RELATIVE FORMULÆ FOR THE SPECTRAL INVARIANTS OF THE b-CALCULUS AND GENERALIZED APS BOUNDARY PROBLEMS OF DIRAC OPERATORS

Paul Loya and Jinsung Park

We relate zeta determinants of Dirac operators with generalized APS boundary conditions for compact manifolds with boundary and parallel b-zeta determinants of perturbed Dirac operators on the corresponding complete manifolds with cylindrical end. We also derive, without invertibility conditions, corresponding relative formulæ for the (b-) zeta determinants.

1. Introduction

Of special geometric and topological importance are those boundary problems of generalized APS type, as can be seen in the Atiyah–Patodi–Singer index theorem [**1**] (cf. Subsection 1.1). In their paper, they found a connection of their theorem with the index of the Dirac operator on a corresponding manifold with cylindrical end. Some years later, Melrose [**18**] worked out this connection and developed the b-calculus to give his "direct proof" of the APS index theorem. Recently, both the index and the eta invariant of Dirac operators with generalized APS conditions for manifolds with boundary were connected with parallel invariants of associated perturbed Dirac operators on the corresponding manifolds with cylindrical end (see Melrose and Piazza [**19**], Loya and Melrose [**12**], and Loya [**11**]). The purpose of this paper is to derive a similar connection for the ζ-determinant. To motivate this connection, we begin by reviewing the connection between the b-calculus and boundary value problems for the index and the eta invariant.

1.1. The b-calculus and the index theorem. We first state our assumptions. Let $\mathcal{D} : C^\infty(X, E) \longrightarrow C^\infty(X, F)$ be a compatible Dirac type operator associated to a $\mathbb{Z}_2$-graded Hermitian Clifford module $E \oplus F$ over a compact Riemannian manifold X with boundary Y. We assume that all the geometric structures are of product type on a collar $[-1, 0]_u \times Y$ of the boundary $\{u = 0\} = \partial X = Y$. Therefore, on this collar we assume that $E \cong E|_{u=0}$, $F \cong F|_{u=0}$, and

$$\mathcal{D} = G(\partial_u + D_Y),$$

where $G : E|_{u=0} \longrightarrow F|_{u=0}$ is a unitary isomorphism and D_Y is a Dirac operator on Y. Let Π_+, Π_-, and Π_0 denote the orthogonal projections

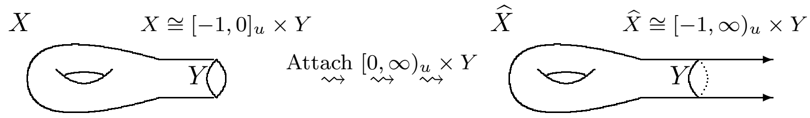

Figure 1. Creating a manifold with cylindrical end.

of $L^2(Y, E_0)$, where $E_0 := E|_{u=0}$, onto the positive, negative, and zero eigenspaces, respectively, of D_Y.

Let us assume for the moment that X is even-dimensional. Let T be a unitary involution (that is, $T^2 = \mathrm{Id}$) on $V := \ker D_Y$. Then T has ± 1 eigenvalues. We define $\mathcal{D}_T$ as the Dirac operator $\mathcal{D}$ with domain

$$\mathrm{dom}(\mathcal{D}_T) := \{\phi \in H^1(X, E) \mid \Pi_-^T\big(\phi|_{u=0}\big) = 0\}, \tag{1}$$

where $\Pi_-^T := \Pi_- + \Pi_{-T}$ with $\Pi_L := \frac{\mathrm{Id}+L}{2}\Pi_0$ for any involution L on V. Such a boundary condition is called a *generalized APS boundary condition.*

Let $\widehat{X}$ be the manifold formed by gluing the infinite cylinder $[0,\infty)_u \times Y$ to the end of the collar $[-1,0]_u \times Y$ of X (see Figure 1):

$$\widehat{X} := X \sqcup_{\partial X} \big([0,\infty)_u \times Y\big).$$

All the geometric structures on X extend naturally to the manifold $\widehat{X}$. We use the same notations for these extended objects on $\widehat{X}$ as for the original objects on X, except we denote the extended Dirac operator by $\widehat{\mathcal{D}}$.

Given a self-adjoint involution T on V, following Melrose and Piazza [**19**], in Section 2 we show how to construct a corresponding b-smoothing operator $\widehat{T} \in \Psi_b^{-\infty}(\widehat{X}, E, F)$ such that the L^2 based operator

$$\widehat{\mathcal{D}} + \widehat{T} : H^1(\widehat{X}, E) \longrightarrow L^2(\widehat{X}, F) \tag{2}$$

is "linked" to the operator $\mathcal{D}_T$ on the compact manifold X. More precisely, in [**19**, **11**], the index theoretic properties of X and $\widehat{X}$ were linked as follows. The operators $\widehat{\mathcal{D}} + \widehat{T}$ and $\mathcal{D}_T$ have the same index theoretic properties:

(a) $\ker(\widehat{\mathcal{D}} + \widehat{T}) \cong \ker \mathcal{D}_T$ and $\ker(\widehat{\mathcal{D}} + \widehat{T})^* \cong \ker(\mathcal{D}_T)^*$.
(b) $\widehat{\mathcal{D}} + \widehat{T} : H^1(\widehat{X}, E) \longrightarrow L^2(\widehat{X}, F)$ and $\mathcal{D}_T : \mathrm{dom}(\mathcal{D}_T) \longrightarrow L^2(X, F)$ are Fredholm with equal indices (by (a)).
(c) The following index formula holds:

$$\mathrm{ind}(\widehat{\mathcal{D}} + \widehat{T}) = \mathrm{ind}\,\mathcal{D}_T = \int_X \mathrm{AS} + \frac{1}{2}[\eta(D_Y) + \mathrm{sign}\, T],$$

where AS is the Atiyah-Singer index density and where $\eta(D_Y)$ is the eta invariant of D_Y (cf. Subsection 1.2). Note that Y is a "right boundary"; this accounts for the $+$ instead of $-$ in front of the eta term.

