Analytic Surgery of the ζ-determinant of the Dirac operator

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We review the work of the authors and their collaborators on the decomposition of the ζ-determinant of the Dirac operator into the contributions coming from different parts of a manifold.

1. Introduction

The main theme of our lectures is to discuss how the decomposition of a manifold (space-time) affects the structure of the ζ-determinant, which is a delicate spectral invariant. This subject has been studied by many authors from many different perspectives (see for instance [11], [12], [14], [15], [19], [25], [27] , [28], [35], [36] and infinitely many others). They have used many different technical approaches introducing incredible amount of beautiful and difficult mathematics. These notes are meant to be an introduction to the authors' perspective onto the subject. The focus here is on ideas rather than on rigorous arguments. Most of the results have been published in recent papers by the authors and their collaborators and we give precise bibliographical references. However, let us stress that due to enormously rich literature we do not attempt to be as complete as possible. We want to apologize for not mentioning many important works, that have made an enormous impact on this area of mathematics and mathematical physics.

In Section 2 we study the properties of the ζ-determinant of the Dirac operator on a closed manifold using the Heat Equation method. We present here standard material, that is described in many great sources. In Section 3 we describe the adjustment we have to make in order to study Dirac operators on a manifold with boundary. We explain our choice of the space of the boundary conditions and show that there is a natural notion of the determinant related to this space. We discuss the projective equality of this new determinant to the ζ-determinant of the boundary problems for the Dirac operators established in the recent work of Scott and Wojciechowski (see [41], see also [42] for the additional discussion). In Section 4 we outline our method of analyzing the decomposition of the ζ-determinant. Section 5 deals with the boundary contributions which appear when we split a manifold along the submanifold of codimension 1. Then in Section 6 we explain how to use the adiabatic approach in order to separate the contributions coming from different parts of the manifold and the boundary contributions. This decomposition is completed in Section 7. In Section 8 we present the "adiabatic" decomposition formulas for the ζ-determinant of the Dirac Laplacians. Let us point out that formulas (54) and (55) are new, while the complete proof of the formula (56) was given in a recent paper by the authors (see [32], see also [31]). In Section 9 we discuss the decomposition of the "phase"of the ζ-determinant, the η-invariant. Here we make more comments concerning the analysis on a manifold with boundary. We explain why there are no analytical problems with the definition of the ζ-determinant on $Gr^*_\infty(\mathcal{D})$, the Grassmannian of the boundary conditions we discuss in this paper. Then we present the proof of the decomposition formula for the η-invariant. The disadvantage of our method is that it does not tell us anything about the integer contribution. Additional study is needed to

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detect the integer contribution, which is responsible for some intriguing topological phenomena. Due to the lack of expertise and space in this article we do not discuss this topic. Instead of that, we refer to a beautiful, recent work of Kirk and Lesch [21]. In the last Section, we discuss the invariance of the ratio of the ζ-determinants of two elliptic problems with respect to the length of the collar neighborhood of the boundary. It is well-known, that in general, the ζ-determinant changes when we stretch the collar. We discuss here the case in which the ratio of the determinants of two Atiyah–Patodi–Singer problems remains constant. The proof is based on the results of the work of Scott and Wojciechowski discussed in Section 3 (see [41]).

In the reminder of the Introduction we introduce the main hero of the lectures - the ζ-determinant of the Dirac operator. We follow here a beautiful exposition given by Singer in [46].

In many important problems of quantizing gauge theories, as well as in some mathematical problems, it is necessary to discuss directly a regularized determinant of an elliptic operator. The *Heuristic Approach* to the determinant in this context was first proposed by mathematicians for the case of a positive definite second-order elliptic differential operator

$$L : C^{\infty}(M; S) \to C^{\infty}(M; S)$$

acting on sections of a smooth vector bundle S over a closed manifold M. The operator L has a discrete spectral resolution and therefore formally has determinant equal to the infinite product of its eigenvalues. The starting point in defining a regularized product is the following formula for an invertible finite-rank linear operator T:

$$\ln \det T = -\frac{d}{ds}\{\mathrm{Tr}\ T^{-s}\}|_{s=0} \quad . \tag{1}$$

For large $Re(s)$ the ζ-function of the operator L is just the trace occurring on the right side of (1)

$$\zeta_L(s) = \mathrm{Tr}\ L^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \mathrm{Tr}\ e^{-tL} dt. \tag{2}$$

It is a holomorphic function of s for $Re(s) > \frac{dim\ M}{2}$ and has a meromorphic extension to the whole complex plane with only simple poles (see [43]). In particular $s = 0$ is not a pole. Hence $\zeta_L'(0) = \frac{d}{ds}\{\zeta_L(s)\}|_{s=0}$ is well-defined and we may define the ζ-*determinant* by

$$\det{}_{\zeta} L = e^{-\zeta_L'(0)} \quad . \tag{3}$$

This definition was introduced in 1971, in a famous paper of Ray and Singer [38], in order to define *Analytic Torsion*, the analytical counterpart to the topological invariant *Franz-Reidemeister Torsion*. The equality of the two torsions was subsequently proved independently by Jeff Cheeger and Werner Müller (see [13], [26]). Since then, there have been numerous applications of the ζ-determinant in physics and mathematics, beginning with the 1977 Hawking paper [20] on quantum gravity.

For positive-definite operators of Laplace type over a closed manifold the ζ-determinant provides a generally satisfactory regularization method. Though the fundamental multiplicative property of the determinant no longer holds; if L_1 and L_2 denote two positive elliptic operator of positive order on a Hilbert space H then in general

$$\det{}_{\zeta} L_1 L_2 \neq \det{}_{\zeta} L_1 \cdot \det{}_{\zeta} L_2 \quad .$$

We refer to other talks in the Meeting for a discussion of the so-called Multiplicative Anomaly. In many physical applications, however, such as the quantization of Fermions, one encounters the more problematic task of defining the determinant of a first-order *Dirac operator*. These are not positive operators, and now the gauge anomalies may arise due to the phase of the determinant (see [1]). For a Dirac operator $\mathcal{D} : C^{\infty}(M; S) \to C^{\infty}(M; S)$ acting on sections of a bundle of Clifford modules over a closed (odd-dimensional) manifold M one proceeds in the way outlined below. The operator $\mathcal{D}$ is an elliptic self-adjoint first-order operator and hence has infinitely many positive and negative eigenvalues. Let $\{\lambda_k\}_{k \in \mathbf{N}}$ denote the set of positive eigenvalues and $\{-\mu_k\}_{k \in \mathbf{N}}$ denote the set of negative eigenvalues. Once again, $\zeta_{\mathcal{D}}(s) = \mathrm{Tr}\,(\mathcal{D}^{-s})$ is well-defined and holomorphic for $\mathrm{Re}(s) > \dim M$ and we have

$$\zeta_{\mathcal{D}}(s) = \sum_k \lambda_k^{-s} + \sum_k (-1)^{-s}\mu_k^{-s}$$
$$= \sum_k \left(\frac{\lambda_k^{-s} - \mu_k^{-s}}{2} + \frac{\lambda_k^{-s} + \mu_k^{-s}}{2}\right)$$
$$+(-1)^{-s}\sum_k \left(\frac{\lambda_k^{-s} + \mu_k^{-s}}{2} - \frac{\lambda_k^{-s} - \mu_k^{-s}}{2}\right) \ ,$$

which can be written as

$$\zeta_{\mathcal{D}}(s) = (-1)^{-s}\frac{\zeta_{\mathcal{D}^2}(s/2) - \eta_{\mathcal{D}}(s)}{2} \qquad (4)$$
$$+\frac{\zeta_{\mathcal{D}^2}(s/2) + \eta_{\mathcal{D}}(s)}{2} \ ,$$

where $\eta_{\mathcal{D}}(s) = \sum_k \lambda_k^{-s} - \sum_k \mu_k^{-s}$ is the η-function of the operator $\mathcal{D}$ introduced by Atiyah, Patodi and Singer (see [2]). Once again, it is holomorphic for $Re(s)$ large and has a meromorphic extension to the whole complex plane with only simple poles. There is no pole at $s = 0$ and therefore we can study the derivative of $\zeta_{\mathcal{D}}(s)$ at $s = 0$. We have

$$\zeta'_{\mathcal{D}}(0) = \frac{\zeta'_{\mathcal{D}^2}(0)}{2} + \frac{d}{ds}\{(-1)^{-s}\}|_{s=0}\frac{\zeta_{\mathcal{D}^2}(0) - \eta_{\mathcal{D}}(0)}{2} \ .$$

The ambiguity in defining $(-1)^{-s}$ (i.e. a choice of spectral cut) now leads to an ambiguity in the phase of the ζ-determinant. We have

$$(-1)^{-s} = e^{\pm i\pi s} \ ,$$

and we pick the " $-$ " sign. This leads to the following formula for the ζ-determinant of the Dirac operator $\mathcal{D}$:

$$\det\nolimits_{\zeta}\mathcal{D} = e^{\frac{i\pi}{2}(\zeta_{\mathcal{D}^2}(0) - \eta_{\mathcal{D}}(0))} \cdot e^{-\frac{1}{2}\zeta'_{\mathcal{D}^2}(0)} \ . \qquad (5)$$

Remark 1.1. We refer to Section 7 of [41] for a discussion of the choice of sign of the phase of the ζ-determinant.

We need to study more closely the regularization process used to make the definition (5). This will be done in the next Section, where the Heat Equation enters the scene.

2. ζ-determinant and Heat Equation

We use the *Heat Equation* method to make sense of the ζ-determinant. We recall the standard material (see [17] for details). In this Section, we assume that $\mathcal{D}$ has the trivial kernel for convenience. The key are the following formulas:

$$\zeta_{\mathcal{D}^2}(s) = \mathrm{Tr}(D^2)^{-s} = \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\mathrm{Tr}\, e^{-t\mathcal{D}^2} dt \quad (6)$$

for $Re(s) > \frac{dim\ M}{2}$ and

$$\eta_{\mathcal{D}}(s) = \mathrm{Tr}\ D(D^2)^{-\frac{s-1}{2}}$$
$$= \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^{\infty} t^{\frac{s-1}{2}}\mathrm{Tr}\ \mathcal{D}e^{-t\mathcal{D}^2} dt \ ,$$

for $Re(s) > \frac{1+dim\ M}{2}$.

We prove the second equality in (6). The proof of the first one is completely analogous. We have

$$\int_0^{\infty} t^{\frac{s-1}{2}}\mathrm{Tr}\ \mathcal{D}e^{-t\mathcal{D}^2} dt = \sum_{-\infty}^{+\infty}\int_0^{\infty} t^{\frac{s-1}{2}}\lambda_k e^{-t\lambda_k^2} dt$$
$$= \sum_{-\infty}^{+\infty}\lambda_k(\lambda_k^2)^{-\frac{s+1}{2}}\int_0^{\infty}(t\lambda_k^2)^{\frac{s-1}{2}}e^{-t\lambda_k^2}d(t\lambda_k^2)$$
$$= \sum_{-\infty}^{+\infty}\mathrm{sign}\ \lambda_k \cdot |\lambda_k|^{-s} \cdot \int_0^{\infty} r^{\frac{s-1}{2}}e^{-r}dr$$
$$= \Gamma\left(\frac{s+1}{2}\right)\eta_{\mathcal{D}}(s).$$

These formulas hold for s making the operators $(D^2)^{-s}$ and $D(D^2)^{-\frac{s-1}{2}}$ operators of trace class. Now we expand the ζ-function and η-function to the whole complex plane. We use here the well-known fact that the trace $\mathrm{Tr}\ e^{-t\mathcal{D}^2}$ has an asymptotic expansion of the form

$$\mathrm{Tr}\ e^{-t\mathcal{D}^2} = t^{-\frac{n}{2}}\sum_{k=0}^{N} t^k a_k + O(t^{N+1-\frac{n}{2}}) \ . \qquad (7)$$

A more general formula (proved in [17] Section 1.9.) gives the following expansion:

$$\text{Tr } Ae^{-t\mathcal{D}^2} = \sum_{k=0}^{N} t^{\frac{k-n-a}{2}} b_k + O(t^{N+1-\frac{a+n}{2}}) \tag{8}$$

where A denotes a differential operator of order a. The coefficients a_k and b_k are the integrals of the local densities

$$a_k = \int_M \alpha_k(x)dx \quad \text{and} \quad b_k = \int_M \beta_k(x)dx \ ,$$

where $\alpha_k(x)$ is constructed from the coefficients of $\mathcal{D}$ at the point $x \in M$ and $\beta_k(x)$ is constructed from coefficients of A and $\mathcal{D}$ at x . Moreover,

$$\beta_k(x) = 0 \quad \text{for} \quad k + a \quad \text{odd} \ .$$

Now we see how to extend $\zeta_{\mathcal{D}^2}(s)$ to the whole complex plane.

$$\begin{aligned} &\int_0^\infty t^{s-1} \text{Tr } e^{-t\mathcal{D}^2} dt \\ &= \int_0^1 t^{s-1} \text{Tr } e^{-t\mathcal{D}^2} dt + \int_1^\infty t^{s-1} \text{Tr } e^{-t\mathcal{D}^2} dt \\ &= \int_0^1 t^{s-1} t^{-\frac{n}{2}} \sum_{k=0}^{N} t^k a_k dt + \int_1^\infty t^{s-1} \text{Tr } e^{-t\mathcal{D}^2} dt \\ &\quad + O(t^{s+N+1-\frac{n}{2}}) \ . \end{aligned}$$

The second and the third term on the right side above provide us with h, a holomorphic function of s for $Re(s) > \frac{n}{2} - N - 1$ and we obtain

$$\int_0^\infty t^{s-1} \text{Tr } e^{-t\mathcal{D}^2} dt = \sum_{k=0}^{N} \frac{a_k}{s + k - \frac{n}{2}} + h(s). \tag{9}$$

It follows that $\zeta_{\mathcal{D}^2}(s)$ has a meromorphic extension to the whole complex plane $\mathbf{C}$ with simple poles at $s_k = \frac{n}{2} - k$, with residue equal to

$$Res_{s=\frac{n}{2}-k} \zeta_{\mathcal{D}^2}(s) = \frac{a_k}{\Gamma(\frac{n}{2} - k)} \ .$$

Let us observe a simple corollary of this analysis:

Lemma 2.1. *The point $s = 0$ is never a pole and $\zeta_{\mathcal{D}^2}(0) = 0$ for n odd, and it is equal to $a_{\frac{n}{2}}$ for n even.*

The reason for the regularity here is that in the neighborhood of $s = 0$, $\zeta_{\mathcal{D}^2}(s)$ can be represented in the form

$$\zeta_{\mathcal{D}^2}(s) = \frac{1}{\Gamma(s)} \left(\frac{a_{\frac{n}{2}}}{s} + h_1(s) \right) , \tag{10}$$

where h_1 is holomorphic in a neighborhood of $s = 0$, and the singularity vanishes since

$$\Gamma(s) = \frac{1}{s} + \gamma + s \cdot h_2(s) \ ,$$

where h_2 is a holomorphic function near $s = 0$ and γ denotes the Euler constant. Unfortunately this is not the case when we discuss the η-function. The pole of Tr $\mathcal{D}e^{-t\mathcal{D}^2}$ is not cancelled out by the corresponding pole of $\Gamma(s)$. A more subtle argument has to be used. However the result holds and in fact is true.

Theorem 2.2. *(see [4] and [17]). Let $\eta_{\mathcal{D}}(s;x)$ denote the local η-density*

$$\eta_{\mathcal{D}}(s;x) = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{s-1} \text{tr } \mathcal{F}(t;x,x) dt \ ,$$

where $\mathcal{F}(t;x,y)$ denotes the kernel of the operator $\mathcal{D}e^{-t\mathcal{D}^2}$. For each $x \in M$ the function $\eta_{\mathcal{D}}(s;x)$ is a holomorphic function of s for $Re(s) > -2$.

