{1}{2}}) \tag{1.2}$$

where $\lambda_n = s_n(1-s_n)$ is a small eigenvalue in $[0, \frac{3}{16}]$ of the Laplacian $\Delta_0$ acting on $L^2(X_\Gamma)$, and $\mathrm{li}(x) := \int_2^x \frac{1}{\log t}\, dt$. Recalling the leading term of $\mathrm{li}(x)$ is $\frac{x}{\log x}$ as $x \to \infty$, it is easy to see that PGT (1.2) is a refinement of (1.1) for a special case of co-finite $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$.

Since the size of the error term in (1.2) is given by the small eigenvalues of $\Delta_0$, and the estimate of the small eigenvalue for a specific arithmetic case is one of the main problems in analytic number theory, there have been many works, for instance the work of Iwaniec [17] and Luo-Sarnak [18], to obtain the optimal size of the error term for such a specific arithmetic discrete subgroup $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$.

Comparing PGT in (1.1) with the one in (1.2), one can try to obtain the corresponding error terms in (1.1) as in (1.2) for a $d$-dimensional locally symmetric space $X_\Gamma$ of rank 1. A plausible approach for this is to use the Selberg zeta function $Z_\Gamma(s)$ following Hejhal [13, 14] or Randol [24] as they did for hyperbolic Riemann surfaces, then one may believe that the poles of $\frac{d}{ds}\log Z_\Gamma(s)$ over the strip $\frac{1}{2}(d-1) < \mathrm{Re}(s) \leq (d-1)$ would provide us with error terms for PGT as in (1.2). However, this approach using the Selberg zeta function $Z_\Gamma(s)$ provide us with only the error terms corresponding to the poles in the strip $(d-2) < \mathrm{Re}(s) \leq (d-1)$, hence the Selberg zeta function $Z_\Gamma(s)$ is not sufficient to obtain the expected error terms if $d > 3$. In the section 5 of [10], this is explained in the view point of the Tauberian theorem of Wiener-Ikehara. On the other hand, reflecting on this section of [10], one can see that a meromorphic extension of the Ruelle zeta function

$$R_\Gamma(s) := \prod_{\gamma\in \mathrm{P}\Gamma_{\mathrm{h}}} \left(1 - e^{-s\,l(C_\gamma)}\right)^{-1} \qquad \text{for} \quad \mathrm{Re}(s) > (d-1)$$

can be used to obtain such error terms in (1.1). Here $\mathrm{P}\Gamma_{\mathrm{h}}$ denotes the set of conjugacy classes of a primitive hyperbolic element $\gamma$ in $\Gamma$. This is also pointed out in the last part of the section 4 of [8]. We also refer to the work of Parry-Pollicott [22] where they used the Ruelle zeta function for an axiom A flow to derive PGT.

The main purpose of this paper is to refine the result of Gangolli-Warner [10] following suggestion of Fried [8] for real hyperbolic manifold $X_\Gamma$ with cusps. Here a real hyperbolic manifold $X_\Gamma$ with cusps is given as follows: Let us put $\Gamma \subset G = \mathrm{SO}_0(d,1)$ be a discrete co-finite torsion free subgroup of $G$. Now let us denote by $\mathcal{P}_\Gamma$ the set of $\Gamma$-conjugacy classes of $\Gamma$-cuspidal parabolic subgroups in $G$. We assume that the discrete subgroup $\Gamma$ satisfies the condition

$$\Gamma_P := \Gamma \cap P = \Gamma \cap N(P) \qquad \text{for} \quad P \in \mathcal{P}_\Gamma$$

where $N(P)$ denotes the nilpotent part of $P$. Now the resulting manifold $X_\Gamma = \Gamma\backslash G/K$ is a $d$-dimensional real hyperbolic manifold with cusps. In our approach, a main ingredient to obtain the error terms of PGT over $X_\Gamma$ is the Ruelle zeta function $R_\Gamma(s)$. Recently in [11], it is shown that $R_\Gamma(s)$ can be extended as a meromorphic function over $\mathbb{C}$ with precise description of the locations of zeros and poles. For our purpose concerning PGT, we need the following basic fact of $R_\Gamma(s)$.

**Theorem 1.1.** *For a $d$-dimensional real hyperbolic manifold $X_\Gamma$ with cusps, a meromorphic extension over $\mathbb{C}$ of the Ruelle zeta function $R_\Gamma(s)$ has an expression*

$$R_\Gamma(s) = \frac{P_\Gamma(s)}{Q_\Gamma(s)}$$

*where $P_\Gamma(s)$, $Q_\Gamma(s)$ are entire functions of order $d$ over $\mathbb{C}$.*

A proof of Theorem 1.1 is given in the section 2. Actually we will prove Theorem 1.1 for the Ruelle zeta function twisted by a special unitary representation $\chi$ of $\Gamma$. This can be used for PGT in a fixed homology class, which would be a refinement of [1], [7], [23] for real hyperbolic manifolds with cusps. A detail of this application will be given elsewhere.

