ζ-DETERMINANTS OF LAPLACIANS WITH NEUMANN AND DIRICHLET BOUNDARY CONDITIONS

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ABSTRACT. In this note, we derive a formula for the ratio of the ζ-determinants of the Laplacian with Neumann and Dirichlet boundary conditions over a noncompact manifold with an infinite cylindrical end and a compact boundary in terms of the ζ-determinant of the Dirichlet to Neumann map.

1. INTRODUCTION

The powerful technique of ζ-regularized determinants entered mathematics in the seminal paper of Ray and Singer [25], and subsequently entered the physics world in quantum field theory, which uses ζ-regularization to renormalize divergent quantities such as vacuum energies and effective actions. In fact, at the one-loop order, any such QFT can be reduced to the theory of determinants. We refer the reader to the works of Dowker and Critchley [5], Hawking [10], Elizalde *et al.* [6], and Kirsten [11] for recent reviews. Because of their increasingly important rôle in mathematics and physics, over the past several years there has been intense research to study functional ζ-determinants of Laplace type operators over a variety of compact and noncompact space-time configurations. Of great practical significance is the Laplacian with Dirichlet and Neumann boundary conditions. The purpose of this paper is to study the ratio of the ζ-regularized determinants of the Laplacian with Dirichlet and Neumann boundary conditions over a noncompact space-time configuration given by a manifold with cylindrical end and compact boundary.

We now describe our situation more precisely. Let X be a Riemannian manifold, of arbitrary positive dimension, with cylindrical end and compact boundary Y, that is,

$$X = \Big((-\infty, 0] \times Z\Big) \cup M,$$

where M is a compact manifold with two compact boundary components of codimension one, Z and Y, and where we assume that M has a tubular neighborhood $[-1, 0]_u \times Y$ of Y. See Figure 1 for an example of a two-dimensional manifold with cylindrical end and compact boundary. Thus, X is just a certain type of noncompact manifold with compact boundary. Let Δ be a Laplace-type operator acting on $C^\infty(X, E)$ with E a Hermitian vector bundle over X, where Laplace-type means that the principal symbol of Δ is the Riemannian metric and Δ is nonnegative on smooth sections compactly supported on the interior of X. We assume that Δ is of product-type over the cylindrical end $(-\infty, 0] \times Z$ and over the tubular neighborhood $[-1, 0]_u \times Y$:

$$\Delta \;=\; -\partial_u^2 + \Delta_Z \quad \text{over} \quad (-\infty, 0]_u \times Z, \tag{1.1}$$

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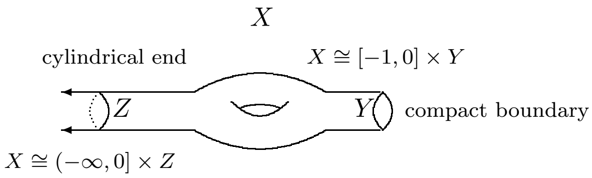

FIGURE 1. X is a noncompact manifold that has both a cylindrical end and a compact boundary.

where Δ_Z is a Laplace-type over the compact boundaryless manifold Z, and

$$\Delta \;=\; -\partial_u^2 + \Delta_Y \quad \text{over} \quad [-1,0]_u \times Y, \tag{1.2}$$

where Δ_Y is a Laplace-type over the compact boundaryless manifold Y.

Since X has a boundary, we need to impose a boundary condition for Δ. In this paper, we consider the two most common boundary conditions, the Neumann and Dirichlet conditions:

$$\Delta_N := \Delta : \mathrm{dom}(\Delta_N) \longrightarrow L^2(X,E),$$

where

$$\mathrm{dom}(\Delta_N) := \{\phi \in H^2(X,E) \mid (\partial_u \phi)|_Y = 0\},$$

and similarly, we define the Dirichlet Laplacian Δ_D with domain

$$\mathrm{dom}(\Delta_D) := \{\phi \in H^2(X,E) \mid \phi|_Y = 0\}.$$

We assume that the Neumann and Dirichlet Laplacians Δ_N and Δ_D are nonnegative in the sense that

$$(L\phi,\phi) \geq 0 \quad \text{for all } \phi \in \mathrm{dom}(L) \qquad \text{with} \quad L = \Delta_N, \Delta_D \tag{1.3}$$

where $(\ ,\ )$ denotes the L^2-inner product, and that the Dirichlet problem is uniquely solvable in the following sense: For each $\varphi \in C^\infty(Y, E|_Y)$ there is a unique *bounded* solution $\phi \in C^\infty(X,E)$ such that $\Delta\phi = 0$ and $\phi|_Y = \varphi$. The uniqueness of bounded solutions implies $\ker \Delta_D = \{0\}$. For example, the product-type conditions (1.1) and (1.2) and the nonnegativity condition (1.3) are both satisfied by the scalar Laplacian operator corresponding to a Riemannian metric on X that is of product-type over the cylindrical end and the tubular neighborhood of Y.