Remark 2.3. (1) One can view this result as the odd-dimensional variant of the *"Local Index Theorem"* for *compatible Dirac operators*.

(2) It follows that the following equality holds for any compatible Dirac operator:

$$\eta_{\mathcal{D}}(0) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} \text{Tr } \mathcal{D}e^{-t\mathcal{D}^2} dt. \tag{11}$$

To get a useful local invariant out of the η-function , we have to study the variation of the

η-invariant (i.e. $\frac{d}{dr}\eta_{\mathcal{D}_r}(0)$). Let us assume that $\{\mathcal{D}_r\}_{(-\varepsilon,+\varepsilon)}$ is a smooth family of compatible Dirac operators. For simplicity we also assume that $\mathcal{D}_r$ is an invertible operator for any r . We have to differentiate the *Heat Operator* $e^{-t\mathcal{D}^2}$. In order to do this we introduce *Duhamel's Principle*.

Duhamel's Principle
Let A and B denote self-adjoint operators acting on a separable Hilbert space $\mathcal{H}$. The following equality holds (under the appropriate technical assumptions):

$$e^{-tA} - e^{-tB} = \int_0^t e^{-sA}(B-A)e^{-(t-s)B}ds. \quad (12)$$

An immediate consequence of Duhamel's principle that we need is the following Proposition:

Proposition 2.4. *The following equality holds:*

$$\frac{d}{dr}\{\mathrm{Tr}\ \mathcal{D}_r e^{-t\mathcal{D}_r^2}\}|_{r=0} \quad (13)$$

$$= \mathrm{Tr}\ \dot{\mathcal{D}}_0 e^{-t\mathcal{D}_0^2} - 2t\cdot\ \mathrm{Tr}\ \dot{\mathcal{D}}_0\mathcal{D}_0^2 e^{-t\mathcal{D}_0^2} \quad ,$$

where $\dot{\mathcal{D}}_0 = \frac{d}{dr}\mathcal{D}_r|_{r=0}$.

Proof. We have

$$\frac{d}{dr}\{\mathrm{Tr}\,\mathcal{D}_r e^{-t\mathcal{D}_r^2}\}|_{r=0} = \lim_{\delta\to 0}\mathrm{Tr}\frac{\mathcal{D}_\delta e^{-t\mathcal{D}_\delta^2} - \mathcal{D}_0 e^{-t\mathcal{D}_0^2}}{\delta}$$

$$= \lim_{\delta\to 0}\mathrm{Tr}\frac{\mathcal{D}_\delta - \mathcal{D}_0}{\delta}e^{-t\mathcal{D}_\delta^2} + \lim_{\delta\to 0}\mathrm{Tr}\,\mathcal{D}_0\frac{e^{-t\mathcal{D}_\delta^2} - e^{-t\mathcal{D}_0^2}}{\delta}$$

$$= \mathrm{Tr}\,\dot{\mathcal{D}}_0 e^{-t\mathcal{D}_0^2}$$

$$+ \lim_{\delta\to 0}\{\mathrm{Tr}\ \mathcal{D}_0\int_0^t e^{-s\mathcal{D}_0^2}\frac{\mathcal{D}_0^2 - \mathcal{D}_\delta^2}{\delta}e^{-(t-s)\mathcal{D}_0^2}ds$$

$$+\mathrm{Tr}\,\mathcal{D}_0\int_0^t \frac{(e^{-s\mathcal{D}_\delta^2} - e^{-s\mathcal{D}_0^2})(\mathcal{D}_0^2 - \mathcal{D}_\delta^2)}{\delta}e^{-(t-s)\mathcal{D}_0^2}ds\}\ .$$

The last term on the right side is of order $O(\delta)$ and we obtain

$$\mathrm{Tr}\ \dot{\mathcal{D}}_0 e^{-t\mathcal{D}_0^2}$$

$$-\mathrm{Tr}\ \mathcal{D}_0\int_0^t e^{-s\mathcal{D}_0^2}(\dot{\mathcal{D}}_0\mathcal{D}_0 + \mathcal{D}_0\dot{\mathcal{D}}_0)e^{-(t-s)\mathcal{D}_0^2}ds$$

$$= \mathrm{Tr}\ \dot{\mathcal{D}}_0 e^{-t\mathcal{D}_0^2} - 2t\cdot\mathrm{Tr}\ \dot{\mathcal{D}}_0\mathcal{D}_0^2 e^{-t\mathcal{D}_0^2}\ .$$

□

We can now discuss two formulas for the variation of the η-invariant. The first follows from formulas (11) and (13). We have

$$\frac{d}{dr}\{\eta_{\mathcal{D}_r}(0)\}|_{r=0} = \frac{1}{\sqrt{\pi}}\left\{\int_0^\infty \frac{1}{\sqrt{t}}\mathrm{Tr}\ \dot{\mathcal{D}}_0 e^{-t\mathcal{D}_0^2}dt\right.$$

$$\left. -2\int_0^\infty \sqrt{t}\cdot\mathrm{Tr}\ \dot{\mathcal{D}}_0\mathcal{D}_0^2 e^{-t\mathcal{D}_0^2}dt\right\}$$

$$= \frac{2}{\sqrt{\pi}}\cdot\int_0^\infty \frac{d}{dt}\{\sqrt{t}\cdot\mathrm{Tr}\ \dot{\mathcal{D}}_0 e^{-t\mathcal{D}_0^2}\}dt$$

$$= -\frac{2}{\sqrt{\pi}}\cdot\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\cdot\mathrm{Tr}\ \dot{\mathcal{D}}_0 e^{-\varepsilon\mathcal{D}_0^2}\quad .$$

Another formula for $\frac{d}{dr}\{\eta_{\mathcal{D}_r}(0)\}|_{r=0}$ is the result of the asymptotic expansion of $\mathrm{Tr}\ \dot{\mathcal{D}}_0 e^{-t\mathcal{D}_0^2}$ (see (8)). Assume, for instance, that $\dot{\mathcal{D}}$ is of order 1, then

$$\mathrm{Tr}\ \dot{\mathcal{D}}_0 e^{-t\mathcal{D}_0^2} = \sum_{k=0}^N t^{\frac{k-n-1}{2}}b_k + O(t^{N-\frac{n-1}{2}})\quad .$$

We differentiate

$$\frac{d}{dr}\left\{\frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty t^{\frac{s-1}{2}}\mathrm{Tr}\ \mathcal{D}_r e^{-t\mathcal{D}_r^2}dt\right\}|_{r=0}$$

$$= \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty t^{\frac{s-1}{2}}\mathrm{Tr}\ \dot{\mathcal{D}}_0 e^{-t\mathcal{D}_0^2}dt$$

$$+\frac{2}{\Gamma(\frac{s+1}{2})}\int_0^\infty t^{\frac{s+1}{2}}\frac{d}{dt}\mathrm{Tr}\ \dot{\mathcal{D}}_0 e^{-t\mathcal{D}_0^2}\}dt$$

$$= \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty t^{\frac{s-1}{2}}\mathrm{Tr}\ \dot{\mathcal{D}}_0 e^{-t\mathcal{D}_0^2}dt$$

$$+\frac{2}{\Gamma(\frac{s+1}{2})}(t^{\frac{s+1}{2}}\mathrm{Tr}\ \dot{\mathcal{D}}_0 e^{-t\mathcal{D}_0^2}]_0^\infty)$$
$$-\frac{s+1}{\Gamma(\frac{s+1}{2})}\int_0^\infty t^{\frac{s-1}{2}}\mathrm{Tr}\ \dot{\mathcal{D}}_0 e^{-t\mathcal{D}_0^2}dt$$
$$=\frac{2}{\Gamma(\frac{s+1}{2})}(t^{\frac{s-1}{2}}\mathrm{Tr}\ \dot{\mathcal{D}}_0 e^{-t\mathcal{D}_0^2}]_0^\infty)$$
$$-\frac{s}{\Gamma(\frac{s+1}{2})}\int_0^\infty t^{\frac{s-1}{2}}\mathrm{Tr}\ \dot{\mathcal{D}}_0 e^{-t\mathcal{D}_0^2}dt \quad .$$

The first term on the last line is equal to 0 for $Re(s)$ large enough and does not affect the meromorphic extension of $\frac{d}{dr}\{\eta_{\mathcal{D}_r}(s)\}|_{r=0}$. The second term gives us what we need

$$-\lim_{s\to 0}\ \frac{s}{\Gamma(\frac{s+1}{2})}\int_0^\infty \mathrm{Tr}\ t^{\frac{s-1}{2}}\dot{\mathcal{D}}_0 e^{-t\mathcal{D}_0^2}dt$$

$$=-\lim_{s\to 0}\ \frac{s}{\Gamma(\frac{s+1}{2})}\int_0^1 t^{\frac{s-1}{2}}\sum_{k=0}^N t^{\frac{k-n-1}{2}}b_k dt$$
$$=-\frac{2}{\sqrt{\pi}}\cdot\lim_{s\to 0}\ s\cdot\sum_{k=0}^N\frac{b_k}{s+k-n}=-\frac{2b_n}{\sqrt{\pi}} \quad .$$

In particular the variation disappears if $n = dim\ M$ is even by the theorem 1.13.2 in [17]. We get the same result in the case of $\dot{\mathcal{D}}$ of order 0, i.e.

$$\frac{d}{dr}\{\eta_{\mathcal{D}_r}(0)\}|_{r=0}=-\frac{2c_n}{\sqrt{\pi}} \quad .$$

where $\{c_k\}$ is the set of new coefficients.

Now, let us discuss the last ingredient in the ζ-determinant of the Dirac operator - the modulus of $\det_\zeta\mathcal{D}$ - the (square root of the) determinant of $\mathcal{D}^2$. We have already written the formula

$$\det{}_\zeta\mathcal{D}^2=e^{-\zeta'_{\mathcal{D}^2}(0)} \quad .$$

Let us remind the reader that $\frac{d}{ds}\zeta_{\mathcal{D}^2}(s)|_{s=0}$ is given by the formula

$$\frac{d}{ds}\zeta_{\mathcal{D}^2}(s)|_{s=0}=\int_0^\infty\frac{1}{t}\mathrm{Tr}\ e^{-t\mathcal{D}^2}dt \tag{14}$$

under the assumption $ker(\mathcal{D})=0$ and $dim\ M$ is odd. Let us explain how to interpret formula (14). The trace $\mathrm{Tr}\ e^{-t\mathcal{D}^2}$ has an asymptotic expansion given by (7), which leads to a meromorphic extension of the ζ-function to the whole complex plane. Lemma 2.1 tells us that $\zeta_{\mathcal{D}^2}(s)$ is holomorphic in the neighborhood of $s=0$, hence the derivative with respect to s exists. Let $\kappa_{\mathcal{D}^2}(s)$ denote the integral $\int_0^\infty t^{s-1}\mathrm{Tr}\ e^{-t\mathcal{D}^2}dt$. The formula (10) gives us the expansion of $\kappa_{\mathcal{D}^2}(s)$ in the neighborhood of $s=0$. We have

$$\kappa_{\mathcal{D}^2}(s)=\frac{a_{\frac{n}{2}}}{s}+h_1(s) \quad .$$

Now, the derivative of the ζ-function at $s=0$ is obtained as follows:

$$\zeta'_{\mathcal{D}^2}(0)=\frac{d}{ds}\frac{\kappa_{\mathcal{D}^2}(s)}{\Gamma(s)}|_{s=0}$$

$$=\frac{d}{ds}(\frac{a_{\frac{n}{2}}+s(\kappa_{\mathcal{D}^2}(s)-\frac{a_{\frac{n}{2}}}{s})}{1+s\gamma+s^2h(s)})|_{s=0}$$
$$=(\kappa_{\mathcal{D}^2}(s)-\frac{a_{\frac{n}{2}}}{s})|_{s=0}-\gamma a_{\frac{n}{2}}.$$

If n is odd then the coefficient $a_{\frac{n}{2}}=0$ and we can ("formally") write

$$-\ln\det{}_\zeta\mathcal{D}^2=\kappa_{\mathcal{D}^2}(s)|_{s=0}=\int_0^\infty\frac{1}{t}\mathrm{Tr}\ e^{-t\mathcal{D}^2}dt. \tag{15}$$

It is worth mentioning that the variation of $\det_\zeta\mathcal{D}^2$ is by no means a local invariant. Assume that we have a family of invertible Dirac operators $\{\mathcal{D}_r\}$, then we can use Duhamel's Principle as in the case of the η-invariant. We obtain

$$\frac{d}{dr}\{\ln\det{}_\zeta\mathcal{D}_r^2\}|_{r=0}=-\frac{d}{dr}\int_0^\infty\frac{1}{t}\mathrm{Tr}\ e^{-t\mathcal{D}_r^2}dt\Big|_{r=0}$$

$$=2\int_0^\infty\mathrm{Tr}\ \dot{\mathcal{D}}_0\mathcal{D}_0 e^{-t\mathcal{D}_0^2}\ dt$$

$$=-2\int_0^\infty\frac{d}{dt}\ \mathrm{Tr}\ \dot{\mathcal{D}}_0\mathcal{D}_0^{-1}\{e^{-t\mathcal{D}_0^2}\}\ dt\ .$$

This gives us the formula

$$\frac{d}{dr}\{\ln \det{}_\zeta \mathcal{D}^2\}|_{r=0} = 2\cdot \lim_{\varepsilon\to 0} \mathrm{Tr}\ \dot{\mathcal{D}}_0\mathcal{D}_0^{-1}e^{-\varepsilon\mathcal{D}_0^2}. \quad (16)$$

This formula allows us to see that $\det_\zeta\mathcal{D}^2$ is actually a highly non-local invariant as it involves the kernel of the operator $\mathcal{D}_0^{-1}$.

To give a simple example let us consider the family $\{\Delta_r = \mathcal{D}^2e^{r\alpha}\}_{0\le r\le 1}$, where $\alpha : C^\infty(M;S) \to C^\infty(M;S)$ is an operator with smooth kernel. We repeat the computations which lead to (16) and obtain

$$\frac{d}{dr}\{\ln \det{}_\zeta\Delta_r\} = \mathrm{Tr}\alpha \quad .$$

which implies

$$\ln \det{}_\zeta\Delta_1 - \ln \det{}_\zeta\Delta_0 = \int_0^1 \mathrm{Tr}\ \alpha dr = \mathrm{Tr}\ \alpha.$$

We have proved the equality

$$\det{}_\zeta\mathcal{D}^2e^\alpha = \det{}_\zeta\mathcal{D}^2\cdot\det{}_{Fr}e^\alpha. \quad (17)$$

On the other hand let us discuss the η-invariant for the family $\{\mathcal{D}_r = \mathcal{D} + r\alpha\}$. We have

$$\frac{d}{dr}\{\eta_{\mathcal{D}_r}(0)\} = -\frac{2}{\sqrt{\pi}}\cdot \lim_{\varepsilon\to 0}\sqrt{\varepsilon}\cdot\mathrm{Tr}\ \dot{\mathcal{D}}_0e^{-\varepsilon\mathcal{D}_0^2}$$

$$= -\frac{2}{\sqrt{\pi}}\cdot \lim_{\varepsilon\to 0}\sqrt{\varepsilon}\cdot\mathrm{Tr}\ \alpha e^{-\varepsilon\mathcal{D}_0^2}$$

$$= -\frac{2}{\sqrt{\pi}}\cdot \lim_{\varepsilon\to 0}\sqrt{\varepsilon}\cdot\mathrm{Tr}\ \alpha = 0\ ,$$

and as a result

$$\eta_{\mathcal{D}+\alpha}(0) = \eta_{\mathcal{D}}(0). \quad (18)$$

3. Determinants of Dirac operators on a manifold with boundary

In this Section we discuss the determinants of Dirac operators on a manifold with boundary. The new ingredient is that, in order to get a nice elliptic operator out of $\mathcal{D}$, we have to consider the boundary conditions. The choice of boundary condition determines the domain of the operator $\mathcal{D}$. We will not discuss here the most general space of elliptic, self-adjoint boundary conditions for $\mathcal{D}$ introduced in the recent work of Kirk and Lesch (see [21]). We stick to the more conventional Grassmannian of the boundary conditions of Atiyah–Patodi–Singer type. We avoid also a discussion of the case of non-product metric structures in the neighborhood of the boundary, which rises to the table many unpleasant analytical issues.