To state the main result about PGT, we introduce some notations. Let us consider an Iwasawa decomposition $G = NAK$ with a decomposition of Lie algebra $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$. The subgroup $M = \mathrm{SO}(d-1)$ is defined to be a maximal subgroup of $K = \mathrm{SO}(d)$ which commutes $A$. Let us denote by $\sigma_k$ the fundamental representation of $M$ acting on $\wedge^k \mathbb{R}^{d-1} \otimes \mathbb{C}$. When $d = 2n+1$, $j = n$, $\sigma_n$ denotes a direct sum of half spin representations $\sigma_+ \oplus \sigma_-$. We denote by $\Delta_k$ the Laplacian acting on the space of $k$-forms over $X_\Gamma$, which decomposes into the subspaces where the principal series representation $\pi_{\sigma_j,\lambda}$ acts for $j = (k-1), k$. By Theorem 1.1 and a modification of the proof in [13], [14], we prove

**Theorem 1.2.** *For a $d$-dimensional real hyperbolic manifold $X_\Gamma$ with cusps, we have PGT* with error terms,

$$\pi_\Gamma(x) = \sum_{\frac{3}{2}d_0 < s_n(k) \le 2d_0} (-1)^k \mathrm{li}(x^{s_n(k)}) + \mathrm{O}\Big(x^{\frac{3}{2}d_0} (\log x)^{-\frac{1}{2}}\Big) \tag{1.3}$$

*where $d_0 = \frac{d-1}{2}$ and $(s_n(k)-k)(2d_0-k-s_n(k))$ is a small eigenvalue in $[0, \frac{3}{4}d_0^2]$ of $\Delta_k$ on $\pi_{\sigma_k,\lambda_n(k)}$ with $s_n(k) = d_0 + i\lambda_n(k)$ or $s_n(k) = d_0 - i\lambda_n(k)$ in $(\frac{3}{2}d_0, 2d_0]$.*

A proof of Theorem 1.2 is given in the section 3. Theorem 1.2 is a refinement of the corresponding result of Gangolli-Warner [10] with error terms for real hyperbolic manifolds with cusps. According to the last part of the section 4 of [8], it seems that Fried also obtained the corresponding result to Theorem 1.2 for compact case in his unpublished note. A new feature in Theorem 1.2 is a signature $(-1)^k$ of the terms $\mathrm{li}(x^{s_n(k)})$ depending on $k$. By Theorem 4.6 in [11] and (2.1), for a small eigenvalue corresponding to $s_n(k) \in (2d_0-1, 2d_0]$, we have $k = 0$, hence when $d = 2$ the signature $(-1)^k$ is always $(-1)^0 = 1$ as we expect from (1.2).

**Remark 1.3.** In general, some parts of cuspidal and residual spectrum can appear as small eigenvalues of $\Delta_k$ (see the chapter 8 of [26] or [27]). We also refer to the section 6 of [3] for an example with small eigenvalues of $\Delta_0$. It is interesting to generalize this result to the case of $\Delta_k$ for $k \ge 1$ since this would imply the leading terms of (1.3) are effective.

**Remark 1.4.** Recently a related work to Theorem 1.2 was given in [21] where the author derived $\Omega_{\pm}$-estimate for error terms of PGT using the Selberg zeta function under some condition for the scattering determinant. According to the aforementioned remark, this approach seems to be problematic to obtain $\Omega_{\pm}$-estimate for error terms in principle.

# 2 Order of the Ruelle zeta function

## 2.1 Twisted Ruelle zeta function

For a special unitary representation $(\chi, V_\chi)$ of $\Gamma = \pi_1(X_\Gamma)$, we define a twisted Ruelle zeta function

$$R_\chi(s) := \prod_{\gamma \in \mathrm{P}\Gamma_{\mathrm{h}}} \det\Big(\mathrm{Id} - \chi(\gamma) e^{-s\, l(C_\gamma)}\Big)^{-1} \qquad \text{for} \quad \mathrm{Re}(s) > (d-1).$$

Here the notation “det" denotes the determinant taken over $V_\chi$. The Selberg zeta function attached to $\sigma_k$ is defined by

$$Z_\chi(\sigma_k, s) := \exp\Big( -\sum_{\gamma \in \Gamma_{\mathrm{h}}} \mathrm{tr}\, \chi(\gamma) j(\gamma)^{-1} D(\gamma)^{-1} \overline{\mathrm{tr}\, \sigma_k(m_\gamma)} e^{-(s-\frac{d-1}{2}) l(C_\gamma)} \Big)$$

for $\mathrm{Re}(s) > (d-1)$. Here $\Gamma_{\mathrm{h}}$ denotes the set of the $\Gamma$-conjugacy classes of the hyperbolic elements in $\Gamma$, $j(\gamma)$ denotes the positive integer such that $\gamma = \gamma_0^{j(\gamma)}$ with a primitive $\gamma_0 \in \mathrm{P}\Gamma_{\mathrm{h}}$. We may assume that a hyperbolic element $\gamma \in \Gamma$ has the form $a_\gamma m_\gamma \in A^+ M$ and

$$D(\gamma) = D(a_\gamma m_\gamma) = a_\gamma^\rho \left| \det\left(\mathrm{Ad}(a_\gamma m_\gamma)^{-1} - \mathrm{Id}|_{\mathfrak{n}}\right) \right|$$

where $a_\gamma^\rho = \exp(d_0 t_\gamma H)$ if $a_\gamma = \exp(t_\gamma H)$ for a normalized $H \in \mathfrak{a}$.

For $\mathrm{Re}(s) > (d-1)$, the following relationship of $R_\chi(s)$ between $Z_\chi(\sigma_k, s)$'s holds,

$$R_\chi(s) = \prod_{k=0}^{d-1} Z_\chi(\sigma_k, s+k)^{(-1)^{k+1}} \tag{2.1}$$

where if $d = 2n+1$, $Z_\chi(\sigma_n, s+n)$ denotes $Z_\chi(\sigma_n^+, s+n) Z_\chi(\sigma_n^-, s+n)$ with the half spinor representations $\sigma_n^\pm$ of $M = \mathrm{SO}(2n)$. In [11], it is shown that $Z_\chi(\sigma_k, s)$ has a meromorphic extension over $\mathbb{C}$ with a precise description of locations of zeros and poles. Hence the Ruelle zeta function $R_\chi(s)$ has a meromorphic extension over $\mathbb{C}$ by (2.1).