To orient the reader to the various "b-"regularizations used throughout this paper, assume just for the moment that $Z = \varnothing$ so that the cylindrical end of X is actually fictitious and X is a compact manifold with boundary. Then focusing on Neumann Laplacian Δ_N, the functional determinant of Δ_N is by definition

$$\begin{aligned} \det{}_\zeta \Delta_N &:= \exp\Big(-\frac{d}{ds}\Big|_{s=0} \zeta(s,\Delta_N)\Big), \\ \zeta(s,\Delta_N) &:= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\, \mathrm{Tr}(\Pi_0^\perp e^{-t\Delta_N})\, dt, \end{aligned} \tag{1.4}$$

where $\Pi_0^\perp$ is the orthogonal projection off the zero modes of Δ_N and $e^{-t\Delta_N}$ is the heat operator of Δ_N. This definition of the ζ-determinant was introduced in Ray and Singer's paper [25] and is valid whether or not Δ_N has zero modes; in the case when Δ_N has zero modes, some authors denote the above determinant with a prime: $\det'_\zeta \Delta_N$. In order to generalize this to the noncompact case, we shall present an alternative, but equivalent, definition of the zeta function $\zeta(s,\Delta_N)$. Consider the integrals

$$(1.5)\quad I_1(s) := \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\,\mathrm{Tr}(e^{-t\Delta_N})\;dt\;,\quad I_2(s) := \frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}\,\mathrm{Tr}(e^{-t\Delta_N})\;dt.$$

It is well-known that the trace of the heat operator satisfies (cf. [6], [11])

$$\mathrm{Tr}(e^{-t\Delta_N}) \sim \sum_{k=0}^{\infty} a_k\, t^{(k-n)/2} \qquad \text{as } t\to 0, \tag{1.6}$$

where $n = \dim X$, and

$$\mathrm{Tr}(e^{-t\Delta_N}) \sim b_0 \qquad \text{as } t\to\infty, \tag{1.7}$$

where $b_0 = \dim\ker\Delta_N$. The expansion (1.6) shows that the function $I_1(s)$ in (1.5) has a meromorphic extension to $\mathbb{C}$, and the expansion (1.7) shows that the function $I_2(s)$ in (1.5) has a meromorphic extension to $\mathbb{C}$ (in fact, the zero of $\frac{1}{\Gamma(s)}$ at $s=0$ cancels the pole of the integral at $t=\infty$, so $I_2(s)$ is an entire function). Moreover, it is a straightforward exercise to prove that

$$\zeta(s,\Delta_N) \equiv I_1(s) + I_2(s). \tag{1.8}$$

Splitting the zeta function in this way has certain advantages; for example, it allows us to separate the small and long-time behavior of the heat operator, which allows us via (1.6) and (1.7) to immediately get the meromorphic structures of $I_1(s)$ and $I_2(s)$ separately, and hence of $\zeta(s,\Delta_N)$. Another advantage is that the right-hand side of (1.8) bypasses the explicit use of the orthogonal projection $\Pi_0^\perp$ off the zero modes of Δ_N in (1.4). (Of course, the zero modes are still present in $I_1(s)$ and $I_2(s)$ but these cancel out when taking the sum $I_1(s)+I_2(s)$.)

Back to the general situation, since X is not compact in the case when $Z\neq\varnothing$ (which we are mostly interested in), as explained in Section 2 the heat operators $e^{-t\Delta_N}$ and $e^{-t\Delta_D}$ are not of trace class essentially because the traces diverges over the infinite cylindrical end. To define their corresponding ζ-functions, it is therefore necessary to introduce an appropriate regularization of the trace. Two natural regularizations of the trace include the relative trace used by, for instance, Bruneau [3], Carron [4], and Müller [23], and the b-trace introduced by Melrose [22], both of which "remove" in slightly different ways the divergent parts of the heat traces. We shall use Melrose's b-trace, denoted by ${}^b\mathrm{Tr}$, throughout this paper, an introduction of which is given in Section 2. In particular, focusing on the Neumann Laplacian for the moment, ${}^b\mathrm{Tr}(e^{-t\Delta_N})$ is well-defined and moreover, following the motivating example in (1.8) we can define the corresponding ${}^b\zeta$-function ${}^b\zeta(s,\Delta_N)$ as the sum of the meromorphic extensions of the functions

$$\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\;{}^b\mathrm{Tr}(e^{-t\Delta_N})\;dt, \tag{1.9}$$

defined a priori for $\Re s \gg 0$, and

$$\frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}\;{}^b\mathrm{Tr}(e^{-t\Delta_N})\;dt, \tag{1.10}$$

defined a priori for $\Re s \ll 0$; then, see Section 2, ${}^b\zeta(s,\Delta_N)$ is regular at $s=0$. The ${}^b\zeta$-determinant of Δ_N is defined, just as in the motivating example (1.4), as

$$\det{}_{{}^b\zeta}\Delta_N := \exp\Big(-\frac{d}{ds}\Big|_{s=0}{}^b\zeta(s,\Delta_N)\Big).$$