An unexpected advantage of the fact that we discuss boundary problems is that in our situation det_ζ is in fact equal (up to a scalar) to the true Fredholm determinant.

Let M denote an odd-dimensional compact manifold with boundary Y and $\mathcal{D} : C^\infty(M;S) \to C^\infty(M;S)$ a compatible Dirac operator acting on sections of S , a bundle of Clifford modules over M . Assume that the Riemannian metric on M and the Hermitian structure on S are products in a certain collar neighborhood of the boundary. Let us fix a parameterization $N = [0,1] \times Y$ of the collar. Then, in N, the operator $\mathcal{D}$ has the form

$$\mathcal{D} = G(\partial_u + B)\ , \quad (19)$$

where $G : S|Y \to S|Y$ is a unitary bundle isomorphism (Clifford multiplication by the unit normal vector) and $B : C^\infty(Y;S|Y) \to C^\infty(Y;S|Y)$ is the corresponding Dirac operator on Y, an elliptic self-adjoint operator of first order. Furthermore, G and B do not depend on the normal coordinate u and they satisfy the identities

$$G^2 = -Id \quad \text{and} \quad GB = -BG\ . \quad (20)$$

Since Y has dimension $2m$ the bundle $S|Y$ decomposes into its positive and negative chirality components $S|Y = S^+ \bigoplus S^-$ and we have a corresponding splitting of the operator B into $B^\pm : C^\infty(Y; S^\pm) \to C^\infty(Y; S^\mp)$, where $(B^+)^* = B^-$. The operator (19) can be rewritten in the form

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \left(\partial_u + \begin{pmatrix} 0 & B^- \\ B^+ & 0 \end{pmatrix} \right) . \qquad (21)$$

In order to obtain an unbounded Fredholm operator with sufficient regularity properties we have to impose a boundary condition on the operator $\mathcal{D}$. Let $\Pi_>$ denote the spectral projection of B onto the subspace of $L^2(Y; S|Y)$ spanned by the eigenvectors corresponding to the nonnegative eigenvalues of B. It is well known that $\Pi_>$ is an elliptic boundary condition for the operator $\mathcal{D}$ (see [2], [9]). The meaning of ellipticity is described below. We introduce the unbounded operator $\mathcal{D}_{\Pi_>}$ equal to the operator $\mathcal{D}$ with domain

$$\text{dom } \mathcal{D}_{\Pi_>} = \{ s \in H^1(M; S) \ ; \ \Pi_>(s|Y) = 0 \} ,$$

where H^1 denotes the first Sobolev space. Then the operator

$$\mathcal{D}_{\Pi_>} = \mathcal{D} : \text{dom}(\mathcal{D}_{\Pi_>}) \to L^2(M; S)$$

is a Fredholm operator with kernel and cokernel consisting only of smooth sections.

The orthogonal projection $\Pi_>$ is a pseudodifferential operator of order 0 (see [9]). Let us point out that we can take any pseudodifferential operator R of order 0 with principal symbol equal to the principal symbol of $\Pi_>$ and obtain an operator $\mathcal{D}_R$ which satisfies the aforementioned properties. In the following, however, we concentrate on the specific subset of the space of self-adjoint elliptic boundary conditions. There exists another pseudodifferential projection on Y , which is in fact the central object in the theory of elliptic boundary value problems. Let us briefly explain this point. In contrast to the case of an elliptic operator on a closed manifold, the operator $\mathcal{D}$ has an infinite-dimensional space of solutions. More precisely, the space

$$\{ s \in C^\infty(M : S) \ ; \ \mathcal{D}s = 0 \ \ in \ M \setminus Y \}$$

is infinite-dimensional. We introduce the Calderon projection, which is the projection onto $\mathcal{H}(\mathcal{D})$ of the *Cauchy Data space* of the operator $\mathcal{D}$

$$\mathcal{H}(\mathcal{D}) = \{ f \in C^\infty(Y; S|Y) \ ; \ \exists \ s \in C^\infty(M; S)$$
$$\text{s.t. } \ \mathcal{D}(\mathrm{s}) = 0 \text{ in } \mathrm{M} \setminus \mathrm{Y} \text{ and } \mathrm{s}|\mathrm{Y} = \mathrm{f} \} \ .$$

The projection $P(\mathcal{D})$ is a pseudodifferential operator with principal symbol equal to the symbol of $\Pi_>$. It is also an orthogonal projection in the case of a Dirac operator on an odd-dimensional manifold (see [9]). The operator $\mathcal{D}$ has the *Unique Continuation Property*, and hence we have an one to one correspondence between solutions of the operator $\mathcal{D}$ and the traces of solutions on the boundary Y. This roughly explains why only the projection $\mathcal{P}_R$ onto the kernel of the boundary conditions R matters. If the difference $\mathcal{P}_R - P(\mathcal{D})$ is an operator of order -1 , then it follows, that by choosing the domain of the operator $\mathcal{D}_R$ as above, we throw away almost all solutions of the operator $\mathcal{D}$ on $M \setminus Y$, with the possible exception of a finite dimensional subspace. The above condition on $\mathcal{P}_R$ also allows us to construct a parametrix for the operator $\mathcal{D}_R$, hence we obtain regularity of the solutions of the operator $\mathcal{D}_R$. We refer to [9] for more details.

This explains why in [41] we restricted ourselves to the study of the Grassmannian $Gr^*_\infty(\mathcal{D})$ of all orthogonal pseudodifferential projections P such that

$$P - P(\mathcal{D}) \ \text{ is a smoothing operator} \qquad (22)$$
$$\text{and } \quad -GPG = Id - P \ .$$

The first condition implies the ellipticity of the operator $\mathcal{D}_P$ and the second guarantees self-adjointness. The spectral projection $\Pi_>$ is an element of $Gr^*_\infty(\mathcal{D})$ if and only if $ker \ B = \{0\}$.

Remark 3.1. Again let us point out that the space $Gr^*_\infty(\mathcal{D})$ is far from being the space of all elliptic boundary conditions for the Dirac operator $\mathcal{D}$. An important example is given by the condition determined by chirality (see (21)). The operator $P_\pm = \frac{1}{2}(Id \mp i\Gamma)$ is the orthogonal projection of $S|_Y$ onto $S^\pm$ and provides $\mathcal{D}$ with a (local) *chiral* elliptic boundary condition. This means that the operator $\mathcal{D}_\pm = \mathcal{D}$ with domain

$$\text{dom } \mathcal{D}_\pm = \{s \in H^1(M;S) \mid P_\pm(s|_Y) = 0\},$$

is Fredholm and that its kernel and cokernel consist of smooth sections only. The operators $\mathcal{D}_\pm$ are not self-adjoint, but we have the equalities

$$\mathcal{D}_+^* = \mathcal{D}_- \text{ and index } \mathcal{D}_\pm = 0. \tag{23}$$

It is not difficult to see that $\Delta_\pm = \mathcal{D}_\mp\mathcal{D}_\pm$ is equal to the operator $\mathcal{D}^2$ with Dirichlet (resp. Neumann) condition on S^+ and Neumann (resp. Dirichlet) condition on S^-.

For any $P \in Gr_\infty^*(\mathcal{D})$ the operator $\mathcal{D}_P$ has a discrete spectrum nicely distributed along the real line. It was shown by the second author that $\eta_{\mathcal{D}_P}(s)$ and $\zeta_{\mathcal{D}_P^2}(s)$ are well-defined functions, holomorphic for $Re(s)$ large and having meromorphic extensions to the whole complex plane with only simple poles. In particular both functions are holomorphic in a neighborhood of $s=0$. Therefore $\det_\zeta \mathcal{D}_P$ is a well-defined, smooth function on $Gr_\infty^*(\mathcal{D})$ (see [52]). We will discuss the regularity of η-function of the operator $\mathcal{D}_P$, with $P \in Gr_\infty^*(\mathcal{D})$ in Section 9. Now we discuss the "true" determinant, which lives on the space $Gr_\infty^*(\mathcal{D})$.

The determinant line bundle over the space of Fredholm operators was first introduced in a seminal paper of Quillen [37]. An equivalent better suited to our purposes was subsequently given by Segal (see [45]), and we follow his approach. Let $\text{Fred}(\mathcal{H})$ denote the space of Fredholm operators on a separable Hilbert space $\mathcal{H}$. First we work in the connected component $\text{Fred}_0(\mathcal{H})$ of this space parameterizing operators of *index* zero. For $A \in \text{Fred}_0(\mathcal{H})$ define

$$\text{Fred}_A = \{S \in \text{Fred}(\mathcal{H})\ ;\ S - A \text{ is trace} - \text{class}\} .$$

Fix a trace-class operator $\mathcal{A}$ such that $S = A + \mathcal{A}$ is an invertible operator. Then the determinant line of A is defined as

$$\text{Det } A = \text{Fred}_A \times \mathbf{C}/_\cong \tag{24}$$

where the equivalence relation is defined by

$$(R,z) = ((RS^{-1})S, z) \simeq (S, z{\cdot}\det_{Fr}(RS^{-1})) \quad .$$

The Fredholm determinant of the operator RS^{-1} is well-defined, as it is of the form $Id_\mathcal{H}$ plus a trace class operator. Denoting the equivalence class of a pair (R,z) by $[R,z]$, complex multiplication is defined on Det A by

$$\lambda{\cdot}[R,z] = [R, \lambda z]. \tag{25}$$

The *canonical determinant element* is defined by

$$\det A := [A, 1], \tag{26}$$

and is non-zero if and only if A is invertible. The complex lines fit together over $\text{Fred}_0(\mathcal{H})$ to define a complex line bundle $\mathcal{L}$, the determinant line bundle. To see this, observe first that over the open set $U_\mathcal{A}$ in $\text{Fred}_0(\mathcal{H})$ defined by

$$U_\mathcal{A} = \{F \in \text{Fred}_0(\mathcal{H})\ ;\ F + \mathcal{A} \text{ is invertible}\},$$

the assignment $F \to \det F$ defines a trivializing (non-vanishing) section of $\mathcal{L}_{|U_\mathcal{A}}$. The transition map between the canonical determinant elements over $U_\mathcal{A} \cap U_\mathcal{B}$ is the smooth (holomorphic) function

$$g_{\mathcal{AB}}(F) = \det_{Fr}((F+\mathcal{A})(F+\mathcal{B})^{-1}) \quad .$$

This defines $\mathcal{L}$ globally as a complex line bundle over $\text{Fred}_0(\mathcal{H})$, endowed with the canonical section $A \to \det A$. If ind $A = d$ we define Det A to be the determinant line of $A \oplus 0$ as an operator $\mathcal{H} \longrightarrow \mathcal{H} \oplus \mathbf{C}^d$ if $d > 0$, or $\mathcal{H} \oplus \mathbf{C}^{-d} \longrightarrow \mathcal{H}$

if $d < 0$ and the construction extends in the obvious way to the other components of $\mathrm{Fred}(\mathcal{H})$. Note that the canonical section is zero outside of $\mathrm{Fred}_0(\mathcal{H})$.

We use this construction in order to define the determinant line bundle over $Gr_\infty(\mathcal{D})$. For each projection $P \in Gr_\infty(\mathcal{D})$ we have the (Segal) determinant line $\mathrm{Det}(P(\mathcal{D}), P)$ of the operator

$$\mathcal{S}(P) = PP(\mathcal{D}) : \mathcal{H}(\mathcal{D}) \to \mathrm{Ran}\ P \quad ,$$

and the determinant line Det $\mathcal{D}_P$ of the boundary-value problem $\mathcal{D}_P$: dom $(\mathcal{D}_P) \longrightarrow L^2(M; S)$. These lines fit together in the manner explained above to define determinant line bundles $\mathrm{DET}_{P(\mathcal{D})}$ and DET $\mathcal{D}$, respectively, over the Grassmannian (some care has to be taken as the operator acts between two different Hilbert spaces, but with the obvious notational modifications we once again obtain well-defined determinant line bundles). The topology of the Grassmannians (see [8], [16]) implies that the bundle $\mathrm{DET}_{P(\mathcal{D})}$ is a non-trivial line bundle over $Gr_\infty(\mathcal{D})$, but when restricted to the Grassmannian $Gr^*_\infty(\mathcal{D})$ it is canonically trivial. The canonical section becomes a function in this trivialization. We call this function *Canonical Determinant* and we denote its value at P by $\det_{\mathcal{C}}\mathcal{D}_P$.

