FUNCTIONAL DETERMINANTS FOR GENERAL SELF-ADJOINT EXTENSIONS OF LAPLACE-TYPE OPERATORS RESULTING FROM THE GENERALIZED CONE

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ABSTRACT. In this article we consider the zeta regularized determinant of Laplace-type operators on the generalized cone. For *arbitrary* self-adjoint extensions of a matrix of singular ordinary differential operators modelled on the generalized cone, a closed expression for the determinant is given. The result involves a determinant of an endomorphism of a finite-dimensional vector space, the endomorphism encoding the self-adjoint extension chosen. For particular examples, like the Friedrich's extension, the answer is easily extracted from the general result. In combination with [13], a closed expression for the determinant of an arbitrary self-adjoint extension of the full Laplace-type operator on the generalized cone can be obtained.

1. INTRODUCTION

Motivated by endeavors to give answers to some fundamental questions in quantum field theory there has been significant interest in the problem of calculating the determinants of second order Laplace-type elliptic differential operators; see for example [6, 59, 95, 96, 99]. In case the operator Δ in question has regular coefficients and is acting on sections of a vector bundle over a smooth compact manifold, it will have a discrete eigenvalue spectrum $\lambda_1 \leq \lambda_2 \leq ... \to \infty$. If all eigenvalues are different from zero the determinant, formally defined by $\det \Delta = \prod_i \lambda_i$, is generally divergent. In order to make sense out of it different procedures like Pauli-Villars regularization [92] or dimensional regularization [103] have been invented. Mathematically the probably most pleasing regularization is the zeta function prescription introduced by Ray and Singer [97] (see also [49, 71]) in the context of analytic torsion; see i.e. [7, 8, 9, 88, 89].

In this method, one uses the zeta function $\zeta(s, \Delta)$ associated with the spectrum λ_i of Δ. In detail, for the real part of s large enough one has

$$\zeta(s, \Delta) = \sum_{i=1}^{\infty} \lambda_i^{-s}.$$

In the briefly described smooth setting, one can show that $\zeta(s, \Delta)$ is analytic about $s = 0$ [66, 100, 107], which allows to define a zeta regularized determinant via

$$\det_\zeta(\Delta) = e^{-\zeta'(0, \Delta)}.$$

This definition has been used extensively in quantum field theory, see i.e. [19, 23, 52, 53, 54, 55, 71, 73], as well as in the context of the Reidemeister-Franz torsion [97, 98]. In particular, in one dimension rather general and elegant results may be

obtained, which has attracted the interest of mathematicians especially in the last decade or so [21, 22, 51, 60, 61, 82, 83, 84]. In higher dimensions known results are restricted to highly symmetric configurations [13, 14, 16, 23, 44, 45, 46, 50] or conformally related ones [10, 11, 16, 47, 48].

Whereas most analysis has been done in the smooth setting, relevant situations do not fall into this category. For example, in order to compute quantum corrections to classical solutions in Euclidean Yang-Mills theory [26, 102] singular potentials need to be considered. They also serve for the description of physical systems like the Calogero Model [3, 4, 27, 28, 29, 56, 91] and conformal invariant quantum mechanical models [2, 20, 30, 31, 40, 62, 94]. More recently they became popular among physicists working on space-times with horizons. There, for a variety of black holes, singular potentials are used to describe the dynamics of quantum particles in the asymptotic near-horizon region [5, 37, 64, 67, 87].

A similar situation occurs when manifolds are allowed to have conical singularities [32, 35]. Under these circumstances, in general, $\zeta'(0, \Delta)$ will not be defined, although for special instances this definition still makes sense; nearly all of the literature has concentrated on these special instances. In order to describe these instances in more detail, let us consider a bounded generalized cone. As we will see below, the Laplacian on a bounded generalized cone has the form

$$\Delta = -\frac{\partial^2}{\partial r^2} + \frac{1}{r^2} A_\Gamma,$$

where A_Γ is defined on the base of the cone. If A_Γ has eigenvalues in the interval $[\frac{3}{4}, \infty)$ only, one can show that Δ is essentially self-adjoint and no choices for self-adjoint extensions exist. Spectral functions, in particular the determinant, have been analyzed in detail in [13]. In case A_Γ has one or more eigenvalues in the interval $[-\frac{1}{4}, \frac{3}{4})$ different self-adjoint extensions exist; see for example [86]. Most literature is concerned with the so-called Friedrich's extension [17, 24, 25, 35, 38, 39, 42, 43, 81, 82, 101] and homogeneous or scale-invariant extensions [35, 81, 85]. Exceptions are [56, 57, 58] where general self-adjoint extensions associated with one eigenvalue in $[-\frac{1}{4}, \frac{3}{4})$ have been considered. Only recently, properties of spectral functions for arbitrary self-adjoint extensions over the generalized cone have been understood [76]; a summary of the results is given in Section 2. In particular, the zeta function is shown to have a logarithmic branch point at $s = 0$, in addition to the standard simple pole at $s = 0$. A natural construct for the determinant is to subtract off these singular terms and to consider the derivative of the finite remainder. This also is explained in Section 2.

The details of the singular behavior as $s \to 0$, as well as of the finite terms, strongly depend on the self-adjoint extension. In Section 3 we therefore briefly review the construction of self-adjoint extensions on the generalized cone using the Hermitian symplectic extension theory [69, 70, 72, 77, 78, 79, 80, 90, 93]. This, finally, provides the set-up for the analysis of the zeta function for arbitrary self-adjoint extensions. Even in the most general case eigenvalues are determined by an implicit or transcendental equation, a perfect starting point for the contour integration method described in detail in [12, 13, 14, 73, 74, 75]. This method allows us to find the determinant for arbitrary self-adjoint extensions, the main result, see Theorem 2.3, being derived in Section 4. In Section 5 we apply the answer for the general case to certain natural self-adjoint extensions. The conclusions provide a brief summary.

2. Zeta functions on generalized cones and their ζ-determinants

In this section we review the notion of Laplace-type operators over generalized cones and we discuss the pathological properties of their zeta functions, which may have poles of arbitrary multiplicity and countably many logarithmic singularities. We state a natural procedure to define the ζ-regularized determinant and finally, we state the main formulas of this paper.

2.1. Generalized cones and regular singular operators.

Let Γ be a smooth $(n-1)$-dimensional compact manifold (with or without boundary). Then the generalized cone with base Γ, also called a cone over Γ, is the n-dimensional manifold

$$M = [0, R]_r \times \Gamma,$$

where $R > 0$ and the metric of M is of the type $dr^2 + r^2 h$ with h a metric over Γ. Let E be a Hermitian vector bundle over M and let

$$\Delta_M : C_c^\infty(M \setminus \{0\} \times \Gamma, E) \to C_c^\infty(M \setminus \{0\} \times \Gamma, E)$$

be a Laplace-type operator with the Dirichlet condition at $r = R$ having the form

$$\Delta_M = -\partial_r^2 - \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_\Gamma,$$

where Δ_Γ is a Laplace-type operator acting on $C^\infty(\Gamma, E_\Gamma)$ where $E_\Gamma := E|_\Gamma$; if Γ has a boundary we put Dirichlet conditions (for example) at $\partial\Gamma$. By introducing a Liouville transformation, we can write Δ_M in an equivalent way that is more convenient for analysis. Writing $\phi \in L^2(M, E, r^{n-1}drdh)$ as

$$(2.1) \qquad\qquad \phi = r^{-\frac{n-1}{2}}\widetilde{\phi},$$

where $\widetilde{\phi} := r^{\frac{n-1}{2}}\phi$, we have

$$\int_M \langle\phi, \psi\rangle \, r^{n-1}drdh = \int_M \langle\widetilde{\phi}, \widetilde{\psi}\rangle \, drdh,$$

and a short computation shows that

$$\Delta_M\phi = \left(-\partial_r^2 - \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_\Gamma\right)\phi = r^{-\frac{n-1}{2}}\Delta\,\widetilde{\phi},$$

where

$$(2.2) \qquad\qquad \Delta := -\partial_r^2 + \frac{1}{r^2}A_\Gamma$$

with $A_\Gamma := \Delta_\Gamma + \frac{n-1}{2}\left(\frac{n-1}{2} - 1\right)$. In conclusion: Under the isomorphism (2.1), $L^2(M, E, r^{n-1}drdh)$ is identified with $L^2(M, E)$ with the standard measure $drdh$, and Δ_M is identified with the operator Δ in (2.2). It turns out that for analytical purposes, the operator Δ is somewhat more natural to work with. Notice that if Δ_Γ happens to be nonnegative, then

$$A_\Gamma = \Delta_\Gamma + \frac{n-1}{2}\left(\frac{n-1}{2} - 1\right) \geq \Delta_\Gamma - \frac{1}{4} \geq -\frac{1}{4},$$

where we used the fact that the function $x(x-1)$ has the minimum value $-\frac{1}{4}$ (when $x = \frac{1}{2}$). In fact, it is both a necessary and sufficient condition that $A_\Gamma \geq -\frac{1}{4}$ in order that Δ_M (or Δ) be bounded below [17, 24, 25]. For this reason, we henceforth *assume* that $A_\Gamma \geq -\frac{1}{4}$. The operator Δ is called a second order *regular singular operator* [17].