Similarly, one can define the ${}^b\zeta$-determinant $\det_{{}^b\zeta}\Delta_D$. The *relative invariant problem*, in this context is to find a formula for the ratio

$$\frac{\det_{{}^b\zeta}\Delta_N}{\det_{{}^b\zeta}\Delta_D} = \boxed{?} \tag{1.11}$$

in terms of recognizable data. In our problem, this data involves the *Dirichlet to Neumann operator*! This is the operator $\mathcal{N}$ over Y defined by

$$\mathcal{N}\varphi := (\partial_u \phi)|_Y \qquad \text{for} \quad \varphi \in C^\infty(Y, E|_Y)$$

where ϕ is the bounded solution of the *Dirichlet problem*, $\Delta\phi = 0$, $\phi|_Y = \varphi$. (This operator is a specific case of the Agranovich-Dynin operator [1] for complementary elliptic boundary conditions.) It is easy to check that $\mathcal{N}$ is a pseudodifferential operator of order 1, which may have negative eigenvalues (see Park and Wojciechowski [24]) but is always bounded from below, and in the case that Δ is the scalar Laplacian, $\mathcal{N}$ is nonnegative. Hence, we can define its ζ-regularized determinant, $\det_\zeta \mathcal{N}$. We remark that besides its appearance in our main theorem, the Dirichlet to Neumann map is perhaps most well-known in the study of *inverse problems*; see Uhlmann [29] for a recent review and his joint work [12] for a recent development. We also remark that there are other relative invariant problems of great interest in addition to (1.11), especially dealing Dirac operators; see Scott [26], Scott and Wojciechowski [27], and Loya and Park [18].

To state our main result, we need two more maps, L and $\widetilde{L}$ over Y, dealing with the L^2 and the bounded solutions, respectively, of Δ_N. Let $\{u_j\}$ be an orthonormal basis for the kernel of Δ_N on $L^2(X, E)$ and let $\{U_j\}$ be a basis of the bounded solutions $\Delta_N U_j = 0$ such that at $\{-\infty\} \times Z$ on the cylinder, $\{U_j(-\infty)\}$ are orthonormal in $L^2(Z, E|_Z)$. Here, $U_j(-\infty) := \lim_{u\to-\infty} U_j(u,z)$ is well defined: Solving the equation $\Delta_N U_j = 0$ where $\Delta_N = -\partial_u^2 + \Delta_Z$ over $(-\infty, 0]_u \times Z$ (see (1.1)) and using that U_j is bounded, it follows that U_j has the following expression over $(-\infty, 0]_u \times Z$,

$$U_j|_{(-\infty,0]_u\times Z} = \sum_{\lambda_k \geq 0} a_{jk} e^{\lambda_k u}\, \phi_k,$$

where $\{(\phi_k, \lambda_k^2)\}$ denotes the spectral resolution of the Laplacian Δ_Z. Hence,

$$U_j(-\infty) := \lim_{u\to-\infty} U_j(u,z) = \sum_{\lambda_k = 0} a_{jk}\, \phi_k \in L^2(Z, E|_Z).$$

Elliptic theory on manifolds with cylindrical ends shows that the sets $\{u_j\}$ and $\{U_j\}$ are finite [22, Ch. 5]. Let $v_j := u_j|_Y$ and $V_j := U_j|_Y$ be the restrictions of u_j and U_j, respectively, to the boundary $\{0\} \times Y$. In [15] it is shown that

$$L := \sum_j v_j \otimes v_j^* \quad , \quad \widetilde{L} := \sum_j V_j \otimes V_j^*,$$

where $v_j^* = (\,\cdot\,, v_j)_Y$ and $V_j^* = (\,\cdot\,, V_j)_Y$ with $(\,\cdot\,,\,\cdot\,)_Y$ denoting the L^2 inner product over Y, are nonnegative linear operators on $V = \operatorname{span}\{v_j,\ V_j\} \subset L^2(Y, E|_Y)$. Since the set $\{v_j,\ V_j\}$ is a linearly independent set spanning V, the operator

$$L + \widetilde{L} : V \longrightarrow V$$

is positive. In particular, $\det(L + \widetilde{L})$ is nonzero. Our main result is

Theorem 1.1. *The following relative formula holds:*

$$\frac{\det_{{}^b\zeta}\Delta_N}{\det_{{}^b\zeta}\Delta_D} = \frac{\det_\zeta \mathcal{N}}{\det(L+\widetilde{L})}. \tag{1.12}$$

As mentioned in the lines above the formula (1.9), in order to define the b-traces of $e^{-t\Delta_N}$ and $e^{-t\Delta_D}$, we remove their components that give rise to divergent traces, but it turns out that these components are the same for $e^{-t\Delta_N}$, $e^{-t\Delta_D}$ (see Remark 2.1 in Section 2). Hence the left side of (1.12) in fact does not depend on the regularization of b-trace. If Z, the cross section of the cylindrical end, is empty, then X is just a compact manifold with boundary Y. In this case, the $\widetilde{L}$ term vanishes, so we get the following corollary for free.