### 2.2 Divisors of $Z_\chi(\sigma_k, s)$

In order to prove Theorem 1.1, it is sufficient to prove the same for $Z_\chi(\sigma_k, s)$ by (2.1). For $Z_\chi(\sigma_k, s)$, from Theorem 4.6 in [11], we have

**Proposition 2.1.** *For $0 \le k \le [d_0]$, there exist two entire functions $F_k(s), G_k(s)$ of order $d$ respectively such that $Z_\chi(\sigma_k, s)F_k(s)G_k(s)^{-1}$ is an entire function over $\mathbb{C}$ with zeros at*

(1) $s_j = d_0 \pm i\lambda_j$ *of order* $m_j$ *where* $\lambda_j^2 + (d_0 - k)^2$ *is an eigenvalue with multiplicity* $m_j$ *of* $\Delta_k$ *on* $\pi_{\sigma_k,\lambda_j}$ *and*

(2) $s_\ell = d_0 + q_\ell$ *of order* $b_\ell \binom{d-1}{k}$ *where* $\det C_\chi^k(\sigma_k, s)$ *has a pole at* $s = q_\ell$ *of order* $b_\ell$ *with* $\mathrm{Re}(q_\ell) < 0$.

*Here $C_\chi^k(\sigma_k, s)$ denotes $\tau_k$-isotypic component of the scattering operator $C_\chi(\sigma_k, s)$ for the fundamental representation $\tau_k$ of $K$ acting on $\wedge^k\mathbb{R}^d \otimes \mathbb{C}$.*

Let $S_1, S_2$ denote respectively the sets consisting of the zeros of $Z_\chi(\sigma_k, s)F_k(s)G_k(s)^{-1}$ appearing in $(1)$, $(2)$ of Proposition 2.1. By the result in [6], we have

$$\sum_{s_j \in S_1} |s_j|^{-d-\epsilon} < \infty$$

for any $\epsilon > 0$. Hence, by Theorem 2.6.5 in [2], the following canonical product over $S_1$ is an entire function of order $d$ over $\mathbb{C}$,

$$P_{S_1}(s) = \prod_{j=1}^{\infty} E\Big(\frac{s}{s_j}, d\Big)$$

where

$$E(u, d) = (1 - u)\exp\Big(u + \frac{u^2}{2} + \ldots + \frac{u^d}{d}\Big).$$

The set $S_2$ is a subset of the set of the zeros of an entire function $B(s)$ (see (2.2)) of order $d$ by Proposition 2.2. Noting that $\det C_\chi^k(\sigma_k, s)$ has finitely many poles over the half plane $\mathrm{Re}(s) > 0$, by the Hadamard's factorization theorem for $B(s)$, the following canonical product over $S_2$ is also an entire function of order $d$ over $\mathbb{C}$,

$$P_{S_2}(s) = \prod_{\ell=1}^{\infty} E\Big(\frac{s}{s_\ell}, d\Big).$$

Therefore, we see that $Z_\chi(\sigma_k, s)F_k(s)G_k(s)^{-1}P_{S_1}(s)^{-1}P_{S_2}(s)^{-1}$ is an entire function and has no zeros over $\mathbb{C}$. Hence, by the Hadamard's factorization theorem there exists a polynomial $g(s)$ such that

$$Z_\chi(\sigma_k, s)F_k(s)G_k(s)^{-1}P_{S_1}(s)^{-1}P_{S_2}(s)^{-1} \equiv \exp(g(s)) \qquad \text{for } s \in \mathbb{C}.$$

On the other hand, from the proof of Theorem 4.6 in [11], we know

$$\frac{d}{ds}\log\big(Z_\chi(\sigma_k,s)F_k(s)G_k(s)^{-1}\big)\to 0 \qquad \text{as } \ s\to\infty.$$

Hence the order of $g(s)$ should be $d$. This completes the proof of Theorem 1.1.

### 2.3 Order of $\det C^k_\chi(\sigma_k, s)$

In this subsection, we prove

**Proposition 2.2.** *A meromorphic extension over* $\mathbb{C}$ *of* $\det C^k_\chi(\sigma_k,s)$ *defined a priori for* $\mathrm{Re}(s) > (d-1)$ *has an expression*

$$\det C^k_\chi(\sigma_k,s) = \frac{A(s)}{B(s)} \tag{2.2}$$

*where* $A(s)$, $B(s)$ *are entire functions of order* $d$ *over* $\mathbb{C}$.

In the remaining part of this subsection, we present detail of a proof of Proposition 2.2. In fact, the following proof is a simple modification of the proof of Selberg in [25] to a higher dimensional case where we employ the Colin de Verdiere's method for the analytic continuation of the resolvent of the Pseudo-Laplacian in [4].

We decompose $X_\Gamma$ as

$$X_\Gamma = X_0 \cup C_1 \cup \ldots \cup C_p$$

where $X_0$ is a compact manifold with boundaries $T^{d-1}$'s and $C_j \cong [r_j,\infty)\times T^{d-1}$. The metric $dg^2$ over $C_j$ induced from the normalized Cartan-Killing form has the form,

$$dg^2|_{C_j} = dr^2 + e^{-2r}dw^2$$

where $dw^2$ is the metric over the flat torus $T^{d-1}$ induced from the Cartan-Killing form.

Let us recall that there exists a Γ-cuspidal parabolic subgroup $P_j$ which fixes the infinity of $C_j$. Let $V_j$ be a maximal subspace of $V_\chi$ where $\chi|_{\Gamma\cap P_j}$ acts trivially. For simplicity, we assume that $\dim V_j = 1$ for the following proof. When $\chi$ is trivial, this condition is satisfied. The general case can be proved by a minor modification.