Let $\{\lambda_\ell\}$ denote the spectrum of A_Γ. Then Weyl's alternative [106] immediately shows that 0 is in the limit case if and only if $-1/4 \leq \lambda_\ell < 3/4$ [105]. Consider only those eigenvalues in $[-\frac{1}{4}, \frac{3}{4})$:

$$(2.3) \qquad -\frac{1}{4} = \underbrace{\lambda_1 = \lambda_2 = \cdots = \lambda_{q_0}}_{=-\frac{1}{4}} < \underbrace{\lambda_{q_0+1} \leq \lambda_{q_0+2} \leq \cdots \leq \lambda_{q_0+q_1}}_{-\frac{1}{4} < \lambda_\ell < \frac{3}{4}},$$

where each eigenvalue is counted according to its multiplicity. Then, as a consequence of von Neumann's theory of self-adjoint extensions the self-adjoint extensions of Δ are in a one-to-one correspondence to the Lagrangian subspaces in $\mathbb{C}^{2q}$ where $q = q_0 + q_1$ and where $\mathbb{C}^{2q}$ has the symplectic form described in (3.3) [33, 34, 35, 65, 81, 86, 80]. A concrete description of these Lagrangian subspaces is as follows (see Proposition 3.2). A subspace $L \subset \mathbb{C}^{2q}$ is Lagrangian if and only if there exists $q \times q$ complex matrices $\mathcal{A}$ and $\mathcal{B}$ such that the rank of the $q \times 2q$ matrix $\begin{pmatrix} \mathcal{A} & \mathcal{B} \end{pmatrix}$ is q, $\mathcal{A}' \mathcal{B}^*$ is self-adjoint where $\mathcal{A}'$ is the matrix $\mathcal{A}$ with the first q_0 columns multiplied by -1, and

$$(2.4) \qquad L = \{v \in \mathbb{C}^{2q} \,|\, \begin{pmatrix} \mathcal{A} & \mathcal{B} \end{pmatrix} v = 0\}.$$

Given such a subspace $L \subset \mathbb{C}^{2q}$ there exists a canonically associated domain $\mathfrak{D}_L \subset H^2(M, E)$ such that

$$\Delta_L := \Delta : \mathfrak{D}_L \longrightarrow L^2(M, E)$$

is self-adjoint (see Proposition 3.3).

2.2. **Exotic zeta functions.** Δ_L has pure discrete spectrum [81], and hence, if $\{\mu_j\}$ denotes the spectrum of Δ_L, then we can form the zeta function

$$\zeta(s, \Delta_L) := \sum_{\mu_j \neq 0} \frac{1}{\mu_j^s}.$$

The meromorphic structure of $\zeta(s, \Delta_L)$ (or the corresponding heat trace) has been extensively studied for special self-adjoint extensions, as for example the Friedrichs extension [13, 17, 24, 25, 35, 38, 39, 42, 43, 63, 101], which corresponds to taking $\mathcal{A} = 0$ and $\mathcal{B} = \mathrm{Id}$ in (2.4) [17], and the homogeneous or scale-invariant extensions [35, 81, 85], which corresponds to taking $\mathcal{A}$ and $\mathcal{B}$ to be diagonal matrices with 0's and 1's along the diagonal such that the first q_0 entries along the diagonal of $\mathcal{B}$ are 1's and $\mathcal{A} + \mathcal{B} = \mathrm{Id}$ [85]. In these cases, the zeta function has the "regular" meromorphic structure; that is, the same structure as on a smooth manifold with one exception, $\zeta(s, \Delta_L)$ might have a pole at $s = 0$. For general self-adjoint extensions, the meromorphic structure has been studied in [56, 57, 58, 76, 86]. The papers [56, 57, 58] are devoted to one-dimensional Laplace-type operators over the unit interval and [76, 86] study the general case of operators over manifolds. The papers [56, 57, 58, 86] show that $\zeta(s, \Delta_L)$ has, in addition to the "regular" poles, additional simple poles at "unusual" location. In [76] it was shown that the zeta function $\zeta(s, \Delta_L)$ has, in the general case, in addition to the "unusual" poles, meromorphic structures that remained unobserved and which are unparalleled in the zeta function literature such as poles of arbitrary order and logarithmic singularities.

The main result of [76] not only states the existence of such exotic singularities but it also gives an algebraic-combinatorial algorithm that finds these singularities explicitly. Although the algorithm is described in detail there, we have to provide

a summary in order to set up the notation used in the rest of the paper. The algorithm is described as follows.

Step 1: Let $\mathcal{A}$ and $\mathcal{B}$ be as in (2.4) and define the function

$$(2.5) \qquad p(x,y) := \det \begin{pmatrix} & & \mathcal{A} & & & \mathcal{B} \\ x\,\mathrm{Id}_{q_0} & 0 & 0 & 0 & \\ 0 & \tau_1\,y^{2\nu_1} & 0 & 0 & \\ & & & & \mathrm{Id}_q \\ 0 & 0 & \ddots & 0 & \\ 0 & 0 & 0 & \tau_{q_1}\,y^{2\nu_{q_1}} & \end{pmatrix},$$

where Id_k denotes the $k \times k$ identity matrix and where

$$\nu_j := \sqrt{\lambda_{q_0+j} + \frac{1}{4}}\,, \qquad \tau_j = 2^{2\nu_j}\frac{\Gamma(1+\nu_j)}{\Gamma(1-\nu_j)}, \qquad j = 1,\dots,q_1,$$

with q_0, q_1, λ_j as in (2.3). Expanding the determinant, we can write $p(x,y)$ as a finite sum

$$p(x,y) = \sum a_{j\alpha}\, x^j\, y^{2\alpha},$$

where the α's are linear combinations of $\nu_1,\dots,\nu_{q_1}$ and the $a_{j\alpha}$'s are constants. Let α_0 be the smallest of all α's with $a_{j\alpha} \neq 0$ and let j_0 be the smallest of all j's amongst the $a_{j\alpha_0} \neq 0$. Then factoring out the term $a_{j_0\alpha_0}\, x^{j_0}\, y^{2\alpha_0}$ in $p(x,y)$ we can write $p(x,y)$ in the form

$$(2.6) \qquad p(x,y) = a_{j_0\alpha_0}\, x^{j_0}\, y^{2\alpha_0}\left(1 + \sum b_{k\beta}\, x^k\, y^{2\beta}\right)$$

for some constants $b_{k\beta}$ (equal to $a_{k\beta}/a_{j_0\alpha_0}$).

Step 2: Second, putting $z = \sum b_{k\beta}\, x^k\, y^{2\beta}$ into the power series $\log(1+z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} z^k$ and *formally* expanding, we can write

$$(2.7) \qquad \log\left(1 + \sum b_{k\beta} x^k y^{2\beta}\right) = \sum c_{\ell\xi}\, x^\ell\, y^{2\xi}$$

for some constants $c_{\ell\xi}$. By construction, the ξ's appearing in (2.7) are nonnegative, countable, and approach $+\infty$ unless $\beta = 0$ is the only β in (2.6), in which case only $\xi = 0$ occurs in (2.7). Also, for a fixed ξ, the ℓ's with $c_{\ell\xi} \neq 0$ are bounded below.

