Functional determinants for the Dirac equation with mixed pseudodifferential boundary conditions over finite cylinders

by

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Abstract.

In this note, we explicitly compute the functional determinant of a Dirac Laplacian with nonlocal *pseudodifferential* boundary conditions over a finite cylinder in terms of the ζ-function of the Dirac operator on the cross section and the pseudodifferential operators defining the boundary conditions. In particular, this result reduces to our previous formula [14] for the special case of generalized APS conditions. To prove our main result, we use the gluing and comparison formulæ established by the authors in [15, 16].

1. Introduction

Recent advances in quantum field theory have necessitated the explicit evaluation of functional determinants of Dirac operators over a variety of space-time configurations. In fact, at the one-loop order, any such theory can be reduced to the theory of determinants. We refer the reader to the highly acclaimed works of Dowker and Critchley [5] and Hawking [11]. See also Elizalde *et al.* [8], Kirsten [12], and Scott and Wojciechowski [23], for recent reviews. Because of their increasingly important rôle in mathematical physics, over the past several years there has been intense research to compute functional ζ-determinants of Dirac Laplacians. Of great significance is the Dirac Laplacian with spectral pseudodifferential boundary conditions; the celebrated Atiyah-Patodi-Singer (henceforth APS) boundary conditions being the most well-known example. Such boundary conditions arise in, for instance, one-loop quantum cosmology [3, 4, 6], spectral branes [26] and the study of Dirac fields in the background of a magnetic flux [2].

However, only recently was the open problem of explicitly computing the ζ-determinant of a Dirac Laplacian with APS conditions over a finite cylinder solved [14]. One reason this problem withstood the efforts of existing mathematical methods is that it is impossible to find explicit formulæ for the eigenvalues of such a Dirac operator. For this reason, we had to attack the problem using the method of *adiabatic decomposition* pioneered by Douglas and Wojciechowski [7] for the eta invariant and by the second author and Wojciechowski [19] for the ζ-determinant. The purpose of this current paper is twofold. First, we extend the result of [14] to a general class of pseudodifferential conditions that generalize the APS condition up to operators of *arbitrary* finite rank. To compute the ζ-determinant in this generalized framework, which in some sense possesses eigenvalues that are even more enigmatic, we use the gluing and comparison formulæ for ζ-determinants proved by the authors in [15, 16] to break up this general framework into tangible parts which can be explicitly computed. The second purpose of this paper is to elucidate the effectiveness of these gluing and comparison formulæ to compute ζ-determinants that have eluded explicit evaluations due to the perplexity of their eigenvalues. This also exemplifies the aim of gluing and comparison formulæ: Breaking up complex problems into simpler more tractable ones.

We now describe our set up. Fix $R > 0$ and let $\mathcal{D} : C^\infty(N_R, S) \to C^\infty(N_R, S)$ be a Dirac type operator where $N_R = [-R, R] \times Y$ is a finite cylinder with $R > 0$, Y a closed compact Riemannian manifold (of arbitrary dimension), and S a Clifford bundle over N_R. We assume that $\mathcal{D}$ is of product form

$$\mathcal{D} = G(\partial_u + D_Y) \tag{1.1}$$

where G is a unitary bundle isomorphism of S and D_Y is a Dirac operator acting on $C^\infty(Y, S)$ such that $G^2 = -\mathrm{Id}$ and $GD_Y = -D_Y G$. Furthermore, we assume that

$$\dim\Big(\ker(G + i) \cap \ker(D_Y)\Big) = \dim\Big(\ker(G - i) \cap \ker(D_Y)\Big). \tag{1.2}$$

Let Π_+, Π_-, and Π_0 denote the orthogonal projections onto the positive, negative, and zero eigenspaces of D_Y. Since N_R has boundaries, we have to impose boundary conditions. Let $\mathrm{Gr}^*_\infty(D_Y)$ denote the space of pairs $(\mathcal{P}_1, \mathcal{P}_2)$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are orthogonal pseudodifferential projections on $L^2(Y, S)$ such that

$$\mathcal{P}_1 - \Pi_+, \quad \mathcal{P}_2 - \Pi_- \quad \text{are smoothing operators,}$$

and for $i = 1, 2$,

$$\mathcal{P}_i G = G(\mathrm{Id} - \mathcal{P}_i), \qquad D_Y \mathcal{P}_i = \mathcal{P}_i D_Y.$$

An important class of such boundary conditions are the renowned generalized APS spectral conditions [1], which are defined as follows. Let σ_1, σ_2 be involutions (that is, $\sigma_i^2 = \mathrm{Id}$) over $\ker(D_Y)$ such that $\sigma_1 G = -G\sigma_1$ and $\sigma_2 G = -G\sigma_2$. Note that such involutions exist because of the assumption (1.2). Then

$$\Pi_+^{\sigma_1} := \Pi_+ + \frac{1+\sigma_1}{2}\Pi_0, \qquad \Pi_-^{\sigma_2} := \Pi_- + \frac{1+\sigma_2}{2}\Pi_0 \tag{1.3}$$

are called *generalized APS spectral projections*, and $(\Pi_+^{\sigma_1}, \Pi_-^{\sigma_2}) \in \mathrm{Gr}_\infty^*(D_Y)$. These generalized APS boundary conditions were considered in our paper [14], but elements of $\mathrm{Gr}_\infty^*(D_Y)$ are much more general and can differ from APS projections by operators of *arbitrary* finite rank. Let $P = (\mathcal{P}_1, \mathcal{P}_2) \in \mathrm{Gr}_\infty^*(D_Y)$ and impose boundary conditions for $\mathcal{D}$ at $\{-R\} \times Y$ and $\{R\} \times Y$ via

$$\mathcal{P}_1 \quad \text{at } \{-R\} \times Y, \qquad\qquad \mathcal{P}_2 \quad \text{at } \{R\} \times Y.$$

We denote by $\mathcal{D}_P$ the resulting operator with these boundary conditions, that is,

$$\mathcal{D}_P := \mathcal{D} \,:\, \mathrm{dom}(\mathcal{D}_P) \longrightarrow L^2(N_R, S) \tag{1.4}$$

where

$$\mathrm{dom}(\mathcal{D}_P) := \Big\{ \phi \in H^1(N_R, S) \;\big|\; \mathcal{P}_1(\phi|_{u=-R}) = 0 \;,\; \mathcal{P}_2(\phi|_{u=R}) = 0 \Big\}.$$

By the fundamental work of Seeley [24, 25], the spectrum of the Dirac operator $\mathcal{D}_P$ consists of discrete real eigenvalues $\{\lambda_k\}$. The ζ-function of $\mathcal{D}_P^2$ is defined by

$$\zeta_{\mathcal{D}_P^2}(s) = \sum_{\lambda_k \neq 0} \lambda_k^{-2s},$$

which is a priori defined for $\Re(s) \gg 0$, and by the work of Grubb [9],[10] and Wojciechowski [27] (cf. Lei [13] and Loya and Park [16]), has a meromorphic extension to $\mathbb{C}$ with 0 as a regular point. The ζ-determinant of $\mathcal{D}_P^2$ is defined by

$$\det{}_\zeta \mathcal{D}_P^2 := \exp(-\zeta'_{\mathcal{D}_P^2}(0)).$$

This definition first appeared in Ray and Singer's seminal paper [20] on the analytic torsion. Since we imposed nonlocal pseudodifferential boundary conditions, it is impossible to compute the eigenvalues $\{\lambda_k\}$ of $\mathcal{D}_P$ explicitly, so there is no direct way to compute the ζ-determinant $\det_\zeta \mathcal{D}_P^2$ from the eigenvalues. However, we shall give two derivations of the formula for $\det_\zeta \mathcal{D}_P^2$:

(1) Using the gluing formula proved in [15].
(2) Using the comparison/relative invariant formula proved in [16].