Let us choose $\phi(r) \in C^\infty_0(\mathbb{R})$ such that $\phi(r) = 0$ for $r \le r_0$ and $\phi(r) = 1$ for $r \ge r_0 + 1$ where $r_0 = \max\{r_1,\ldots,r_p\}$. For $j \in \{1,2,\ldots,p\}$ and $s \in \mathbb{C}$, we put

$$\Theta_j(x,s) = \begin{cases} 0 & \text{if } \ x \in X_\Gamma - C_j, \\ \phi(r)e^{(d_0+s)r}v & \text{if } \ x=(r,y)\in C_j \end{cases}$$

for a fixed $v \in V_j \otimes \mathcal{H}(\sigma_k,\tau_k)$ where $\mathcal{H}(\sigma_k,\tau_k)$ denotes the $\tau_k$-isotypic component of $\mathcal{H}(\pi(\sigma_j,\lambda_j))$ for a representation $\pi(\sigma_j,\lambda_j)$ of $P_j$. We refer to the section 2.6 of [11] for more detail. Let us put

$$\Phi_j(x,s) := \big(\Delta_k - (s-k)(2d_0-k-s)\big)\Theta_j(x,s)$$

where $\Delta_k$ denotes the twisted Laplacian acting on $\Omega^k(X_\Gamma, V_\chi)$. Note that $\Phi_j \in \Omega^k(X_\Gamma, V_\chi)$ has a compact support, and the generalized eigensection $E_j(x,s)$ is given by

$$E_j(x,s) = \Theta_j(x,s) - \big(\overline{\Delta}_k - (s-k)(2d_0-k-s)\big)^{-1}\Phi_j(x,s)$$

for $\mathrm{Re}(s) > d_0$. Here $\overline{\Delta}_k$ denotes the self adjoint extension of $\Delta_k$ acting on $\Omega_0^k(X_\Gamma, V_\chi)$ to its $L^2$-closure denoted by $\mathrm{H} := L^2(X_\Gamma, \wedge^k(T^*X_\Gamma)\otimes V_\chi)$. (Although H depends on $k, \chi$, we omit these indices for simplicity.)

For $a > r_0$, let

$$\mathrm{H}_a := \big\{\, \Phi \in H^1(X_\Gamma, \wedge^k(T^*X_\Gamma)\otimes V_\chi) \mid \Phi_j^0|_{(a,\infty)} \equiv 0,\ \ j = 1, \dots, p \,\big\}$$

where $\Phi_j^0(r) := \int_{T^{d-1}} \Phi(r,y)\,dy$ for $(r,y) \in (r_j,\infty)\times T^{d-1}$. Let $\mathcal{H}_a$ be the $L^2$-closure of $\mathrm{H}_a$ in H. Let us recall that $\Delta_k$ has an form

$$\Delta_k = -\mathrm{D}^2 + \lambda_k$$

for a self-adjoint elliptic differential operator D and a constant $\lambda_k$ by Proposition 1.1 in [20]. Now let us consider the quadratic form $Q_a$ on $\mathrm{H}_a$ given by

$$Q_a(\Phi) = ||\mathrm{D}\Phi||^2, \qquad \text{for } \ \Phi \in \mathrm{H}_a.$$

This quadratic form $Q_a$ is closed and therefore it is represented by a self-adjoint operator $\mathrm{D}_a^2$ on $\mathcal{H}_a$, and we put $\widetilde{\Delta}_k = -\mathrm{D}_a^2 + \lambda_k$ on $\mathcal{H}_a$. Then $\widetilde{\Delta}_k$ has a pure point spectrum and the resolvent $R_k(s) = \big(\widetilde{\Delta}_k - (s-k)(2d_0-k-s)\big)^{-1}$ is a compact operator on $\mathcal{H}_a$. For $a > r_0 + 2$, observing $\Phi_j \in \mathrm{H}_a$, we put

$$F_j(x,s) := \Theta_j(x,s) - R_k(s)\Phi_j(x,s). \tag{2.3}$$

The zeroth Fourier coefficient $F_{ij}^0(x,s) := F_i^0(x,s)|_{C_j}$ has the form

$$F_{ij}^0(x,s) = \begin{cases} \delta_{ij}\, e^{(d_0+s)r}\, v & \text{if } \quad r > a, \\ e^{(d_0+s)r}A_{ij}(s)v + e^{(d_0-s)r}B_{ij}(s)v & \text{if } \ a_j \le r \le a. \end{cases} \tag{2.4}$$

We put $C_j^0 \cong [a,\infty)\times T^{d-1}$ to be the subset of $C_j$ and $X_0^0 = X_\Gamma - \cup_{j=1}^p C_j^0$. We consider a Neumann Laplacian $\Delta_k^n$, which is the self-adjoint extension of $\Delta_k$ acting on $C_c^\infty(X_0^0) \oplus \oplus_{j=1}^p C_c^\infty(C_j^0)$ with the Neumann boundary conditions. We repeat the above construction of $\widetilde{\Delta}_k$ for $\Delta_k^n$ and denote by $\widetilde{\Delta}_k^n$ the resulting operator. Then $\widetilde{\Delta}_k^n$ satisfies the usual Weyl law because the restrictions of $\widetilde{\Delta}_k^n$ over each components satisfy this, for instance, over $C_j^0$ it follows from [6]. By the min-max principle, $j$-th eigenvalue of $\widetilde{\Delta}_k^n$ is not larger than $j$-th eigenvalue of $\widetilde{\Delta}_k$. Therefore, we can see

$$\#\big\{\, \mu_n \in \widetilde{S}_k \mid 0 < \mu_n \le \lambda \,\big\} \le C\lambda^{\frac{d}{2}} \tag{2.5}$$

where $\widetilde{S}_k = \{\mu_n \,|\, n \in \mathbb{N}\}$ denotes the spectrum of $\widetilde{\Delta}_k$. Note that there is no zero eigenvalue of $\widetilde{\Delta}_k$ since the corresponding harmonic form can not belong to $\mathcal{H}_a$. By (2.5),

$$\sum_{n=1}^{\infty} \mu_n^{-(\frac{d}{2}+\epsilon)} < \infty$$

for a $\epsilon > 0$. We put

$$\widehat{S}_k := \{\, \lambda_\ell, \mid \lambda_\ell \text{ is a root of } \mu_n = (s-k)(2d_0-k-s) \text{ for some } n \in \mathbb{N} \,\}.$$