Step 3: Third, for each ξ appearing in (2.7), define

$$(2.8) \qquad p_\xi := \min\{\ell \leq 0 \,|\, c_{\ell\xi} \neq 0\} \qquad \text{and} \qquad \ell_\xi := \min\{\ell > 0 \,|\, c_{\ell\xi} \neq 0\},$$

whenever the sets $\{\ell \leq 0 \,|\, c_{\ell\xi} \neq 0\}$ and $\{\ell > 0 \,|\, c_{\ell\xi} \neq 0\}$, respectively, are nonempty. Let $\mathscr{P}$, respectively $\mathscr{L}$, denote the set of ξ values for which the respective sets are nonempty. The following theorem is our main result [76, Th. 2.1].

Theorem 2.1. *The ζ-function $\zeta(s,\Delta_L)$ extends from $\Re s > \frac{n}{2}$ to a meromorphic function on $\mathbb{C} \setminus (-\infty, 0]$. Moreover, $\zeta(s,\Delta_L)$ can be written in the form*

$$\zeta(s,\Delta_L) = \zeta_{\mathrm{reg}}(s,\Delta_L) + \zeta_{\mathrm{sing}}(s,\Delta_L),$$

where $\zeta_{\mathrm{reg}}(s,\Delta_L)$ has possible "regular" poles at the "usual" locations $s = \frac{n-k}{2}$ with $s \notin -\mathbb{N}_0$ for $k \in \mathbb{N}_0$ and at $s = 0$ if $\dim\Gamma > 0$, and where $\zeta_{\mathrm{sing}}(s,\Delta_L)$ has the

following expansion:

$$(2.9) \quad \zeta_{\mathrm{sing}}(s, \Delta_L) = \frac{\sin(\pi s)}{\pi} \Bigg\{ (j_0 - q_0) e^{-2s(\log 2 - \gamma)} \log s$$

$$+ \sum_{\xi \in \mathscr{P}} \frac{f_\xi(s)}{(s + \xi)^{|p_\xi| + 1}} + \sum_{\xi \in \mathscr{L}} g_\xi(s) \log(s + \xi) \Bigg\},$$

where j_0 appears in (2.6) and $f_\xi(s)$ and $g_\xi(s)$ are entire functions of s such that

$$f_\xi(-\xi) = (-1)^{|p_\xi| + 1} c_{p_\xi \xi} \frac{|p_\xi|!}{2^{|p_\xi|}} \xi$$

and

$$g_\xi(s) = \begin{cases} c_{\ell_0, 0} \frac{2^{\ell_0}}{(\ell_0 - 1)!} s^{\ell_0} + \mathcal{O}(s^{\ell_0 + 1}) & \text{if } \xi = 0, \\ -c_{\ell_\xi \xi} \frac{\xi \, 2^{\ell_\xi}}{(\ell_\xi - 1)!} (s + \xi)^{\ell_\xi - 1} + \mathcal{O}((s + \xi)^{\ell_\xi}) & \text{if } \xi > 0. \end{cases}$$

Remark 2.2. *The expansion (2.9) means that for any $N \in \mathbb{N}$,*

$$\zeta_{\mathrm{sing}}(s, \Delta_L) = \frac{\sin(\pi s)}{\pi} \Bigg\{ (j_0 - q_0) e^{-2s(\log 2 - \gamma)} \log s + \sum_{\xi \in \mathscr{P}, \xi \leq N} \frac{f_\xi(s)}{(s + \xi)^{|p_\xi| + 1}}$$

$$+ \sum_{\xi \in \mathscr{L}, \xi \leq N} g_\xi(s) \log(s + \xi) \Bigg\} + F_N(s),$$

where $F_N(s)$ is holomorphic for $\Re s \geq -N$. Note that the leading terms as $s \to 0$ are contained in $\zeta_{\mathrm{reg}}(s, \Delta_L)$ and the first term of $\zeta_{\mathrm{sing}}(s, \Delta_L)$.

2.3. **ζ-determinant formulæ.** For a general self-adjoint extension, Theorem 2.1 shows that the $\zeta(s, \Delta_L)$ may not only have a simple pole at $s = 0$ (from $\zeta_{\mathrm{reg}}(s, \Delta_L)$) but also a logarithmic singularity at $s = 0$. Needless to say, the zeta function is rarely regular at $s = 0$ except for special self-adjoint extensions. In particular, the usual definition of the zeta-regularized determinant is ill-defined via taking the derivative of $\zeta(s, \Delta_L)$ at $s = 0$. However, we can still associate a natural definition of a determinant by subtracting off the singularities. Thus, let us define

$$\zeta_0(s, \Delta_L) := \zeta(s, \Delta_L) - \frac{c}{s} - (j_0 - q_0) s \log s,$$

where $c = \mathrm{Res}_{s=0} \zeta_{\mathrm{reg}}(s, \Delta_L)$. The term c/s cancels the possible pole of $\zeta_{\mathrm{reg}}(s, \Delta_L)$ at $s = 0$ and by the explicit formula (2.9) for $\zeta_{\mathrm{sing}}(s, \Delta_L)$, the term $(j_0 - q_0) s \log s$ cancels the logarithmic singularity of $\zeta_{\mathrm{sing}}(s, \Delta_L)$ at $s = 0$ up to a term that is $\mathcal{O}(s^2 \log s)$ at $s = 0$. It follows that $\lim_{s \to 0^+} \zeta_0'(s, \Delta_L)$ exists. Therefore, we can define

$$\det_\zeta(\Delta_L) := \exp\left(-\lim_{s \to 0^+} \zeta_0'(s, \Delta_L) \right)$$

This definition of course agrees with the standard definition in case $\zeta(s, \Delta_L)$ is regular at $s = 0$. In Theorem 2.3 below, we find an explicit formula for this determinant. Because of some unyielding constants, it is elegant to write our main formula as a relative formula in terms of the Neumann extension. The *Neumann extension* is given by choosing $\mathcal{A}$ and $\mathcal{B}$ to be the diagonal matrices with the $q_0 +$

$1, \ldots, q$ entries in $\mathcal{A}$ equal to 1 and the $1, \ldots, q_0$ entries in $\mathcal{B}$ equal to 1 with the rest of the entries 0. By Corollary 4.7 (or [85]), we find the explicit formula

$$(2.10) \qquad \det_\zeta(\Delta_\mathcal{N}) = (2\pi R)^{\frac{q}{2}} \prod_{j=1}^{q_1} \frac{2^{\nu_j} R^{-\nu_j}}{\Gamma(1-\nu_j)} \cdot \det_\zeta(\widetilde{\Delta})$$

where $\widetilde{\Delta}$ is the (essentially self-adjoint) operator obtained by projecting Δ onto the eigenvalues of A_Γ in $[\frac{3}{4}, \infty)$ (see (3.1) for a more precise definition of $\widetilde{\Delta}$). The determinant $\det_\zeta(\widetilde{\Delta})$ is given explicitly in Equation (9.8) of [13] when $R = 1$, with a similar formula holding for arbitrary $R > 0$. We refer the reader to [13] for the appropriate details on $\det_\zeta(\widetilde{\Delta})$. The following theorem is our main result.

Theorem 2.3. *For a Lagrangian $L \subset \mathbb{C}^{2q}$ such that the operator obtained by projecting Δ onto the eigenvalues of A_Γ in $[-\frac{1}{4}, \frac{3}{4})$ is invertible, we have*

$$\frac{\det_\zeta(\Delta_L)}{\det_\zeta(\Delta_\mathcal{N})} = \frac{(-2e^\gamma)^{q_0-j_0}}{a_{j_0\alpha_0}} \det \begin{pmatrix} \mathcal{A} & & \mathcal{B} & \\ \mathrm{Id}_{q_0} & 0 & (\log R)\mathrm{Id}_{q_0} & 0 \\ 0 & \mathbf{R}^{2\nu} & 0 & \mathrm{Id}_{q_1} \end{pmatrix},$$

where $a_{j_0\alpha_0}$ is the coefficient in (2.6) and $\mathbf{R}^{2\nu}$ is the $q_1 \times q_1$ diagonal matrix with entries $R^{2\nu_\ell}$ for $1 \le \ell \le q_1$.

Combining this formula with (2.10), we get an explicit formula for $\det_\zeta(\Delta_L)$.

The next result follows from an application of Theorem 2.3 to a particular class of matrices $\mathcal{A}$ and $\mathcal{B}$.