Now we give more precise description of $\det_{\mathcal{C}}\mathcal{D}_P$. Simon Scott showed that elements of $Gr^*_\infty(\mathcal{D})$ are in one to one correspondence with the unitary elliptic operators $T : C^\infty(Y; S^+) \to C^\infty(Y; S^-)$, which satisfy an additional condition (see [39]). Namely, let us introduce the operator $V_> = (B^+B^-)^{-1}B^+$. We assume that

$$T - V_> \quad \text{is a smoothing operator.}$$

The correspondence is as follows: if we fix the operator T as above then the corresponding projection is

$$T \to P = \frac{1}{2}\begin{pmatrix} Id_{F^+} & T^{-1} \\ T & Id_{F^-} \end{pmatrix} \quad . \tag{27}$$

Let us stress that the invertibility assumption on the tangential operator B can be easily relaxed when we discuss this construction (see Section 7.3 of [41] for the details). Let us also point out that this fixes the isomorphism of $Gr^*_\infty(\mathcal{D})$ with $U^\infty(F^-)$ the group of unitary operators on the sections of S^- of the form Id_{F^-} plus *smoothing operator* . Let $K : C^\infty(Y; S^+) \to C^\infty(Y; S^-)$ be a unitary operator such that $\mathcal{H}(\mathcal{D}) = \textit{graph}\ K$. Then the operator

$$U(P) = \begin{pmatrix} Id_{F^+} & 0 \\ 0 & TK^{-1} \end{pmatrix} \tag{28}$$

has the property

$$P = U(P)P(\mathcal{D})U(P)^{-1}$$

and it defines an isomorphism $P \to TK^{-1}$ between $Gr^*_\infty(\mathcal{D})$ and $U^\infty(F^-)$. Now we have a well-defined operator

$$U(P)^{-1}\mathcal{S}(P) : \mathcal{H}(\mathcal{D}) \to \mathcal{H}(\mathcal{D}) \quad .$$

It is of the form $Id_{\mathcal{H}(\mathcal{D})}$ plus *smoothing operator*, hence it has a well-defined Fredholm determinant and straightforward computations show that

$$\det{}_{Fr} U(P)^{-1}\mathcal{S}(P) = \det{}_{Fr}\left(\frac{Id + KT^{-1}}{2}\right).$$

All this was explained in Section 1 of [41]. The study of the preferred trivialization, defined by means of the operator $U(P)$, now shows that we have the equality

$$\det{}_{\mathcal{C}}\mathcal{D}_P = \det{}_{Fr} U(P)^{-1}\mathcal{S}(P) \quad . \tag{29}$$

The question arises: Is $det_{\mathcal{C}}$ related to det_ζ? A positive answer was given in work of Scott and Wojciechowski, as the main result of [41] is

Theorem 3.2. *The following equality holds over* $Gr^*_\infty(\mathcal{D})$*:*

$$\det{}_\zeta\mathcal{D}_P = \det{}_\zeta\mathcal{D}_{P(\mathcal{D})} \cdot \det{}_{\mathcal{C}}\mathcal{D}_P \quad . \tag{30}$$

To prove Theorem 3.2 we study the variation of the determinants. More precisely, we fix two projections $P_1, P_2 \in Gr^*_\infty(\mathcal{D})$ such that the operators $\mathcal{D}_{P_i}$ are invertible. Next, we choose a family of unitary operators of the form

$$\left\{ \begin{pmatrix} Id_{F^+} & 0 \\ 0 & g_r \end{pmatrix} \right\}_{0 \le r \le 1} \quad ,$$

where $g_r : F^- \to F^-$ is a unitary operator, and such that $g_r - Id_{F^-}$ is an operator with a smooth kernel for any r , and $g_0 = Id_{F^-}$. We define two families of boundary conditions:

$$P_{i,r} = \begin{pmatrix} Id_{F^+} & 0 \\ 0 & g_r \end{pmatrix} P_i \begin{pmatrix} Id_{F^+} & 0 \\ 0 & g_r^{-1} \end{pmatrix} \quad ,$$

and study the relative variation:

$$\frac{d}{dr}\{\ln \det \mathcal{D}_{P_{1,r}} - \ln \det \mathcal{D}_{P_{2,r}}\}|_{r=0} \tag{31}$$

for both the *Canonical determinant* and the ζ-determinant. Of course we face the technical problem of dealing with a family of unbounded operators with varying domain. To circumvent this, and to make sense of the variation with respect to the boundary condition we follow Douglas and Wojciechowski [16] and apply their *"Unitary Trick"*. It is not difficult to define an extension of our family of unitary operators on the boundary sections to a family $\{U_r\}$ of unitary operators acting on $L^2(M;S)$ (see Section 9 for more details). The operator $\mathcal{D}_{P_{i,r}}$ is unitarily equivalent to the operator $(\mathcal{D}_r)_{P_i}$, where

$$\mathcal{D}_r = U_r^{-1} \mathcal{D} U_r \quad .$$

Both the ζ-determinant and the canonical determinant are invariant under this unitary twist which allows us to show that both determinants have variation given by the same expression

$$\frac{d}{dr}\{\ln \det \mathcal{D}_{P_{1,r}} - \ln \det \mathcal{D}_{P_{2,r}}\}|_{r=0} = \text{Tr } \dot{D}_0(\mathcal{D}_{P_1}^{-1} - \mathcal{D}_{P_2}^{-1}) \quad , \tag{32}$$

where $\dot{\mathcal{D}}_0$ denotes the operator $\frac{d}{dr}\mathcal{D}_r|_{r=0}$. Now we use the fact that the set of projections $P \in Gr^*_\infty(\mathcal{D})$, for which the operator $\mathcal{D}_P$ is invertible is actually path connected (see Section 7.2 of [41]) and integrate the equality

$$\frac{d}{dr}\{\ln \det{}_\zeta \ \mathcal{D}_{P_{1,r}} - \ln \det{}_\zeta \ \mathcal{D}_{P_{2,r}}\}|_{r=0}$$
$$= \tfrac{d}{dr}\{\ln \det{}_\mathcal{C} \ \mathcal{D}_{P_{1,r}} - \ln \det{}_\mathcal{C} \ \mathcal{D}_{P_{2,r}}\}|_{r=0} \quad ,$$

in order to obtain formula (30) of Theorem 3.2.

The reader might think that formula (32) is incorrect as the variation of the phase of the ζ-determinant is not present. However, we will see in Section 9 that the variation of the η-invariant in our situation does depend only on $\{g_r\}$, and not on the choice of the base projection, hence the variation here is the same at P_1 as it is at P_2. We learn more about the properties of the η-invariant on the Grassmannian in Section 9.

4. An outline of the method

The idea to use the adiabatic limit in this particular way belongs to Singer (see [47]). There are three basic ingredients which we use in our approach to the decomposition of the ζ-determinant.

First, we rely heavily on the assumption that metric structures are product near the boundary. This implies that the operator $\mathcal{D}$ has a cylindrical form in the collar neighborhood of the boundary. The determinant is expressed via different *Heat Operators* determined by $\mathcal{D}$ and those operators are not local. The crucial quantity here is

$$\int_M \text{tr } \mathcal{E}(t;x,x)dx \quad ,$$

where $\mathcal{E}(t;x,y)$ denotes the kernel of such an operator. We know the construction of $\mathcal{E}(t;x,y)$ on a closed manifold, hence in the interior of M. The product structure gives also the explicit formulas for the kernel $\mathcal{E}(t;x,y)$ on the cylinder. The problem is to paste those kernels in order to get a kernel on M. Moreover, the endomorphism $\mathcal{E}(t;x,y)$ is not determined via coefficients of $\mathcal{D}$ at x and

y only but depends on global information from the whole manifold M. Now, the construction of the kernel $\mathcal{E}(t;x,y)$ on a closed manifold M is standard and described in many different places. What is important for us is that the estimates, obvious in the case of flat space, hold also in the case of a general manifold.

Proposition 4.1. *Let $\mathcal{D}$ be a Dirac operator on a closed manifold and $\mathcal{E}(t;x,y)$ and $\mathcal{F}(t;x,y)$ denote the kernels of the operators $e^{-t\mathcal{D}^2}$ and $\mathcal{D}e^{-t\mathcal{D}^2}$. Then there exist positive constants c_1 and c_2 such that*

$$\|\mathcal{E}(t;x,y)\| \leq c_1 t^{-\frac{n}{2}} e^{-c_2\frac{d^2(x,y)}{t}} ,$$

$$\|\mathcal{F}(t;x,y)\| \leq c_1 t^{-\frac{n+1}{2}} e^{-c_2\frac{d^2(x,y)}{t}}$$

for any $x,y \in M$ and any $t > 0$.

We show that those estimates extend easily to our situation. We refer to [9], [16] and [48] for additional information on this subject and more comprehensive bibliography as the literature on this topic is extremely rich.

Second, we use Duhamel's Principle to paste kernels. The Duhamel Principle shows explicitly that the heat kernel on M splits into interior part, cylindrical part and the error term. It also provides us the tools to study the error term.

Third, we assume that the tangential operator B is invertible. This allows us to make assumptions concerning the behavior of the eigenvalues of the boundary problems, which eventually allows us to discard the *large time contribution.* We also rule out the existence of the L^2-solutions of $\mathcal{D}$ on manifolds with cylindrical ends, which enter the picture during our analysis. These assumptions secure the non-existence of the "small" eigenvalues in the situations we study.

We will not discuss here the scattering theory, which enters the picture in the case of the non-invertible tangential operator. The situation is as follows. Assume that we have a given decomposition of a closed manifold M into two submanifolds M_1 and M_2 along the submanifold Y of codimension 1. As we stretch the collar neighborhood around Y, the eigenvalues of the Dirac operator $\mathcal{D}$ change. The assumptions we made above guarantee that they stay bounded away from 0 . However if the tangential operator is non-invertible, we have to deal with small eigenvalues of $\mathcal{D}$. They fall in two different categories. We have finitely many eigenvalues which decay exponentially with respect to the length R of the cylinder $[-R,R]\times Y$ joining M_1 and M_2. These eigenvalues are constructed from L^2 solutions of $\mathcal{D}$ on M_1 and M_2 with the cylinders of infinite length attached. There are also an infinite family of eigenvalues of size $\frac{1}{R}$, which can be constructed from the eigenvalues of the corresponding Dirac operators on the circles of large radius determined by the *Scattering Theory* defined by $\mathcal{D}$. This type of analysis was applied to the Atiyah–Patodi–Singer problem by Werner Müller (see [27], see also [28] for related results). Following Müller the authors were able to establish a decomposition formula for the ζ-determinant in the case of non-invertible tangential operator. We refer to the recent paper [33] (see also [34]) for more details.

5. Cylinder, Duhamel's Principle and Heat Kernels on a manifold with boundary

We start with the infinite cylinder $[0,\infty)\times Y$ and the operator $\mathcal{D}^2 = -\partial_u^2 + B^2$, subject to the boundary condition at $u=0$. We collect several explicit formulas for the kernels of the heat operators determined by $\mathcal{D}^2$ subject to different boundary conditions. Then we show how Duhamel's Principle leads to the splitting of the trace of the heat operator on a manifold M onto interior contribution, cylinder contribution and the error term.

We start with the Dirichlet condition. We introduce the operator $\Delta_d = \mathcal{D}^2$ with the domain

$$\operatorname{dom}\Delta_d = \{s \in C^\infty([0,\infty)\times Y; S); s|_{u=0} = 0\} \ .$$

It has a unique closed self-adjoint extension and therefore $e^{-t\Delta_d}$ is well-defined and its kernel is given by the formula

$$\mathcal{E}_d(t;(u,x),(v,y)) = \tag{33}$$

$$\frac{1}{\sqrt{4\pi t}}\{e^{-\frac{(u-v)^2}{4t}} - e^{-\frac{(u+v)^2}{4t}}\}\ e^{-tB^2}(t;x,y),$$

where $e^{-tB^2}(t;x,y)$ denotes the kernel of the operator e^{-tB^2}. Similarly, to discuss the Neumann condition we introduce $\Delta_n = \mathcal{D}^2$ with domain

$$\mathrm{dom}\,\Delta_n = \{s \in C^\infty([0,\infty)\times Y;S); (\partial_u s)|_{u=0} = 0\}.$$

The corresponding heat kernel is given by the formula

$$\mathcal{E}_n(t;(u,x),(v,y)) = \tag{34}$$

$$\frac{1}{\sqrt{4\pi t}}\{e^{-\frac{(u-v)^2}{4t}} + e^{-\frac{(u+v)^2}{4t}}\}\ e^{-tB^2}(t;x,y) \quad .$$

Finally let us note the formula for the kernel of the operators $e^{-t\Delta_\pm}$ (see Remark 3.1)

$$\begin{aligned}&\mathcal{E}_\pm(t;(u,x),(v,y)) = \\ &\frac{1}{\sqrt{4\pi t}}\{e^{-\frac{(u-v)^2}{4t}} \mp e^{-\frac{(u+v)^2}{4t}}\}e^{-tB^2}(t;x,y)P_+ \\ &+\frac{1}{\sqrt{4\pi t}}\{e^{-\frac{(u-v)^2}{4t}} \pm e^{-\frac{(u+v)^2}{4t}}\}e^{-tB^2}(t;x,y)P_- \ .\end{aligned}$$

The formulas for the Atiyah–Patodi–Singer condition are more complicated. It follows from (21) that the operator B has a symmetric spectrum. Let $\{\mu_n\}_{n\in\mathbf{N}}$ denote the set of positive eigenvalues and $\{\phi_n\}$ the set of corresponding eigenspinors, then the negative eigenvalues are $\{-\mu_n\}$ with the corresponding eigenspinors $\{G\phi_n\}$. The heat kernel of $\mathcal{D}^2_{\Pi_>}$ on the cylinder has the form

$$\sum_{n\in\mathbf{N}} g_n(t;u,v)\phi_n(x)\otimes\phi_n(y)$$

$$+\sum_{n\in\mathbf{N}} g_{-n}(t;u,v)G\phi_n(x)\otimes G\phi_n(y) \quad .$$

Recall the formulas for the functions $g_n(t;u,v)$ (see for instance [9], (22.33) and (22.35))

$$g_n(t;u,v) = \frac{e^{-\mu_n^2 t}}{2\sqrt{\pi t}}\cdot\{e^{-\frac{(u-v)^2}{4t}} - e^{-\frac{(u+v)^2}{4t}}\} \tag{35}$$

for $n > 0$, and

$$g_n(t;u,v) = \frac{e^{-(-\mu_n)^2 t}}{2\sqrt{\pi t}}\cdot\{e^{-\frac{(u-v)^2}{4t}} + e^{-\frac{(u+v)^2}{4t}}\}+$$

$$(-\mu_n)e^{-(-\mu_n)(u+v)}\cdot\mathrm{erfc}\left(\frac{u+v}{2\sqrt{t}} - (-\mu_n)\sqrt{t}\right)$$

for $n < 0$ where

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-r^2}dr < \frac{2}{\sqrt{\pi}}e^{-x^2} \quad .$$

Note that all those kernels satisfy the estimates from Proposition 4.1.

The interior heat kernel is defined by the kernel of the double of the Dirac operator $\mathcal{D}$. Let us recall that this operator has a natural double $\tilde{\mathcal{D}}$, which leaves on $\tilde{M}$, the double of a manifold M. Let $\tilde{\mathcal{E}}(t;x,y)$ denote the kernel of the operator $e^{-t\tilde{\mathcal{D}}^2}$ and let $\mathcal{E}_{cyl}(t;x,y)$ denote one of the kernels on the cylinder discussed above. We use them to construct the kernel of the operator $e^{-t\mathcal{D}^2_{\Pi.}}$ (and the operators $e^{-t\Delta_d}$, $e^{-t\Delta_n}$, $e^{-t\Delta_\pm}$) on the manifold M. Roughly speaking we glue cylinder kernel and interior kernel together.