As before, by Theorem 2.6.5 in [2], the canonical product over the set $\widehat{S}_k$,

$$P_{\widehat{S}_k}(s) = \prod_{\ell=1}^{\infty} E\Big(\frac{s}{\lambda_\ell}, d\Big)$$

is an entire function of order $d$ over $\mathbb{C}$. Putting $P(s) := (s-d_0)P_{\widehat{S}_k}(s)$ (we omit the index $k$ of $P(s)$ for simplicity), it is easy to see that $P(s)R_k(s)\mathbf{\Phi}_j(x,s)$ is entire. Moreover, from (2.3) and (2.4), it follows that $P(s)A_{ij}(s)$, $P(s)B_{ij}(s)$ are also entire functions over $\mathbb{C}$.

Recall that $\widetilde{\Delta}_k\mathbf{\Phi} = \Delta_k\mathbf{\Phi}$ for $\mathbf{\Phi} \in \mathrm{H}_a \subset \mathrm{H}$. Then it is easy to see that for $\mathbf{\Psi} \in \mathrm{H}$,

$$\begin{aligned} &\langle R_k(s)\mathbf{\Phi}_j(x,s), \mathbf{\Psi}(x)\rangle \\ =&\sum_{n=1}^{\infty} \big(\mu_n(\mu_n - (s-k)(2d_0-k-s))\big)^{-1} \langle \Delta_k\mathbf{\Phi}_j, \mathbf{\Psi}_n\rangle \cdot \langle \mathbf{\Psi}_n, \mathbf{\Psi}\rangle \end{aligned} \qquad (2.6)$$

where $\mathbf{\Psi}_n$ denote the eigenfunction of $\widetilde{\Delta}_k$ corresponding to $\mu_n$ for $n \in \mathbb{N}$. We put

$$E_n(s) := \prod_{(\lambda_\ell-k)(2d_0-k-\lambda_\ell)\neq\mu_n} E\Big(\frac{s}{\lambda_\ell}, d\Big).$$

By Theorem 1.11 in [12], there exist $C_1, C_2 > 0$ such that for $z \in \mathbb{C}$,

$$|E_n(s)| \le C_1 \exp\big(C_2|s|^d \log^+|s|\big) \qquad (2.7)$$

where $\log^+ x = \log x$ for $x \ge 1$ and $\log^+ x = 0$ for $x \le 1$. By (2.6) and (2.7), there exist $C_1, C_2 > 0$ such that for $\mathbf{\Psi} \in \mathrm{H}$,

$$|P(s)| \cdot |\,\langle R_k(s)\mathbf{\Phi}_j(x,s), \mathbf{\Psi}(x)\rangle\,| \;\le\; C_1 \exp\big(C_2|s|^d \log^+|s|\big)\, ||\mathbf{\Psi}||.$$

In particular, this implies that for $\mathbf{\Psi}$ with support in $(a_j, a) \times T^{d-1} \subset C_j$,

$$|P(s)| \cdot |\,\langle F_j(x,s), \mathbf{\Psi}(x)\rangle\,| \;\le\; C_1 \exp\big(C_2|s|^d \log^+|s|\big)\, ||\mathbf{\Psi}||.$$

Choosing $\mathbf{\Psi}(x) = e^{(d_0+\bar{s})r}\frac{d}{dr}\big(e^{-2\bar{s}r}g(r)\big)v$ for $g(r) \in C_0^\infty((a_j,a))$, we conclude

$$|P(s)A_{ij}(s)| \le C_3 \exp\big(C_2|s|^d \log^+|s|\big) \qquad (2.8)$$

for a constant $C_3 > 0$ and in a similar way,

$$|P(s)B_{ij}(s)| \le C_3 \exp\big(C_2|s|^d \log^+|s|\big). \tag{2.9}$$

Let $\psi_j(x)$ be the characteristic function of $[a,\infty)\times T^{d-1}$ and set

$$G_i(x,s) := F_i(x,s) + \sum_{j=1}^{p} \psi_j(x)\Big(e^{(d_0+s)r_j}A_{ij}(s)v + e^{(d_0-s)r_j}B_{ij}(s)v - \delta_{ij}e^{(d_0+s)r_j}v\Big)$$

where $r_j$ denotes the $r$-coordinate over $C_j$. For $\mathcal{A}(s) = \big(A_{ij}(s)\big)$, $\mathcal{B}(s) = \big(B_{ij}(s)\big)$, and $p\times p$ matrices $\mathcal{E}(x,s)$, $\mathcal{G}(x,s)$ whose $j$-th rows are given by $E_j(x,s)$, $G_j(x,s)$ respectively, we have

$$\mathcal{E}(x,s) = \mathcal{A}(s)^{-1}\mathcal{G}(x,s), \qquad C^k_\chi(\sigma_k,s) = \mathcal{A}(s)^{-1}\mathcal{B}(s).$$

Therefore,

$$\det C^k_\chi(\sigma_k,s) = \frac{\det \mathcal{B}(s)}{\det \mathcal{A}(s)} = \frac{P(s)^p \det \mathcal{B}(s)}{P(s)^p \det \mathcal{A}(s)}.$$

Putting $A(s) = P(s)^p \det \mathcal{A}(s)$, $B(s) = P(s)^p \det \mathcal{B}(s)$, by (2.8), (2.9), these are entire functions of order $d$ over $\mathbb{C}$. This completes the proof of Proposition 2.2.