Theorem 2.4. *Let $q - r = \mathrm{rank}(\mathcal{A})$ and assume that $\mathcal{A}$ has r rows and columns identically zero. Let $i_1, \ldots, i_q$ be a permutation of the numbers $1, \ldots, q$ such that the rows and columns $i_1, \ldots, i_r$ of $\mathcal{A}$ are zero. Choose $j_0 \in \{0, 1, \ldots, q_0\}$ such that*

$$1 \le i_1 < i_2 < \cdots < i_{j_0} \le q_0 < i_{j_0+1} < \cdots < i_r \le q.$$

Let I_r denote the $q \times q$ matrix which is zero everywhere except along the diagonal where the entries $i_1, \ldots, i_r$ equal 1, and let I_{q-r} denotes the $q \times q$ matrix which is zero everywhere except along the diagonal where the entries $i_{r+1}, \ldots, i_q$ equal 1. Then for a Lagrangian L having $\mathcal{A}$ as a first component and satisfying the condition in Theorem 2.3, we have:

$$\frac{\det_\zeta(\Delta_L)}{\det_\zeta(\Delta_\mathcal{N})} = (-2e^\gamma)^{q_0-j_0} \prod_{j=j_0+1}^{r} \left[2^{-2\nu_{i_j}} \frac{\Gamma(1-\nu_{i_j})}{\Gamma(1+\nu_{i_j})} \right] \det \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ I_r & I_{q-r} \end{pmatrix}^{-1} \times$$

$$\det \begin{pmatrix} \mathcal{A} & & \mathcal{B} & \\ \mathrm{Id}_{q_0} & 0 & (\log R)\mathrm{Id}_{q_0} & 0 \\ 0 & \mathbf{R}^{2\nu} & 0 & \mathrm{Id}_{q_1} \end{pmatrix}.$$

See Section 5 for more special cases including one-dimensional operators.

3. The Hermitian symplectic theory of self-adjoint extensions

In this section we briefly explain the correspondence between self-adjoint extensions and the Lagrangian subspaces described by (2.4). This correspondence is a direct consequence of von Neumann's classical theory of self-adjoint extensions; a partial list of relevant references is [33, 34, 35, 65, 81, 86, 80, 69, 70, 72, 77, 78, 79, 80, 90, 93, 104, 105].

3.1. **Reduction to the model problem.** Let $\{\lambda_\ell\}$ denote the set of all eigenvalues of A_Γ and let E_ℓ denote the span of the λ_ℓ-th eigenvector. Let Π and $\Pi^\perp$ denote, respectively, the orthogonal projections of $L^2(\Gamma, E_\Gamma)$ onto $W := \bigoplus_{-\frac{1}{4} \leq \lambda_\ell < \frac{3}{4}} E_\ell \cong \mathbb{C}^q$ and $W^\perp$. Using the isometry between

$$L^2([0, R] \times \Gamma, E) \cong L^2([0, R], L^2(\Gamma, E_\Gamma)),$$

we obtain the corresponding projections on $L^2([0, R] \times \Gamma, E)$, which we denote with the same notations Π and $\Pi^\perp$. Since A_Γ preserves W and $W^\perp$, we can write

$$\Delta = \mathcal{L} \oplus \widetilde{\Delta},$$

where

$$(3.1) \qquad \widetilde{\Delta} := \Pi^\perp \Delta \Pi^\perp = -\partial_r^2 + \frac{1}{r^2} A_\Gamma\big|_{W^\perp},$$

and $\mathcal{L}$ is the (matrix) ordinary differential operator

$$\mathcal{L} := \Pi \Delta \Pi = -\frac{d^2}{dr^2} + \frac{1}{r^2} A,$$

where A is the $q \times q$ diagonal matrix

$$A = \begin{pmatrix} -\frac{1}{4}\mathrm{Id}_{q_0} & & & & 0 & & \\ & \lambda_{q_0+1} & 0 & 0 & \cdots & 0 \\ & 0 & \lambda_{q_0+2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda_{q_0+3} & \cdots & 0 \\ & & & & \ddots & \\ & 0 & 0 & 0 & \ddots & 0 \\ & 0 & 0 & 0 & \cdots & \lambda_{q_0+q_1} \end{pmatrix};$$

here we write A with respect to the basis of $W = \bigoplus_{-\frac{1}{4} \leq \lambda_\ell < \frac{3}{4}} E_\ell \cong \mathbb{C}^q$. It is well-known that the operator $\widetilde{\Delta}$ is essentially self-adjoint [17, 18, 24, 25, 86] Therefore, the various self-adjoint extensions of Δ are simply the various self-adjoint extensions of the "toy model operator" $\mathcal{L}$, which we now study.

3.2. **Self-adjoint extensions of the model operator.** The key to determining the self-adjoint extensions of $\mathcal{L}$ is to first characterize the *maximal domain* of $\mathcal{L}$:

$$\mathfrak{D}_{\max} := \left\{ \phi \in L^2([0, R], \mathbb{C}^q) \,\middle|\, \mathcal{L}\phi \in L^2([0, R], \mathbb{C}^q) \text{ and } \phi(R) = 0 \right\},$$

which is the largest set of L^2 functions on which $\mathcal{L}$ can act and stay within L^2. As an immediate consequence of Cheeger [34, 35] we have

Proposition 3.1. $\phi \in \mathfrak{D}_{\max}$ *if and only if* $\phi(R) = 0$ *and* ϕ *has the following form:*

$$(3.2) \qquad \phi = \sum_{\ell=1}^{q_0} \left\{ c_\ell(\phi)\, r^{\frac{1}{2}} e_\ell + c_{q+\ell}(\phi)\, r^{\frac{1}{2}} \log r e_\ell \right\}$$

$$+ \sum_{\ell=1}^{q_1} \left\{ c_{q_0+\ell}(\phi)\, r^{\nu_\ell + \frac{1}{2}} e_{q_0+\ell} + c_{q+q_0+\ell}(\phi)\, r^{-\nu_\ell + \frac{1}{2}} e_{q_0+\ell} \right\} + \widetilde{\phi},$$

where

$$\nu_\ell := \sqrt{\lambda_{q_0+\ell} + \frac{1}{4}} \; > \; 0,$$

e_ℓ *is the column vector with* 1 *in the* ℓ*-th slot and* 0*'s elsewhere, the* $c_j(\phi)$*'s are constants, and the* $\widetilde{\phi}$ *is continuously differentiable on* $[0, R]$ *such that* $\widetilde{\phi}(r) = \mathcal{O}(r^{\frac{3}{2}})$ *and* $\widetilde{\phi}'(r) = \mathcal{O}(r^{\frac{1}{2}})$ *near* $r = 0$*, and* $\mathcal{L}\widetilde{\phi} \in L^2([0, R], \mathbb{C}^q)$.

We next want to formulate the correspondence between self-adjoint extensions and Lagrangian subspaces with respect to a suitable symplectic form. Let

$$J := \begin{pmatrix} 0 & -\mathrm{Id}_q \\ \mathrm{Id}_q & 0 \end{pmatrix},$$

and recall that

$$\mathbb{C}^{2q} \times \mathbb{C}^{2q} \ni (v, w) \mapsto \langle Jv, w \rangle \in \mathbb{C}$$

is the standard Hermitian symplectic form on $\mathbb{C}^{2q}$; that is, this form is Hermitian antisymmetric and nondegenerate. Now defining $T : \mathbb{C}^{2q} \to \mathbb{C}^{2q}$ by

$$T(v_1, \ldots, v_{2q}) = (-v_1, \ldots, -v_{q_0}, v_{q_0+1}, \ldots, v_{2q})$$

and putting $\vec{\phi} = (c_1(\phi), c_2(\phi), \ldots, c_{2q}(\phi))^t$, $\vec{\psi} = (c_1(\psi), c_2(\psi), \ldots, c_{2q}(\psi))^t$, one has

$$(3.3) \qquad \langle \mathcal{L}\phi, \psi \rangle - \langle \phi, \mathcal{L}\psi \rangle = \langle JT\vec{\phi}, T\vec{\psi} \rangle =: \omega(\vec{\phi}, \vec{\psi}),$$

where

$$\omega(v, w) := \langle JTv, Tw \rangle \quad \text{for all} \quad v, w \in \mathbb{C}^{2q}$$

defines a symplectic form on $\mathbb{C}^{2q}$. We say that a subspace $L \subset \mathbb{C}^{2q}$ is *Lagrangian* (with respect to ω) if

$$\{w \in \mathbb{C}^{2q} \mid \omega(v, w) = 0 \text{ for all } v \in L\} = L.$$