Corollary 1.2. *For a compact manifold with boundary, we have*

$$\frac{\det_\zeta\Delta_N}{\det_\zeta\Delta_D} = \frac{\det_\zeta \mathcal{N}}{\det L}. \tag{1.13}$$

Remark 1.3. Under the condition that $\mathcal{N}$ is positive, the equality (1.13) was proved by Park and Wojciechowski [24] — in this case there is no term $\det L$. The proof in [24] is in principle similar to Forman's proof [7], but to overcome certain trace class issues, the method of comparison with the model problem was employed; this method has also been used in [15], [16], [17], [20].

The main body of this paper consists of the following three sections: In Section 2, we review the b-trace and then the ${}^b\zeta$-determinant along with its gluing formula for manifolds with an infinite cylindrical end and a compact boundary. In Section 3, we compute the ζ-determinant of the Laplacian over a finite cylinder with the Dirichlet and Neumann boundary conditions over each boundary. In Section 4, we prove Theorem 1.1 combining results presented in the previous sections.

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2. Gluing formula of the ${}^b\zeta$-determinant

We give an elementary introduction to Melrose's b-trace introduced in [22]. To see the necessity for a regularized trace, we begin by describing the heat operator $e^{-t\Delta_N}$ on the cylindrical end $(-\infty,0]_u\times Z$. Restricting the heat kernel to the diagonal, taking the fiber-wise trace, and looking at it on the cylinder, one can show that [22, Ch. 8]

$$\operatorname{tr} e^{-t\Delta_N}|_{\mathrm{Diag}} = \frac{1}{\sqrt{4\pi t}}\operatorname{tr} e^{-t\Delta_Z}(z,z) + H_N(t,u,z) \quad \text{over} \quad (-\infty,0]_u\times Z, \tag{2.1}$$

where z is the Z variable, Δ_Z is a Laplace-type operator over Z, and for fixed $t>0$, $H_N(t,u,z)=\mathcal{O}(e^{-|u|})$ as $u\to-\infty$. Since $H_N(t,u,z)=\mathcal{O}(e^{-|u|})$, the integral of $H_N(t,u,z)$ exists over $(-\infty,0]_u\times Z$. Unfortunately, the first term on the right in (2.1) is constant with respect to u, so is not integrable on the infinite cylinder. In particular, the trace given by the standard integral formula

$$\int_X \operatorname{tr} e^{-t\Delta_N}|_{\mathrm{Diag}} \tag{2.2}$$

is not defined. However, Melrose [22] defined another notion of trace called the b-trace described as follows. Let ϕ be a locally integrable function on X and suppose that on the infinite cylinder $(-\infty,0]_u\times Y$, we can write $\phi(u,z)=\varphi(z)+\psi(u,z)$

where $\varphi(z)$ is constant in u and $\psi(u,z)$ is integrable (cf. (2.1)). Then the function $\varphi(z)$ is exactly the obstruction to ϕ being integrable on X. We define the *b-integral* of ϕ by throwing out this obstruction and keeping the integrable part:

$$^b\!\!\int_X \phi := \int_M \phi + \int_{(-\infty,0]_u\times Z} \psi(u,z)\,du\,dz,$$

where dz is the measure on Z. From the decomposition (2.1), we see that

$$^b\mathrm{Tr}\, e^{-t\Delta_N} := {}^b\!\!\int_X \mathrm{tr}\, e^{-t\Delta_N}|_{\mathrm{Diag}},$$

is well-defined; $^b\mathrm{Tr}\, e^{-t\Delta_N}$ is called the *b-trace* of $e^{-t\Delta_N}$.

In a similar way, the Dirichlet heat kernel has the following form over the cylinder:

$$\mathrm{tr}\, e^{-t\Delta_D}|_{\mathrm{Diag}} = \frac{1}{\sqrt{4\pi t}}\,\mathrm{tr}\, e^{-t\Delta_Z}(z,z) + H_D(t,u,z) \quad \text{over} \quad (-\infty,0]_u\times Z, \tag{2.3}$$

where $H_D(t,u,z) = \mathcal{O}(e^{-|u|})$ as $u\to-\infty$. In particular, just as for the Neumann Laplacian, the b-trace of the Dirichlet Laplacian, $^b\mathrm{Tr}\, e^{-t\Delta_D}$, is also well-defined.

Remark 2.1. Note that the term $\frac{1}{\sqrt{4\pi t}}\,\mathrm{tr}\, e^{-t\Delta_Z}(z,z)$, which leads to a divergent trace integral as discussed around (2.2), is the same as for the Neumann case. This accounts for the fact that the left-hand side in Equation (1.12) of Theorem 1.1 is independent of the regularization of the b-trace.