See Section 2 for more on these results. The formula for $\det_\zeta \mathcal{D}_P^2$ is described as follows. Since the $\mathcal{P}_i$'s are orthogonal projectors such that $\mathcal{P}_i G = G(\mathrm{Id} - \mathcal{P}_i)$ by assumption, with respect to the decomposition

$$L^2(Y, S) = L^2(Y, S^+) \oplus L^2(Y, S^-) \tag{1.5}$$

with $S^\pm$ denoting the $(\pm i)$-eigenspaces of G in S (recall that $G^2 = -\mathrm{Id}$), we can write

$$\mathcal{P}_i = \frac{1}{2}\begin{pmatrix} \mathrm{Id} & \kappa_i^{-1} \\ \kappa_i & \mathrm{Id} \end{pmatrix}$$

where $\kappa_i : L^2(Y,S^+) \to L^2(Y,S^-)$ are corresponding isometries. In particular,

$$U_P := -\kappa_1\kappa_2^{-1} : L^2(Y,S^-) \longrightarrow L^2(Y,S^-)$$

is a unitary operator. Let $\widehat{U}_P$ denote the restriction of U_P to the orthogonal complement of its (-1)-eigenspace. Let $W := \mathrm{Im}(\mathrm{Id}-\mathcal{P}_1) \cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}_2)$. Then our assumptions on $(\mathcal{P}_1,\mathcal{P}_2)$ imply that W is a finite-dimensional vector space and that $D_Y : \mathrm{Im}(\mathcal{P}_i) \longrightarrow \mathrm{Im}(\mathcal{P}_i)$ and $D_Y : \mathrm{Im}(\mathrm{Id}-\mathcal{P}_i) \longrightarrow \mathrm{Im}(\mathrm{Id}-\mathcal{P}_i)$. We define a map T_P over W by

$$T_P := \begin{cases} \dfrac{\sinh(2R\,D_Y)}{D_Y} & \text{on} \quad W \cap \ker(D_Y)^{\perp}, \\ 2R & \text{on} \quad W \cap \ker(D_Y). \end{cases}$$

We also define $T_{\mathcal{P}_1}$ and $T_{\mathcal{P}_2}$ over the finite-dimensional vector spaces $\mathrm{Im}(\Pi_+) \cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}_1)$ and $\mathrm{Im}(\Pi_+) \cap \mathrm{Im}(\mathcal{P}_2)$, respectively, by

$$T_{\mathcal{P}_1} := e^{4RD_Y} \quad , \quad T_{\mathcal{P}_2} := e^{4RD_Y}.$$

The following theorem is the main result of this note.

Theorem 1.1. *The following equality holds:*

$$\mathrm{det}_\zeta \mathcal{D}_P^2 = e^{2CR}\, 2^{\zeta_{D_Y^2}(0)+h_Y} \frac{(\det T_P)^2}{(\det T_{\mathcal{P}_1})^2 (\det T_{\mathcal{P}_2})^2} \cdot \mathrm{det}_F \left(\frac{2\mathrm{Id} + \widehat{U}_P + \widehat{U}_P^{-1}}{4} \right)$$

where $C = -(2\sqrt{\pi})^{-1}(\Gamma(s)^{-1}\Gamma(s-1/2)\zeta_{D_Y^2}(s-1/2))'(0)$ *with* $\zeta_{D_Y^2}(s)$ *the* ζ*-function of* D_Y^2, $h_Y = \dim\ker(D_Y)$ *and* det_F *denotes the Fredholm determinant.*

More explicitly, if we let E_μ denote the eigenspace of D_Y associated to the eigenvalue $\mu \in \mathrm{spec}(D_Y)$, then we can write

$$\begin{aligned} (1.6) \quad \mathrm{det}_\zeta \mathcal{D}_P^2 = e^{2CR}\, 2^{\zeta_{D_Y^2}(0)+h_Y} & \left(\prod_{\substack{\mu>0 \\ E_\mu \cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}_1) \neq 0}} e^{-4\mu R} \right) \cdot \left(\prod_{\substack{\mu>0 \\ E_\mu \cap \mathrm{Im}(\mathcal{P}_2) \neq 0}} e^{-4\mu R} \right) \cdot \\ & \left(\prod_{\substack{\mu \in \mathrm{spec}(D_Y) \\ E_\mu \cap W \neq 0}} \frac{\sinh^2(2\mu R)}{\mu^2} \right) \cdot \mathrm{det}_F \left(\frac{2\mathrm{Id} + \widehat{U}_P + \widehat{U}_P^{-1}}{4} \right), \end{aligned}$$

where if $\mu = 0$ in the product in the second line, we replace $\frac{\sinh^2(2\mu R)}{\mu^2}$ by its limit as $\mu \to 0$, that is, $(2R)^2$. In particular, when $\mathcal{P}_1 = \Pi_+^{\sigma_1}$ and $\mathcal{P}_2 = \Pi_-^{\sigma_2}$, the generalized APS spectral projectors in (1.3), then Theorem 1.1 reduces to the main result of [14]:

$$(1.7) \quad \mathrm{det}_\zeta \mathcal{D}^2_{\Pi_+^{\sigma_1},\Pi_-^{\sigma_2}} = e^{2CR}\, 2^{\zeta_{D_Y^2}(0)+h_Y} (2R)^{2h} \det{}^* \left(\frac{2\mathrm{Id} - (\sigma_1\sigma_2)_- - (\sigma_1\sigma_2)_-^{-1}}{4} \right)$$

where $(\sigma_1\sigma_2)_-$ is the restriction of the unitary map $\sigma_1\sigma_2$ to $L^2(Y,S^-) \cap \ker(D_Y)$, h is the number of $(+1)$-eigenvalues of $(\sigma_1\sigma_2)_-$, and where $\det^*(L)$ denotes the determinant of $(L|_{\ker(L)^\perp})$ for an operator L over a finite-dimensional vector space.

The structure of this paper is as follows. In Section 2, we review the gluing formula from [15] and the comparison/relative invariant formula from [16], which we shall use in the subsequent sections. In Section 3, we derive new formulæ for ζ-determinant ratios of Dirac operators with boundary conditions of special types.

Finally, in Section 4 we combine these special ζ-determinant ratios and the gluing and comparison formulæ from [15, 16] to derive our main Theorem 1.1.

2. The gluing and comparison formulæ from [15, 16]

In this section we review the gluing formula from [15] and the comparison/relative invariant formula from [16].

Let $\mathcal{D}$ be a Dirac type operator acting on $C^\infty(M,S)$ where M is a closed compact Riemannian manifold of arbitrary dimension and S is a Clifford bundle over M. Suppose that $M = M_- \cup M_+$ is partitioned into a union of manifolds with a common boundary $Y = \partial M_- = \partial M_+$. We assume that all geometric structures are of product type over a tubular neighborhood N of Y where $\mathcal{D}$ takes the product form (1.1). By restriction of $\mathcal{D}$, we obtain Dirac type operators $\mathcal{D}_\pm$ over $M_\pm$. We impose the boundary conditions given by the orthogonalized Calderón projectors $\mathcal{C}_\pm$ for $\mathcal{D}_\pm$ and we denote by $\mathcal{D}_{\mathcal{C}_\pm}$ the resulting operators,

$$\mathcal{D}_{\mathcal{C}_\pm} = \mathcal{D}_\pm \,:\, \mathrm{dom}(\mathcal{D}_{\mathcal{C}_\pm}) := \{\, \phi \in H^1(M_\pm, S) \mid \mathcal{C}_\pm(\phi|_Y) = 0 \,\} \longrightarrow L^2(M_\pm, S).$$

Here, we recall that the Calderón projectors $\mathcal{C}_\pm$ are the projectors defined intrinsically as the unique orthogonal projectors onto the closures in $L^2(Y,S)$ of the infinite-dimensional *Cauchy data spaces* of $\mathcal{D}_\pm$:

$$\{\, \phi|_Y \mid \phi \in C^\infty(M_\pm, S)\ ,\ \mathcal{D}_\pm \phi = 0 \,\} \subset C^\infty(Y,S).$$