Self-adjoint extensions of $\mathcal{L}$ are then in one-to-one correspondence with Lagrangian subspaces of $(\mathbb{C}^{2q}, \omega)$ in the sense that given any Lagrangian subspace $L \subset \mathbb{C}^{2q}$ and defining

$$\mathfrak{D}_L := \{\phi \in \mathfrak{D}_{\max} \mid \vec{\phi} \in L\},$$

the operator

$$\mathcal{L}_L := \mathcal{L} : \mathfrak{D}_L \longrightarrow L^2([0, R], \mathbb{C}^q)$$

is self-adjoint and any self-adjoint extension of $\mathcal{L}$ is of the form $\mathcal{L}_L$ for some Lagrangian subspace $L \subset \mathbb{C}^{2q}$. The fact that any Lagrangian subspace $L \subset \mathbb{C}^{2q}$ with respect to the standard symplectic form can be described by a system of equations

$$(3.4) \qquad L = \left\{ v \in \mathbb{C}^{2q} \mid \begin{pmatrix} \mathcal{A} & \mathcal{B} \end{pmatrix} v = 0 \right\} \subset \mathbb{C}^{2q},$$

where $\mathcal{A}$ and $\mathcal{B}$ are $q \times q$ matrices such that $\begin{pmatrix} \mathcal{A} & \mathcal{B} \end{pmatrix}$ has full rank and $\mathcal{A}\mathcal{B}^*$ is self-adjoint translates into the following result when the symplectic form ω is used.

Proposition 3.2. *The set in (3.4) is a Lagrangian subspace of $(\mathbb{C}^{2q}, \omega)$ if and only if the rank of $\begin{pmatrix} \mathcal{A} & \mathcal{B} \end{pmatrix}$ is q and $\mathcal{A}'\mathcal{B}^*$ is self-adjoint where $\mathcal{A}'$ is the matrix $\mathcal{A}$ with the first q_0 columns of $\mathcal{A}$ multiplied by -1.*

The following proposition concludes our summary of basically known results.

Proposition 3.3. *The self-adjoint extensions of Δ are in one-to-one correspondence with Lagrangian subspaces of $(\mathbb{C}^{2q}, \omega)$. More, precisely, self-adjoint extensions are of the form*

$$\Delta_L = \mathcal{L}_L \oplus \widetilde{\Delta},$$

where

$$\mathcal{L}_L := \mathcal{L} : \mathfrak{D}_L \to L^2([0, R], \mathbb{C}^q) , \quad \mathfrak{D}_L := \{\phi \in \mathfrak{D}_{\max} \mid \vec{\phi} \in L\}.$$

Here, $L \subset \mathbb{C}^{2q}$ is given by (3.4) where $\mathcal{A}$ and $\mathcal{B}$ are $q \times q$ matrices such that $\begin{pmatrix} \mathcal{A} & \mathcal{B} \end{pmatrix}$ has rank q and $\mathcal{A}'\mathcal{B}^*$ is self-adjoint.

4. PROOF OF THEOREM 2.3

In this section we prove Theorem 2.3 using the contour integration method [12, 13, 14, 73, 74, 75]. We begin by reducing our computation to the model operator.

4.1. **Reduction to the model problem.** From the results in Section 3 it is clear that the zeta function of Δ_L splits according to

$$(4.1) \qquad \zeta(s, \Delta_L) = \zeta_{\mathrm{reg}}(s, \Delta_L) + \zeta_{\mathrm{sing}}(s, \Delta_L),$$

where

$$\zeta_{\mathrm{reg}}(s, \Delta_L) := \zeta(s, \widetilde{\Delta}) \quad \text{and} \quad \zeta_{\mathrm{sing}}(s, \Delta_L) := \zeta(s, \mathcal{L}_L).$$

The properties of $\widetilde{\Delta}$, including the spectral functions, have been studied extensively, see for example [13, 35, 39, 43]. In particular, $\zeta_{\mathrm{reg}}(s, \Delta_L)$ has possible poles at the usual locations $s = \frac{n-k}{2}$ with $s \notin -\mathbb{N}_0$ for $k \in \mathbb{N}_0$ and at $s = 0$ if $\dim \Gamma > 0$. The residue of $\zeta_{\mathrm{reg}}(s, \Delta_L)$ at $s = 0$ is given by

$$c := \mathrm{Res}_{s=0} \zeta_{\mathrm{reg}}(s, \Delta_L) = -\frac{1}{2} \mathrm{Res}_{s=-\frac{1}{2}} \zeta(s, A_\Gamma).$$

In particular, this vanishes if $\zeta(s, A_\Gamma)$ is in fact analytic at $s = -\frac{1}{2}$. Furthermore, the determinant

$$\mathrm{det}_\zeta(\widetilde{\Delta}) := \exp\left(-\frac{d}{ds}\Big|_{s=0}\left\{\zeta(s, \widetilde{\Delta}) - \frac{c}{s}\right\}\right)$$

is thoroughly studied in [13]. The meromorphic structure of the singular function $\zeta_{\mathrm{sing}}(s, \Delta_L) := \zeta(s, \mathcal{L}_L)$ has the properties stated in Theorem 2.1, which was proved in [76]. In particular,

$$\zeta_0(s, \mathcal{L}_L) := \zeta(s, \mathcal{L}_L) - (j_0 - q_0)s \log s,$$

is differentiable at $s = 0$ and so

$$\mathrm{det}_\zeta(\mathcal{L}_L) := \exp\left(-\lim_{s_0 \to 0^+} \frac{d}{ds}\Big|_{s_0=0} \zeta_0(s, \mathcal{L}_L)\right)$$

is defined. Also, by (4.1), we have

$$\mathrm{det}_\zeta(\Delta_L) = \mathrm{det}_\zeta(\mathcal{L}_L) \cdot \mathrm{det}_\zeta(\widetilde{\Delta})$$

Therefore, we have reduced to computing $\mathrm{det}_\zeta(\mathcal{L}_L)$. We shall compute this in Proposition 4.5, but first we need to review some fundamental results from [76].

4.2. **Properties of the implicit eigenvalue equation.** In order to analyze $\mathrm{det}_\zeta(\mathcal{L}_L)$, we need to understand the behavior of the eigenvalue equation for $\mathcal{L}_L$. In order to write down the eigenvalue equation, we need some notation. Define the $q \times q$ matrices

$$J_+(\mu) := \begin{pmatrix} J_0(\mu R)\mathrm{Id}_{q_0} & 0 & \cdots & 0 \\ 0 & 2^{\nu_1}\Gamma(1+\nu_1)\mu^{-\nu_1}J_{\nu_1}(\mu R) & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & & 0 \\ 0 & 0 & \cdots & 2^{\nu_{q_1}}\Gamma(1+\nu_{q_1})\mu^{-\nu_{q_1}}J_{\nu_{q_1}}(\mu R) \end{pmatrix}$$

and

$$J_-(\mu) := \begin{pmatrix} \tilde{J}_0(\mu R)\mathrm{Id}_{q_0} & 0 & \cdots & 0 \\ 0 & 2^{-\nu_1}\Gamma(1-\nu_1)\mu^{\nu_1}J_{-\nu_1}(\mu R) & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & & 0 \\ 0 & 0 & \cdots & 2^{-\nu_q}\Gamma(1-\nu_q)\mu^{\nu_{q_1}}J_{-\nu_{q_1}}(\mu R) \end{pmatrix}$$

where $J_v(z)$ denotes the Bessel function of the first kind and

$$(4.2) \qquad \widetilde{J}_0(\mu r) := \frac{\pi}{2} Y_0(\mu r) - (\log \mu - \log 2 + \gamma) J_0(\mu r),$$

with $Y_0(z)$ the Bessel function of the second kind. Now we define

$$(4.3) \qquad F(\mu) := \det \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ J_+(\mu) & J_-(\mu) \end{pmatrix}.$$

Then $F(\mu)$ is an even function of μ. Indeed, to see this observe that, by definition, $F(\mu)$ is expressed in terms of $\mu^v J_{-v}(\mu R)$ with appropriate v's and the function $\widetilde{J}_0(\mu R)$. The following equation [1, p. 360]

$$(4.4) \qquad z^{-v} J_v(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{v+2k} k! \, \Gamma(v+k+1)}$$

shows that $\mu^{-v} J_v(\mu R)$ is even while the equality [1, p. 360]:

$$(4.5) \qquad \frac{\pi}{2} Y_0(z) = \left(\log z - \log 2 + \gamma \right) J_0(z) - \sum_{k=1}^{\infty} \frac{H_k(-\frac{1}{4} z^2)^k}{(k!)^2},$$

where $H_k := 1 + \frac{1}{2} + \cdots + \frac{1}{k}$, and the definition of $\widetilde{J}_0(\mu r)$ in (4.2) show that $\widetilde{J}_0(\mu R)$ is even.