By the work in Melrose [22], the b-trace of $e^{-t\Delta_N}$ has the usual short-time asymptotic expansion:

$$^b\mathrm{Tr}\, e^{-t\Delta_N} \sim \sum_{k=0}^{\infty} a_k\, t^{(k-n)/2} \qquad \text{as } t\to 0, \tag{2.4}$$

where $n=\dim X$, and the long-time asymptotic expansion (see [8, Appendix]):

$$^b\mathrm{Tr}\, e^{-t\Delta_N} \sim \sum_{k=0}^{\infty} b_k\, t^{-k/2} \qquad \text{as } t\to \infty, \tag{2.5}$$

where $b_0 = \dim\ker\Delta_N + \frac{p}{2} - \frac{1}{4}\dim\ker\Delta_Z$ with p the dimension of the extended L^2 kernel of Δ_N. The heat kernel for the Dirichlet Laplacian Δ_D has similar expansions, the main difference being that b_0 in (2.5) is equal to $-\frac{1}{4}\dim\ker\Delta_Z$ in this case (because the Dirichlet problem is uniquely solvable). From (2.4) and (2.5) it follows that $^b\zeta(s,\Delta_N)$ and $^b\zeta(s,\Delta_D)$ (defined via (1.9) and (1.10)) are regular at $s=0$, so their corresponding $^b\zeta$-determinants are well-defined.

We now discuss the gluing formula of $\det_{^b\zeta}\Delta_N$ in our context. We consider a hypersurface H in X of the form $\{s\}\times Z$ or $\{r\}\times Y$ where $s\in(-\infty,0)$, $r\in(-1,0)$. We decompose X into two parts X_+ and X_- along H, the right and left sides of H. For the restriction of Δ_N over X_+, X_-, we impose Dirichlet boundary conditions over H and denote by Δ_+, Δ_- the resulting operators. Note that Δ_- is defined over the noncompact manifold X_- with cylindrical end and one boundary component. Hence, as for Δ_N, we have to use the $^b\zeta$-determinant for Δ_- rather than the ordinary ζ-determinant. Then the gluing problem in this context is to find a formula for the ratio

$$\frac{\det_{^b\zeta}\Delta_N}{\det_\zeta\Delta_+\cdot\det_{^b\zeta}\Delta_-} = \boxed{?}$$

in terms of recognizable data. To describe the right side, we need to introduce some notations. First, we consider the *Dirichlet to Neumann operators* $\mathcal{N}_\pm$ for $\Delta_\pm$; that is, we consider the solutions $\phi_\pm$ of the Dirichlet problems for $\Delta_\pm$ with the boundary data φ. Then the operators $\mathcal{N}_\pm$ are defined by $\mathcal{N}_\pm\varphi = \mp(\partial_u\phi_\pm)|_H$. Now we define

$$\mathcal{R}\varphi = \mathcal{N}_-\varphi + \mathcal{N}_+\varphi \qquad \text{for} \quad \varphi \in C^\infty(H, E|_H).$$

Then the operator $\mathcal{R}$ is a nonnegative pseudodifferential operator of order 1 over H (this can be proved as in [15, Appendix]). In particular, we can define its ζ-determinant, $\det_\zeta \mathcal{R}$. Second, we recall that $\{u_j\}$ is an orthonormal basis for the kernel of Δ_N on $L^2(X,E)$ and $\{U_j\}$ is a basis of the bounded solutions $\Delta_N U_j = 0$ such that at $\{-\infty\}\times Z$ on the cylinder, $\{U_j(-\infty)\}$ are orthonormal in $L^2(Z, E|_Z)$. We put $v_j(H) := u_j|_H$ and $V_j(H) := U_j|_H$, the restrictions of u_j and U_j, respectively, to the cutting hypersurface H. As before, we define

$$L(H) := \sum_j v_j(H)\otimes v_j(H)^* \quad , \quad \widetilde{L}(H) := \sum_j V_j(H)\otimes V_j(H)^*,$$

which are nonnegative linear operators on $\operatorname{span}\{v_j(H),\ V_j(H)\} \subset L^2(H, E|_H)$. Now we can state the gluing formula for $\det_{{}^b\zeta}\Delta_N$:

Theorem 2.2. *The following gluing formula holds:*

$$\frac{\det_{{}^b\zeta}\Delta_N}{\det_\zeta\Delta_+\cdot\det_{{}^b\zeta}\Delta_-} = 2^{-\zeta_{\Delta_H}(0)-h_H}\cdot\frac{\det_\zeta\mathcal{R}}{\det\big(L(H)+\widetilde{L}(H)\big)}$$

where $\zeta_{\Delta_H}(s)$ *is the* ζ*-function of* $\Delta_H := \Delta|_H$ *and* $h_H := \dim\ker\Delta_H$*.*

This theorem can be proved in essentially the same way as in [15], so we will not repeat its proof here. We remark that Theorem 2.2 is a generalization of the result of Burghelea, Friedlander, and Kappeler [2] for compact manifolds; cf. also Levit and Smilansky [14], Carron [4], Hassell and Zelditch [9], Lee [13], and Hassell [8], and Vishik [30] for related results dealing with the analytic torsion. We also refer to Mazzeo and Piazza [21] for an overview of gluing problems. Finally, we remark that there are gluing formulas similar to that in Theorem 2.2 in other contexts, see [16],[17] [19], [20] for some recent developments.