We introduce a smooth, increasing function $\rho(a,b) : [0,\infty) \to [0,1]$ equal to 0 for $0 \le u \le a$ and equal to 1 for $b \le u$. We use $\rho(a,b)(u)$ to define

$$\phi_1 = 1 - \rho\left(\frac{5}{7},\frac{6}{7}\right),\ \psi_1 = 1 - \psi_2,$$

and

$$\phi_2 = \rho\left(\frac{1}{7},\frac{2}{7}\right),\ \psi_2 = \rho\left(\frac{3}{7},\frac{4}{7}\right).$$

We extend those functions to the symmetric functions on the whole real line. All those functions are constant outside the interval $[-1,1]$ and we use them to define the corresponding functions on a manifold M. Now we define $Q(t;x,y)$, a *"Parametrix"* for the real heat kernel $\mathcal{E}(t;x,y)$, by

$$Q(t;x,y) = \phi_1(x)\mathcal{E}_{cyl}(t;x,y)\psi_1(y) \tag{36}$$

$$+\phi_2(x)\tilde{\mathcal{E}}(t;x,y)\psi_2(y) \quad .$$

A standard computation shows that

$$\mathcal{E}(t;x,y) = Q(t;x,y) + (\mathcal{E} * \mathcal{C})(t;x,y), \tag{37}$$

where $\mathcal{E} * \mathcal{C}$ is a convolution given by

$$(\mathcal{E}*\mathcal{C})(t;x,y) = \int_0^t ds \int_{M_R} dz\, \mathcal{E}(s;x,z)\mathcal{C}(t-s;z,y),$$

and the correction term $\mathcal{C}(t;x,y)$ is given by the formula

$$\begin{aligned}\mathcal{C}(t;x,y) = &-\frac{\partial^2\phi_1}{\partial u^2}(x)\mathcal{E}(t;x,y)\psi_1(y)\\ &-\frac{\partial\phi_1}{\partial u}(x)\frac{\partial\mathcal{E}}{\partial u}(t;x,y)\psi_1(y)\\ &-\frac{\partial^2\phi_2}{\partial u^2}(x)\mathcal{E}(t;x,y)\psi_2(y)\\ &-\frac{\partial\phi_2}{\partial u}(x)\frac{\partial\mathcal{E}}{\partial u}(t;x,y)\psi_2(y).\end{aligned}$$

The choice of cut-off functions implies the following result:

Lemma 5.1. *The "error" term $\mathcal{C}(t;x,y)$ vanishes outside the cylinder $[\frac{1}{7},\frac{6}{7}]\times Y$ and is equal to 0 for $d(x,y) < \frac{1}{7}$. Therefore, there exist positive constants c_1, c_2 such that*

$$\|\mathcal{C}(t;x,y)\| \le c_1 e^{-c_3\frac{d^2(x,y)}{t}}. \tag{38}$$

We define the series

$$Q(t;x,y) + \sum_{n=1}^{\infty}(Q * \mathcal{C}_n)(t;x,y), \tag{39}$$

where

$$\mathcal{C}_1 = C \quad \text{and} \quad \mathcal{C}_{n+1}(t;x,y) = \mathcal{C}_n * C \quad .$$

The elementary estimate

$$\|\mathcal{C}_n(t;x,y)\| \le \frac{c_1 \mathrm{vol}(y)t^{n-1}}{(n-1)!}e^{-c_2\frac{d^2(x,y)}{t}} \tag{40}$$

implies the absolute convergence of (39) and now the equality

$$\mathcal{E}(t;x,y) = Q(t;x,y) + \sum_{n=1}^{\infty}(Q * \mathcal{C}_n)(t;x,y)$$

is obvious. Proposition 4.1, jointly with (38) and (40) gives us the following estimates on the kernels of the heat operators:

Proposition 5.2. *Let $\mathcal{E}(t;x,y)$ denote the kernel of one of the operators $e^{-t\mathcal{D}^2_{\Pi_>}}$, $e^{-t\Delta_d}$, $e^{-t\Delta_n}$, $e^{-t\Delta_\pm}$ on a manifold M and let us denote by $\mathcal{F}(t;x,y)$ the kernel $\mathcal{D}\mathcal{E}(t;x,y)$. Assume that the corresponding Laplacian is an invertible operator. Then there exist positive constants c_1, c_2, c_3 such that*

$$\|\mathcal{E}(t;x,y)\| \le c_1 t^{-\frac{n}{2}} e^{c_2 t} e^{-c_3\frac{d^2(x,y)}{t}}, \tag{41}$$

$$\|\mathcal{F}(t;x,y)\| \le c_1 t^{-\frac{n+1}{2}} e^{c_2 t} e^{-c_3\frac{d^2(x,y)}{t}} \quad .$$

6. Duhamel's principle and the Adiabatic Limit

Now we want to analyze the behavior of the heat kernels in the adiabatic limit. We start with the manifold M with collar neighborhood $N = [0,1]\times Y$ and we replace M by M_R, which is M with N replaced by $N_R = [0,R]\times Y$, a collar of length R. To study the behavior of the heat kernels on M_R we need the uniform estimates corresponding to the one we have given in Proposition 4.1. To get them we use Duhamel's Principle as in the previous Section, but now we take the parameter R into account.

More precisely, first we get the heat kernel $\tilde{\mathcal{E}}_R(t;x,y)$ of the operator $\tilde{\mathcal{D}}^2_R$ on a manifold $\tilde{M}_R$. We obtain this kernel by gluing together the heat kernel of $\mathcal{D}^2$ on the cylinder $(-\infty,+\infty)\times Y$ (restricted to $[-R,R]\times Y$) to the two copies of the heat kernel of $\mathcal{D}^2$ on M (one for each end). The method described in Section 5 works in this case and the resulting kernel $\tilde{\mathcal{E}}_R(t;x,y)$ satisfies the estimate (41).

Now, we paste kernels together, but this time we make our parametrix dependent on R. We use the function $\rho(a,b)$ to define

$$\phi_1 = 1-\rho\left(\frac{5}{7}R,\frac{6}{7}R\right),\ \psi_1 = 1-\psi_2,$$

and

$$\phi_2 = \rho\left(\frac{1}{7}R,\frac{2}{7}R\right),\ \psi_2 = \rho\left(\frac{3}{7}R,\frac{4}{7}R\right),$$

and introduce the corresponding functions on a manifold M_R. We define $Q_R(t;x,y)$ a *"parametrix"* for the heat kernel $\mathcal{E}_R(t;x,y)$ (where again $\mathcal{E}_{cyl}(t;x,y)$ denotes the heat kernel of one of our boundary problems)

$$Q_R(t;x,y) = \phi_1(x)\mathcal{E}_{cyl}(t;x,y)\psi_1(y) \tag{42}$$

$$+\phi_2(x)\tilde{\mathcal{E}}_R(t;x,y)\psi_2(y)\ .$$

Again, we have

$$\mathcal{E}_R(t;x,y) = Q_R(t;x,y) + (\mathcal{E}_R * \mathcal{C}_R)(t;x,y), \tag{43}$$

where $\mathcal{E}_R * \mathcal{C}_R$ is a convolution and the correction term $\mathcal{C}_R(t;x,y)$ is given by the formula from the previous Section. The only difference is that cut-off functions depends on R . The crucial result is

Theorem 6.1. *The error term $\mathcal{C}_R(t;x,y)$ is equal to 0 outside the cylinder $[\frac{1}{7}R,\frac{6}{7}R]\times Y$. Moreover, it is equal to 0 if the distance between x and y is smaller than $\frac{R}{7}$. As a result, there exist positive constants c_1,c_2,c_3, such that the following estimate holds:*

$$\|\mathcal{E}_R(t;x,y)\| \le c_1 t^{-\frac{n}{2}} e^{c_2 t} e^{-c_3 \frac{d^2(x,y)}{t}}\ . \tag{44}$$

Moreover, the error term satisfies the estimate

$$\|(\mathcal{E}_R * \mathcal{C}_R)(t;x,x)\| \le c_1 e^{c_2 t} e^{-c_3\frac{R^2}{t}}. \tag{45}$$

The proof goes exactly as before. We only sketch the proof of (45). In the following we use the vanishing of $\mathcal{C}_R(t-s;z,x)$ for $d(x,z) > \frac{R}{7}$:

$$\|(\mathcal{E}_R * \mathcal{C}_R)(t;x,x)\|$$

$$= \left\| \int_0^t ds \int_{M_R} \mathcal{E}_R(s;x,z)\mathcal{C}_R(t-s;z,x)dz \right\|$$

$$= \left\| \int_0^t ds \int_{[\frac{R}{7},\frac{6R}{7}]\times Y} \mathcal{E}_R(s;x,z)\mathcal{C}_R(t-s;z,x)dz \right\|$$

$$\le c_1 e^{c_2 t} \int_0^t ds \int_{[\frac{R}{7},\frac{6R}{7}]\times Y} e^{-c_3\frac{td^2(x,z)}{s(t-s)}} dz$$

$$\le c_1 e^{c_2 t} \int_0^t ds \int_{[\frac{R}{7},\frac{6R}{7}]\times Y} e^{-c_4\frac{R^2}{t}} dz$$

$$\le c_1 e^{c_2 t} R\cdot\mathrm{vol}(Y) e^{-c_4\frac{R^2}{t}} \int_0^t ds < c_5 e^{c_2 t} e^{-c_6\frac{R^2}{t}}.$$

Now we are able to show that the error contribution to the ζ-determinant for the *"small"* time interval, meaning $[0,R^{1-\varepsilon}]$, disappears in the adiabatic limit, i.e

Corollary 6.2. *The following equality holds for small $\varepsilon > 0$:*

$$\lim_{R\to\infty} \int_0^{R^{1-\varepsilon}} \frac{dt}{t} \int_{M_R} \mathrm{tr}\ (\mathcal{E}_R * \mathcal{C}_R)(t;x,x)dx = 0. \tag{46}$$

Proof. The result is an immediate consequence of Theorem 6.1, because

$$\left|\int_0^{R^{1-\varepsilon}} \frac{dt}{t}\int_{M_R} \operatorname{tr}(\mathcal{E}_R * \mathcal{C}_R)(t;x,x)dx\right|$$

$$\leq c_1 \int_0^{R^{1-\varepsilon}} \frac{e^{c_2 t}}{t} dt \int_{M_R} e^{-c_3\frac{R^2}{t}}dx$$

$$\leq c_1 e^{c_2 R^{1-\varepsilon}} e^{\frac{-c_3}{2R^{1+\varepsilon}}} \int_0^{R^{1-\varepsilon}} \frac{e^{-c_3\frac{R^2}{2t}}}{t} dt \int_{M_R} dx$$

$$\leq c_4 R e^{c_2 R^{1-\varepsilon}} e^{\frac{-c_3}{2R^{1+\varepsilon}}} \int_0^{R^{1-\varepsilon}} \frac{e^{-c_3\frac{R^2}{2t}}}{t} dt$$

$$\leq c_5 e^{-c_6 R^{\varepsilon}},$$

and (46) follows easily.

□

The meaning of the result is that as we take the adiabatic limit the error contribution to the determinant can be neglected and we are left only with the interior contribution and the cylinder contribution. This, however holds only for a small time interval. We will show in the next Section that the large time contribution coming from the time interval $[R^{1-\varepsilon}, +\infty)$ can be neglected.

7. The small eigenvalues and the large time contribution

In this Section we explain why, in the adiabatic limit, we can forget the contribution coming from the large time interval. Once again we discuss only the simplest possible situation in which we do not have to deal with small eigenvalues. We make the assumption that the tangential operator B is invertible. This condition implies that there exists a constant $b > 0$ such that we have only finitely many eigenvalues in the interval $[-b, b]$ for R sufficiently large. To simplify further, in this exposition we make one more assumption. We introduce the manifold $M_\infty = ((-\infty, 0]\times Y)\cup M$. The bundle S and operator $\mathcal{D}$ extend naturally to M_∞ and we assume that

$$\ker_{L^2}\mathcal{D} = \{s \in L^2(M_\infty; S); \mathcal{D}s = 0\} = \{0\}\ . \quad (47)$$

Assumption (47) greatly simplifies the analysis of the *Adiabatic Decomposition* of the ζ-determinant. The reason is that the operator $\mathcal{D}$ on M_∞ has a unique closed, self-adjoint extension, which we denote by $\mathcal{D}_\infty$. This is a Fredholm operator (see Section 6 of [16]) and (47) implies that the kernel of $\mathcal{D}_\infty$ is equal to $\{0\}$. This implies the existence of a positive constant b such that for any spinor s on M_∞ we have

$$(\mathcal{D}^2 s; s) \geq b\|s\|^2. \quad (48)$$

Let $\Delta_{R,\pm}$ denote the operator $\Delta_\pm$ on the manifold M_R. We introduce similar notation for the other boundary conditions. We also consider the operator $\tilde{\mathcal{D}}_R$, the Dirac operator on $\tilde{M}_R$ the double of M_R. The operator $\tilde{\mathcal{D}}_R$ is the natural double of $\mathcal{D}_R$ which is the Dirac operator $\mathcal{D}$ extended to M_R .

Proposition 7.1. *Let us assume that (47) holds. Then there exists R_0 such that*

$$\mu > \frac{b}{2}$$

for any eigenvalue μ of the operator $\Delta_{R,\pm}$, $\Delta_{R,d}$, $\Delta_{R,n}$, $\mathcal{D}^2_{R,\Pi_>}$ or $\tilde{\mathcal{D}}^2_R$ and for any $R > R_0$.

We do not present the proof of this result. It is not difficult but long and technical. The idea behind the result is easy to understand, however. Let λ_0 denote the smallest eigenvalue of the operator B^2 and $\mu = \mu(R) < \lambda_0$ denote an eigenvalue of one of the aforementioned Laplacians, with the corresponding eigensection ϕ. Assume that $\|\phi\| = 1$. We can extend ϕ to the spinor ϕ_∞ on M_∞, which belongs to the domain of $\mathcal{D}^2_\infty$. Moreover we can choose ϕ_∞ in a such a way that the L^2-norm of ϕ_∞ restricted to the cylinder $M_\infty \setminus M_R$ is bounded as follows:

$$\|\phi_\infty|_{M_\infty\setminus M_R}\|^2_{L^2} \leq c_1 e^{-c_2 R}, \quad (49)$$

for suitable positive constants c_1, c_2. Now the statement of Proposition 7.1 is an obvious consequence of min-max principle. We refer to [16] (Theorem 6.1) and [50] (Proposition 2.1). A more general result was published in [27], Proposition 8.14.

All this implies that, in our *"simple"* case, we can ignore the large time contribution in the adiabatic limit.

Proposition 7.2. *Let us assume (47), then for any $\varepsilon > 0$ the following equality holds:*

$$\lim_{R\to\infty} \int_{R^{\varepsilon}}^{\infty} \frac{1}{t}\cdot \mathrm{Tr}\ e^{-t\Delta_R} dt = 0\,. \tag{50}$$

Proof. Assume that $R > R_0$ and let $\{\mu_k\}_{k=1}^{\infty}$ denote the set of eigenvalues of Δ_R. We have

$$\int_{R^{\varepsilon}}^{\infty} \frac{1}{t}\cdot \mathrm{Tr}\ e^{-t\Delta_R} dt = \int_{R^{\varepsilon}}^{\infty} \frac{1}{t}\cdot \sum_{k=1}^{\infty} e^{-t\mu_k} dt$$

$$= \int_{R^{\varepsilon}}^{\infty} \frac{1}{t}\cdot \sum_{k=1}^{\infty} e^{-(t-1)\mu_k} e^{-\mu_k} dt$$

$$< \int_{R^{\varepsilon}}^{\infty} \frac{1}{t} e^{-(t-1)\frac{b}{2}} \cdot \mathrm{Tr}\ e^{-\Delta_R} dt \quad ,$$

where b is the constant from Proposition 7.1. We now have

$$\int_{R^{\varepsilon}}^{\infty} \frac{1}{t}\cdot \mathrm{Tr}\ e^{-t\Delta_R} dt$$

$$< \int_{R^{\varepsilon}}^{\infty} \frac{1}{t} e^{-(t-1)\frac{b}{2}} \cdot \mathrm{Tr}\ e^{-\Delta_R} dt < c_6 R^{1-\varepsilon} \cdot e^{-c_7 R^{\varepsilon}},$$

and the Proposition follows easily. □

8. The decomposition of the ζ-determinant of the Dirac Laplacian

At last we are ready to discuss the decomposition of the ζ-determinant. The manifold M is now an odd-dimensional closed manifold and we assume that it has a decomposition $M_1 \cup M_2$, where M_1 and M_2 are compact manifolds with boundary such that

$$M = M_1 \cup M_2\ ,\ M_1 \cap M_2 = Y = \partial M_1 = \partial M_2 . \tag{51}$$

In this set-up N denotes $N = [-1, 1] \times Y$, the bicollar neighborhood of Y in M, and $N_R = [-R, R] \times Y$ is the corresponding neighborhood of Y in M_R. We denote by $\mathcal{D}$ the Dirac operator on M and $\mathcal{D}_i = \mathcal{D}|_{M_i}$. We want to find a formula for the quotient

$$\frac{\det_{\zeta} \mathcal{D}_R^2}{\det_{\zeta} \Delta_{1,R,d} \cdot \Delta_{2,R,d}} \tag{52}$$

or alternatively the difference

$$\ln\ \det_{\zeta} \mathcal{D}_R^2 - \ln\ \det_{\zeta} \Delta_{1,R,d} - \ln\ \det_{\zeta} \Delta_{1,R,d}$$

$$= -\int_0^{\infty} \frac{dt}{t} \{ \mathrm{Tr}\, e^{-t\mathcal{D}_R^2} - \mathrm{Tr}\, e^{-t\Delta_{1,R,d}} - \mathrm{Tr}\, e^{-t\Delta_{2,R,d}} \}.$$