The importance of $F(\mu)$ lies in the following Proposition.

Proposition 4.1. μ^2 *is an eigenvalue of* $\mathcal{L}_L$ *if and only if* $F(\mu) = 0$. *Moreover,*

$$F(0) = \det \begin{pmatrix} \mathcal{A} & & \mathcal{B} & \\ \mathrm{Id}_{q_0} & 0 & (\log R)\mathrm{Id}_{q_0} & 0 \\ 0 & \mathbf{R}^{\nu} & 0 & \mathbf{R}^{-\nu} \end{pmatrix},$$

where $\mathbf{R}^{\pm \nu}$ *are the* $q_1 \times q_1$ *diagonal matrices with entries* $R^{\pm \nu_\ell}$ *for* $1 \le \ell \le q_1$.

The first statement is straightforward to prove by solving the equation $(\mathcal{L}_L - \mu^2)\phi = 0$ for ϕ and using the fact that $L = \left\{ v \in \mathbb{C}^{2q} \,|\, \begin{pmatrix} \mathcal{A} & \mathcal{B} \end{pmatrix} v = 0 \right\}$ and that $\phi \in \mathfrak{D}_L$. The details are provided in Proposition 4.2 of [76]. The formula for $F(0)$ follows directly from Equations (4.2), (4.4) and (4.5).

The following lemma analyzes the asymptotics of $F(\mu)$ as $|\mu| \to \infty$ and is proved in Proposition 4.3 of [76].

Lemma 4.2. *Let* $\Upsilon \subset \mathbb{C}$ *be a sector (closed angle) in the right-half plane. Then we can write*

$$(4.6) \quad F(ix) = (2\pi R)^{-\frac{q}{2}} \prod_{j=1}^{q_1} 2^{-\nu_j} \Gamma(1-\nu_j) \, x^{|\nu|-\frac{q}{2}} \, e^{qxR} \left(\widetilde{\gamma} - \log x \right)^{q_0} \times$$

$$p\left(\left(\widetilde{\gamma} - \log x \right)^{-1}, x^{-1} \right) \left(1 + f(x) \right),$$

where $\widetilde{\gamma} = \log 2 - \gamma$, $p(x,y)$ *is the function in* (2.5), *and where as* $|x| \to \infty$ *with* $x \in \Upsilon$, $f(x)$ *is a power series in* x^{-1} *with no constant term.*

Using this lemma, we prove the following Proposition.

Proposition 4.3. *Let* $\Upsilon \subset \mathbb{C}$ *be a sector in the right-half plane. Then we can write*

$$(4.7) \qquad F(ix) = C x^{|\nu|-\frac{q}{2}-2\alpha_0} e^{qxR} (\widetilde{\gamma} - \log x)^{q_0 - j_0} \left(1 + G(x) \right),$$

where

$$(4.8) \qquad C = a_{j_0 \alpha_0} (2\pi R)^{-\frac{q}{2}} \prod_{j=1}^{q_1} 2^{-\nu_j} \Gamma(1 - \nu_j),$$

with $a_{j_0 \alpha_0}$ the coefficient in (2.6), and $G(x) = \mathcal{O}\left(\frac{1}{\log x}\right)$ and $G'(x) = \mathcal{O}\left(\frac{1}{x(\log x)^2}\right)$ as $|x| \to \infty$ with $x \in \Upsilon$.

Proof. Recall that α_0 is the smallest of all α's with $a_{j\alpha} \neq 0$ and j_0 is the smallest of all j's amongst the $a_{j\alpha_0} \neq 0$ in the expression

$$p(x, y) = \sum a_{j\alpha} \, x^j \, y^{2\alpha},$$

which is obtained by expanding the determinant in the definition of $p(x, y)$. Factoring out $a_{j_0 \alpha_0} x^{j_0} y^{2\alpha_0}$ in $p(x, y)$ we can write $p(x, y)$ in the form (see (2.6))

$$p(x, y) = a_{j_0 \alpha_0} \, x^{j_0} \, y^{2\alpha_0} \left(1 + \sum b_{k\beta} \, x^k \, y^{2\beta}\right),$$

where we may assume that all $b_{k\beta} \neq 0$. By definition of α_0, all the β's in this expression are nonnegative real numbers and the k's can be nonpositive or nonnegative integers except when $\beta = 0$, when the k's can only be positive by definition of j_0. Now observe that

$$(4.9) \qquad p\Big(\big(\widetilde{\gamma} - \log x\big)^{-1}, x^{-1}\Big) = a_{j_0 \alpha_0} \big(\widetilde{\gamma} - \log x\big)^{-j_0} x^{-2\alpha_0} \big(1 + g(x)\big),$$

where $g(x) = \sum b_{k\beta} \big(\widetilde{\gamma} - \log x\big)^{-k} x^{-2\beta}$. Notice that as $x \to \infty$,

$$\big(\widetilde{\gamma} - \log x\big)^{-k} = \mathcal{O}\left(\frac{1}{\log x}\right) \quad \text{for } k > 0,$$

and, because $\log x$ increases slower than any positive power of x,

$$\big(\widetilde{\gamma} - \log x\big)^{-k} x^{-2\beta} = \mathcal{O}\left(\frac{1}{\log x}\right) \quad \text{for } k \in \mathbb{Z} \text{ and } \beta > 0.$$

Therefore, $g(x) = \mathcal{O}\left(\frac{1}{\log x}\right)$. A similar argument shows that $g'(x) = \mathcal{O}\left(\frac{1}{x(\log x)^2}\right)$. Finally, replacing the formula (4.9) into the formula (4.6), we obtain

$$F(ix) \sim C x^{|\nu| - \frac{q}{2} - 2\alpha_0} e^{qxR} (\widetilde{\gamma} - \log x)^{q_0 - j_0} \Big(1 + g(x)\Big)\Big(1 + f(x)\Big)$$

$$\sim C x^{|\nu| - \frac{q}{2} - 2\alpha_0} e^{qxR} (\widetilde{\gamma} - \log x)^{q_0 - j_0} \Big(1 + G(x)\Big),$$

where C is given in (4.8) and $G(x) = f(x) + g(x) + f(x)\, g(x)$. The "big-$\mathcal{O}$" properties of $g(x)$ we discussed above and the fact that $f(x)$ is a power series in x^{-1} with no constant term shows that $G(x)$ has the desired properties. $\qquad\square$

4.3. **Computation of** $\det_\zeta(\mathcal{L}_L)$.

In order to facilitate the computation, we first need to establish the following

Lemma 4.4. *For any constants c and $|t|$ such that $\log |t| > c$, we have*

$$\int_{|t|}^{\infty} x^{-2s-1} \frac{1}{c - \log x} \, dx = e^{-2sc} \log s + e^{-2sc} \Big(\gamma + \log(2(\log|t| - c)) + \mathcal{O}(s)\Big),$$

where $\mathcal{O}(s)$ is an entire function of s that is $\mathcal{O}(s)$ at $s = 0$.