The *gluing problem* for the ζ-determinant is to describe the "defect"

$$\frac{\det_\zeta \mathcal{D}^2}{\det_\zeta \mathcal{D}^2_{\mathcal{C}_+} \cdot \det_\zeta \mathcal{D}^2_{\mathcal{C}_-}} = \boxed{?}$$

in terms of recognizable data. To describe the solution in [15], we need to introduce some notations. With respect to the decomposition as in (1.5), the Calderón projectors $\mathcal{C}_\pm$ have the matrix forms

$$\mathcal{C}_\pm = \frac{1}{2}\begin{pmatrix} \mathrm{Id} & \kappa_\pm^{-1} \\ \kappa_\pm & \mathrm{Id} \end{pmatrix} \tag{2.1}$$

where the maps $\kappa_\pm : L^2(Y,S^+) \to L^2(Y,S^-)$ are corresponding isometries, so that $\mathcal{U} := -\kappa_- \kappa_+^{-1}$ is a unitary operator over $L^2(Y,S^-)$. Furthermore, $\mathcal{U}$ is of Fredholm determinant class. We denote by $\widehat{\mathcal{U}}$ the restriction of $\mathcal{U}$ to the orthogonal complement of its (-1)-eigenspace. We also put

$$\mathcal{L} := \sum_{k=1}^{h_M} \gamma U_k \otimes \gamma U_k \,:\, \gamma \ker(\mathcal{D}) \longrightarrow \gamma \ker(\mathcal{D})$$

where $h_M = \dim \ker(\mathcal{D})$, γ is the restriction map from M to Y, and $\{U_k\}$ is an orthonormal basis of $\ker(\mathcal{D})$. Then $\mathcal{L}$ is a positive operator on the finite-dimensional vector space

$$\gamma \ker(\mathcal{D}) \equiv \mathrm{Im}(\mathcal{C}_-) \cap \mathrm{Im}(\mathcal{C}_+).$$

We now have all the ingredients to state the following *gluing formula* [15]:

$$\frac{\det_\zeta \mathcal{D}^2}{\det_\zeta \mathcal{D}^2_{\mathcal{C}_-} \cdot \det_\zeta \mathcal{D}^2_{\mathcal{C}_+}} = 2^{-\zeta_{D_Y^2}(0) - h_Y} (\det \mathcal{L})^{-2} \det_F \Big(\frac{2\mathrm{Id} + \widehat{\mathcal{U}} + \widehat{\mathcal{U}}^{-1}}{4} \Big) \tag{2.2}$$

where $h_Y = \dim \ker(D_Y)$ and $\det_F$ denotes the Fredholm determinant. There is a similar formula for manifolds with cylindrical ends [17].

We now explain the comparison/relative invariant formula proved in [16] for $(M_-, \mathcal{D}_-)$. To this end, we consider the space $\mathrm{Gr}^*_\infty(\mathcal{D}_-)$, which consists of orthogonal projections $\mathcal{P}$ such that $G\mathcal{P} = (\mathrm{Id} - \mathcal{P})G$ and $\mathcal{P} - \mathcal{C}_-$ are smoothing operators. Let us fix $\mathcal{P} \in \mathrm{Gr}^*_\infty(\mathcal{D}_-)$ and let $\kappa_{\mathcal{P}} : L^2(Y, S^+) \to L^2(Y, S^-)$ be the map that determines $\mathcal{P}$ as $\kappa_\pm$ does $\mathcal{C}_\pm$ in (2.1). Let $\mathcal{D}_{\mathcal{P}}$ denote the operator $\mathcal{D}_-$ on M_- with the boundary condition given by $\mathcal{P}$. Let P_W be the orthogonal projection of $L^2(Y, S)$ onto the finite-dimensional vector space

$$W := \gamma \ker(\mathcal{D}_{\mathcal{P}}) \equiv \mathrm{Im}(\mathcal{C}_-) \cap \mathrm{Im}(\mathrm{Id} - \mathcal{P}).$$

Then we introduce a linear map

$$L := -P_W\, G\, \mathcal{R}^{-1}\, G\, P_W \;:\; \gamma \ker(\mathcal{D}_{\mathcal{P}}) \longrightarrow \gamma \ker(\mathcal{D}_{\mathcal{P}}) \tag{2.3}$$

where $\mathcal{R}$ is the sum of the Dirichlet to Neumann maps on an extension of M_- defined as follows. Fix *any* invertible extension $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ to a manifold $\widetilde{M}$ that contains M_-. (The *double* of $\mathcal{D}$ would do nicely.) Then for any $\varphi \in C^\infty(Y, S)$, there are unique $\phi_1 \in C^\infty(M_-, S)$ and $\phi_2 \in C^\infty(\widetilde{M} \setminus M_-, S)$ that are continuous at Y with value φ such that $\widetilde{\mathcal{D}}^2 \phi_i = 0$, $i = 1, 2$, off of Y. Then

$$\mathcal{R}\,\varphi := \partial_u \phi_1\Big|_Y - \partial_u \phi_2\Big|_Y. \tag{2.4}$$

In [16], we prove that L is a positive operator so that $\det L$ is a positive real number. Now the main result of [16] states that

$$\frac{\det_\zeta \mathcal{D}^2_{\mathcal{P}}}{\det_\zeta \mathcal{D}^2_{\mathcal{C}_-}} = (\det L)^2 \cdot \det_F\Big(\frac{2\mathrm{Id} + \widehat{U} + \widehat{U}^{-1}}{4}\Big) \tag{2.5}$$

where $\widehat{U}$ is the restriction of $U := \kappa_- \kappa_{\mathcal{P}}^{-1}$ to the orthogonal complement of its (-1)-eigenspace. The formula (2.5) extends the groundbreaking work of Scott [21] for the invertible case (cf. Scott and Wojciechowski [22]), and has recently been further extended to noncompact manifolds whose boundaries are manifolds with multi-cylindrical ends [18].

3. The ζ-determinant for special boundary conditions

If $(\mathcal{P}_1, \mathcal{P}_2) \in \mathrm{Gr}^*_\infty(D_Y)$, then for the sake of clarity we shall denote the operator $\mathcal{D}_P$ in the introduction with these boundary conditions by $\mathcal{D}_{\mathcal{P}_1, \mathcal{P}_2}$. Thus, over $N_R = [-R, R] \times Y$ we impose boundary conditions at $\{-R\} \times Y$ and $\{R\} \times Y$ via

$$\mathcal{P}_1 \quad \text{at } \{-R\} \times Y, \qquad\qquad \mathcal{P}_2 \quad \text{at } \{R\} \times Y,$$

and $\mathcal{D}_{\mathcal{P}_1, \mathcal{P}_2}$ is the operator with domain

$$\mathrm{dom}(\mathcal{D}_{\mathcal{P}_1, \mathcal{P}_2}) := \Big\{ \phi \in H^1(N_R, S) \;\big|\; \mathcal{P}_1(\phi|_{u=-R}) = 0\ ,\ \mathcal{P}_2(\phi|_{u=R}) = 0 \Big\}. \tag{3.1}$$

Let $\mathcal{P}$ be a projection on $L^2(Y, S)$ with $(\mathcal{P}, \mathrm{Id} - \mathcal{P}) \in \mathrm{Gr}^*_\infty(D_Y)$. By definition of $\mathrm{Gr}^*_\infty(D_Y)$, the image of $\mathcal{P}\Pi_0$ is a Lagrangian subspace in $\mathrm{Im}(\Pi_0) = \ker(D_Y)$. Let σ be the involution over $\mathrm{Im}(\Pi_0)$ such that $\frac{1+\sigma}{2}\Pi_0 = \mathcal{P}\Pi_0$. Recalling $\Pi^\sigma_+ := \Pi_+ + \frac{\mathrm{Id}+\sigma}{2}\Pi_0$, it follows that $(\mathcal{P}, \mathrm{Id} - \Pi^\sigma_+) \in \mathrm{Gr}^*_\infty(D_Y)$. Recall that we can write

$$\mathcal{P} = \frac{1}{2}\begin{pmatrix} \mathrm{Id} & \kappa_{\mathcal{P}}^{-1} \\ \kappa_{\mathcal{P}} & \mathrm{Id} \end{pmatrix}, \qquad \Pi^\sigma_+ = \frac{1}{2}\begin{pmatrix} \mathrm{Id} & \kappa_\sigma^{-1} \\ \kappa_\sigma & \mathrm{Id} \end{pmatrix}$$

for corresponding isometries $\kappa_{\mathcal{P}}, \kappa_\sigma : L^2(Y, S^+) \to L^2(Y, S^-)$, and define $\widehat{U}_{\mathcal{P}}$ as the restriction of $U_{\mathcal{P}} := \kappa_{\mathcal{P}} \kappa_\sigma^{-1}$ over $L^2(Y, S^-)$ to the orthogonal complement of

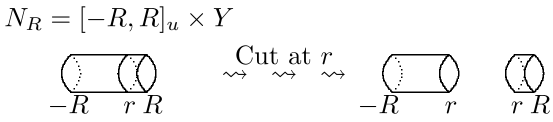

FIGURE 1. Cutting N_R at r into two pieces.

its (-1)-eigenspace. Note that $-\kappa_{\mathcal{P}}$ and $-\kappa_\sigma$ are the isometries corresponding to $\mathrm{Id} - \mathcal{P}$ and $\mathrm{Id} - \Pi_+^\sigma$, respectively. We begin by computing the following ratio.