The connections (b) and (c) were first observed in Melrose and Piazza's paper [**19**]. This theorem holds even when $\dim X$ is odd, but in this case

$\int_X$ AS vanishes. As a trivial corollary, we get the following relative formula: For any two such maps T and S on V, we have

$$\operatorname{ind}(\widehat{\mathcal{D}}+\widehat{T})-\operatorname{ind}(\widehat{\mathcal{D}}+\widehat{S})=\operatorname{ind}\mathcal{D}_T-\operatorname{ind}\mathcal{D}_S=\frac{1}{2}\,[\operatorname{sign}T-\operatorname{sign}S\,]. \tag{3}$$

1.2. The b-calculus and the eta invariant. We now review the connection between the eta invariants of $\widehat{\mathcal{D}}+\widehat{T}$ and $\mathcal{D}_T$ established in [**11**, **12**]. Henceforth we assume that $E=F$ and X can be of arbitrary dimension. Then G is a unitary isomorphism on E_0 only, since $E=F$. Moreover, Clifford algebra and self-adjointness considerations impose the relations

$$G^2=-\mathrm{Id}\quad,\quad G^*=-G\quad,\quad G\,D_Y=-D_YG. \tag{4}$$

The last equality in (4) implies that G acts on V. We denote the set of unitary involutions T on V such that $GT=-T\,G$ by $\mathcal{L}(V)$ and for such a T, we denote the $+1$ eigenspace of T by Λ_T. There is a distinguished subspace Λ_σ of V defined by

$$\Lambda_\sigma:=\left\{\Pi_0\left(\phi|_{u=0}\right)\mid\phi\in H^1(X,E),\ \mathcal{D}\phi=0,\ \Pi_-\left(\phi|_{u=0}\right)=0\right\}.$$

If Π_σ is the orthogonal projection onto Λ_σ, then $\sigma:=2\Pi_\sigma-\mathrm{Id}$, acting on V, is in $\mathcal{L}(V)$ with associated $+1$ eigenspace Λ_σ. The unitary map σ is called the *scattering matrix* and Λ_σ is called the *scattering Lagrangian.*

We now recall the definition of the eta invariant of $\mathcal{D}_T$ for $T\in\mathcal{L}(V)$ (cf. Douglas and Wojciechowski [**6**] and Grubb and Seeley [**7**]). Since $T\in\mathcal{L}(V)$, it turns out that the operator $\mathcal{D}_T$ is self-adjoint and has real discrete spectrum. If $\{\lambda_j\}$ are the eigenvalues of $\mathcal{D}_T$, then the eta function of $\mathcal{D}_T$,

$$\eta_{\mathcal{D}_T}(s):=\sum_{\lambda_j\neq 0}\frac{\operatorname{sign}\lambda_j}{|\lambda_j|^s},$$

extends from $\operatorname{Re}s\gg 0$ to be a meromorphic function of $s\in\mathbb{C}$ that is regular at $s=0$. The eta invariant is by definition $\eta(\mathcal{D}_T):=\eta_{\mathcal{D}_T}(0)$.

The operator $\widehat{\mathcal{D}}+\widehat{T}$ over the noncompact manifold $\widehat{X}$ has a corresponding invariant, called the b-eta invariant, which we denote by ${}^b\eta(\widehat{\mathcal{D}}+\widehat{T})$. In [**11**, **12**], the eta invariant theoretic properties of X and $\widehat{X}$ were linked as follows (for odd-dimensional manifolds, but the even-dimensional case is similar). For $T\in\mathcal{L}(V)$, the perturbed operator $\widehat{\mathcal{D}}+\widehat{T}$ in (2) and the operator $\mathcal{D}_T$ with domain (1) have the same eta invariant theoretic properties:

(a) ${}^b\eta(\widehat{\mathcal{D}}+\widehat{T})=\eta(\mathcal{D}_T)$.
(b) The following formula holds:

$$ {}^b\eta(\widehat{\mathcal{D}}+\widehat{T})=\eta(\mathcal{D}_T)=\eta(\mathcal{D}_{-\sigma})+m(\Lambda_T,\Lambda_\sigma),$$

where σ is the scattering matrix.

Here, the "m-function" was introduced by Lesch and Wojciechowski in [**10**] (cf. [**4**]): For $T, S \in \mathcal{L}(V)$,

$$m(\Lambda_T, \Lambda_S) := -\frac{1}{i\pi} \sum_{\substack{e^{i\theta} \in \operatorname{spec}(-[(i+G)/2i]TS) \\ \theta \in (-\pi,\pi)}} i\theta.$$

As a trivial corollary, we get the following relative formula:

$$ {}^b\eta(\widehat{\mathcal{D}} + \widehat{T}) - {}^b\eta(\widehat{\mathcal{D}} - \widehat{\sigma}) = \eta(\mathcal{D}_T) - \eta(\mathcal{D}_{-\sigma}) = m(\Lambda_T, \Lambda_\sigma). \tag{5}$$

This formula is related to the gluing problem for the (b-)eta invariant, which has been investigated by many authors, see for instance, Dai and Freed [**5**], Bunke [**4**], Mazzeo and Melrose [**16**], Hassell, Mazzeo, and Melrose [**8**], Wojciechowski [**24**], Müller [**20**], Brüning and Lesch [**3**], Kirk and Lesch [**9**], Loya and Park [**13**, **14**]; see the survey article by Mazzeo and Piazza [**17**] for more on this topic.

1.3. The b-calculus and the ζ-determinant. To paraphrase the previous two sections: $\widehat{\mathcal{D}} + \widehat{T}$ and $\mathcal{D}_T$ have identical index and eta invariant theoretic properties; moreover, we have exact (no integer ambiguities) relative invariant formulæ (3) and (5). The purpose of this paper is to investigate the ζ-determinant connection of X and $\widehat{X}$, which we now explain. Recall that if $\{\lambda_j\}$ are the eigenvalues of $\mathcal{D}_T$, then the zeta function of $\mathcal{D}_T^2 := (\mathcal{D}_T)^2$ is

$$\zeta_{\mathcal{D}_T^2}(s) := \sum_{\lambda_j \neq 0} \lambda_j^{-2s}, \tag{6}$$

which is defined a priori for $\operatorname{Re} s \gg 0$ and has an analytic continuation to the whole complex plane with 0 as a regular point [**7**]. Then the ζ-determinant of $\mathcal{D}_T^2$ is by definition

$$\det{}_\zeta \mathcal{D}_T^2 := \exp\Big(-\frac{d}{ds}\Big|_{s=0} \zeta_{\mathcal{D}_T^2}(s)\Big). \tag{7}$$

Since $(\widehat{\mathcal{D}} + \widehat{T})^2$ has discrete and continuous spectrum, it does not have a ζ-function as in (6). However, there is a natural generalization called the b-zeta function ${}^b\zeta_{(\widehat{\mathcal{D}}+\widehat{T})^2}(s)$, see Section 2, and then the b-zeta determinant $\det_{{}^b\zeta}(\widehat{\mathcal{D}} + \widehat{T})^2$ can be defined in terms of ${}^b\zeta_{(\widehat{\mathcal{D}}+\widehat{T})^2}(s)$ by the formula (7).