It follows from the analysis presented in the previous Sections that as $R \to \infty$ we can neglect the error terms and study only the cylinder contribution to the difference $\mathrm{Tr}\ e^{-t\mathcal{D}_R^2} - \mathrm{Tr}\ e^{-t\Delta_{1,R,d}} - \mathrm{Tr}\ e^{-t\Delta_{2,R,d}}$. Moreover, on the cylinder the heat kernel of the operator $\mathcal{D}_R^2$ can be replaced by the heat kernel determined by the operator $-\partial_u^2 + B^2$. Modulo terms which disappear as $R \to \infty$ we now have the equality

$$\mathrm{Tr}\ e^{-t\mathcal{D}_R^2} - \mathrm{Tr}\ e^{-t\Delta_{1,R,d}} - \mathrm{Tr}\ e^{-t\Delta_{2,R,d}}$$

$$= \int_{-R}^{R} \frac{1}{\sqrt{4\pi t}} \mathrm{Tr}\ e^{-tB^2} du$$

$$- \int_0^R \frac{1}{\sqrt{4\pi t}} (1 - e^{-\frac{u^2}{t}}) \mathrm{Tr}\ e^{-tB^2} du$$

$$- \int_{-R}^{0} \frac{1}{\sqrt{4\pi t}} (1 - e^{-\frac{u^2}{t}}) \mathrm{Tr}\ e^{-tB^2} du$$

$$= \frac{2}{\sqrt{4\pi t}} \int_0^R e^{-\frac{u^2}{t}} du \cdot \mathrm{Tr}\ e^{-tB^2}$$

$$= \mathrm{Tr}\ e^{-tB^2} \cdot \frac{1}{\sqrt{\pi}} \cdot \int_0^{\frac{R}{\sqrt{t}}} e^{-v^2} dv.$$

We obtain $\frac{1}{2}\mathrm{Tr}\ e^{-tB^2}$ as $R \to \infty$ and modulo minor technicalities we have proved

$$\lim_{R\to\infty}\{\ln \det{}_\zeta \mathcal{D}_R^2 - \ln \det{}_\zeta \Delta_{1,R,d} \quad (53)$$
$$-\ln \det{}_\zeta \Delta_{1,R,d}\} = \tfrac{1}{2}\cdot\ln \det{}_\zeta B^2,$$

which yields our first adiabatic decomposition result

Theorem 8.1. *The following equality holds under the assumptions we have made:*

$$\lim_{R\to\infty} \frac{\det_\zeta \mathcal{D}_R^2}{\det_\zeta \Delta_{1,R,d}\cdot\Delta_{2,R,d}} = \sqrt{\det_\zeta B^2}. \quad (54)$$

We work out the case of the Neumann condition in the same way. The only difference is the sign of the contribution and therefore we obtain

$$\lim_{R\to\infty} \frac{\det_\zeta \mathcal{D}_R^2}{\det_\zeta \Delta_{1,R,n}\cdot\Delta_{2,R,n}} = \frac{1}{\sqrt{\det_\zeta B^2}}. \quad (55)$$

This method also works in the case of the chiral boundary condition. In this case the Neumann contribution cancels out the Dirichlet contribution and as the result we have the formula

$$\lim_{R\to\infty} \frac{\det_\zeta \mathcal{D}_R^2}{\det_\zeta \Delta_{1,R,+}\cdot\Delta_{2,R,+}} = 1. \quad (56)$$

This formula was somehow the first we noticed and we used it to obtain the corresponding result for the Atiyah–Patodi–Singer condition (see [32])

Theorem 8.2. *The following equality holds in the case of Atiyah–Patodi–Singer condition:*

$$\lim_{R\to\infty} \frac{\det_\zeta \mathcal{D}_R^2}{\det_\zeta \mathcal{D}^2_{1,R,\Pi_<}\cdot\det_\zeta \mathcal{D}^2_{2,R,\Pi_>}} = 2^{-\zeta_{B^2}(0)}. \quad (57)$$

We refer to [32] for the details of the proof. This ends the discussion of the decomposition of the ζ-determinant of the Dirac Laplacian in case we do not have to deal with the small eigenvalues.

9. The splitting of the η-invariant

Here we discuss the decomposition of the phase of the ζ-determinant. Let us first observe that the η-function of the Atiyah–Patodi–Singer boundary problem shares the properties of the η-function of the Dirac operator on a closed manifold. This is due to the fact that the boundary does not create any new singularities of the η-function. The singularities are created by the small time asymptotics of the trace Tr $\mathcal{D}e^{-t\mathcal{D}^2}$. Once again using Duhamel's principle we see that we have to study the trace of the Heat Kernel on the cylinder. The kernel has the form

$$G(\partial_u + B)\mathcal{E}_>(t;(u,x),(v,y)) \quad ,$$

where $\mathcal{E}_>(t;(u,x),(v,y))$ is the kernel of the operator on the cylinder (see the formulas (35)). We only need to notice that it has the form

$$\sum g_n(t;u,v)\phi_n(x)\otimes\phi_n(y) \quad ,$$

where $\{\phi_n\}$ is the orthonormal basis of eigenspinors of B. Let us recall that we can choose this basis in such a way that $G\phi_n = \phi_{-n}$. Now it follows from (20) that the trace of

$$G(\partial_u + B)\mathcal{E}_>(t;(u,x),(v,y))$$

in the Y-direction is equal to 0.

It is not difficult to see that in fact not only the η-invariant, but the ζ-determinant is well-defined on the whole Grassmannian $Gr^*_\infty(\mathcal{D})$. This happens because the η- and ζ-functions behave nicely on this particular space of boundary conditions. We start with a more precise description of the *Unitary Twist*, which we already encountered in Section 3.

Lemma 9.1. *For any $P \in Gr^*_\infty(\mathcal{D})$ there exists a smooth path $\{g_u\}_{0\le u\le 1}$ of unitary operators on $L^2(Y;S|Y)$ which satisfies*

$$Gg_u = g_uG \ \text{ and } \ g_u - Id \ \text{ has a smooth kernel},$$

such that $g_1 = Id$ *and the path* $\{P_u = g_u \Pi_> g_u^{-1}\} \subset Gr^*_\infty(\mathcal{D})$ *connects* $P_0 = P$ *with* $P_1 = \Pi_>$.

We can always assume that the path $\{g_u\}$ is constant on subintervals $[0, 1/4]$ and $[3/4, 1]$. We introduce U, a unitary operator on $L^2(M; S)$. The operator U is equal to the Id on the complement of the collar N and

$$U|_{\{u\}\times Y} = g_u.$$

The following Lemma introduces the *Unitary Twist*, which allows us to replace the operator $\mathcal{D}_P$ by a modified operator $\mathcal{D} + \mathcal{R}$ subject to the boundary condition $\Pi_>$. This makes an explicit construction of the heat kernels on a cylinder possible.

Lemma 9.2. *The operators* $\mathcal{D}_P$ *and* $\mathcal{D}_{U,\Pi_>} = (U^{-1}\mathcal{D}U)_{\Pi_>}$ *are unitarily equivalent operators.*

Proof. Let $\{f_k; \mu_k\}_{k\in\mathbf{Z}}$ denote a spectral resolution of the operator $\mathcal{D}_P$. This means that for each k we have

$$\mathcal{D}f_k = \mu_k f_k \quad and \quad P(f_k|Y) = 0 \quad .$$

This implies

$$U^{-1}\mathcal{D}U(U^{-1}f_k) = \mu_k(U^{-1}f_k)$$

and

$$\Pi_\sigma((U^{-1}f_k)|Y) = g_0^{-1}P(f_k|Y) = 0 \quad .$$

hence $\{U^{-1}f_k; \mu_k\}$ is a spectral resolution of

$(U^{-1}\mathcal{D}U)_{\Pi_>}$.

□

In the collar N , we have the formulas

$$U^{-1}\mathcal{D}U = \mathcal{D} + GU^{-1}\frac{\partial U}{\partial u} + GU^{-1}[B, U] \ ,$$

and

$$U^{-1}\mathcal{D}^2U = \mathcal{D}^2 - 2U^{-1}\frac{\partial U}{\partial u}\partial_u$$

$$-U^{-1}\frac{\partial^2 U}{\partial u^2} + U^{-1}[B^2, U] \ ,$$

which, restricted to the collar $[0, 1/4] \times Y$, give

$$U^{-1}\mathcal{D}U = \mathcal{D} + GU^{-1}[B, U], \tag{58}$$

and

$$U^{-1}\mathcal{D}^2U = \mathcal{D}^2 + U^{-1}[B^2, U] \ .$$

It follows from Lemma 9.2 that we can study the operator $\mathcal{D}_{U,\Pi_>}$ instead of the operator $\mathcal{D}_P$. Again, it is enough to study the small time asymptotics of the trace of the Heat operator on the cylinder. This all tells you that up to an exponentially small error we have to study the trace of the operator kernel of the operator

$$(G(\partial_u + B) + \mathcal{K}_1)e^{-t(-\partial_u^2 + B^2 + \mathcal{K}_2)_{\Pi_>}} \tag{59}$$

where

$$\mathcal{K}_1 = GU^{-1}[B, U] \quad and \quad \mathcal{K}_2 = U^{-1}[B^2, U] \quad ,$$

to study meromorphic extension of the η-function, and simply the trace

$$Tr \ e^{-t(-\partial_u^2 + B^2 + \mathcal{K}_2)_{\Pi_>}}$$

to learn about ζ-function. Let us observe that $\mathcal{K}_1$ anticommutes and $\mathcal{K}_2$ commutes with the involution G. The point is that $\mathcal{K}_1$ and $\mathcal{K}_2$ are smoothing in the Y-direction. Hence when we study the trace $\mathrm{Tr}\ e^{-t\mathcal{D}^2_{U,\Pi_>}}$ we can easily show that

$$|\mathrm{Tr}\ e^{-t\mathcal{D}^2_{U,\Pi_>}} - \mathrm{Tr}\ e^{-t\mathcal{D}^2_{\Pi_>}}| < c\sqrt{t}. \tag{60}$$

Modulo exponentially small term this difference is equal to the sum given by Duhamel's principle. The first term here is

$$\mathrm{Tr}(\mathcal{E}_{\Pi_>} * \mathcal{K}_2\mathcal{E}_{\Pi_>})(t) =$$

$$\int_0^t \mathrm{Tr}\mathcal{E}_{\Pi_>}(s)\mathcal{K}_2\mathcal{E}_{\Pi_>}(t-s)ds = t\cdot\mathrm{Tr}\ \mathcal{K}_2\mathcal{E}_{\Pi_>}(t).$$

The operator $\mathcal{K}_2$ smoothes things out in the Y-

direction so the only singularity left is in the normal direction and we have

$$|\mathrm{Tr}\ \mathcal{K}_2\mathcal{E}_{\Pi_>}(t)| < \frac{c}{\sqrt{t}} \quad ,$$

and now (60) follows. Details are given in [52].

Straightforward computations show that

$$\zeta_{\mathcal{D}^2_{\Pi_>}}(0) = 0$$

(see [23] and [32]). This fact combined with the estimate (60) gives the following result:

Proposition 9.3. *([23]).*

$$\zeta_{\mathcal{D}^2_P}(0) = 0$$

for any $P \in Gr^*_\infty(\mathcal{D})$, such that $\mathcal{D}_P$ is invertible.

Proof. We have

$$\zeta_{\mathcal{D}^2_P}(0) = \zeta_{\mathcal{D}^2_P}(0) - \zeta_{\mathcal{D}^2_{\Pi_>}}(0)$$

$$= \lim_{s\to 0}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\mathrm{Tr}(e^{-t\mathcal{D}^2_P} - e^{-t\mathcal{D}^2_{\Pi_>}})dt$$

$$= \lim_{s\to 0} s\int_0^1 t^{s-1}\mathrm{Tr}(e^{-t\mathcal{D}^2_P} - e^{-t\mathcal{D}^2_{\Pi_>}})dt.$$

Now we use (60) to end the proof, i.e.

$$|\zeta_{\mathcal{D}^2_P}(0) - \zeta_{\mathcal{D}^2_{\Pi_>}}(0)| \le c\lim_{s\to 0} s\int_0^1 t^{s-\frac{1}{2}}dt = 0.$$

□

Similarly we show

$$|\mathrm{Tr}\mathcal{D}_P e^{-t\mathcal{D}^2_P} - \mathrm{Tr}\mathcal{D}_{\Pi_>}e^{-t\mathcal{D}^2_{\Pi_>}}| \le c, \tag{61}$$

which implies the following result:

Theorem 9.4. *For any $P \in Gr^*_\infty(\mathcal{D})$ the function $\eta_{\mathcal{D}_P}$ is a holomorphic function of s for $Re(s) > -1$. In particular we have the equality*

$$\eta_{\mathcal{D}_P}(0) = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{1}{\sqrt{t}}\mathrm{Tr}\mathcal{D}_P e^{-t\mathcal{D}^2_P}dt. \tag{62}$$

Moreover, let $\{P_r\}$ denote a smooth family of projections from $Gr^*_\infty(\mathcal{D})$. The variation of the η-invariant of the family $\{\mathcal{D}_{P_r}\}$ is given by the formula

$$\frac{d}{dr}\{\eta_{\mathcal{D}_{P_r}}(0)\}|_{r=0} = \tag{63}$$

$$-\tfrac{2}{\sqrt{\pi}}\cdot\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\cdot\mathrm{Tr}\ \dot{\mathcal{D}}_{P_0}e^{-\varepsilon\mathcal{D}^2_{P_0}}$$

where $\dot{\mathcal{D}}_0 = \frac{d}{dr}\{U_r^{-1}\mathcal{D}U_r\}|_{r=0}$.

It follows that the ζ-determinant gives a well-defined smooth function

$$P \to \det{}_\zeta\mathcal{D}_P$$

on the Grassmannian $Gr^*_\infty(\mathcal{D})$.

Let us discuss the decomposition of the η-invariant and its dependence on the choice of the boundary conditions on M_1 and M_2. If we fix Atiyah–Patodi–Singer conditions on both M_1 and M_2, then we know that there is no boundary contribution. We repeat the analysis from the case of the Dirac Laplacian. This gives

Theorem 9.5. *The following formula holds under the assumptions we have made on the operator $\mathcal{D}$:*

$$\lim_{R\to\infty}\{\eta_{\mathcal{D}_R}(0) - \eta_{\mathcal{D}_{1,R,\Pi_<}}(0) - \eta_{\mathcal{D}_{2,R,\Pi_>}}(0)\} = 0. \tag{64}$$

Theorem 9.5 corresponds to the results on the decomposition of the modulus of the ζ-determinant. However, the η-invariant is a much more rigid invariant than $\det_\zeta\mathcal{D}^2$. First of all the variation of the η-invariant is given by the local formula. This gives an immediate corollary. Namely (64) holds independently of R . It is easy to understand what is going on. We replace small part of the cylinder (away from the boundary) by a longer piece. In this way we change the operator $\mathcal{D}_R$. Now the variation of the η-invariant feels only what is going on at the given point and from the local point of view the process corresponds to the study of the operator $G(\partial_u + B)$ on the manifold $S^1_R \times Y$, where S^1_R is the circle with radius R. This operator has a symmetric spectrum. If $\phi(u,y)$ is an eigenspinor corresponding to the eigenvalue λ, then

$$\bar{\phi}(u,y) = G\phi(2\pi R - u, y)$$

is the eigensection corresponding to the eigenvalue $-\lambda$. Therefore for any R the η-function on S^1_R disappears. This gives

$$\eta_{\mathcal{D}_R}(0) - \eta_{\mathcal{D}_{1,R,\Pi_<}}(0) - \eta_{\mathcal{D}_{2,R,\Pi_>}}(0) = 0. \qquad (65)$$

Now we want to relax the assumptions on the small eigenvalues. There is no problem with the situation in which the operators $\mathcal{D}_i$ have nontrivial L^2-kernel when extended to $M_{i,\infty}$. We simply modify the operator $\mathcal{D}_i$ by adding the orthogonal projection onto the L^2-kernel. The only thing which may vary during this process is that a finite number of eigenvalues might change the sign, or become zero modes. For this reason we can only discuss the equalities *mod* $\mathbf{Z}$. The computation of the integer contribution has to be done separately and uses different methods. Similar modification lead to the relaxing of the condition of the invertibility of the tangential operator B .