# 3 Proof of Theorem 1.2

## 3.1 Counting function $\psi_n(x)$

Let us introduce

$$\psi_0(x) = \sum_{\gamma\in\Gamma_h, N(\gamma)\le x} \Lambda(\gamma)$$

where $\Lambda(\gamma) = l(C_{\gamma_0})$ with $\gamma = \gamma_0^n$ for $\gamma_0 \in \mathrm{P}\Gamma_h$, and

$$\psi_n(x) = \int_0^x \psi_{n-1}(t)\,dt \qquad \text{for } n = 1,2,3,\ldots.$$

Using Theorem A in [16], we can derive

$$\psi_n(x) = \frac{1}{n!}\sum_{\gamma\in\Gamma_h, N(\gamma)\le x} \Lambda(\gamma)\big(x - N(\gamma)\big)^n. \tag{3.1}$$

Now we relate $\psi_n(x)$ to the Ruelle zeta function $R_\Gamma(s)$ as follows. First of all, we have

$$\frac{d}{ds}\log R_\Gamma(s) = -\sum_{\gamma\in\Gamma_h} \Lambda(\gamma)N(\gamma)^{-s} \qquad \text{for } \mathrm{Re}(s) > (d-1). \tag{3.2}$$

Now, using (3.1), (3.2), and Theorem B in [16], for $n\ge 1$ we obtain

$$\psi_n(x) = -\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=c} \frac{x^{s+n}}{s(s+1)(s+2)\ldots(s+n)}\frac{d}{ds}\log R_\Gamma(s)\,ds \tag{3.3}$$

where $c > (d-1)$.

### 3.2 Asymptotics of $\psi_{2d_0}(x)$

For $T \gg 0$, let $R(T)$ be a closed domain given by

$$R(T) = \{ \, s \in \mathbb{C} \, \big| \, |s| \le T, \ \text{Re}(s) \le d_0 \, \} \bigcup \{ \, s \in \mathbb{C} \, \big| \, d_0 \le \text{Re}(s) \le c, \ -\widetilde{T} \le \text{Im}\, s \le \widetilde{T} \, \}$$

where $\widetilde{T} = \sqrt{T^2 - d_0^2}$. We may assume:

($*$) There is no zero or pole of the integrand of (3.3) over the boundary of $R(T)$.

Now we apply the Cauchy integral formula over $R(T)$ to obtain

$$\begin{aligned} &\frac{1}{2\pi i} \int_{c-i\widetilde{T}}^{c+i\widetilde{T}} \frac{x^{s+2d_0}}{s(s+1)(s+2)\cdots(s+2d_0)} \frac{d}{ds} \log R_\Gamma(s)\, ds \\ &= -\frac{1}{2\pi i} \Big( \int_{c+i\widetilde{T}}^{d_0+i\widetilde{T}} + \int_{C_T} + \int_{d_0-i\widetilde{T}}^{c-i\widetilde{T}} \Big) \frac{x^{s+2d_0}}{s(s+1)(s+2)\cdots(s+2d_0)} \frac{d}{ds} \log R_\Gamma(s)\, ds \\ &\quad + \sum_{z \in R(T)} \text{Res}_{s=z} \Big( \frac{x^{s+2d_0}}{s(s+1)(s+2)\cdots(s+2d_0)} \frac{d}{ds} \log R_\Gamma(s) \Big) \end{aligned} \tag{3.4}$$

where $C_T$ denotes the circular part of the boundary of $R(T)$ with the anti-clockwise orientation.

We compare (3.3) for $n = 2d_0$ with the left hand side of (3.4). By Section 4.2 in [11], we know that $\frac{d}{ds} \log R_\Gamma(s)$ is bounded over the line $\text{Re}(s) = c > 2d_0$. Hence, it is easy to see

$$\psi_{2d_0}(x) = -\frac{1}{2\pi i} \int_{c-i\widetilde{T}}^{c+i\widetilde{T}} \frac{x^{s+2d_0}}{s(s+1)(s+2)\cdots(s+2d_0)} \frac{d}{ds} \log R_\Gamma(s)\, ds + \mathrm{O}(x^{c+2d_0} T^{-2d_0}). \tag{3.5}$$

The asymptotics of the first term on the right hand side of (3.5) is the same as the one of the right hand side of (3.4). From now on, we analyze this part:

- For the integral $\int_{C_T} \cdot \, ds$, we have

$$\Big| \int_{C_T} \frac{x^{s+2d_0}}{s(s+1)(s+2)\cdots(s+2d_0)} \frac{d}{ds} \log R_\Gamma(s)\, ds \Big| \ \le \ x^{3d_0} T^{-d} \int_{C_T} \Big| \frac{d}{ds} \log R_\Gamma(s) \Big| \, |ds|. \tag{3.6}$$

By Theorem 1.1 and Proposition 7 of [8],

$$\int_{C_T} \Big| \frac{d}{ds} \log R_\Gamma(s) \Big| \, |ds| \le \int_{|s|=T} \Big| \frac{d}{ds} \log R_\Gamma(s) \Big| \, |ds| \le C T^d \log T \tag{3.7}$$

for a constant $C$. Note that the parameter $T$ in (3.7) should satisfy the condition $(*)$. By (3.6) and (3.7),

$$\int_{C_T} \frac{x^{s+2d_0}}{s(s+1)(s+2)\cdots(s+2d_0)} \frac{d}{ds} \log R_\Gamma(s)\, ds = \mathrm{O}(x^{3d_0} \log T). \tag{3.8}$$