Now the question arises: Given $T \in \mathcal{L}(V)$, what is the "defect" of

$$\frac{\det_{{}^b\zeta}(\widehat{\mathcal{D}} + \widehat{T})^2}{\det_\zeta \mathcal{D}_T^2} = \boxed{\ ?\ }$$

One may conjecture that there is no defect (that is, the ratio is unity) in view of the fact that there are no "defects" for the index and the eta invariant. The first main result of this paper shows that this is not the case. To detail this theorem, recall that $\ker(\widehat{\mathcal{D}} + \widehat{T})|_X \equiv \ker \mathcal{D}_T$. On the finite-dimensional vector space $\gamma_0 \ker(\widehat{\mathcal{D}}+\widehat{T}) = \gamma_0 \ker \mathcal{D}_T$, where γ_0 is the restriction map from X to $\{u=0\}$, we define

$$\widehat{\mathcal{L}}_T := \sum \gamma_0 \widehat{\phi}_k \otimes \gamma_0 \widehat{\phi}_k, \qquad \mathcal{L}_T := \sum \gamma_0 \phi_k \otimes \gamma_0 \phi_k,$$

where $\{\widehat{\phi}_k\}$ and $\{\phi_k\}$ are orthonormal bases for the kernels of $\widehat{\mathcal{D}}+\widehat{T}$ and $\mathcal{D}_T$, respectively. We can now state our first result.

Theorem 1.1. *For any $T \in \mathcal{L}(V)$, the perturbed Dirac operator $\widehat{\mathcal{D}}+\widehat{T}$ and the operator $\mathcal{D}_T$ have the following relation:*

$$\frac{\det_{{}^b\zeta}(\widehat{\mathcal{D}}+\widehat{T})^2}{\det_\zeta \mathcal{D}_T^2} = 2^{-\frac{1}{2}\zeta_{D_Y^2}(0)} \left(\frac{\det \widehat{\mathcal{L}}_T}{\det \mathcal{L}_T}\right)^{-2} \left(\frac{(\det_{{}^b\zeta}\Delta_d)(\det_{{}^b\zeta}\Delta_\nu)}{4}\right)^{h_Y/2} \tag{8}$$

where $h_Y = \dim\ker D_Y$, $\zeta_{D_Y^2}(s)$ is the ζ-function of D_Y^2, and where Δ_d and Δ_ν are perturbed one-dimensional Dirichlet and Neumann Laplacians (defined independent of $T \in \mathcal{L}(V)$), respectively, acting on scalar functions over the half-line $[0,\infty)$ (see Equation (16) in Section 4).

We remark that the value of the right-hand side of (8) varies with T and is maximized when $T=\sigma$. To see this, by Proposition 4.1, we have

$$\left(\frac{\det \widehat{\mathcal{L}}_T}{\det \mathcal{L}_T}\right)^{-2} = \left(\prod_{k=1}^{\dim\ker(\widehat{\mathcal{D}}+\widehat{T})} \int_X |\widehat{\phi}_k|^2\, dg\right)^{-2}. \tag{9}$$

By Theorem 2.2, $\dim\ker(\widehat{\mathcal{D}}+\widehat{T}) = \dim(\Lambda_T \cap \Lambda_\sigma) + \dim\ker \mathcal{D}_{-\sigma}$, so the value of (9) can be changed by varying T so that the number of elements of the intersection $\Lambda_T \cap \Lambda_\sigma$ increases or decreases. In particular, (9) has a maximum when $T=\sigma$ since $\int_X |\widehat{\phi}_k|^2 dg < 1$ (because $\int_{\widehat{X}} |\widehat{\phi}_k|^2 dg = 1$). Therefore, the right-hand side of (8) can vary with T and is maximized when $T=\sigma$ since $2^{-\frac{1}{2}\zeta_{D_Y^2}(0)}$ and $\det_{{}^b\zeta}\Delta_d \cdot \det_{{}^b\zeta}\Delta_\nu$ are independent of T.

Next, we extend the relative formulæ (3) and (5) to the ζ-determinant. For a linear operator L over a finite-dimensional vector space, we define $\det^*(L) := \det(L|_{\ker(L)^\perp})$. The second main result of this paper is the following relative formula.

Theorem 1.2. *Given $T \in \mathcal{L}(V)$, the following formulas hold:*

$$\frac{\det_{{}^b\zeta}(\widehat{\mathcal{D}}+\widehat{T})^2}{\det_{{}^b\zeta}(\widehat{\mathcal{D}}-\widehat{\sigma})^2} = \left(\frac{\det \widehat{\mathcal{L}}_T}{\det \widehat{\mathcal{L}}_{-\sigma}}\right)^{-2} \det{}^*\Big(\frac{2\mathrm{Id}-\sigma^+T^- - T^+\sigma^-}{4}\Big),$$

$$\frac{\det_\zeta \mathcal{D}_T^2}{\det_\zeta \mathcal{D}_{-\sigma}^2} = \left(\frac{\det \mathcal{L}_T}{\det \mathcal{L}_{-\sigma}}\right)^{-2} \det{}^*\Big(\frac{2\mathrm{Id}-\sigma^+T^- - T^+\sigma^-}{4}\Big).$$

In the case that $\ker \mathcal{D}_T = \ker \mathcal{D}_{-\sigma} = 0$, the second formula in Theorem 1.2 can be derived from the papers of Scott and Wojciechowski [**22**], [**23**]. We emphasize that the term $(\det \mathcal{L}_{-\sigma})^2/(\det \mathcal{L}_T)^2$ in this formula is new,

b-calculus and BVP relative formulæ (no integer ambiguities)
$\operatorname{ind}(\widehat{\mathcal{D}}+\widehat{T}) = \operatorname{ind}\mathcal{D}_T$
${}^b\eta(\widehat{\mathcal{D}}+\widehat{T}) = \eta(\mathcal{D}_T)$
$\dfrac{\det_{{}^b\zeta}(\widehat{\mathcal{D}}+\widehat{T})^2}{\det_\zeta \mathcal{D}_T^2} = 2^{-\frac{1}{2}\zeta_{D_Y^2}(0)} \left(\dfrac{\det \widehat{\mathcal{L}}_T}{\det \mathcal{L}_T}\right)^{-2} \left(\dfrac{(\det_{{}^b\zeta}\Delta_d)(\det_{{}^b\zeta}\Delta_\nu)}{4}\right)^{h_Y/2}$

Figure 2. b-calculus and BVP relative invariant formulæ.

and this factor is nontrivial in general. For reference we summarize the relationships between the b-calculus and BVPs in Figure 2.