Lemma 3.1. *We have*

$$\frac{\det_\zeta \mathcal{D}^2_{\mathcal{P},\mathrm{Id}-\Pi_+^\sigma}}{\det_\zeta \mathcal{D}^2_{\mathcal{P},\mathrm{Id}-\mathcal{P}}} = \left(\prod_{\substack{\mu>0 \\ E_\mu \cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}) \neq 0}} \frac{e^{4\mu R} \sinh^2(2\mu R)}{\mu^2} \right) \cdot \det_F \left(\frac{2\mathrm{Id} + \widehat{U}_{\mathcal{P}} + \widehat{U}_{\mathcal{P}}^{-1}}{4} \right),$$

where E_μ denotes the eigenspace of D_Y associated to the eigenvalue $\mu \in \mathrm{spec}(D_Y)$.

Proof. We give two proofs, first using the gluing formula (2.2) then using the comparison formula (2.5), in order to demonstrate the effectiveness of these formulæ.

Gluing proof of Lemma 3.1: Let us decompose N_R into two parts $[-R, r] \times Y$ and $[r, R] \times Y$ as shown in Figure 1. Then the restrictions of $\mathcal{D}_{\mathcal{P},\mathrm{Id}-\Pi_+^\sigma}$ over the decomposed parts $[-R, r] \times Y$ and $[r, R] \times Y$ define two Dirac type operators with boundary conditions given by $\mathcal{P}$ at $\{-R\} \times Y$ and $\mathrm{Id} - \Pi_+^\sigma$ at $\{R\} \times Y$. It is easy to check that the Calderón projections of these two operators are, respectively, just $\mathcal{C}_- = \mathrm{Id} - \mathcal{P}$ and $\mathcal{C}_+ = \Pi_+^\sigma$ over $\{r\} \times Y$. We denote the operators over the decomposed parts $[-R, r] \times Y$ and $[r, R] \times Y$, with the new boundary conditions given by these Calderón projectors over $\{r\} \times Y$, by $\mathcal{D}_{\mathcal{P},\mathrm{Id}-\mathcal{P}}(r)$ and $\mathcal{D}_{\Pi_+^\sigma,\mathrm{Id}-\Pi_+^\sigma}(r)$, respectively. Applying the gluing formula (2.2) to this situation, we obtain

$$\begin{aligned} &\frac{\det_\zeta \mathcal{D}^2_{\mathcal{P},\mathrm{Id}-\Pi_+^\sigma}}{\det_\zeta \mathcal{D}^2_{\mathcal{P},\mathrm{Id}-\mathcal{P}}(r) \cdot \det_\zeta \mathcal{D}^2_{\Pi_+^\sigma,\mathrm{Id}-\Pi_+^\sigma}(r)} \\ &\qquad = 2^{-\zeta_{D_Y^2}(0) - h_Y} (\det \mathcal{L}_r)^{-2} \det_F \left(\frac{2\mathrm{Id} + \widehat{U}_{\mathcal{P}} + \widehat{U}_{\mathcal{P}}^{-1}}{4} \right), \end{aligned} \tag{3.2}$$

where $\zeta_{D_Y^2}(s)$ is the ζ-function of D_Y^2 and $h_Y = \dim \ker(D_Y)$, where we used that the $\widehat{\mathcal{U}}$ in (2.2) for this situation is $-(-\kappa_{\mathcal{P}})(\kappa_\sigma^{-1}) = \kappa_{\mathcal{P}} \kappa_\sigma^{-1} =: \widehat{U}_{\mathcal{P}}$, and where

$$\mathcal{L}_r := \sum_{k=1}^{h_{\mathcal{P}}} \gamma_r U_k \otimes \gamma_r U_k \; : \; \gamma_r \ker(\mathcal{D}_{\mathcal{P},\mathrm{Id}-\Pi_+^\sigma}) \longrightarrow \gamma_r \ker(\mathcal{D}_{\mathcal{P},\mathrm{Id}-\Pi_+^\sigma})$$

where $h_{\mathcal{P}} = \dim \ker(\mathcal{D}_{\mathcal{P},\mathrm{Id}-\Pi_+^\sigma})$, γ_r is the restriction map from N_R to $\{r\} \times Y$, and $\{U_k\}$ is an orthonormal basis of $\ker(\mathcal{D}_{\mathcal{P},\mathrm{Id}-\Pi_+^\sigma})$. From the main result of [14] (that is, the formula (1.7)), we have

$$\det_\zeta \mathcal{D}^2_{\Pi_+^\sigma,\mathrm{Id}-\Pi_+^\sigma}(r) = e^{C(R-r)} 2^{\zeta_{D_Y^2}(0) + h_Y}$$

where $C = -(2\sqrt{\pi})^{-1}(\Gamma(s)^{-1}\Gamma(s-1/2)\zeta_{D_Y^2}(s-1/2))'(0)$. Thus, as $r \to R$, $\det_\zeta \mathcal{D}^2_{\Pi^\sigma_+,\mathrm{Id}-\Pi^\sigma_+}(r) \to 2^{\zeta_{D_Y^2}(0)+h_Y}$, so taking $r \to R$ in (3.2), we see that

$$\frac{\det_\zeta \mathcal{D}^2_{\mathcal{P},\mathrm{Id}-\Pi^\sigma_+}}{\det_\zeta \mathcal{D}^2_{\mathcal{P},\mathrm{Id}-\mathcal{P}}} = (\det \mathcal{L}_R)^{-2} \det{}_F \left(\frac{2\mathrm{Id} + \widehat{U}_{\mathcal{P}} + \widehat{U}_{\mathcal{P}}^{-1}}{4} \right).$$

It remains to compute $(\det \mathcal{L}_R)^{-2}$. To do so, we note that

$$\ker(\mathcal{D}_{\mathcal{P},\mathrm{Id}-\Pi^\sigma_+}) \equiv \mathrm{Im}(\mathrm{Id}-\mathcal{P}) \cap \mathrm{Im}(\Pi^\sigma_+) = \mathrm{Im}(\mathrm{Id}-\mathcal{P}) \cap \mathrm{Im}(\Pi_+),$$

since $\mathrm{Im}(\mathrm{Id}-\mathcal{P})$ and $\mathrm{Im}(\Pi^\sigma_+)$ have zero intersection in $\ker(D_Y)$ by definition of σ. Let $\{\psi_\mu\}$ be an orthonormal basis of $\mathrm{Im}(\mathrm{Id}-\mathcal{P}) \cap \mathrm{Im}(\Pi_+)$ where $\psi_\mu \in E_\mu$. (Here, we recall that $D_Y\mathcal{P} = \mathcal{P}D_Y$, so D_Y preserves $\mathrm{Im}(\mathrm{Id}-\mathcal{P})$ and obviously $\mathrm{Im}(\Pi_+)$ as well. It follows that D_Y can be diagonalized within the finite-dimensional space $\mathrm{Im}(\mathrm{Id}-\mathcal{P}) \cap \mathrm{Im}(\Pi_+)$. This elementary fact will be used quite often in the sequel.) Then $\phi_\mu = e^{-\mu u}\psi_\mu \in \ker(\mathcal{D}_{\mathcal{P},\mathrm{Id}-\Pi^\sigma_+})$ and

$$\|\phi_\mu\|^2 = \int_{-R}^{R} e^{-2\mu u}\, du = \frac{e^{2\mu R} - e^{-2\mu R}}{2\mu} = \frac{\sinh(2\mu R)}{\mu}.$$