• Let us deal with the integral $\int_{c+i\widetilde{T}}^{d_0+i\widetilde{T}} \cdot\, ds$. From Theorem 1.1 and Theorem 4.6 in [11], it follows that for $s = \sigma + it$ with $\sigma \geq d_0$, $|t| \gg 0$,

$$\frac{d}{ds} \log R_\Gamma(s) = \mathrm{O}\big(t^{2d_0}\big) + \sum_{|t-\lambda_j|\leq 1} |s - d_0 - i\lambda_j|^{-1} \tag{3.9}$$

where $d_0 + i\lambda_j$ is a pole of $R_\Gamma(s)$ along the critical line $\mathrm{Re}(s) = d_0$. Then, by the fact that $\frac{d}{ds} \log R_\Gamma(s)$ is bounded over the line $\mathrm{Re}(s) = c > 2d_0$, the equality (3.9), and the Phragmén-Lindelöf theorem, for $s = \sigma + it$ with $\sigma \geq d_0 + u$ and $u > 0$, $|t| \gg 0$,

$$\frac{d}{ds} \log R_\Gamma(s) = \mathrm{O}\Big(\frac{t^{2d_0}}{u}\Big). \tag{3.10}$$

For a fixed $\epsilon > 0$, we decompose $\int_{c+i\widetilde{T}}^{d_0+i\widetilde{T}} \cdot\, ds$ into $\int_{d_0+\epsilon+i\widetilde{T}}^{d_0+i\widetilde{T}} \cdot\, ds$ and $\int_{c+i\widetilde{T}}^{d_0+\epsilon+i\widetilde{T}} \cdot\, ds$. For the first one, using (3.9) (as in Proposition 6.14 of [14]) we have

$$\int_{d_0+\epsilon+i\widetilde{T}}^{d_0+i\widetilde{T}} \frac{x^{s+2d_0}}{s(s+1)(s+2)\cdots(s+2d_0)} \frac{d}{ds} \log R_\Gamma(s)\, ds = \mathrm{O}\big(x^{3d_0+\epsilon} T^{-1}\big). \tag{3.11}$$

For the second one, by (3.10) we get

$$\int_{c+i\widetilde{T}}^{d_0+\epsilon+i\widetilde{T}} \frac{x^{s+2d_0}}{s(s+1)(s+2)\cdots(s+2d_0)} \frac{d}{ds} \log R_\Gamma(s)\, ds = \mathrm{O}\big(x^{c+2d_0} T^{-1} (\log x)^{-1} \epsilon^{-1}\big). \tag{3.12}$$

Combining (3.11) and (3.12), we conclude

$$\begin{aligned} &\int_{c+i\widetilde{T}}^{d_0+i\widetilde{T}} \frac{x^{s+2d_0}}{s(s+1)(s+2)\cdots(s+2d_0)} \frac{d}{ds} \log R_\Gamma(s)\, ds \\ &\qquad = \mathrm{O}\big(x^{3d_0+\epsilon} T^{-1}\big) + \mathrm{O}\big(x^{c+2d_0} T^{-1} (\log x)^{-1} \epsilon^{-1}\big). \end{aligned} \tag{3.13}$$

The other integral $\int_{d_0-i\widetilde{T}}^{c-i\widetilde{T}} \cdot\, ds$ can be treated in a similar way and gives us the same estimate as (3.13).

We split the remaining terms given by the residues over $R(T)$ in (3.4) into three parts.

- For the poles $s_n(k)$'s of $-\frac{d}{ds}\log R_\Gamma(s)$ lying in $(d_0, 2d_0]$, we have

$$\sum_{s_n(k)\in(d_0,2d_0]} \frac{(-1)^k}{s_n(k)(s_n(k)+1)\cdots(s_n(k)+2d_0)} x^{2d_0+s_n(k)} \tag{3.14}$$

where $s_n(k)$ is related to the small eigenvalue $(s_n(k)-k)(2d_0-k-s_n(k))$ in $[0, \frac{3}{4}d_0^2]$ of $\Delta_k$ on $\pi_{\sigma_k,\lambda_n(k)}$ with $s_n(k) = d_0 + i\lambda_n(k)$ or $s_n(k) = d_0 - i\lambda_n(k)$ in $(d_0, 2d_0]$.

- For the poles $s_n(0)$'s of $-\frac{d}{ds}\log R_\Gamma(s)$ along the critical line $\mathrm{Re}(s) = d_0$, we have

$$\mathrm{O}\Big(x^{3d_0}\int_{-T}^{T} t^{-d} dN(t)\Big) = \mathrm{O}\big(x^{3d_0}\log T\big) \tag{3.15}$$

where $N(t)(= \mathrm{O}(t^d))$ denotes the counting function of $s_n(0) = d_0 + it_n$ over the critical line $\mathrm{Re}(s) = d_0$.

- To deal with the poles of $\frac{1}{s(s+1)(s+2)\cdots(s+2d_0)}\frac{d}{ds}\log R_\Gamma(s)$ in $R(T, d_0) = R(T)\cap \{s \mid \mathrm{Re}(s) < d_0\}$, we define $R(s)$ to be a meromorphic function of order $d$ given by removing the zeros and poles of $R_\Gamma(s)$ in the half plane $\mathrm{Re}(s) \geq d_0$ using the canonical product. Then we have

$$\sum_{z\in R(T,d_0)} \mathrm{Res}_{s=z}\Big(\frac{x^{s+2d_0}}{s(s+1)(s+2)\cdots(s+2d_0)}\frac{d}{ds}\log R_\Gamma(s)\Big) = \frac{1}{2\pi i}\int_{|s|=T} \frac{x^{s+2d_0}}{s(s+1)(s+2)\cdots(s+2d_0)}\frac{d}{ds}\log R(s)\,ds. \tag{3.16}$$