We remark that although the eta invariant and ζ-determinant are nonlocal quantities, the ζ-determinant is more nonlocal in the following variational sense (see Propositions (2.9) and (2.10) of [**2**]): the variation of the eta is "local" in that it only involves finitely many terms of the local symbol expansions of the original operator while the variation of the ζ-determinant is "nonlocal" because the variation involves the inverse of the operator. With this in mind, we remark that the first two lines of Figure 2 hold, basically, because the index and the variation of the eta invariant are "local" objects so these invariants are not able to distinguish between X and $\widehat{X}$. Because the ζ-determinant and its variation are "nonlocal", the last line of Figure 2 shows that the ζ-determinant is able to distinguish between X and $\widehat{X}$.

This paper is organized as follows. In Section 2 we explicitly construct the b-smoothing perturbation $\widehat{T}$ corresponding to the matrix $T \in \mathcal{L}(V)$. In Section 3 we derive gluing formulæ for the ζ-determinants on X and $\widehat{X}$ from the gluing theorems proved in [**13**, **14**]. Lastly, in Section 4 we use these gluing theorems to prove Theorems 1.1 and 1.2.

The authors thank the referee for corrections and helpful suggestions, all of which considerably improved this paper.

2. Perturbed Dirac operators and the b-zeta function

Let us henceforth fix $T \in \mathcal{L}(V)$. In this section, we construct the perturbation $\widehat{T}$ in Theorem 1.1 and we review the b-trace and the b-zeta determinant.

2.1. Perturbations of Dirac operators. We first define an auxiliary b-smoothing operator acting on scalar functions on the half-line $[0,\infty)$. Let $\chi \in C^\infty(\mathbb{R})$ be a cut-off function, where $\chi(u) = 1$ for $u \geq 2$ and $\chi(u) = 0$ for $u \leq 1$. Let $\varrho \geq 0$ be a smooth compactly supported even function on $\mathbb{R}$ with $\varrho(0) > 0$. Then $\widehat{\varrho}(\tau)$ is an even entire function — throughout the rest of this paper, the functions χ and ϱ shall remain fixed. Define an operator Q acting on Schwartz functions over $[0,\infty)$ by

$$Q\psi := \frac{1}{2\pi}\chi(u)\int_{\mathbb{R}} e^{iu\tau}\,\widehat{\varrho}(\tau)\,\widehat{\chi\psi}(\tau)\,d\tau, \tag{10}$$

where $\widehat{\chi\psi}$ is the Fourier transform of $\chi\psi$:

$$\widehat{\chi\psi}(\tau) = \int_{\mathbb{R}} e^{-iu\tau}\,\chi(u)\,\psi(u)\,du.$$

Since ϱ is compactly supported, $\widehat{\varrho}(\tau)$ vanishes to infinite order as $|\tau| \to \infty$ for $|\operatorname{Im}\tau|$ within any fixed bound and therefore, Q is by definition a *b-pseudodifferential operator of order $-\infty$* (a "b-smoothing operator"). Moreover, since ϱ is even, $\widehat{\varrho}(\tau)$ is also even, so $K_Q(u,u') = \overline{K_Q(u',u)}$, which implies that Q is formally self-adjoint. The following lemma, proved in [**11**, **12**], summarizes one of the main properties of the operator Q.

Lemma 2.1. *If W is a subspace of $V := \ker D_Y$, then given any $w \in W$, the boundary value problem*

$$v \in H^1([0,\infty), V) \quad , \quad \Big(\partial_u + Q^2 T\Big) v = 0 \quad , \quad v|_{u=0} = w,$$

has a non-trivial solution if and only if $w \in \Lambda_T \cap W$, in which case, the solution is unique and also takes values in $\Lambda_T \cap W$.

As in Melrose and Piazza [**19**], we define $\widehat{T} : L^2(\widehat{X}, E) \longrightarrow H^\infty(\widehat{X}, E)$, which is completely supported on the cylindrical end $[0,\infty)_u \times Y$, by

$$\widehat{T} := G\,Q^2\,T, \tag{11}$$

where Q is in (10). Note that T is a finite rank smoothing operator acting over the cross section Y while Q acts over the half-line $[0,\infty)_u$. Then $\widehat{T} \in \Psi_b^{-\infty}(\widehat{X}, E)$, the space of b-pseudodifferential operator of order $-\infty$, by definition of this space [**11**, **18**]. The following theorem, which is derived from Lemma 2.1 and proved in [**11**, **12**], gives one of the main properties of the perturbation $\widehat{T}$.

Theorem 2.2. *The operator $\widehat{\mathcal{D}} + \widehat{T} : H^1(\widehat{X}, E) \longrightarrow L^2(\widehat{X}, E)$ is Fredholm, and $\ker(\widehat{\mathcal{D}} + \widehat{T})$ is canonically isomorphic to $\ker \mathcal{D}_T$ by $\ker(\widehat{\mathcal{D}} + \widehat{T})|_X \equiv \ker \mathcal{D}_T$. Moreover, these are canonically isomorphic to $(\Lambda_T \cap \Lambda_\sigma) \oplus \ker \mathcal{D}_{-\sigma}$.*

We remark that the operator $\widehat{\mathcal{D}}$ is Fredholm if and only if $V = 0$ [**18**], so $\widehat{\mathcal{D}}$ alone is almost never Fredholm. As this theorem shows, the main impetus for introducing perturbations is to achieve the Fredholm property.

2.2. The b-zeta function. Recall that the heat operator $e^{-t(\widehat{\mathcal{D}}+\widehat{T})^2}$ is b-trace class [**18**, Ch. 4] with a long time expansion [**11**, **12**, **19**]:

$$ {}^b\mathrm{Tr}\, e^{-t(\widehat{\mathcal{D}}+\widehat{T})^2} \sim b_0 + b_1 e^{-\varepsilon t} \qquad \text{as } t \to \infty, \tag{12}$$

where $b_0 = \dim \ker(\widehat{\mathcal{D}}+\widehat{T})$ and where $\varepsilon > 0$. The reason for the exponential decay is that $\widehat{\mathcal{D}}+\widehat{T}$ is Fredholm so has discrete spectrum near 0. Also, there

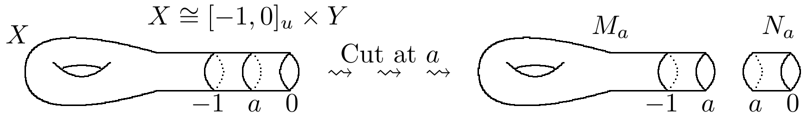

Figure 3. Cutting X into two peices.