It follows that

$$\mathcal{L}_R = \sum_{\substack{\mu>0 \\ E_\mu \cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}) \neq 0}} \frac{\phi_\mu(R)}{\|\phi_\mu\|} \otimes \frac{\phi_\mu(R)}{\|\phi_\mu\|} = \sum_{\substack{\mu>0 \\ E_\mu \cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}) \neq 0}} \frac{e^{-2\mu R}}{\|\phi_\mu\|^2}\, \psi_\mu \otimes \psi_\mu.$$

Hence,

$$(\det \mathcal{L}_R)^{-2} = \prod_{\substack{\mu>0 \\ E_\mu \cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}) \neq 0}} \left(\frac{e^{-2\mu R}}{\|\phi_\mu\|^2} \right)^{-2} = \prod_{\substack{\mu>0 \\ E_\mu \cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}) \neq 0}} e^{4\mu R} \frac{\sinh^2(2\mu R)}{\mu^2}.$$

This completes the Gluing proof of Lemma 3.1.

Comparison proof of Lemma 3.1: We shall apply the comparison formula (2.5) to the pair $(\mathcal{D}^2_{\mathcal{P},\mathrm{Id}-\Pi^\sigma_+}, \mathcal{D}^2_{\mathcal{P},\mathrm{Id}-\mathcal{P}})$. Here we regard $\mathrm{Id}-\mathcal{P}$ as the Calderón projector at the boundary $\{R\} \times Y$ of the operator

$$D : \mathrm{dom}(D) \longrightarrow L^2(N_R, S)$$

where

$$\mathrm{dom}(D) := \Big\{ \phi \in H^1(N_R, S) \;\Big|\; \mathcal{P}(\phi|_{u=-R}) = 0 \Big\}.$$

Then $D_{\mathrm{Id}-\Pi^\sigma_+} = \mathcal{D}_{\mathcal{P},\mathrm{Id}-\Pi^\sigma_+}$ and $D_{\mathcal{C}_-} = \mathcal{D}_{\mathcal{P},\mathrm{Id}-\mathcal{P}}$, so by the comparison formula (2.5),

$$\frac{\det_\zeta \mathcal{D}^2_{\mathcal{P},\mathrm{Id}-\Pi^\sigma_+}}{\det_\zeta \mathcal{D}^2_{\mathcal{P},\mathrm{Id}-\mathcal{P}}} = (\det L)^2 \cdot \det{}_F \left(\frac{2\mathrm{Id} + \widehat{U}_{\mathcal{P}} + \widehat{U}_{\mathcal{P}}^{-1}}{4} \right),$$

where we used that the U in (2.5) for this situation is $(-\kappa_{\mathcal{P}})(-\kappa_\sigma^{-1}) = \kappa_{\mathcal{P}}\kappa_\sigma^{-1} =: U_{\mathcal{P}}$, and where L is the map over $\{R\} \times Y$ defined in (2.3):

$$L = -P_W\, G\, \mathcal{R}^{-1}\, G\, P_W \;:\; W \longrightarrow W$$

with

$$W := \gamma_R \ker(\mathcal{D}_{\mathcal{P},\mathrm{Id}-\Pi^\sigma_+}) \equiv \mathrm{Im}(\mathrm{Id}-\mathcal{P}) \cap \mathrm{Im}(\Pi^\sigma_+) = \mathrm{Im}(\mathrm{Id}-\mathcal{P}) \cap \mathrm{Im}(\Pi_+).$$

To determine $(\det L)^2$, we need to find $\mathcal{R}$. An invertible extension $\widetilde{D}$ of D is just $G(\partial_u + D_Y)$ over $[-R, 2R] \times Y$ with boundary condition $\mathcal{P}$ at $-R$ and $\mathrm{Id} - \mathcal{P}$ at $2R$. Let $\{\psi_\mu\}$ be an orthonormal basis of $W = \mathrm{Im}(\mathrm{Id} - \mathcal{P}) \cap \mathrm{Im}(\Pi_+)$ where $\psi_\mu \in E_\mu$, and for each such μ, define $\varphi_\mu := G\psi_\mu \in GW$. Since $GD_Y = -D_Y G$ and $\mathcal{P}G = G(\mathrm{Id} - \mathcal{P})$, it follows that $\varphi_\mu = G\psi_\mu \in E_{-\mu} \cap \mathrm{Im}(\mathcal{P})$. Using this, it is straightforward to check that

$$\phi_1 := \frac{\sinh(\mu(u+R))}{\sinh(2\mu R)}\,\varphi_\mu \qquad \text{over} \quad [-R, R] \times Y,$$

and

$$\phi_2 := e^{\mu(u-R)}\varphi_\mu \qquad \text{over} \quad [R, 2R] \times Y,$$

are continuous at $u = R$ with value φ_μ and satisfy $\widetilde{D}^2\phi_i = 0$, $i = 1, 2$. Thus, by the definition of $\mathcal{R}$ in (2.4), we have

$$\mathcal{R}\varphi_\mu := \partial_u\phi_1\Big|_{u=R} - \partial_u\phi_2\Big|_{u=R} = \left(\mu\frac{\cosh(2\mu R)}{\sinh(2\mu R)} - \mu\right)\varphi_\mu = \frac{\mu e^{-2\mu R}}{\sinh(2\mu R)}\varphi_\mu.$$

Therefore,

$$-G\mathcal{R}G\psi_\mu = -G\mathcal{R}\varphi_\mu = -G\frac{\mu e^{-2\mu R}}{\sinh(2\mu R)}\varphi_\mu = \frac{\mu e^{-2\mu R}}{\sinh(2\mu R)}\psi_\mu.$$

It follows that

$$L = -P_W\, G\,\mathcal{R}^{-1}\, G\, P_W = \sum_{\substack{\mu>0 \\ E_\mu\cap\mathrm{Im}(\mathrm{Id}-\mathcal{P})\neq 0}} \frac{e^{2\mu R}\sinh(2\mu R)}{\mu}\,\psi_\mu\otimes\psi_\mu.$$

Hence,

$$(\det L)^2 = \prod_{\substack{\mu>0 \\ E_\mu\cap\mathrm{Im}(\mathrm{Id}-\mathcal{P})\neq 0}} e^{4\mu R}\frac{\sinh^2(2\mu R)}{\mu^2},$$

which completes the Comparison proof of Lemma 3.1. $\square$

Next, we compute a related ζ-determinant ratio.

Lemma 3.2. *With the notations above, the following equality holds:*

$$\frac{\det_\zeta \mathcal{D}^2_{\Pi_+^\sigma,\mathrm{Id}-\mathcal{P}}}{\det_\zeta \mathcal{D}^2_{\Pi_+^\sigma,\mathrm{Id}-\Pi_+^\sigma}} = \left(\prod_{\substack{\mu>0 \\ E_\mu\cap\mathrm{Im}(\mathrm{Id}-\mathcal{P})\neq 0}} \frac{e^{-4\mu R}\sinh^2(2\mu R)}{\mu^2}\right)\cdot \det_F\left(\frac{2\mathrm{Id} + \widehat{U}_{\mathcal{P}} + \widehat{U}_{\mathcal{P}}^{-1}}{4}\right).$$

Proof. Observe that $\mathrm{Id} - \Pi_+^\sigma$ is the Calderón projector at $\{R\}\times Y$ of the operator

$$D \,:\, \mathrm{dom}(D) \longrightarrow L^2(N_R, S)$$

where

$$\mathrm{dom}(D) := \Big\{\phi \in H^1(N_R, S) \;\big|\; \Pi_+^\sigma(\phi|_{u=-R}) = 0\Big\}.$$