3. ζ-determinants over finite cylinders

Let $Y_r := [-r, 0]_u \times Y$ and over Y_r, consider the Laplace type operator

$$-\partial_u^2 + \Delta_Y$$

where Δ_Y is the Laplace type operator over Y. We impose the Dirichlet (resp. Neumann) boundary condition at $\{-r\}\times Y$ (resp. $\{0\}\times Y$) and denote by Δ_r^c the resulting operator. First, we have

Proposition 3.1. *The following equality holds:*

$$\det_\zeta\Delta_r^c = 2^{h_Y}\cdot\exp(Cr)\cdot\det{}_F^*\left(\mathrm{Id} + e^{-2r\sqrt{\Delta_Y}}\right) \tag{3.1}$$

where $h_Y = \dim\ker(\Delta_Y)$*,* $C = -(2\sqrt{\pi})^{-1}\frac{d}{ds}\big|_{s=0}\big(\Gamma(s)^{-1}\Gamma(s-1/2)\zeta_{\Delta_Y}(s-1/2)\big)$ *and* $\det_F^*$ *denotes the Fredholm determinant over* $\ker(\Delta_Y)^\perp$ *.*

Proof. Let us denote the spectrum of Δ_Y by $\{\, \mu_l \;:\; l \in \mathbb{N}\,\}$. Then we have

$$\mathrm{spec}(\Delta^c_r) = \Big\{\ \mu_l + \frac{\pi^2(k+1/2)^2}{r^2}\ \Big|\ l,k \in \mathbb{N}\ \Big\}.$$

This implies that

$$\zeta_{\Delta^c_r}(s) \;=\; \sum_{l=h_Y+1}^{\infty}\sum_{k=1}^{\infty}\Big(\mu_l + \frac{\pi^2(k+1/2)^2}{r^2}\Big)^{-s} \;+\; h_Y (r/\pi)^{2s}\zeta(2s,1/2), \tag{3.2}$$

where $\zeta(s,a)$ is the Hurwitz zeta function defined by (see [31])

$$\zeta(s,a) = \sum_{k=0}^{\infty}(k+a)^{-s} \quad \text{for } 0 < a < 1,$$

with the properties $\zeta(0,a) = \frac{1}{2} - a$ and $\frac{d}{ds}\big|_{s=0}\zeta(s,a) = \log(\Gamma(a)) - \frac{1}{2}\log(2\pi)$. We can rewrite the first term of the right side of (3.2) as

$$\frac{1}{2}\frac{1}{\Gamma(s)}\sum_{l=h_Y+1}^{\infty}\mu_l^{-s}\int_0^{\infty}\sum_{k\in\mathbb{Z}}\exp\bigg(-\Big(1+\Big(\frac{\pi(k+1/2)}{r\sqrt{\mu_l}}\Big)^2\Big)x\bigg)\, x^{s-1}\, dx. \tag{3.3}$$

Recalling the Poisson summation formula

$$\sum_{k\in\mathbb{Z}} e^{-a^2(k+b)^2} \;=\; \sum_{k\in\mathbb{Z}}\frac{\sqrt{\pi}}{a}e^{-\frac{\pi^2k^2}{a^2}}\cdot e^{2\pi i k b}$$

where a, b are positive real numbers, we see that (3.3) is same as

$$\begin{aligned}
&\frac{1}{2}\frac{1}{\Gamma(s)}\sum_{l=h_Y+1}^{\infty}\mu_l^{-s}\int_0^{\infty}\bigg(\sum_{k\in\mathbb{Z}}\frac{r\sqrt{\mu_l}}{\sqrt{\pi x}}\exp\bigg(-\frac{(r\sqrt{\mu_l}k)^2}{x}+\pi i k\bigg)e^{-x}\bigg)x^{s-1}\, dx\\
&=\frac{r}{\sqrt{\pi}}\frac{1}{\Gamma(s)}\sum_{l=h_Y+1}^{\infty}\mu_l^{-s+1/2}\int_0^{\infty}\bigg(\sum_{k\in\mathbb{N}}\exp\bigg(-\frac{(r\sqrt{\mu_l}k)^2}{x}+\pi i k\bigg)e^{-x}\bigg)x^{s-3/2}\, dx\\
&\quad+\frac{1}{2}\frac{r}{\sqrt{\pi}}\frac{1}{\Gamma(s)}\sum_{l=h_Y+1}^{\infty}\mu_l^{-s+1/2}\Gamma(s-1/2).
\end{aligned}$$

Now observe that the function

$$\int_0^{\infty}\bigg(\sum_{k\in\mathbb{N}}\exp\bigg(-\frac{(r\sqrt{\mu_l}k)^2}{x}+\pi i k\bigg)e^{-x}\bigg)x^{s-3/2}\, dx$$

is regular at $s = 0$ and that

$$\frac{1}{2}\frac{r}{\sqrt{\pi}}\frac{1}{\Gamma(s)}\sum_{l=h_Y+1}^{\infty}\mu_l^{-s+1/2}\Gamma(s-1/2) = \frac{1}{2}\frac{r}{\sqrt{\pi}}\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta_{\Delta_Y}\Big(s-1/2\Big).$$

Therefore, taking the derivative of $\zeta_{\Delta_r^c}(s)$ at $s=0$ in (3.2) and using the equality $\frac{d}{ds}\big|_{s=0}(1/\Gamma(s)) = 1$, we obtain