is the usual short time asymptotic expansion [**11**, **12**, **19**]:

$$
{}^b\mathrm{Tr}\, e^{-t(\widehat{\mathcal{D}}+\widehat{T})^2} \sim \sum_{k=0}^{\infty} a_k\, t^{k-\frac{n}{2}} + \sum_{k=1}^{\infty} a'_k\, t^k \qquad \text{as } t \to 0, \tag{13}
$$

where $n = \dim \widehat{X}$. Using (12) and (13), a straightforward computation shows that

$$
{}^b\zeta_{(\widehat{\mathcal{D}}+\widehat{T})^2}(s) = \frac{1}{\Gamma(s)} \Big(\int_0^1 + \int_1^\infty \Big)\, t^{s-1}\, {}^b\mathrm{Tr}\, e^{-t(\widehat{\mathcal{D}}+\widehat{T})^2}\, dt,
$$

where the first integral is defined a priori for $\mathrm{Re}\, s \gg 0$ and the second one a priori for $\mathrm{Re}\, s \ll 0$, extend to be meromorphic functions on $\mathbb{C}$ that are regular at $s = 0$. In particular, the b-zeta determinant

$$
\det{}_{{}^b\zeta}(\widehat{\mathcal{D}} + \widehat{T})^2 := \exp\Big(- \frac{d}{ds}\, {}^b\zeta_{(\widehat{\mathcal{D}}+\widehat{T})^2}(s)\big|_{s=0} \Big)
$$

is well-defined.

3. Gluing formulæ for the ζ-determinant

In this section, applying the gluing formulae of the ζ-determinant in [**13**, **14**], we prove two propositions which will be used in the proof of Theorems 1.1 and 1.2.

3.1. Gluing formulæ for X. Let $-1 < a < 0$. We shall apply the gluing formula in Theorem 1.1 in [**13**] to the operator $\mathcal{D}_T$ over the manifold X, which we decompose at $u = a$:

$$
X = M_a \cup N_a \quad , \quad M_a,\, N_a = \text{ the left, right half of } \{u = a\} \text{ in } X;
$$

see Figure 3. Let $\mathcal{C}_{M_a}$ and $\mathcal{C}_{N_a}$ denote the Calderón projectors of $\mathcal{D}_T|_{M_a}$ and $\mathcal{D}_T|_{N_a}$, respectively, at the hypersurface $\{u = a\}$. Since $\mathcal{D}_T$ has the boundary condition $\Pi_-^T = \Pi_- + \frac{\mathrm{Id}-T}{2}\Pi_0$ at $\{u = 0\}$, one can check that $\mathcal{C}_{N_a} = \Pi_+^{-T} = \Pi_+ + \frac{\mathrm{Id}+T}{2}\Pi_0$, which is independent of a. Thus, $\mathcal{D}_{\mathcal{C}_{N_a}}$ is just the operator $G(\partial_u + D_Y)$ over $[a, 0] \times Y$ with boundary conditions Π_+^{-T} at $\{u = a\}$ and Π_-^T at $\{u = 0\}$. By Theorem 1.1 in [**15**], we have

$$
\det{}_\zeta \mathcal{D}^2_{\mathcal{C}_{N_a}} = e^{-Ca}\, 2^{\zeta_{D_Y^2}(0) + h_Y}, \tag{14}
$$

where $C = -(2\sqrt{\pi})^{-1}(\Gamma(s)^{-1}\Gamma(s-1/2)\zeta_{D_Y^2}(s-1/2))'(0)$. Now we recall that the Calderón projectors $\mathcal{C}_{M_a}, \Pi_+^{-T}$ have the forms

$$\mathcal{C}_{M_a} = \frac{1}{2}\begin{pmatrix} \mathrm{Id} & \kappa_a^{-1} \\ \kappa_a & \mathrm{Id} \end{pmatrix}, \qquad \Pi_+^{-T} = \frac{1}{2}\begin{pmatrix} \mathrm{Id} & \kappa_T^{-1} \\ \kappa_T & \mathrm{Id} \end{pmatrix} \tag{15}$$

with respect to $L^2(Y, E_0) = L^2(Y, E^+) \oplus L^2(Y, E^-)$ where $E^\pm$ are the subbundles of E_0 consisting of the $(\pm i)$-eigensections of G. Let $U_{T,a} := -\kappa_a \kappa_T^{-1}$, which is a unitary operator on $L^2(Y, E^-)$, and let $\widehat{U}_{T,a}$ denote the restriction of $U_{T,a}$ to the orthogonal complement of its (-1)-eigenspace. Finally let $\mathcal{L}_{T,a} := \sum \gamma_a\phi_k \otimes \gamma_a\phi_k$ with γ_a the restriction map from X to $\{u = a\}$ and $\{\phi_k\}$ an orthonormal basis for $\ker \mathcal{D}_T$. Then by Theorem 1.1 in [**13**] and the formula (14), we obtain

Proposition 3.1. *The following equality holds:*

$$\frac{\det_\zeta \mathcal{D}_T^2}{\det_\zeta \mathcal{D}_{\mathcal{C}_{M_a}}^2} \;=\; e^{-Ca}\,(\det \mathcal{L}_{T,a})^{-2} \det\nolimits_F\Big(\frac{2\mathrm{Id} + \widehat{U}_{T,a} + \widehat{U}_{T,a}^{-1}}{4}\Big).$$

3.2. Gluing formulæ for $\widehat{X}$. Again assume that $-1 < a < 0$. We now apply the above argument to $\widehat{\mathcal{D}} + \widehat{T}$ over $\widehat{X}$ separated at $u = a$:

$$\widehat{X} = M_a \cup \widehat{N}_a \quad , \quad M_a,\, \widehat{N}_a = \text{ the left, right half of } \{u = a\} \text{ in } \widehat{X}.$$

Let $\mathcal{C}_{M_a}$ (just as before) and $\mathcal{C}_{\widehat{N}_a}$ denote the Calderón projectors of $(\widehat{\mathcal{D}} + \widehat{T})|_{M_a} = \mathcal{D}|_{M_a}$ (since $\widehat{T}$ vanishes on X) and $(\widehat{\mathcal{D}} + \widehat{T})|_{\widehat{N}_a}$ at $\{u = a\}$, respectively. Splitting $H^1([a,\infty) \times Y, E)$ into its projections onto $V = \ker D_Y$ and $V^\perp$, it follows that

$$(\widehat{\mathcal{D}} + \widehat{T})\,|_{\widehat{N}_a} = \begin{cases} G(\partial_u + Q^2 T) & \text{over } \; \Pi_0 H^1([a,\infty) \times Y, E), \\ G(\partial_u + D_Y) & \text{over } \; \Pi_0^\perp H^1([a,\infty) \times Y, E). \end{cases}$$

Lemma 3.2. *We have $\mathcal{C}_{\widehat{N}_a} = \Pi_+^{-T}$, which equals $\mathcal{C}_{N_a}$.*

Proof. Lemma 2.1 immediately implies that $\mathcal{C}_{\widehat{N}_a} = \Pi_+^{-T}$ when $a = 0$. To see that this holds even for the nonzero a, we recall that the statement of Lemma 2.1 is independent of the choice of χ, ϱ, which define the operator Q in (10). Therefore, we can show that the same statement is true for nonzero a by shifting χ, ϱ. Hence, the Calderón projector at $\{a\} \times Y$ is also given by Π_+^{-T} even for nonzero a. □

In the next proposition we compute $\det_{{}^b\zeta}(\widehat{\mathcal{D}} + \widehat{T})^2_{\mathcal{C}_{\widehat{N}_a}}$.