Then $D_{\mathrm{Id}-\mathcal{P}} = \mathcal{D}_{\Pi_+^\sigma,\mathrm{Id}-\mathcal{P}}$ and $D_{\mathcal{C}_-} = \mathcal{D}_{\Pi_+^\sigma,\mathrm{Id}-\Pi_+^\sigma}$, so the comparison formula (2.5) applied to this situation gives us

$$\frac{\det_\zeta \mathcal{D}^2_{\Pi_+^\sigma,\mathrm{Id}-\mathcal{P}}}{\det_\zeta \mathcal{D}^2_{\Pi_+^\sigma,\mathrm{Id}-\Pi_+^\sigma}} = (\det L)^2\cdot\det_F\left(\frac{2\mathrm{Id} + \widehat{U}_{\mathcal{P}}^{-1} + \widehat{U}_{\mathcal{P}}}{4}\right),$$

where we used that the U in (2.5) for this situation is $(-\kappa_\sigma)(-\kappa_{\mathcal{P}}^{-1}) = \kappa_\sigma \kappa_{\mathcal{P}}^{-1} = (\kappa_{\mathcal{P}}\kappa_\sigma^{-1})^{-1} = U_{\mathcal{P}}^{-1}$, and where L is the map over $\{R\} \times Y$ defined in (2.3):

$$L := -P_W\, G\, \mathcal{R}^{-1}\, G\, P_W \;:\; W \longrightarrow W$$

with

$$W := \gamma_R \ker(\mathcal{D}_{\Pi_+^\sigma, \mathrm{Id}-\mathcal{P}}) = \mathrm{Im}(\mathrm{Id} - \Pi_+^\sigma) \cap \mathrm{Im}(\mathcal{P}) = \mathrm{Im}(\mathcal{P}) \cap \mathrm{Im}(\Pi_-).$$

To compute $(\det L)^2$ for this example, we proceed in much the same way as for the Comparison proof of Lemma 3.1. An invertible extension $\widetilde{D}$ of D is just $G(\partial_u + D_Y)$ over $[-R, 2R] \times Y$ with boundary condition Π_+^σ at $-R$ and $\mathrm{Id} - \Pi_+^\sigma$ at $2R$. Let $\{\psi_\nu\}$ be an orthonormal basis of $W = \mathrm{Im}(\mathcal{P}) \cap \mathrm{Im}(\Pi_-)$ where $\psi_\nu \in E_\nu$. It is important to note that, in contrast to the Comparison proof of Lemma 3.1, the ν's are *negative* (rather than positive) and that $\psi_\nu \in E_\nu \cap \mathrm{Im}(\mathcal{P})$ (rather than $\mathrm{Im}(\mathrm{Id} - \mathcal{P})$). Following the Comparison proof of Lemma 3.1 almost verbatim, it is straightforward to check that

$$L = -P_W\, G\, \mathcal{R}^{-1}\, G\, P_W = \sum_{\substack{\nu<0 \\ E_\nu \cap \mathrm{Im}(\mathcal{P}) \neq 0}} \frac{e^{2\nu R}\sinh(2\nu R)}{\nu}\, \psi_\nu \otimes \psi_\nu,$$

so

$$(\det L)^2 = \prod_{\substack{\nu<0 \\ E_\nu \cap \mathrm{Im}(\mathcal{P}) \neq 0}} e^{4\nu R} \frac{\sinh^2(2\nu R)}{\nu^2}.$$

Finally, using that G maps $E_\nu \cap \mathrm{Im}(\mathcal{P})$ isomorphically onto $E_{-\nu} \cap \mathrm{Im}(\mathrm{Id} - \mathcal{P})$, where we used that $GD_Y = -D_Y G$ and $\mathcal{P}G = G(\mathrm{Id} - \mathcal{P})$, we finally get

$$(\det L)^2 = \prod_{\substack{\mu>0 \\ E_\mu \cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}) \neq 0}} e^{4(-\mu)R} \frac{\sinh^2(-2\mu R)}{(-\mu)^2} = \prod_{\substack{\mu>0 \\ E_\mu \cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}) \neq 0}} e^{-4\mu R} \frac{\sinh^2(2\mu R)}{\mu^2},$$

which completes our proof. $\square$

We are now ready to prove the following

Lemma 3.3. *With the notations above, the following equality holds:*

$$\det{}_\zeta \mathcal{D}^2_{\mathcal{P}, \mathrm{Id}-\mathcal{P}} = e^{2CR}\, 2^{\zeta_{D_Y^2}(0) + h_Y} \cdot \left(\prod_{\substack{\mu>0 \\ E_\mu \cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}) \neq 0}} e^{-8\mu R} \right),$$

where $C = -(2\sqrt{\pi})^{-1}(\Gamma(s)^{-1}\Gamma(s-1/2)\zeta_{D_Y^2}(s-1/2))'(0)$ with $\zeta_{D_Y^2}(s)$ the ζ-function of D_Y^2 and $h_Y = \dim \ker(D_Y)$.

Proof. Solving for $\det_\zeta \mathcal{D}^2_{\mathcal{P}, \mathrm{Id}-\mathcal{P}}$ in Lemma 3.1, we obtain

$$\det{}_\zeta \mathcal{D}^2_{\mathcal{P}, \mathrm{Id}-\mathcal{P}} = \det{}_\zeta \mathcal{D}^2_{\mathcal{P}, \mathrm{Id}-\Pi_+^\sigma} \cdot \left(\prod_{\substack{\mu>0 \\ E_\mu \cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}) \neq 0}} \frac{e^{-4\mu R}\mu^2}{\sinh^2(2\mu R)} \right) \cdot \det{}_F \left(\frac{2\mathrm{Id} + \widehat{U}_{\mathcal{P}} + \widehat{U}_{\mathcal{P}}^{-1}}{4} \right)^{-1}. \tag{3.3}$$

$$N_R = [-R, R]_u \times Y$$

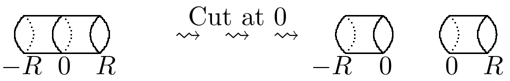

Figure 2. Cutting N_R at 0 into two pieces.

On the other hand, from Lemma 3.2, we know that

$$\frac{\det_\zeta \mathcal{D}^2_{\Pi^\sigma_+,\mathrm{Id}-\mathcal{P}}}{\det_\zeta \mathcal{D}^2_{\Pi^\sigma_+,\mathrm{Id}-\Pi^\sigma_+}} = \left(\prod_{\substack{\mu>0 \\ E_\mu \cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}) \neq 0}} \frac{e^{-4\mu R}\sinh^2(2\mu R)}{\mu^2} \right) \cdot \det_F \left(\frac{2\mathrm{Id} + \widehat{U}_{\mathcal{P}} + \widehat{U}_{\mathcal{P}}^{-1}}{4} \right)$$

and from the main result of [14] (see formula (1.7)), we also have

$$\det_\zeta \mathcal{D}^2_{\Pi^\sigma_+,\mathrm{Id}-\Pi^\sigma_+} = e^{2CR} 2^{\zeta_{D^2_Y}(0)+h_Y}.$$

Hence,

$$\det_\zeta \mathcal{D}^2_{\Pi^\sigma_+,\mathrm{Id}-\mathcal{P}} = e^{2CR} 2^{\zeta_{D^2_Y}(0)+h_Y} \left(\prod_{\substack{\mu>0 \\ E_\mu \cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}) \neq 0}} \frac{e^{-4\mu R}\sinh^2(2\mu R)}{\mu^2} \right) \cdot \det_F \left(\frac{2\mathrm{Id} + \widehat{U}_{\mathcal{P}} + \widehat{U}_{\mathcal{P}}^{-1}}{4} \right).$$