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0}\zeta_{\Delta_r^c}(s) = &\frac{r}{\sqrt{\pi}}\sum_{l=h_Y+1}^{\infty}\sqrt{\mu_l}\int_0^\infty\Big(\sum_{k\in\mathbb{N}}\exp\Big(-\frac{(r\sqrt{\mu_l}k)^2}{x}+\pi ik\Big)e^{-x}\Big)x^{-3/2}\,dx\\ &+\frac{r}{2\sqrt{\pi}}\frac{d}{ds}\Big|_{s=0}\big(\Gamma(s)^{-1}\Gamma(s-1/2)\zeta_{\Delta_Y}(s-1/2)\big)\\ &+h_Y\Big(2\log(r/\pi)\ \zeta(0,1/2)+2\frac{d}{ds}\Big|_{s=0}\zeta(s,1/2)\Big).\end{aligned}$$

Simplifying this expression, we obtain

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0}\zeta_{\Delta_r^c}(s) \ = \ &\sum_{l=h_Y+1}^{\infty}\sum_{k\in\mathbb{N}}\frac{e^{-2r\sqrt{\mu_l}k}}{k}\cdot e^{\pi ik}\\ &+\frac{r}{2\sqrt{\pi}}\frac{d}{ds}\Big|_{s=0}\Big(\Gamma(s)^{-1}\Gamma(s-1/2)\zeta_{\Delta_Y}\Big(s-1/2\Big)\Big)-h_Y\log 2.\end{aligned}$$

This equality immediately implies (3.1). □

Let $\mathcal{N}_r$ denote the Dirichlet to Neumann operator for the operator

$$\Delta_{r,N}^c := -\partial_u^2+\Delta_Y\ :\ \mathrm{dom}(\Delta_{r,N}^c)\longrightarrow L^2(Y_r,E)$$

where

$$\mathrm{dom}(\Delta_{r,N}^c) := \{\ \phi\in H^2(Y_r,E)\mid(\partial_u\phi)|_{\{0\}\times Y}=0\ \}.$$

Now we have

Proposition 3.2. *The following equality holds:*

$$\mathcal{N}_r = \sqrt{\Delta_Y}\ \frac{\mathrm{Id}-e^{-2r\sqrt{\Delta_Y}}}{\mathrm{Id}+e^{-2r\sqrt{\Delta_Y}}}.$$

Proof. Since $\Delta_{r,N}^c$ is of product form, it is enough to know the map $\mathcal{N}_r$ on eigensections of Δ_Y. Let φ_l be an eigensection of Δ_Y corresponding to the eigenvalue μ_l. Let us first consider the case of nonzero μ_l. Then the solution ϕ_l of the Dirichlet problem for $\Delta_{r,N}^c$ with $\phi_l|_{\{-r\}\times Y}=\varphi_l$ is given by

$$\phi_l = \frac{e^{(u+r)\sqrt{\mu_l}}+e^{(-u+r)\sqrt{\mu_l}}}{1+e^{2r\sqrt{\mu_l}}}\varphi_l.$$

Hence,

$$\mathcal{N}_r\varphi_l := -\partial_u\big|_{u=-r}\phi_l = -\sqrt{\mu_l}\frac{1-e^{2r\sqrt{\mu_l}}}{1+e^{2r\sqrt{\mu_l}}}\varphi_l = \sqrt{\mu_l}\frac{1-e^{-2r\sqrt{\mu_l}}}{1+e^{-2r\sqrt{\mu_l}}}\varphi_l.$$

For $\mu_l=0$, it is easy to see that $\mathcal{N}_r\varphi_l=0$. These complete the proof. □

4. Proof of Theorem

Let us decompose X into X_r and $Y_r=[-r,0]\times Y$ with $0<r<1$ as shown in Figure 2. For the restrictions of Δ_N to X_r and Y_r, we impose Dirichlet boundary conditions over $\{-r\}\times Y$ and we denote by Δ_{X_r}, Δ_{Y_r} the resulting operators. Then, by Theorem 2.2, the following equality holds:

$$\frac{\det_{b\zeta}\Delta_N}{\det_{b\zeta}\Delta_{X_r}\cdot\det_\zeta\Delta_{Y_r}} \ = \ 2^{-\zeta_{\Delta_Y}(0)-h_Y}\frac{\det_\zeta\mathcal{R}_r}{\det(L_r+\widetilde{L}_r)} \tag{4.1}$$

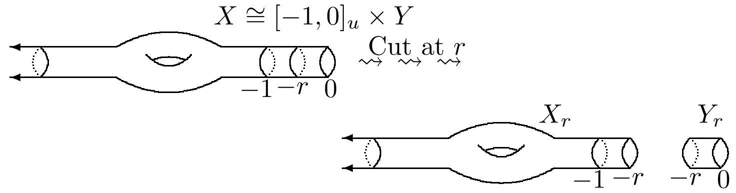

FIGURE 2. Cutting X at r into X_r and $Y_r = [-r, 0] \times Y$.