Proposition 3.3. *Over $H^1([0,\infty))$, define*

$$\begin{aligned} \Delta_d &:= -\,(\partial_u - Q^2)(\partial_u + Q^2) \;\; \textit{with Dirichlet condition at } u = 0 \\ \Delta_\nu &:= -\,(\partial_u + Q^2)(\partial_u - Q^2) \;\; \textit{with Neumann condition at } u = 0. \end{aligned} \tag{16}$$

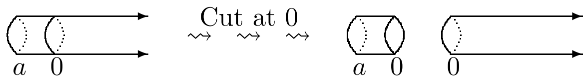

Figure 4. Cutting the cylinder.

These are one-dimensional "perturbed Laplace-type operators". Then

$$\det{}_{{}^b\zeta}(\widehat{\mathcal{D}}+\widehat{T})^2_{\mathcal{C}_{\widehat{N}_a}} = \Big((\det{}_{{}^b\zeta}\Delta_d)(\det{}_{{}^b\zeta}\Delta_\nu)\Big)^{h_Y/2}\cdot e^{-Ca}\cdot 2^{\frac{1}{2}\zeta_{D_Y^2}(0)}.$$

Proof. We start by applying Theorem 1.1 in [**14**] to $\det{}_{{}^b\zeta}(\widehat{\mathcal{D}}+\widehat{T})^2_{\mathcal{C}_{\widehat{N}_a}}$ with the decomposition (see Figure 4):

$$[a,\infty)\times Y = \big([a,0]\times Y\big)\cup\big([0,\infty)\times Y\big).$$

By Lemmas 2.1 and 3.2, it follows that the Calderón projectors at the left and right side of the dividing hypersurface $\{0\}\times Y$ are Π_-^T and Π_+^{-T}, respectively. In particular, the induced operator on $[a,0]\times Y$ is just the operator $G(\partial_u+D_Y)$ over $[a,0]\times Y$ with boundary conditions Π_+^{-T} at $\{u=a\}$ and Π_-^T at $\{u=0\}$, which has ζ-determinant equal to $e^{-Ca}\,2^{\zeta_{D_Y^2}(0)+h_Y}$ (see (14)). Combining this value and Theorem 1.1 in [**14**], one can derive

$$\det{}_{{}^b\zeta}(\widehat{\mathcal{D}}+\widehat{T})^2_{\mathcal{C}_{\widehat{N}_a}} = e^{-Ca}\cdot\det{}_{{}^b\zeta}(\widehat{\mathcal{D}}+\widehat{T})^2_{\Pi_+^{-T}}.$$

Thus, it remains to compute $\det{}_{{}^b\zeta}(\widehat{\mathcal{D}}+\widehat{T})^2_{\Pi_+^{-T}}$ on $[0,\infty)\times Y$. To do so, we observe that

$$\det{}_{{}^b\zeta}(\widehat{\mathcal{D}}+\widehat{T})^2_{\Pi_+^{-T}} = \det{}_{{}^b\zeta}A^2\cdot\det{}_{{}^b\zeta}B^2,$$

where

$$\begin{aligned} A :=& G(\partial_u+Q^2T)_{\Pi_T} \quad \text{over } \Pi_0H^1([0,\infty)\times Y,E),\\ B :=& G(\partial_u+D_Y)_{\Pi_+} \quad \text{over } \Pi_0^\perp H^1([0,\infty)\times Y,E). \end{aligned}$$

By Lemma 2.2 in [**14**], we have $\det{}_{{}^b\zeta}B^2 = 2^{\frac{1}{2}\zeta_{D_Y^2}(0)}$, so it now remains to compute $\det{}_{{}^b\zeta}A^2$. To this end, we recall that

$$\operatorname{dom}(A) = \{\phi\in H^1([0,\infty),V)\mid \Pi_T\big(\phi|_{u=0}\big)=0\}, \tag{17}$$

so

$$\operatorname{dom}(A^2) = \{\phi\in H^2([0,\infty),V)\mid \Pi_T\big(\phi|_{u=0}\big)=0\ ,\ \Pi_T\big(A\phi|_{u=0}\big)=0\}.$$

Now the heat operator e^{-tA^2} takes an initial condition ψ to a function ϕ_t that satisfies

$$(\partial_t+A^2)\phi_t=0\quad;\quad \phi_0=\psi\ \ ,\ \ \Pi_T\big(\phi_t|_{u=0}\big)=0\quad \Pi_T\big(A\phi_t|_{u=0}\big)=0.$$

Near $u=0$, $Q=0$, so at $u=0$ we have $\Pi_TA=\Pi_TG\partial_u=G\Pi_{-T}\partial_u$. Thus,

$$(\partial_t+A^2)\phi_t=0\quad;\quad \phi_0=\psi\ \ ,\ \ \Pi_T\big(\phi_t|_{u=0}\big)=0\ \ ,\ \ \Pi_{-T}\big(\partial_u\phi_t|_{u=0}\big)=0.$$

Since Π_T is the orthogonal projection onto Λ_T and Π_{-T} is the one onto $\Lambda_T^\perp$, we have Dirichlet conditions on Λ_T and Neumann conditions on $\Lambda_T^\perp$. Moreover,

$$A^2 = G(\partial_u + Q^2T)G(\partial_u + Q^2T) = -(\partial_u - Q^2T)(\partial_u + Q^2T)$$
$$= \begin{cases} -(\partial_u - Q^2)(\partial_u + Q^2) & \text{over } \Lambda_T \\ -(\partial_u + Q^2)(\partial_u - Q^2) & \text{over } \Lambda_T^\perp. \end{cases}$$