Substituting this expression into (3.3), using that

$$\det_\zeta \mathcal{D}^2_{\mathcal{P},\mathrm{Id}-\Pi^\sigma_+} = \det_\zeta \mathcal{D}^2_{\Pi^\sigma_+,\mathrm{Id}-\mathcal{P}}, \tag{3.4}$$

concludes the proof of our result once we show that (3.4) holds. In fact, (3.4) holds in the more general setting: $\det_\zeta \mathcal{D}^2_{\mathcal{P},\mathrm{Id}-\mathcal{Q}} = \det_\zeta \mathcal{D}^2_{\mathcal{Q},\mathrm{Id}-\mathcal{P}}$ for all $(\mathcal{P}, \mathrm{Id} - \mathcal{Q}) \in \mathrm{Gr}^*_\infty(D_Y)$. To prove this, we simply observe that

$$G(\partial_u + D_Y)\phi = \lambda\,\phi \quad , \quad \mathcal{P}\phi(-R) = 0 \quad , \quad (\mathrm{Id} - \mathcal{Q})\phi(R) = 0$$

if and only if $\psi(u) := G\,\phi(-u)$ satisfies

$$G(\partial_u + D_Y)\psi = -\lambda\,\psi \quad , \quad (\mathrm{Id} - \mathcal{P})\psi(R) = 0 \quad , \quad \mathcal{Q}\psi(-R) = 0,$$

where we used that $G^2 = -\mathrm{Id}$, $GD_Y = -D_Y G$, $\mathcal{P}G = G(\mathrm{Id} - \mathcal{P})$, and $\mathcal{Q}G = G(\mathrm{Id} - \mathcal{Q})$. It follows that $\mathrm{spec}(\mathcal{D}^2_{\mathcal{P},\mathrm{Id}-\mathcal{Q}}) = \mathrm{spec}(\mathcal{D}^2_{\mathcal{Q},\mathrm{Id}-\mathcal{P}})$, which implies that $\det_\zeta \mathcal{D}^2_{\mathcal{P},\mathrm{Id}-\mathcal{Q}} = \det_\zeta \mathcal{D}^2_{\mathcal{Q},\mathrm{Id}-\mathcal{P}}$. □

4. Proof of Theorem 1.1

As we already mentioned, in order to demonstrate the effectiveness of the gluing formula (2.2) and the comparison formula (2.5), we shall give separate proofs of Theorem 1.1 exploiting both formulæ.

Gluing proof of Theorem 1.1: Recall that the operator $\mathcal{D}_P$ as defined in (1.4) written using the notation (3.1) is

$$\mathcal{D}_{\mathcal{P}_1,\mathcal{P}_2} := \mathcal{D} \,:\, \mathrm{dom}(\mathcal{D}_{\mathcal{P}_1,\mathcal{P}_2}) \longrightarrow L^2(N_R, S)$$

where

$$\mathrm{dom}(\mathcal{D}_{\mathcal{P}_1,\mathcal{P}_2}) := \Big\{ \phi \in H^1(N_R, S) \;\Big|\; \mathcal{P}_1(\phi|_{u=-R}) = 0 \;,\; \mathcal{P}_2(\phi|_{u=R}) = 0 \Big\}.$$

Let us decompose N_R into two parts $[-R,0]\times Y$ and $[0,R]\times Y$ as shown in Figure 2. Then the restrictions of $\mathcal{D}_{\mathcal{P}_1,\mathcal{P}_2}$ over the decomposed parts define two Dirac type operators with boundary conditions given by $\mathcal{P}_1$ at $\{-R\}\times Y$ and $\mathcal{P}_2$ at $\{R\}\times Y$. It is easy to check that the Calderón projections of these operators are, respectively, just $\mathrm{Id}-\mathcal{P}_1$ and $\mathrm{Id}-\mathcal{P}_2$ over $\{0\}\times Y$. We denote the operators over the decomposed parts $[-R,0]\times Y$ and $[0,R]\times Y$ with the new boundary conditions given by these Calderón projectors over $\{0\}\times Y$ by $D_{\mathcal{P}_1,\mathrm{Id}-\mathcal{P}_1}$ and $D_{\mathrm{Id}-\mathcal{P}_2,\mathcal{P}_2}$, respectively. Now applying the gluing formula (2.2) to this situation, we obtain

$$\begin{aligned}(4.1)\qquad &\frac{\det_\zeta \mathcal{D}^2_{\mathcal{P}_1,\mathcal{P}_2}}{\det_\zeta D^2_{\mathcal{P}_1,\mathrm{Id}-\mathcal{P}_1}\cdot \det_\zeta D^2_{\mathrm{Id}-\mathcal{P}_2,\mathcal{P}_2}}\\ &\qquad = 2^{-\zeta_{D_Y^2}(0)-h_Y}(\det\mathcal{L})^{-2}\det_F\left(\frac{2\mathrm{Id}+\widehat{U}_P+\widehat{U}_P^{-1}}{4}\right)\end{aligned}$$

where the $\widehat{\mathcal{U}}$ in (2.2) for this situation is the restriction of $-(-\kappa_1)(-\kappa_2^{-1}) = -\kappa_1\kappa_2^{-1} =: U_P$ over $L^2(Y,S^-)$ to the orthogonal complement of its (-1)-eigenspace, noting that $-\kappa_i$ is the isometry corresponding to $\mathrm{Id}-\mathcal{P}_i$ for $i=1,2$, and where

$$(4.2)\qquad \mathcal{L} := \sum_{k=1}^{h_P}\gamma_0 U_k\otimes\gamma_0 U_k \ :\ \gamma_0\ker(\mathcal{D}_{\mathcal{P}_1,\mathcal{P}_2}) \longrightarrow \gamma_0\ker(\mathcal{D}_{\mathcal{P}_1,\mathcal{P}_2})$$

where $h_P=\dim\ker(\mathcal{D}_{\mathcal{P}_1,\mathcal{P}_2})$, γ_0 is the restriction map from N_R to $\{0\}\times Y$, and $\{U_k\}$ is an orthonormal basis of $\ker(\mathcal{D}_{\mathcal{P}_1,\mathcal{P}_2})$. By Lemma 3.3 (here we need to replace R with $R/2$ since the lengths of $[-R,0]$ and $[0,R]$ are half that of $[-R,R]$, which is the interval considered in Lemma 3.3),

$$\det_\zeta D^2_{\mathcal{P}_1,\mathrm{Id}-\mathcal{P}_1} = e^{CR}\,2^{\zeta_{D_Y^2}(0)+h_Y}\cdot\left(\prod_{\substack{\mu>0\\ E_\mu\cap\mathrm{Im}(\mathrm{Id}-\mathcal{P}_1)\neq 0}} e^{-4\mu R}\right),$$

and

$$\det_\zeta D^2_{\mathrm{Id}-\mathcal{P}_2,\mathcal{P}_2} = e^{CR}\,2^{\zeta_{D_Y^2}(0)+h_Y}\cdot\left(\prod_{\substack{\mu>0\\ E_\mu\cap\mathrm{Im}(\mathcal{P}_2)\neq 0}} e^{-4\mu R}\right).$$

Now to complete the Gluing proof of Theorem 1.1, it remains to compute $(\det\mathcal{L})^{-2}$ over $\gamma_0\ker(\mathcal{D}_{\mathcal{P}_1,\mathcal{P}_2})\equiv\mathrm{Im}(\mathrm{Id}-\mathcal{P}_1)\cap\mathrm{Im}(\mathrm{Id}-\mathcal{P}_2)$. To do so, we note that

$$\ker(\mathcal{D}_{\mathcal{P}_1,\mathcal{P}_2})\equiv\mathrm{Im}(\mathrm{Id}-\mathcal{P}_1)\cap\mathrm{Im}(\mathrm{Id}-\mathcal{P}_2) := W.$$

Let $\{\psi_\mu\}$ be an orthonormal basis of W where $\psi_\mu\in E_\mu$. Then $\phi_\mu := e^{-\mu u}\psi_\mu\in\ker(\mathcal{D}_{\mathcal{P}_1,\mathcal{P}_2})$ and

$$\|\phi_\mu\|^2 = \int_{-R}^{R} e^{-2\mu u}\,du = \frac{\sinh(2\mu R)}{\mu},$$

where if $\mu=0$, then we replace $\frac{\sinh(2\mu R)}{\mu}$ by its limit as $\mu\to 0$, that is, $2R$. It follows that

$$\mathcal{L} = \sum_{\substack{\mu\in\mathrm{spec}(D_Y)\\ E_\mu\cap W\neq 0}}\frac{\phi_\mu(0)}{\|\phi_\mu\|}\otimes\frac{\phi_\mu(0)}{\|\phi_\mu\|} = \sum_{\substack{\mu\in\mathrm{spec}(D_Y)\\ E_\mu\cap W\neq 0}}\frac{1}{\|\phi_\mu\|^2}\,\psi_\mu\otimes\psi_\mu.$$

Hence,

$$(\det \mathcal{L})^{-2} = \prod_{\substack{\mu\in\mathrm{spec}(D_Y)\\ E_\mu\cap W\neq 0}} \left(\frac{1}{\|\phi_\mu\|^2}\right)^{-2} = \prod_{\substack{\mu\in\mathrm{spec}(D_Y)\\ E_\mu\cap W\neq 0}} \frac{\sinh^2(2\mu R)}{\mu^2}.$$

This completes the Gluing proof of Theorem 1.1.