As in the derivation of (3.7), one can show that the right hand side of (3.16) has the same size as (3.7). Hence we have

$$\sum_{z\in R(T,d_0)} \mathrm{Res}_{s=z}\Big(\frac{x^{s+2d_0}}{s(s+1)(s+2)\cdots(s+2d_0)}\frac{d}{ds}\log R_\Gamma(s)\Big) = \mathrm{O}\big(x^{3d_0}\log T\big). \tag{3.17}$$

For a given $x \gg 0$, we choose $T$ satisfying $(*)$ such that $|x^d - T| \leq 1$. Putting $c = d$ and combining (3.4), (3.5), (3.8), (3.13), (3.14), (3.15) and (3.17), we conclude

$$\psi_{2d_0}(x) = \sum_{s_n(k)\in(d_0,2d_0]} \frac{(-1)^k}{s_n(k)(s_n(k)+1)\cdots(s_n(k)+2d_0)} x^{2d_0+s_n(k)} + \mathrm{O}(x^{3d_0}\log x). \tag{3.18}$$

## 3.3 Reduction to $\psi_0(x)$, $\pi_\Gamma(x)$

To derive the asymptotics of $\psi_0(x)$ from $\psi_{2d_0}(x)$, we use

$$\Delta_n^+ f(x) := \int_x^{x+h}\int_{x_{n-1}}^{x_{n-1}+h}\cdots\int_{x_1}^{x_1+h} f^{(n)}(x_0)\,dx_0\cdots dx_{n-1}$$

where $h$ is a constant which will be fixed later. By the mean value theorem,

$$\Delta_{2d_0}^{+}\, x^{2d_0+s_n} = h^{2d_0}\,(s_n+2d_0)(s_n+2d_0-1)\cdots(s_n+1)\,\widetilde{x}^{s_n} \tag{3.19}$$

where $\widetilde{x}\in[x,x+2d_0h]$, hence

$$\psi_0(x) \le h^{-2d_0}\Delta_{2d_0}^{+}\,\psi_{2d_0}(x) \le \psi_0(x+2d_0h). \tag{3.20}$$

By (3.18) and (3.19), we have

$$h^{-2d_0}\Delta_{2d_0}^{+}\,\psi_{2d_0}(x) = \sum_{s_n(k)\in(d_0,2d_0]} \frac{(-1)^k x^{s_n(k)}}{s_n(k)} + \mathrm{O}(h^{-2d_0}x^{3d_0}\log x) + \mathrm{O}(h^{2d_0}). \tag{3.21}$$

The optimal size of the error term in (3.21) is given when $h = x^{\frac{3}{4}}(\log x)^{\frac{1}{4d_0}}$. By (3.20) and (3.21) with $h = x^{\frac{3}{4}}(\log x)^{\frac{1}{4d_0}}$, we obtain

$$\psi_0(x) \le \sum_{s_n(k)\in(\frac{3}{2}d_0,2d_0]} \frac{(-1)^k x^{s_n(k)}}{s_n(k)} + \mathrm{O}\big(x^{\frac{3}{2}d_0}(\log x)^{\frac{1}{2}}\big). \tag{3.22}$$

For the lower bound, we use

$$\Delta_n^{-} f(x) := \int_{x-h}^{x}\int_{x_{n-1}-h}^{x_{n-1}}\cdots\int_{x_1-h}^{x_1} f^{(n)}(x_0)\,dx_0\cdots dx_{n-1}$$

to obtain

$$\psi_0(x-2d_0h) \le h^{-2d_0}\Delta_{2d_0}^{-}\,\psi_{2d_0}(x) \le \psi_0(x).$$

In a similar way, one can show

$$\sum_{s_n(k)\in(\frac{3}{2}d_0,2d_0]} \frac{(-1)^k x^{s_n(k)}}{s_n(k)} + \mathrm{O}\big(x^{\frac{3}{2}d_0}(\log x)^{\frac{1}{2}}\big) \le \psi_0(x). \tag{3.23}$$

Combining (3.22) with (3.23), we conclude that

$$\psi_0(x) = \sum_{s_n(k)\in(\frac{3}{2}d_0,2d_0]} \frac{(-1)^k x^{s_n(k)}}{s_n(k)} + \mathrm{O}\big(x^{\frac{3}{2}d_0}(\log x)^{\frac{1}{2}}\big). \tag{3.24}$$

To obtain the formula for $\pi_\Gamma(x)$, we recall

$$\pi_\Gamma(x) = \int_2^x \frac{1}{\log t}\,d\,\theta(t)$$

where

$$\theta(x) = \sum_{\gamma\in\mathrm{P}\Gamma_h,\,N(\gamma)\le x} \Lambda(\gamma).$$

Combining

$$\psi_0(x) = \theta(x) + \theta(x^{\frac{1}{2}}) + \theta(x^{\frac{1}{3}}) + \cdots$$

with (3.24), we have

$$\pi_\Gamma(x) = \sum_{s_n(k)\in(\frac{3}{2}d_0, 2d_0]} (-1)^k \operatorname{li}(x^{s_n(k)}) + \mathrm{O}\Big(\frac{x^{\frac{3}{2}d_0}}{(\log x)^{\frac{1}{2}}}\Big). \tag{3.25}$$

To obtain the error term in (3.25), we used the following asymptotics with $\alpha = \frac{1}{2}$,

$$\int_2^x \frac{dt}{(\log t)^\alpha} = \frac{x}{(\log x)^\alpha} + \alpha \frac{x}{(\log x)^{\alpha+1}} + \alpha(\alpha+1)\frac{x}{(\log x)^{\alpha+2}} \cdots$$

where $\alpha > 0$. This completes the proof of Theorem 1.2.

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## Author information

Jinsung Park, School of Mathematics
Korea Institute for Advanced Study
Hoegiro 87, Dongdaemun-gu
Seoul 130-722, Korea.
E-mail: `jinsung@kias.re.kr`