Proof. To analyze this integral we make the change of variables $u = \log x - c$ or $x = e^c e^u$, and obtain

$$\int_{|t|}^{\infty} x^{-2s-1} \frac{1}{c - \log x} \, dx = -e^{-2sc} \int_{\log|t|-c}^{\infty} e^{-2su} \frac{du}{u}.$$

Making the change of variables $y = 2su$, we get

$$\int_{|t|}^{\infty} x^{-2s-1} \frac{1}{c - \log x} \, dx = -e^{-2sc} \int_{2s(\log|t|-c)}^{\infty} e^{-y} \frac{dy}{y}$$
$$= e^{-2sc} \text{Ei}\big(-2s(\log|t| - c)\big),$$

where $\text{Ei}(z) := -\int_{-z}^{\infty} e^{-y} \frac{dy}{y}$ is the *exponential integral* (see [1, Ch. 5] or [68, Sec. 8.2]). From [68, p. 877], we have

$$\text{Ei}(z) = \gamma + \log(-z) + \sum_{k=1}^{\infty} \frac{z^k}{k \cdot k!},$$

therefore

$$\int_{|t|}^{\infty} x^{-2s-1} \frac{1}{c - \log x} \, dx = e^{-2sc} \Big(\gamma + \log(2s(\log|t| - c)) + \mathcal{O}(s)\Big)$$
$$= e^{-2sc} \log s + e^{-2sc} \Big(\gamma + \log(2(\log|t| - c)) + \mathcal{O}(s)\Big),$$

where $\mathcal{O}(s)$ is an entire function of s that is $\mathcal{O}(s)$ at $s = 0$. $\qquad\square$

We now compute $\det_\zeta(\mathcal{L}_L)$ explicitly.

Proposition 4.5. *If* $\ker \mathcal{L}_L = \{0\}$,

$$\det_\zeta(\mathcal{L}_L) = \frac{(2\pi R)^{\frac{q}{2}}}{a_{j_0 \alpha_0}} \prod_{j=1}^{q_1} \frac{2^{\nu_j}}{\Gamma(1 - \nu_j)} (-2e^\gamma)^{q_0 - j_0} \times$$

$$\det \begin{pmatrix} \overset{\mathcal{A}}{\overbrace{\text{Id}_{q_0} \quad 0}} & \overset{\mathcal{B}}{\overbrace{(\log R)\text{Id}_{q_0} \quad 0}} \\ 0 \quad \mathbf{R}^\nu & 0 \quad \mathbf{R}^{-\nu} \end{pmatrix}.$$

Proof. First, applying the Argument Principle (which is really a form of Cauchy's formula) [41, p. 123], the ζ-function of $\mathcal{L}_L$ is given by

$$\zeta(s, \mathcal{L}_L) = \frac{1}{2\pi i} \int_\gamma \mu^{-2s} \frac{d}{d\mu} \log F(\mu) d\mu = \frac{1}{2\pi i} \int_\gamma \mu^{-2s} \frac{F'(\mu)}{F(\mu)} d\mu,$$

where γ is a contour in the plane shown in Figure 4.3. Breaking up our integral into three parts, one from t to $i\infty$, another from $-i\infty$ to $-t$, and then another over
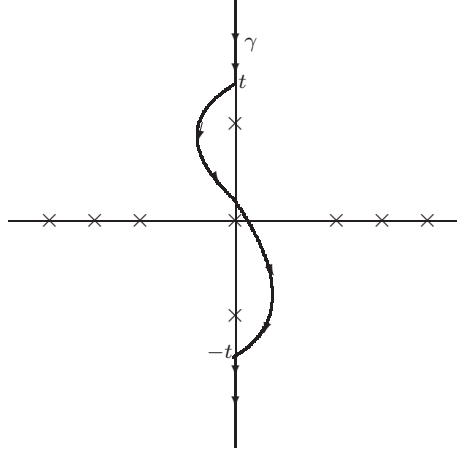
FIGURE 1. The contour γ for the zeta function. The $\times$'s represent the zeros of $F(\mu)$ and squaring these $\times$'s are the eigenvalues of $\mathcal{L}_L$. Here, t is on the imaginary axis and $|t|^2$ is larger than the largest absolute value of a negative eigenvalue of $\mathcal{L}_L$ (if it has any). The contour γ_t goes from t to $-t$.

γ_t, which is the part of γ from t to $-t$, we obtain

$$\zeta(s, \mathcal{L}_L) = \frac{1}{2\pi i} \int_\gamma \mu^{-2s} \frac{d}{d\mu} \log F(\mu) \, d\mu$$

$$= \frac{1}{2\pi i} \left\{ - \int_{|t|}^\infty (ix)^{-2s} \frac{d}{dx} \log F(ix) \, dx + \int_{|t|}^\infty (-ix)^{-2s} \frac{d}{dx} \log F(-ix) \, dx \right\}$$

$$+ \frac{1}{2\pi i} \int_{\gamma_t} \mu^{-2s} \frac{F'(\mu)}{F(\mu)} \, d\mu$$

$$= \frac{1}{2\pi i} \left(- e^{-i\pi s} + e^{i\pi s} \right) \int_{|t|}^\infty x^{-2s} \frac{d}{dx} \log F(ix) \, dx + \frac{1}{2\pi i} \int_{\gamma_t} \mu^{-2s} \frac{F'(\mu)}{F(\mu)} \, d\mu,$$

or,

$$(4.10) \qquad \zeta(s, \mathcal{L}_L) = \frac{\sin \pi s}{\pi} \int_{|t|}^\infty x^{-2s} \frac{d}{dx} \log F(ix) \, dx + \frac{1}{2\pi i} \int_{\gamma_t} \mu^{-2s} \frac{F'(\mu)}{F(\mu)} \, d\mu.$$

The first step to compute $\det_\zeta(\mathcal{L}_L)$ is to construct the analytical continuation of the first integral in (4.10) to $s = 0$; the second term (being entire since it is an integral over a finite contour) is already regular at $s = 0$. To do so, recall Proposition 4.3 (see (4.7)), which states that we can write

$$F(ix) = C x^{|\nu| - \frac{q}{2} - 2\alpha_0} e^{qxR} (\widetilde{\gamma} - \log x)^{q_0 - j_0} \left(1 + G(x) \right),$$

where

$$C = a_{j_0 \alpha_0} (2\pi R)^{-\frac{q}{2}} \prod_{j=1}^{q_1} 2^{-\nu_j} \Gamma(1 - \nu_j),$$

and where $G(x) = \mathcal{O}\left(\frac{1}{\log x}\right)$ and $G'(x) = \mathcal{O}\left(\frac{1}{x(\log x)^2}\right)$ as $|x| \to \infty$. Hence,

$$\int_{|t|}^{\infty} x^{-2s} \frac{d}{dx} \log F(ix)\, dx = \int_{|t|}^{\infty} x^{-2s} \frac{d}{dx} \log\left(1 + G(x)\right) dx$$

$$+ \int_{|t|}^{\infty} x^{-2s} \frac{d}{dx} \log\left(Cx^{|\nu| - \frac{q}{2} - 2\alpha_0} e^{qxR}(\widetilde{\gamma} - \log x)^{q_0 - j_0}\right) dx.$$

The second integral can be computed explicitly:

$$\int_{|t|}^{\infty} x^{-2s} \frac{d}{dx} \log\left(x^{|\nu| - \frac{q}{2} - 2\alpha_0} e^{qxR}(\widetilde{\gamma} - \log x)^{q_0 - j_0}\right) dx$$

$$= \int_{|t|}^{\infty} x^{-2s} \left(\frac{|\nu| - \frac{q}{2} - 2\alpha_0}{x} + qR - \frac{(q_0 - j_0)}{x(\widetilde{\gamma} - \log x)}\right) dx$$

$$= \left(|\nu| - \frac{q}{2} - 2\alpha_0\right) \frac{|t|^{-2s}}{2s} + qR \frac{|t|^{-2s+1}}{2s - 1} + (j_0 - q_0) \int_{|t|}^{\infty} x^{-2s-1} \frac{1}{\widetilde{\gamma} - \log x}\, dx.$$

From Lemma 4.4 we know that

$$\int_{|t|}^{\infty} x^{-2s-1} \frac{1}{\widetilde{\gamma} - \log x}\, dx = e^{-2s\widetilde{\gamma}} \log s + g(s),$$

where $g(s)$ is entire such that

(4.11)
$$g(0) = \gamma + \log(2(\log|t| - \widetilde{\gamma})).$$

Therefore,

$$\zeta(s, \mathcal{L}_L) = \frac{\sin \pi s}{\pi} \left(|\nu| - \frac{q}{2} - 2\alpha_0\right) \frac{|t|^{-2s}}{2s} + \frac{\sin \pi s}{\pi} qR \frac{|t|^{-2s+1}}{2s - 1}$$

$$+ \frac{\sin \pi s}{\pi}(j_0 - q_0) e^{-2s\widetilde{\gamma}} \log s + \frac{\sin \pi s}{\pi}(j_0 - q_0) g(s)$$

$$+ \frac{\sin \pi s}{\pi} \int_{|t|}^{\infty} x^{-2s} \frac{d}{dx} \log\left(1 + G(x)\right) dx + \frac{1}{2\pi i} \int_{\gamma_t} \mu^{-2s} \frac{F'(\mu)}{F(\mu)}\, d\mu.$$