Thus, by definition of Δ_d and Δ_ν,

$$e^{-tA^2} = e^{-t\Delta_d}\ \Pi_T + e^{-t\Delta_\nu}\ \Pi_{-T}.$$

Since $\operatorname{Tr}\Pi_T = \dim\Lambda_T = h_Y/2$ with the same for $\operatorname{Tr}\Pi_{-T}$, we obtain

$${}^b\zeta_{A^2}(s) = \frac{h_Y}{2}\Big({}^b\zeta_{\Delta_d}(s) + {}^b\zeta_{\Delta_\nu}(s)\Big).$$

Using the definition of the b-zeta determinant finishes the proof. $\square$

Now, again applying Theorem 1.1 in [**14**] with Proposition 3.3, we have

Proposition 3.4. *The following equality holds:*

$$\frac{\det_\zeta(\widehat{\mathcal{D}} + \widehat{T})^2}{\det_\zeta \mathcal{D}^2_{\mathcal{C}_{M_a}}} = e^{-Ca}\, 2^{-\frac{1}{2}\zeta_{D_Y^2}(0)} \cdot \left(\frac{(\det_{{}^b\zeta}\Delta_d)(\det_{{}^b\zeta}\Delta_\nu)}{4}\right)^{h_Y/2}$$
$$\cdot\, (\det\widehat{\mathcal{L}}_{T,a})^{-2} \cdot \det{}_F\Big(\frac{2\mathrm{Id} + \widehat{U}_{T,a} + \widehat{U}_{T,a}^{-1}}{4}\Big)$$

where $\widehat{\mathcal{L}}_{T,a} := \sum \gamma_a\widehat{\phi}_k \otimes \gamma_a\widehat{\phi}_k$ *with* $\{\widehat{\phi}_k\}$ *an orthonormal basis for* $\ker(\widehat{\mathcal{D}} + \widehat{T})$.

4. Proof of main theorems

In this final section we put together the results obtained in the previous section to prove Theorems 1.1 and 1.2.

First of all, Theorem 1.1 is easy to prove: Dividing the formulas in Propositions 3.1 and 3.4, we obtain

(18)
$$\frac{\det_{{}^b\zeta}(\widehat{\mathcal{D}} + \widehat{T})^2}{\det_\zeta \mathcal{D}_T^2} = 2^{-\frac{1}{2}\zeta_{D_Y^2}(0)} \left(\frac{\det\widehat{\mathcal{L}}_{T,a}}{\det\mathcal{L}_{T,a}}\right)^{-2} \left(\frac{(\det_{{}^b\zeta}\Delta_d)(\det_{{}^b\zeta}\Delta_\nu)}{4}\right)^{h_Y/2}.$$

It follows that the ratio $\det\widehat{\mathcal{L}}_{T,a}/\det\mathcal{L}_{T,a}$ does not depend on a. In particular, we can take $a \to 0$ in this equality, which completes the proof of Theorem 1.1.

Theorem 1.2 takes a little more work. By Theorem 1.1, we can derive the ratio of $\det_{{}^b\zeta}(\widehat{\mathcal{D}} + \widehat{T})^2$ with $\det_{{}^b\zeta}(\widehat{\mathcal{D}} - \widehat{\sigma})^2$ from the ratio of $\det_\zeta\mathcal{D}_T^2$ with

$\det_\zeta \mathcal{D}^2_{-\sigma}$, so we shall focus on the latter ratio. Applying Proposition 3.1 to T and $-\sigma$, then dividing the resulting formulas, we obtain

$$\begin{aligned}&\frac{\det_\zeta \mathcal{D}^2_T}{\det_\zeta \mathcal{D}^2_{-\sigma}}\\ &= \Big(\frac{\det \mathcal{L}_{T,a}}{\det \mathcal{L}_{-\sigma,a}}\Big)^{-2} \det_F\Big[\Big(\frac{2\mathrm{Id} + \widehat{U}_{T,a} + \widehat{U}^{-1}_{T,a}}{4}\Big)\Big(\frac{2\mathrm{Id} + \widehat{U}_{-\sigma,a} + \widehat{U}^{-1}_{-\sigma,a}}{4}\Big)^{-1}\Big].\end{aligned}$$

We can find the right-hand determinant as follows. First, we can write

$$L^2(Y, E^-) = V^- \oplus (V^\perp)^-,$$

where $(V^\perp)^- = \frac{\mathrm{Id}+iG}{2} V^\perp$; this allows us to consider the above determinant over V^- and $(V^\perp)^-$ separately. Second, we notice that

$$\mathcal{C}_{M_a} = \frac{\mathrm{Id} + \sigma}{2} \quad \text{over} \quad V \quad \Longrightarrow \quad \kappa_{M_a} = \sigma^+ \quad \text{over} \quad V^+$$

because the intersection of the Cauchy data space with V does not change with respect to a by the description of the dynamics of the Cauchy data space in [**21**]. Hence, by definition of $U_{T,a}$ and $U_{-\sigma,a}$, we have

$$U_{T,a} = -\sigma^+ T^- \quad , \quad U_{-\sigma,a} = -\sigma^+(-\sigma^-) = \mathrm{Id} \qquad \text{over} \quad V^-.$$

Third, since $\Pi_+^{-T} = \Pi_+^\sigma$ (which is equal to Π_+) over $V^\perp$, it follows that

$$U_{T,a} = U_{-\sigma,a} \quad \text{over} \quad (V^\perp)^-.$$

Hence,

$$\begin{aligned}&\det_F\Big(\frac{2\mathrm{Id} + \widehat{U}_{T,a} + \widehat{U}^{-1}_{T,a}}{4}\Big)\Big(\frac{2\mathrm{Id} + \widehat{U}_{-\sigma,a} + \widehat{U}^{-1}_{-\sigma,a}}{4}\Big)^{-1}\\ &\qquad\qquad = \det{}^*\Big(\frac{2\mathrm{Id} - \sigma^+T^- - T^+\sigma^-}{4}\Big),\end{aligned}$$

since $(\sigma^+T^-)^{-1} = T^+\sigma^-$. Thus,

$$\frac{\det_\zeta \mathcal{D}^2_T}{\det_\zeta \mathcal{D}^2_{-\sigma}} = \Big(\frac{\det \mathcal{L}_{T,a}}{\det \mathcal{L}_{-\sigma,a}}\Big)^{-2} \cdot \det{}^*\Big(\frac{2\mathrm{Id} - \sigma^+T^- - T^+\sigma^-}{4}\Big).$$

It follows that the ratio $\det \mathcal{L}_{T,a}/\det \mathcal{L}_{-\sigma,a}$ does not depend on a. In particular, we can take $a \to 0$ in this equality, which completes our proof of Theorem 1.2.

Finally, we end our paper with a proof of the following "explicit" formula for the ratio of the kernel determinants.