Let us assume that the operator B has nontrivial kernel. The involution G (see (20)) restricted to $ker(B)$ defines a symplectic structure on this subspace of $L^2(Y;S|Y)$ and the *Cobordism Theorem for Dirac Operators* (see for instance [9], Corollary 21.16) implies

$$\dim \ker(B^+) = \dim \ker(B^-) \quad .$$

This last equality shows the existence of Lagrangian subspaces of $ker(B)$. We choose such a subspace W and let $\sigma : L^2(Y;S|Y) \to L^2(Y;S|Y)$ denote the orthogonal projection of $L^2(Y;S|Y)$ onto W. Let $\Pi_>$ denote the orthogonal projection of $L^2(Y;S|Y)$ onto the subspace spanned by eigenvectors of B corresponding to the positive eigenvalues. Then

$$\Pi_\sigma = \Pi_> + \sigma \in Gr^*_\infty(\mathcal{D}) \qquad (66)$$

gives an element of $Gr^*_\infty(\mathcal{D})$, which is a finite-dimensional perturbation of the Atiyah–Patodi–Singer condition.

Let Π_σ denote a projection given by Formula (66). We repeat our analysis again and obtain the following formula:

$$\eta_{\mathcal{D}}(0) = \eta_{\mathcal{D}_{1_{Id-\Pi_\sigma}}}(0) + \eta_{\mathcal{D}_{2_{\Pi_\sigma}}}(0) \mod \mathbf{Z} \ . \qquad (67)$$

Now we introduce the formula for the variation of the η-invariant under a change of boundary condition. The correct approach to the computation was proposed by Lesch and Wojciechowski (see [24]).

Let $P \in Gr^*_\infty(\mathcal{D})$ and let us choose a path $\{P_r\}_{0\leq r\leq 1} \subset Gr^*_\infty(\mathcal{D})$ such that $P_0 = \Pi_\sigma$ and $P_1 = P$. There exists a smooth family $\{g_r\}$ of unitary operators of the form $Id|_{(S|Y)} +$ *smoothing operator* which commutes with G and such that

$$g_0 = Id \ \text{ and } \ g_1 \Pi_\sigma g_1^{-1} = P \ .$$

Next, we choose a smooth non-increasing function $\gamma(u)$ such that

$$\gamma(u) = 1 \ \text{ for } u < 1/4, \ g(u) = 0 \ \text{ for } u > 3/4,$$

and for each $0 \leq r \leq 1$ use the family

$$g_{r,u} = g_{r\gamma(u)} \ \text{ for } \ 0 \leq u \leq 1 \ , \qquad (68)$$

in order to construct a corresponding unitary operator U_r on M_2 . The operator $\mathcal{D}_{2_{U_{r,\sigma}}}$ is unitarily equivalent to the operator $\mathcal{D}_{2_{P_r}}$. The variation of the η-invariant is given by the standard formula (63), which allows us to prove the next result.

Theorem 9.6. *For any $P \in Gr^*_\infty(\mathcal{D})$, and any path $g = \{g_{r,u}\}$ connecting Π_σ with P, as described above, the following formula holds:*

$$\eta_{\mathcal{D}_{2_P}}(0) - \eta_{\mathcal{D}_{2_{\Pi_\sigma}}}(0)$$

$$= -\frac{1}{\pi}\int_0^1 dr \int_0^1 du \ \mathrm{Tr}\ G\left(g^{-1}\dot{\frac{\partial g}{\partial u}}\right)|_r \mod \mathbf{Z}, \quad (69)$$

where $(g^{-1}\dot{\frac{\partial g}{\partial u}})|_{r_0} = \frac{d}{dr}(g^{-1}\frac{\partial g}{\partial u})|_{r=r_0}$.

Proof. We show that

$$\frac{2}{\sqrt{\pi}}\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\mathrm{Tr}\ (d(\mathcal{D}_{2_{U_r,\sigma}})/dr)|_{r=r_0} e^{-\varepsilon \mathcal{D}^2_{2_{U_{r_0},\sigma}}}, \quad (70)$$

$$= \frac{1}{\pi}\int_0^1 \mathrm{Tr}\ G\left(g^{-1}\dot{\frac{\partial g}{\partial u}}\right)|_{r=r_0} du.$$

We have

$$(U^{-1}\dot{\mathcal{D}}U) = GU^{-1}\dot{\frac{\partial U}{\partial u}} + G[U^{-1}BU, U^{-1}\dot{U}]$$

$$= Gg^{-1}\dot{\frac{\partial g}{\partial u}} + G[g^{-1}Bg, g^{-1}\dot{g}].$$

Thus

$$\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\cdot\mathrm{Tr}\ (d(\mathcal{D}_{2_{U_r,\sigma}})/dr)|_{r=r_0} e^{-\varepsilon \mathcal{D}^2_{2_{U_{r_0},\sigma}}}$$

contains two terms. Let us start with

$$\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\cdot\mathrm{Tr}\ G[g^{-1}Bg, g^{-1}\dot{g}]e^{-\varepsilon \mathcal{D}^2_{2_{U_{r_0},\sigma}}}\ .$$

Once again we use *Duhamel's Principle* and replace $e^{-\varepsilon \mathcal{D}^2_{2_{U_{r_0},\sigma}}}$ by the operator $exp(-t(-\partial_u^2 + B^2 + \mathcal{K}_2)_\sigma)$ on the cylinder. The point here is that the kernel of this operator commutes with G and the operator $[g^{-1}Bg, g^{-1}\dot{g}]$ anticommutes with the involution G. It follows that

$$Tr\ G[g^{-1}Bg, g^{-1}\dot{g}]e^{-\varepsilon \mathcal{D}^2_{U_r,\sigma}} = O(e^{-c/\varepsilon})\ ,$$

and one is left with

$$\frac{2}{\sqrt{\pi}}\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\cdot\mathrm{Tr}\ G\left(U^{-1}\dot{\frac{\partial U}{\partial u}}\right)e^{-\varepsilon \mathcal{D}^2_{2_{U_{r_0},\sigma}}}\ .$$

The term $G(U^{-1}\dot{\frac{\partial U}{\partial u}})|_{r=r_0} = G(g^{-1}\dot{\frac{\partial g}{\partial u}})|_{r=r_0}$ is supported in $[1/4, 3/4]\times Y$, and so we replace the kernel of the operator $e^{-\varepsilon \mathcal{D}^2_{2_{U_{r_0},\sigma}}}$ by the kernel of the operator $exp(-\varepsilon(-\partial_u^2 + B^2))$ on the infinite cylinder $(-\infty, +\infty)\times Y$. Now we have

$$\frac{2\sqrt{\varepsilon}}{\sqrt{\pi}}\cdot\mathrm{Tr}\ G\left(g^{-1}\dot{\frac{\partial g}{\partial u}}\right)|_{r_0} e^{-\varepsilon \mathcal{D}^2_{2_{U_r,\sigma}}}$$

$$= \frac{2\sqrt{\varepsilon}}{\sqrt{\pi}}\int_0^1 du\ \mathrm{Tr}_Y\ G\left(g^{-1}\dot{\frac{\partial g}{\partial u}}\right)|_{r_0} g^{-1}e^{-\varepsilon(-\partial_u^2+B^2)}g$$

$$= \frac{2\sqrt{\varepsilon}}{\sqrt{\pi}}\int_0^1 du\ \mathrm{Tr}_Y\ G\left(g^{-1}\dot{\frac{\partial g}{\partial u}}\right)|_{r_0} e^{-\varepsilon(-\partial_u^2+B^2)}$$

$$= \frac{2\sqrt{\varepsilon}}{\sqrt{\pi}}\frac{1}{\sqrt{4\pi\varepsilon}}\int_0^1 du\ \mathrm{Tr}_Y\ G\left(g^{-1}\dot{\frac{\partial g}{\partial u}}\right)|_{r_0} e^{-\varepsilon B^2}$$

$$= \frac{1}{\pi}\int_0^1 du\mathrm{Tr}_Y\ G\left(g^{-1}\dot{\frac{\partial g}{\partial u}}\right)|_{r_0} e^{-\varepsilon B^2}$$

so that

$$\frac{2}{\sqrt{\pi}}\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\cdot\mathrm{Tr}\ G\left(g^{-1}\dot{\frac{\partial g}{\partial u}}\right)|_{r_0} e^{-\varepsilon \mathcal{D}^2_{2_{U_r,\sigma}}}$$

$$= \frac{1}{\pi}\int_0^1 du\ \mathrm{Tr}_Y\ G\left(g^{-1}\dot{\frac{\partial g}{\partial u}}\right)|_{r_0}.$$

□

Remark 9.7. If we assume that $g_r(u)$ is given by the formula

$$g_r(u) = \begin{pmatrix} Id & 0 \\ 0 & \exp(ir\gamma(u))\Theta \end{pmatrix}\ ,$$

where $\Theta : C^\infty(Y; S^-|Y) \to C^\infty(Y; S^-|Y)$ is a self-adjoint operator with a smooth kernel, then our formula has a very nice and simple form

$$\eta_{\mathcal{D}_{2_P}}(0) - \eta_{\mathcal{D}_{2_{\Pi_\sigma}}}(0)$$

$$= -\frac{1}{\pi}\int_0^1 dr \int_0^1 du\ \gamma'(u)\mathrm{Tr}\ \Theta = \frac{\mathrm{Tr}\ \Theta}{\pi} \qquad (71)$$

mod $\mathbf{Z}$. This is the formula obtained by Lesch and Wojciechowski for the finite-dimensional perturbation of the Atiyah–Patodi–Singer condition (see [24]).

Corollary 9.8. *Let $P_1, P_2 \in Gr^*_\infty(\mathcal{D})$, then*

$$\eta_{\mathcal{D}_{2_{P_1}}}(0) - \eta_{\mathcal{D}_{2_{P_2}}}(0)$$

$$= -\frac{1}{\pi}\int_0^1 dr \int_0^1 du\ \mathrm{Tr}\ G\left(g^{-1}\dot{\frac{\partial g}{\partial u}}\right)|_r \mod \mathbf{Z} \qquad (72)$$

where $\{g_{r,u}\}$ is any family connecting P_1 with P_2 in the way described above (see (68)).

Corollary 9.9. *The variation of the η-invariant $\frac{d}{dr}(\eta_{\mathcal{D}_{2_{P_r}}}(0))|_{r=0}$ does not depend on the choice of the base projection $P = P_0$. It depends only on the family of unitary operators $\{g_r\}$.*

This result plays an important role in the proof of equality of the ζ-determinant and the $\mathcal{C}$-determinant.

Theorem 9.10. *For any $P_1, P_2 \in Gr^*_\infty(\mathcal{D})$ one has the following formula:*

$$\eta_{\mathcal{D}}(0) = \eta_{\mathcal{D}_{1_{Id-P_1}}}(0) + \eta_{\mathcal{D}_{2_{P_2}}}(0) \qquad (73)$$
$$+\eta(P_1, P_2)(0) \mod \mathbf{Z},$$

where $\eta(P_1, P_2)(0)$ denotes the eta-invariant of the operator $G(\partial_u + B)$ on $[0,1]\times Y$ subject to the boundary condition equal to P_1 at $u = 0$ and $Id - P_2$ at $u = 1$.

We need to explain the appearance of the middle term. We start with the equality

$$\eta_{\mathcal{D}}(0) = \eta_{\mathcal{D}_{1_{Id-\Pi_\sigma}}}(0) + \eta_{\mathcal{D}_{2_{\Pi_\sigma}}}(0)$$
$$+\eta(\Pi_\sigma, \Pi_\sigma)(0) \mod \mathbf{Z}.$$

The last term on the right side is equal to 0 by virtue of the natural symmetry described earlier in this Section. Now we vary the boundary conditions replacing $Id - \Pi_\sigma$ by $Id - P_1$ on M_1 and Π_σ by P_1 on the left end of the cylinder. Then we replace Π_σ by P_2 on M_2 and $Id-\Pi_\sigma$ by $Id-P_2$ on the right end of the cylinder. The total variation of the η-invariant under these changes is equal to 0 (mod $\mathbf{Z}$).

10. Some remarks on the dependence on R

In general, as one might expect, the ratios of the ζ-determinant discussed in this paper depend on the length of the cylinders connecting two different parts of the manifolds. We made explicit computations in which the Fredholm determinant shows up and it is easy to see its explicit R-dependence (unpublished work of the authors). However, here we want to study the case in which the ratio is R-independent. This situation brings up another nice adiabatic picture to the story. The approach is based on the work of L. Nicolaescu (see [29]).

We now denote by M_R the manifold

$$M_R = M_2 \cup [-R, 0] \times Y \quad .$$

We have a 1-parameter family of Cauchy data spaces of $\mathcal{D}_R$, $\Lambda^R(D)$. For any non-negative real number ν, we define

$$H_\nu = \mathrm{span}_{L^2}\{\phi \mid B\phi = \lambda\phi \ \text{ and } \ |\lambda| \le \nu\} \quad ,$$
$$H^\nu_< = \mathrm{span}_{L^2}\{\phi \mid B\phi = \lambda\phi \ \text{ and } \ \lambda < \nu\} \quad ,$$
$$H^\nu_> = \mathrm{span}_{L^2}\{\phi \mid B\phi = \lambda\phi \ \text{ and } \ \lambda > \nu\} \quad .$$

It is well known that $\Lambda^0(D)$ and $H^\nu_< \oplus U$ are the Fredholm pair for any $\nu \in \mathbb{R}$, and any finite dimensional subspace $U \subset L^2(Y, S|_Y)$. Hence there exists a number ν_0 such that

$$\Lambda^0(D) \cap H^{\nu_0}_< = 0.$$

The smallest such ν_0 is called the non-resonance level of D. The symplectic reduction of $\Lambda^R(D)$ to H_ν, which is defined by

$$L^R_\nu := \frac{\Lambda^R(D) \cap (H_\nu \oplus H^\nu_<)}{H^\nu_<} \subset H_\nu$$

is the Lagrangian subspace of H_ν. Since the tangential operator B preserves H_ν, we can form the 1-parameter family of finite-dimensional operators

$$e^{-RB_\nu} : H_\nu \quad \to \quad H_\nu,$$

where B_ν is the restriction of B to H_ν. We need the following description of the dynamics of the Cauchy data space $\Lambda^R(D)$:

Proposition 10.1. *([29]). For $\nu \geq \nu_0$, as $R \to \infty$,*

$$\Lambda^R(D) \;\to\; L^\infty_\nu \oplus H^{-\nu}_<,$$

where

$$L^\infty_\nu = \lim_{R\to\infty} L^R_\nu = \lim_{R\to\infty} e^{-RB_\nu} L^R_\nu.$$

Nicolaescu's proposition leads to the following interesting result.