Comparison proof of Theorem 1.1: We now prove Theorem 1.1 using the comparison formula (2.5) applied to the pair $(\mathcal{D}^2_{\mathcal{P}_1,\mathcal{P}_2}, \mathcal{D}^2_{\mathcal{P}_1,\mathrm{Id}-\mathcal{P}_1})$. Here we regard $\mathrm{Id}-\mathcal{P}_1$ as the Calderón projector at the boundary $\{R\}\times Y$ of the operator

$$D \,:\, \mathrm{dom}(D) \longrightarrow L^2(N_R, S)$$

where

$$\mathrm{dom}(D) := \Big\{\, \phi \in H^1(N_R, S) \mid \mathcal{P}_1(\phi|_{u=-R}) = 0 \Big\}.$$

Then $D_{\mathcal{P}_2} = \mathcal{D}_{\mathcal{P}_1,\mathcal{P}_2}$ and $D_{\mathcal{C}_-} = \mathcal{D}_{\mathcal{P}_1,\mathrm{Id}-\mathcal{P}_1}$, so applying the comparison formula (2.5) to this situation, we obtain

$$\frac{\det_\zeta \mathcal{D}^2_{\mathcal{P}_1,\mathcal{P}_2}}{\det_\zeta \mathcal{D}^2_{\mathcal{P}_1,\mathrm{Id}-\mathcal{P}_1}} = (\det L)^2 \cdot \det{}_F\left(\frac{2\mathrm{Id} + \widehat{U}_P + \widehat{U}_P^{-1}}{4}\right), \tag{4.3}$$

where we used that the U in (2.5) for this situation is $(-\kappa_1)\kappa_2^{-1} =: U_P$ noting that $-\kappa_1$ corresponds to $\mathrm{Id}-\mathcal{P}_1$, and where L is the operator defined in (2.3) for this situation, which we will investigate in detail soon. Now, by Lemma 3.3,

$$\det_\zeta \mathcal{D}^2_{\mathcal{P}_1,\mathrm{Id}-\mathcal{P}_1} = e^{2CR}\, 2^{\zeta_{D_Y^2}(0)+h_Y} \cdot \left(\prod_{\substack{\mu>0\\ E_\mu\cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}_1)\neq 0}} e^{-8\mu R}\right),$$

therefore by (4.3),

$$\det_\zeta \mathcal{D}^2_{\mathcal{P}_1,\mathcal{P}_2} = e^{2CR}\, 2^{\zeta_{D_Y^2}(0)+h_Y} \left(\prod_{\substack{\mu>0\\ E_\mu\cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}_1)\neq 0}} e^{-8\mu R}\right)\cdot$$
$$(\det L)^2 \cdot \det{}_F\left(\frac{2\mathrm{Id} + \widehat{U}_P + \widehat{U}_P^{-1}}{4}\right).$$

To compute $\det L$, we use almost the exact same argument found in Lemma 3.2 to show that with $W := \mathrm{Im}(\mathrm{Id}-\mathcal{P}_1)\cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}_2)$,

$$(\det L)^2 = \prod_{\substack{\mu\in\mathrm{spec}(D_Y)\\ E_\mu\cap W\neq 0}} e^{4\mu R}\frac{\sinh^2(2\mu R)}{\mu^2},$$

where in the product, when $\mu = 0$ we replace $\frac{\sinh^2(2\mu R)}{\mu^2}$ by its limit as $\mu \to 0$, that is, $(2R)^2$. Therefore,

$$\det{}_\zeta \mathcal{D}^2_{\mathcal{P}_1,\mathcal{P}_2} = e^{2CR}\, 2^{\zeta_{D_Y^2}(0)+h_Y} \Bigg(\prod_{\substack{\mu>0 \\ E_\mu \cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}_1)\neq 0}} e^{-8\mu R} \Bigg) \cdot \tag{4.4}$$
$$\Bigg(\prod_{\substack{\mu\in \mathrm{spec}(D_Y) \\ E_\mu\cap W\neq 0}} e^{4\mu R} \Bigg) \cdot \Bigg(\prod_{\substack{\mu\in \mathrm{spec}(D_Y) \\ E_\mu\cap W\neq 0}} \frac{\sinh^2(2\mu R)}{\mu^2} \Bigg) \cdot \det{}_F \Bigg(\frac{2\mathrm{Id} + \widehat{U}_P + \widehat{U}_P^{-1}}{4} \Bigg).$$

Since $W = \mathrm{Im}(\mathrm{Id}-\mathcal{P}_1)\cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}_2)$, we have

$$\prod_{\substack{\mu\in \mathrm{spec}(D_Y) \\ E_\mu\cap W\neq 0}} e^{4\mu R} = \Bigg(\prod_{\substack{\mu>0 \\ E_\mu\cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}_1)\cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}_2)\neq 0}} e^{4\mu R} \Bigg) \cdot \Bigg(\prod_{\substack{\mu<0 \\ E_\mu\cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}_1)\cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}_2)\neq 0}} e^{4\mu R} \Bigg)$$
$$= \Bigg(\prod_{\substack{\mu>0 \\ E_\mu\cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}_1)\neq 0}} e^{4\mu R} \Bigg) \cdot \Bigg(\prod_{\substack{\mu>0 \\ E_\mu\cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}_1)\cap \mathrm{Im}(\mathcal{P}_2)\neq 0}} e^{-4\mu R} \Bigg) \cdot \Bigg(\prod_{\substack{\mu<0 \\ E_\mu\cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}_1)\cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}_2)\neq 0}} e^{4\mu R} \Bigg).$$

Since $D_Y G = -G D_Y$ and $\mathcal{P}_i G = G(\mathrm{Id}-\mathcal{P}_i)$, we have

$$E_\mu \cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}_1)\cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}_2) \neq 0 \iff E_{-\mu}\cap \mathrm{Im}(\mathcal{P}_1)\cap \mathrm{Im}(\mathcal{P}_2)\neq 0,$$

as G maps the first space isomorphically onto the second space, therefore

$$\prod_{\substack{\mu\in \mathrm{spec}(D_Y) \\ E_\mu\cap W\neq 0}} e^{4\mu R} = \Bigg(\prod_{\substack{\mu>0 \\ E_\mu\cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}_1)\neq 0}} e^{4\mu R} \Bigg) \cdot \Bigg(\prod_{\substack{\mu>0 \\ E_\mu\cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}_1)\cap \mathrm{Im}(\mathcal{P}_2)\neq 0}} e^{-4\mu R} \Bigg) \cdot \Bigg(\prod_{\substack{\mu>0 \\ E_\mu\cap \mathrm{Im}(\mathcal{P}_1)\cap \mathrm{Im}(\mathcal{P}_2)\neq 0}} e^{-4\mu R} \Bigg)$$
$$= \Bigg(\prod_{\substack{\mu>0 \\ E_\mu\cap \mathrm{Im}(\mathrm{Id}-\mathcal{P}_1)\neq 0}} e^{4\mu R} \Bigg) \cdot \Bigg(\prod_{\substack{\mu>0 \\ E_\mu\cap \mathrm{Im}(\mathcal{P}_2)\neq 0}} e^{-4\mu R} \Bigg).$$

Putting this expression into (4.4) completes the Comparison proof of Theorem 1.1.

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