Proposition 4.1. *We have*

$$\frac{\det \widehat{\mathcal{L}}_T}{\det \mathcal{L}_T} = \prod_{k=1}^{\dim \ker(\widehat{\mathcal{D}}+\widehat{T})} \int_X |\widehat{\phi}_k|^2\, dg.$$

Proof. Recall from Theorem 2.2 that $\ker(\mathcal{D}_T) \equiv \ker(\widehat{\mathcal{D}}+\widehat{T})|_X$; in particular, $\{a_k\,\widehat{\phi}_k\}$ is an orthonormal basis for $\ker\mathcal{D}_T$, where $\{\widehat{\phi}_k\}$ an orthonormal basis for $\ker(\widehat{\mathcal{D}}+\widehat{T})$ and $a_k := 1/\|\widehat{\phi}_k\|_X$ with $\|\widehat{\phi}_k\|_X^2 := \int_X |\widehat{\phi}_k|^2\,dg$. Therefore, setting $v_k := \gamma_0\widehat{\phi}_k$, we have

$$\mathcal{L}_T := \sum_{k=1}^{h_T} a_k^2\, v_k \otimes v_k, \qquad \widehat{\mathcal{L}}_T := \sum_{k=1}^{h_T} v_k \otimes v_k,$$

where $h_T = \dim\ker(\widehat{\mathcal{D}}+\widehat{T}) = \dim\ker\mathcal{D}_T$. Now with respect to the basis $\{v_k\}$, we can write

$$\mathcal{L}_T = \begin{pmatrix} a_1^2\langle v_1, v_1\rangle & a_1^2\langle v_2, v_1\rangle & \cdots & a_1^2\langle v_{h_T}, v_1\rangle \\ a_2^2\langle v_1, v_2\rangle & a_2^2\langle v_2, v_2\rangle & \cdots & a_2^2\langle v_{h_T}, v_2\rangle \\ \vdots & \vdots & \cdots & \vdots \\ a_{h_T}^2\langle v_1, v_{h_T}\rangle & a_{h_T}^2\langle v_2, v_{h_T}\rangle & \cdots & a_{h_T}^2\langle v_{h_T}, v_{h_T}\rangle \end{pmatrix}$$

where $\langle\ ,\ \rangle$ denotes the L^2 inner product on Y. It follows that

$$\det\mathcal{L}_T = a_1^2\cdots a_{h_T}^2\cdot\det\widehat{\mathcal{L}}_T = \Big(\prod_{k=1}^{h_T}\|\widehat{\phi}_k\|_X^2\Big)^{-1}\cdot\det\widehat{\mathcal{L}}_T,$$

and this formula implies our result. □

References

1. M.F. Atiyah, V.K. Patodi, and I.M. Singer, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.
2. ______, *Spectral asymmetry and Riemannian geometry. III*, Math. Proc. Cambridge Philos. Soc. **79** (1976), 71–99.
3. J. Brüning and M. Lesch, *On the η-invariant of certain nonlocal boundary value problems*, Duke Math. J. **96** (1999), 425–468.
4. U. Bunke, *On the gluing formula for the η-invariant*, J. Differential Geometry **18** (1995), 397–448.
5. X. Dai and D. Freed, *η-invariants and determinant lines*, J. Math. Phys. **35** (1994), 5155–5195.
6. R. G. Douglas and K. P. Wojciechowski, *Adiabatic limits of the η-invariants. The odd-dimensional Atiyah-Patodi-Einger problem*, Comm. Math. Phys. **142** (1991), 139–168.
7. G. Grubb and R. T. Seeley, *Zeta and eta functions for Atiyah-Patodi-Singer operators*, J. Geom. Anal. **6** (1996), 31–77.
8. A. Hassell, R. R. Mazzeo and R. B. Melrose, *Analytic surgery and the accumulation of eigenvalues*, Comm. Anal. Geom. **3** (1995), 115–222.
9. P. Kirk and M. Lesch, *The eta invariant, Maslov index, and spectral flow for Dirac-type operators on manifolds with boundary*, Forum Math. **16** (2004), 553–629.
10. M. Lesch and K. P. Wojciechowski, On the η-invariant of generalized Atiyah-Patodi-Singer boundary value problems, Illinois J. Math. **40** (1996), 30–46.

11. P. Loya, *Dirac operators, Boundary Value Problems, and the b-Calculus*, Contemp. Math. **366** (2005), 241–280.
12. P. Loya and R. Melrose, *Fredholm perturbations of Dirac operators on manifolds with corners*, Preprint, 2003.
13. P. Loya and J. Park, *On the gluing problem for the spectral invariants of Dirac operators*, Adv. in Math. to appear.
14. ______, *On the gluing problem for Dirac operators on manifolds with cylindrical ends*, J. Geom. Anal. **15**, no. 2 (2005), 285–319.
15. ______, *The ζ-determinant of generalized APS boundary problems over the cylinder*, J. Phys. A. Math. Gen. **37**, no. 29 (2004), 7381–7392.
16. R. Mazzeo and R. B. Melrose, *Analytic surgery and the eta invariant*, Geom. Funct. Anal. **5** (1995), 14–75.
17. R. Mazzeo and P. Piazza, *Dirac operators, heat kernels and microlocal analysis. II. Analytic surgery*, Rend. Mat. Appl. (7) **18** (1998), 221–288.
18. R.B. Melrose, *The Atiyah-Patodi-Singer Index Theorem*, A.K. Peters, Wellesley, 1993.
19. R. B. Melrose and P. Piazza, *Families of Dirac operators, boundaries and the b-calculus*, J. Differential Geometry **46** (1997), 99–180.
20. W. Müller, *On the L^2-index of Dirac operators on manifolds with corners of codimension two I*, J. Differential Geometry **44** (1996), 97–177.
21. L. Nicolaescu, *The Maslov index, the spectral flow, and decompositions of manifolds* , Duke Math. J. **80** (1995), 485–533.
22. S. Scott, *Zeta determinants on manifolds with boundary*, J. Funct. Anal. **192** (2002), 112–185.
23. S. Scott and K. P. Wojciechowski, *The ζ-determinant and Quillen determinant for a Dirac operator on a manifold with boundary*, Geom. Funct. Anal. **10** (2000), 1202–1236.
24. K. P. Wojciechowski, *The additivity of the η-invariant. The case of a singular tangential operator*, Comm. Math. Phys. **169** (1995), 315–327.

Department of Mathematics, Binghamton University, Vestal Parkway East, Binghamton, NY 13902, U.S.A.
E-mail address: paul@math.binghamton.edu

School of Mathematics, Korea Institute for Advanced Study
207-43, Cheongnyangni 2-dong, Dongdaemun-gu, Seoul 130-722, Korea
E-mail address: jinsung@kias.re.kr