where L_r, $\widetilde{L}_r$ are defined by restricting $\{u_j\}$, $\{U_j\}$ to $\{-r\} \times Y$. Recall that the operator $\mathcal{R}_r$ over $\{-r\} \times Y$ is defined by

$$\mathcal{R}_r := \mathcal{N}_{X_r} + \mathcal{N}_{Y_r}$$

where $\mathcal{N}_{X_r}$, $\mathcal{N}_{Y_r}$ denote the Dirichlet to Neumann operators over X_r, Y_r, respectively. Rewriting (4.1) as

$$\frac{\det_{b\zeta}\Delta_N}{\det_{b\zeta}\Delta_{X_r}} \;=\; 2^{-\zeta_{\Delta_Y}(0) - h_Y} \cdot \det_\zeta \Delta_{Y_r} \cdot \frac{\det_\zeta \mathcal{R}_r}{\det(L_r + \widetilde{L}_r)}, \tag{4.2}$$

let us consider the limit of both sides as $r \to 0$. First, we note that Δ_{X_r} and Δ_D have no kernels by our assumption, hence it follows that

$$\lim_{r \to 0} \det_\zeta \Delta_{X_r} = \det_\zeta \Delta_D. \tag{4.3}$$

Second, by Proposition 3.1 we have

$$\det_\zeta \Delta_{Y_r} = 2^{h_Y} \cdot \exp(Cr) \cdot \det{}^*_F \,(\mathrm{Id} + e^{-2r\sqrt{\Delta_Y}})$$

where $C = -(2\sqrt{\pi})^{-1} \frac{d}{ds}\big|_{s=0} \big(\Gamma(s)^{-1}\Gamma(s - 1/2)\zeta_{\Delta_Y}(s - 1/2)\big)$ and $\det^*_F$ denotes the Fredholm determinant over $\ker(\Delta_Y)^\perp$. Now let us consider the following equalities:

$$\begin{aligned} \lim_{r \to 0} \det{}^*_F \,(\mathrm{Id} + e^{-2r\sqrt{\Delta_Y}}) \cdot \det_\zeta \Delta_Y &= \lim_{r \to 0} \det_\zeta \big((\mathrm{Id} + e^{-2r\sqrt{\Delta_Y}}) \cdot \Delta_Y\big) \\ &= \det_\zeta(2\Delta_Y) = 2^{\zeta_{\Delta_Y}(0)} \cdot \det_\zeta \Delta_Y. \end{aligned}$$

Cancelling $\det_\zeta \Delta_Y$ from both sides, we see that $\lim_{r \to 0} \det^*_F \,(\mathrm{Id} + e^{-2r\sqrt{\Delta_Y}}) = 2^{\zeta_{\Delta_Y}(0)}$. Therefore,

$$\lim_{r \to 0} \det_\zeta \Delta_{Y_r} = \lim_{r \to 0} \; 2^{h_Y} \cdot \exp(Cr) \cdot \det{}^*_F \,(\mathrm{Id} + e^{-2r\sqrt{\Delta_Y}}) = 2^{\zeta_{\Delta_Y}(0) + h_Y}. \tag{4.4}$$

Third, by Proposition 3.2, we have

$$\mathcal{N}_{Y_r} = \sqrt{\Delta_Y} \; \frac{\mathrm{Id} - e^{-2r\sqrt{\Delta_Y}}}{\mathrm{Id} + e^{-2r\sqrt{\Delta_Y}}},$$

which implies

$$\lim_{r \to 0} \mathcal{N}_{Y_r} = 0 \qquad \text{with respect to the operator norm.} \tag{4.5}$$

Now let us observe that $\mathcal{R}_r$ is continuous in r such that $\ker \mathcal{R}_r$ has constant rank as it is given by restricting the harmonic sections of Δ_N to $\{-r\} \times Y$. Moreover, defining $\mathcal{R}_0 := \mathcal{N}_{X_0} = \mathcal{N}$, we can see that $\mathcal{R}_r$ is continuous even at $r = 0$ by (4.5) with $\ker \mathcal{R}_0 \cong \ker \mathcal{R}_r$ for small nonzero r. Hence, we can see that

$$\lim_{r \to 0} \det_\zeta \mathcal{R}_r = \lim_{r \to 0} \det_\zeta(\mathcal{N}_{X_r} + \mathcal{N}_{Y_r}) = \det_\zeta \Big[\lim_{r \to 0}(\mathcal{N}_{X_r} + \mathcal{N}_{Y_r})\Big] = \det_\zeta \mathcal{N}. \tag{4.6}$$

Trivially, as $r \to 0$,

$$\det(L_r + \widetilde{L}_r) \longrightarrow \det(L_0 + \widetilde{L}_0) =: \det(L + \widetilde{L}). \tag{4.7}$$

Combining (4.3), (4.4), (4.6) and (4.7) into the identity (4.2), we conclude that

$$\frac{\det_{{}^b\zeta}\Delta_N}{\det_{{}^b\zeta}\Delta_D} = \frac{\det_\zeta \mathcal{N}}{\det(L + \widetilde{L})}.$$

This completes the proof of Theorem 1.1.

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