Proposition 10.2. *Given a couple of boundary conditions $(P_1, P_2) = (\Pi_> + \sigma_1, \Pi_> + \sigma_2)$ where σ_1, σ_2 are the orthogonal projections to the Lagrangian subspaces L_1, L_2 of $H_0 = \ker B$, we assume that $\ker(\mathcal{D}_R)_{P_1} = \ker(\mathcal{D}_R)_{P_2} = 0$. Then the quotient*

$$\frac{\det_\zeta(\mathcal{D}_R)^2_{P_1}}{\det_\zeta(\mathcal{D}_R)^2_{P_2}}$$

does not depend on R .

Proof. The proof of this proposition is an application of the following Scott–Wojciechowski formula, Proposition 4.1 in [41],

$$\frac{\det_\zeta(\mathcal{D}_R)^2_{P_1}}{\det_\zeta(\mathcal{D}_R)^2_{P_2}} = |\det{}_{Fr} U_{P_2}(U_{P_1})^{-1}\mathcal{S}_R(P_1)\mathcal{S}_R(P_2)^{-1}|^2, \qquad (74)$$

where $\det_{Fr}$ is the Fredholm determinant, and

$$U_{P_2}(U_{P_1})^{-1} : \mathrm{Range}(P_1) \to \mathrm{Range}(P_2)$$

is an unitary map which depends only on P_1, P_2. The operators $U(P)$ and $S(P)$ were introduced in Section 3. By the definition of P_1, P_2, we can decompose $\mathcal{S}_R(P_1)$ into $\Pi_>\mathcal{S}_R(P_1)$ and $\sigma_1\mathcal{S}_R(P_1)$. We can also decompose $\mathcal{S}_R(P_2)^{-1}$ into its restrictions to the images of $\Pi_>$ and σ_2. We denote these maps by $\mathcal{S}_R(P_2)^{-1}\Pi_>$ and $\mathcal{S}_R(P_2)^{-1}\sigma_2$ respectively. Hence the operator $S_{R,1,2} := \mathcal{S}_R(P_1)\mathcal{S}_R(P_2)^{-1}$ has the following form:

$$\begin{pmatrix} \Pi_> S_{R,1,2}\Pi_> & \Pi_> S_{R,1,2}\sigma_2 \\ \sigma_1 S_{R,1,2}\Pi_> & \sigma_1 S_{R,1,2}\sigma_2 \end{pmatrix}.$$

Now we see that $\Pi_>\mathcal{S}_R(P_1)\mathcal{S}_R(P_2)^{-1}\Pi_>$ is the identity map on $H^0_>$ so that it does not depend on R. By definition, $\Pi_>\mathcal{S}_R(P_1)\mathcal{S}_R(P_2)^{-1}\sigma_2$ and $\sigma_1\mathcal{S}_R(P_1)\mathcal{S}_R(P_2)^{-1}\Pi_>$ are the zero maps. Finally we consider the map $\sigma_1\mathcal{S}_R(P_1)\mathcal{S}_R(P_2)^{-1}\sigma_2$. By the Nicolaescu description of the dynamics of $e^{-RB_\nu}L^0_\nu$, $e^{-RB_0}L^0_0$ does not depend on R so that $\sigma_1\mathcal{S}_R(P_1)\mathcal{S}_R(P_2)^{-1}\sigma_2$ is independent of R. Hence $\mathcal{S}_R(P_1)\mathcal{S}_R(P_2)^{-1}$ is independent of R. Now the Proposition follows from (74). □

We can combine Proposition 10.2 with the results of [33] to obtain a very interesting result which corresponds to the Lesch–Wojciechowski formula for the variation of the η-invariant.

We have to introduce elements of *Scattering Theory* in order to present the formula. We introduce the manifolds $M_{2,\infty}$ which are manifolds

M_2 with the semicylinder $(-\infty,0]\times Y$ attached to. Let $\mathcal{D}_{2,\infty}$ denote the natural extension of $\mathcal{D}$ to $M_{2,\infty}$. The operator $\mathcal{D}_{2,\infty}$ over $M_{2,\infty}$ has continuous spectrum equal to $(-\infty,\infty)$. The number $\lambda\in(-\infty,\infty)$ and $\phi\in\ker(B)$ determine a generalized eigensection of $\mathcal{D}_{2,\infty}$, which has the following form on $(-\infty,0]\times Y\subset M_{2,\infty}$ (see (4.24) in [27]):

$$E(\phi,\lambda)=e^{i\lambda u}(\phi+iG\phi)+e^{-i\lambda u}C(\lambda)(\phi+iG\phi)$$
$$+\theta(\phi,\lambda),$$

where $\theta(\phi,\lambda)$ is a square-integrable section of S on $M_{2,\infty}$ which is orthogonal to $\ker(B)$, when restricted to $\{u\}\times Y$, and $C(\lambda)$ is the scattering matrix. We refer to [27] and [28] for the presentation of the necessary material from *Scattering Theory*.

Let $C: W\to W$ denote a unitary operator acting on the finite-dimensional vector space W. We introduce the operator $D(C)$ equal to the differential operator $-i\frac{1}{2}\frac{d}{du}$ acting on $L^2(S^1,E_C)$ where E_C is the flat vector bundle over $S^1=\mathbb{R}/\mathbb{Z}$ defined by the holonomy $\overline{C}$.

Now we define the operators

$$I=(G-i):\ker(B)\to\ker(G+i)\quad,$$

$$P_i=\frac{1}{2}(\sigma_i-1):\ker(B)\to\ker(\sigma_i+1)$$

and

$$S_i(\lambda)=-P_i\circ C(\lambda)\circ I|_{\ker(\sigma_i+1)}\quad.$$

Then $S_1:=S_1(0)$ and $S_2:=S_2(0)$ are the unitary operators acting on the finite-dimensional vector spaces and we have well-defined self-adjoint, elliptic operators $D(S_1),D(S_2)$. The main result of [33] gives the formula

$$\lim_{R\to\infty}\frac{\det_\zeta(\mathcal{D}_R)^2_{P_1}}{\det_\zeta(\mathcal{D}_R)^2_{P_2}}=\frac{\det_\zeta D(S_1)^2}{\det_\zeta D(S_2)^2}\tag{75}$$

under the assumption $\ker(\mathcal{D}_R)_{P_1}=\ker(\mathcal{D}_R)_{P_2}=0$. However we showed that the left side of (75) is independent of R, hence we have the following Corollary of Proposition 10.2:

Corollary 10.3. *Assume that* $\ker(\mathcal{D}_R)_{P_1}=\ker(\mathcal{D}_R)_{P_2}=0$. *Then we have*

$$\frac{\det_\zeta\mathcal{D}^2_{P_1}}{\det_\zeta\mathcal{D}^2_{P_2}}=\frac{\det_\zeta D(S_1)^2}{\det_\zeta D(S_2)^2}\quad.$$

REFERENCES

1. M.F. Atiyah and I.M. Singer, *Dirac operators coupled to vector potentials*, Proc. Nat. Acad. Sci. USA 81 (1984) 2597–2600.
2. M.F. Atiyah, V.K. Patodi and I.M. Singer, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Cambridge Phil. Soc. 77 (1975) 43–69.
3. M.F. Atiyah, V.K. Patodi and I.M. Singer, *Spectral asymmetry and Riemannian geometry. II*, Math. Proc. Cambridge Phil. Soc. 78 (1975) 405–432.
4. J.M. Bismut and D.S. Freed, *The analysis of elliptic families. II. Dirac operators, eta invariants, and the holonomy theorem*, Commun. Math. Phys. 107 (1986) 103–163.
5. B. Bojarski, *The abstract linear conjugation problem and Fredholm pairs of subspaces*, in: Volume in Memoriam I. N. Vekua: Differential and Integral Equations. Boundary Value Problems. Publications of I. N. Vekua Institute of Applied Mathematics, Tibilisi 1979, 45-60.
6. B. Booß–Bavnbek and K.P. Wojciechowski, *Desuspension of splitting elliptic symbols. I*, Ann. Global Anal. Geom. 3 (1985) 349–400.
7. B. Booß–Bavnbek and K.P. Wojciechowski, *Desuspension of splitting elliptic symbols. II*, Ann. Global Anal. Geom. 4 (1986) 337–383.
8. B. Booß–Bavnbek and K.P. Wojciechowski, *Pseudo-differential projections and the topology of certain spaces of elliptic boundary value problems*, Commun. Math. Phys. 121 (1989) 1–9.

9. B. Booß–Bavnbek and K.P. Wojciechowski, Elliptic Boundary Problems for Dirac Operators, Birkhäuser, Boston, 1993.
10. J. Brünning and M. Lesch, *On the η-invariant of certain nonlocal boundary value problems*, Duke Math. J. 96 (1999) 425–468.
11. D. Burghelea, L. Friedlander and T. Kappeler, *Mayer–Vietoris type formula for determinants of elliptic differential operators*, J. Funct. Anal. 107 (1992) 34–65.
12. S.E. Cappell, R. Lee and E.Y. Miller, *Self-adjoint elliptic operators and manifold decompositions. I. Low eigenmodes and stretching*, Commun. Pure Appl. Math. 49 (1996) 825–866.
13. J. Cheeger, *Analytic torsion and the heat equation*, Annals of Math. 109 (1979) 259–322.
14. J. Cheeger, *Spectral geometry of singular Riemannian spaces*, J. Diff. Geom. 18 (1983) 575–657.
15. J. Cheeger, *η-invariants, the adiabatic approximation and conical singularities*, J. Diff. Geom. 26 (1987) 175–221.
16. R.G. Douglas and K.P. Wojciechowski, *Adiabatic limits of the η–invariants. The odd–dimensional Atiyah–Patodi–Singer problem*, Commun. Math. Phys. 142 (1991) 139–168.
17. P.B. Gilkey, Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theory, CRC Press, Boca Raton, 1995.
18. G. Grubb, *Trace expansions for pseudodifferential boundary problems for Dirac-type operators and more general systems*, Arkiv f. Matematik. 37 (1999) 45–86.
19. A. Hassell, R.R. Mazzeo and R.B. Melrose, *Analytic surgery and the accumulation of eigenvalues*, Commun. Anal. Geom. 3 (1995) 115–222.
20. S.W. Hawking, *Zeta function regularization of path integrals in curved spacetime*, Commun. Math. Phys. 55 (1977) 133-148.
21. P. Kirk and M. Lesch, *The η-invariant, Maslov index, and spectral flow for Dirac type operators on manifolds with boundary.* Preprint DG/0012123.
22. S. Klimek and K.P. Wojciechowski, *Adiabatic cobordism theorems for analytic torsion and η–invariant*, J. Funct. Anal. 136 (1996) 269–293.
23. Y. Lee and K.P. Wojciechowski, *Two remarks on Scott–Wojciechowski paper on Determinants (GAFA, vol. 10, 1202-1236).* IUPUI Preprint 01-04.
24. M. Lesch and K.P. Wojciechowski, *On the η–invariant of generalized Atiyah–Patodi–Singer problems*, Illinois J. Math. 40 (1996) 30–46.
25. R.R. Mazzeo and R.B. Melrose, *Analytic surgery and the eta invariant*, Geom. Func. Anal. 5 (1995) 14–75.
26. W. Müller, *Analytic torsion and R-torsion of Riemannian manifolds*, Adv. in Math. 28 (1978) 233–305.
27. W. Müller, *Eta invariants and manifolds with boundary*, J. Diff. Geom. 40 (1994) 311–377.
28. W. Müller, *Relative zeta functions, relative determinants, and scattering theory*, Commun. Math. Phys. 192 (1998) 309–347.
29. L. Nicolaescu, *The Maslov index, the spectral flow, and decompositions of manifolds*, Duke Math. J. 80 (1995) 485–533.
30. R.S. Palais, Seminar on the Atiyah–Singer Index Theorem, Princeton University Press, Princeton, 1965.
31. J. Park and K.P. Wojciechowski, *Relative ζ-determinant and Adiabatic decomposition of the ζ-determinat of the Dirac Laplacian*, Lett. Math. Phys. 52 (2000) 329–337.
32. J. Park and K.P. Wojciechowski, with Appendix by Y. Lee, 2001, *Adiabatic Decomposition of the ζ-determinant of the Dirac Laplacian I. The Case of Invertible Tangential Operator*, Commun. Part. Diff. Eq. (to appear).
33. J. Park and K.P. Wojciechowski, *Scattering Theory and Adiabatic Decomposition of the ζ-determinant of the Dirac Laplacian*, IUPUI Preprint 01-02.
34. J. Park and K.P. Wojciechowski, *Adiabatic decomposition of the ζ-determinant of the Dirac Laplacian II. The case of non-invertible tangential operator.* In preparation (2001).
35. P. Piazza, *Determinant bundles, manifolds with boundary and surgery. I*, Commun. Math. Phys. 178 (1996) 597–626.
36. P. Piazza, *Determinant bundles, manifolds*

with boundary and surgery. II, Commun. Math. Phys. 193 (1998) 105–124.

37. D.G. Quillen, *Determinants of Cauchy–Riemann operators over a Riemann surface*, Funkcionalnyi Analiz i ego Prilozhenya 19 (1985) 37–41.
38. D. Ray and I.M. Singer, *R–torsion and the Laplacian on Riemannian manifolds*, Adv. Math. 7 (1971) 145–210.
39. S.G. Scott, *Determinants of Dirac boundary value problems over odd–dimensional manifolds*, Commun. Math. Phys. 173 (1995) 43–76.
40. S.G. Scott and K.P. Wojciechowski, *ζ-determinant and the Quillen determinant on the Grassmannian of elliptic self-adjoint boundary conditions*, C. R. Acad. Sci., Serie I, 328 (1999) 139–144.
41. S.G. Scott and K.P. Wojciechowski, *The ζ-determinant and Quillen determinant for a Dirac operator on a manifold with boundary*, Geom. Funct. Anal. 10 (1999) 1202–1236.
42. S.G. Scott and K.P. Wojciechowski, *Heat Kernels, Determinants and Elliptic Boundary Problems*, to appear in: Noncommutative Differential Geometry and Its Applications to Physics. Proceedings of the Shonan Kokusaimura June 1999 Workshop.
43. R.T. Seeley, *Complex powers of an elliptic operator*, AMS Proc. Symp. Pure Math. X (1967) 288–307.
44. R.T. Seeley, *Topics in pseudodifferential operators*. In: CIME Conference on Pseudo-Differential operators (Stresa 1968), pp. 167–305. Cremonese 1969.
45. G.B. Segal, *The definition of conformal field theory*, Oxford preprint (1990).
46. I.M. Singer, *Families of Dirac operators with applications to physics*, Asterisque, hors série (1985) 323–340.
47. I.M. Singer, *The η-invariant and the index*, in: Mathematical Aspects of String Theory, ed. S.T. Yau, World Scientific Press, Singapore, 1988, pp. 239–258.
48. M.E. Taylor, Partial Differential Equations. Qualitative Studies of Linear Equations, Springer–Verlag, New York, 1996.
49. K.P. Wojciechowski, *Elliptic operators and relative $K-$homology groups on manifolds with boundary*, C. R. Math. Rep. Acad. Sci. Canada 7 (1985) 149–154.
50. K.P. Wojciechowski, *The additivity of the η-invariant: The case of an invertible tangential operator*, Houston J. Math. 20 (1994) 603–621.
51. K.P. Wojciechowski, *The additivity of the η-invariant. The case of a singular tangential operator*, Commun. Math. Phys. 169 (1995) 315–327.
52. K.P. Wojciechowski, *The ζ-determinant and the additivity of the η-invariant on the smooth, self-adjoint Grassmannian*, Commun. Math. Phys. 201 (1999) 423–444.