Since

$$\frac{\sin \pi s}{\pi}(j_0 - q_0) e^{-2s\widetilde{\gamma}} \log s \equiv (j_0 - q_0) s \log s$$

modulo a function that is $\mathcal{O}(s^2 \log s)$, it follows that

(4.12) $$\zeta_0(s, \mathcal{L}_L) = \zeta(s, \mathcal{L}_L) - (j_0 - q_0) s \log s$$

$$\equiv \frac{\sin \pi s}{\pi} \left(|\nu| - \frac{q}{2} - 2\alpha_0\right) \frac{|t|^{-2s}}{2s} + \frac{\sin \pi s}{\pi} qR \frac{|t|^{-2s+1}}{2s - 1}$$

$$+ \frac{\sin \pi s}{\pi}(j_0 - q_0) g(s) + \frac{\sin \pi s}{\pi} \int_{|t|}^{\infty} x^{-2s} \frac{d}{dx} \log\left(1 + G(x)\right) dx$$

$$+ \frac{1}{2\pi i} \int_{\gamma_t} \mu^{-2s} \frac{F'(\mu)}{F(\mu)}\, d\mu$$

modulo a function that is $\mathcal{O}(s^2 \log s)$. The derivative of the fourth term on the right in (4.12) is equal to

$$(4.13) \quad \cos \pi s \int_{|t|}^{\infty} x^{-2s} \frac{d}{dx} \log \left(1 + G(x)\right) dx$$
$$- \frac{2 \sin \pi s}{\pi} \int_{|t|}^{\infty} x^{-2s} (\log x) \frac{d}{dx} \log \left(1 + G(x)\right) dx.$$

Since $G(x) = \mathcal{O}\left(\frac{1}{\log x}\right)$ and $G'(x) = \mathcal{O}\left(\frac{1}{x(\log x)^2}\right)$ as $|x| \to \infty$ we can put $s = 0$ into the first term in (4.13) and get

$$\int_{|t|}^{\infty} \frac{d}{dx} \log \left(1 + G(x)\right) dx = - \log \left(1 + G(|t|)\right).$$

Also using the asymptotics of $G(x)$ and $G'(x)$, we see that the second term in (4.13) satisfies, for $s \in \mathbb{R}$ with $s \to 0^+$,

$$\frac{2 \sin \pi s}{\pi} \int_{|t|}^{\infty} x^{-2s} (\log x) \frac{d}{dx} \log \left(1 + G(x)\right) dx = \mathcal{O}\left(s \int_{|t|}^{\infty} x^{-2s} \frac{(\log x)}{x(\log x)^2} dx\right)$$
$$= \mathcal{O}\left(s \int_{|t|}^{\infty} x^{-2s} \frac{1}{x(\log x)} dx\right) = \mathcal{O}(s \log s),$$
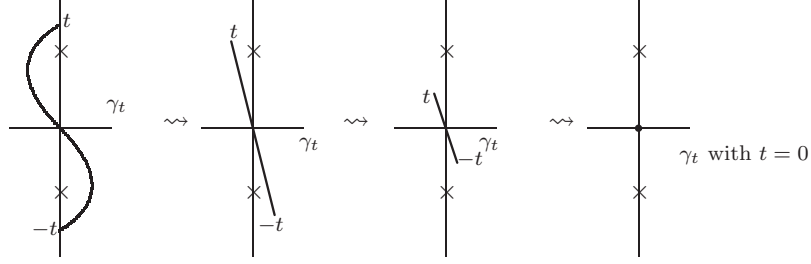
where we used Lemma 4.4 with $c = 0$. In conclusion,

$$\lim_{s \to 0^+} \frac{d}{ds} \left\{ \frac{\sin \pi s}{\pi} \int_{|t|}^{\infty} x^{-2s} \frac{d}{dx} \log \left(1 + G(x)\right) dx \right\} = - \log \left(1 + G(|t|)\right).$$

Now, using that

$$\frac{\sin(\pi s)}{\pi} \Big|_{s=0} = 0 \ , \ \frac{d}{ds} \frac{\sin(\pi s)}{\pi} \Big|_{s=0} = 1 \ , \ \frac{\sin(\pi s)}{\pi s} \Big|_{s=0} = 1 \ , \ \frac{d}{ds} \frac{\sin(\pi s)}{\pi s} \Big|_{s=0} = 0,$$

and the formula (4.11) for $g(0)$, we can take the derivatives of the other terms in (4.12) and set $s = 0$ to conclude that

$$\lim_{s \to 0^+} \zeta_0'(s, \mathcal{L}_L) = - \left(|\nu| - \frac{q}{2} - 2\alpha_0\right) \log |t| - qR|t| + (j_0 - q_0)g(0)$$
$$- \log \left(1 + G(|t|)\right) - \frac{1}{\pi i} \int_{\gamma_t} \log \mu \frac{F'(\mu)}{F(\mu)} d\mu$$
$$= - \left(|\nu| - \frac{q}{2} - 2\alpha_0\right) \log |t| - qR|t| + (j_0 - q_0)\left(\gamma + \log(2(\log |t| - \widetilde{\gamma}))\right)$$
$$- \log \left(1 + G(|t|)\right) - \frac{1}{\pi i} \int_{\gamma_t} \log \mu \frac{F'(\mu)}{F(\mu)} d\mu.$$

FIGURE 2. The contour γ_t as we let $t \to 0$ in $\mathscr{D}$ from the upper half plane.

By definition of $G(x)$, we have

$$\log\Big(1 + G(|t|)\Big) = \log\Big(\frac{F(i|t|)}{C|t|^{|\nu|-\frac{q}{2}-2\alpha_0}e^{q|t|R}(\widetilde{\gamma} - \log|t|)^{q_0-j_0}}\Big)$$

$$= \log\Big(\frac{F(i|t|)}{C(-1)^{q_0-j_0}}\Big) - \log\Big(|t|^{|\nu|-\frac{q}{2}-2\alpha_0}e^{q|t|R}(\log|t| - \widetilde{\gamma})^{q_0-j_0}\Big)$$

$$= \log\Big(\frac{F(i|t|)}{C(-1)^{q_0-j_0}}\Big) - \Big(|\nu| - \frac{q}{2} - 2\alpha_0\Big)\log|t| - qR|t|$$

$$+ (j_0 - q_0)\log(\log|t| - \widetilde{\gamma}).$$

Replacing this expression into the preceding expression for $\lim_{s\to 0^+}\zeta'_0(s, \mathcal{L}_L)$, cancelling appropriate terms, and using that $F(i|t|) = F(t)$ since $t = i|t|$, we obtain

$$\lim_{s\to 0^+}\zeta'_0(s, \mathcal{L}_L) = -\log\Big(\frac{F(t)}{C(-1)^{q_0-j_0}}\Big) + (j_0 - q_0)\Big(\gamma + \log 2\Big) - \frac{1}{\pi i}\int_{\gamma_t}\log\mu\frac{F'(\mu)}{F(\mu)}\,d\mu$$

$$= -\log\Big((-1)^{q_0-j_0}2^{q_0-j_0}e^{(q_0-j_0)\gamma}\frac{F(t)}{C}\Big) - \frac{1}{\pi i}\int_{\gamma_t}\log\mu\frac{F'(\mu)}{F(\mu)}\,d\mu.$$

Therefore,

$$(4.14) \quad \det_\zeta(\mathcal{L}_L) = (-1)^{q_0-j_0}2^{q_0-j_0}e^{(q_0-j_0)\gamma}\frac{F(t)}{C}\cdot\exp\Big(\frac{1}{\pi i}\int_{\gamma_t}\log\mu\frac{F'(\mu)}{F(\mu)}\,d\mu\Big).$$

This formula is derived, a priori, when t is on the upper half part of the imaginary axis. However, the right-hand side is a *holomorphic* function of $t \in \mathscr{D}$, where $\mathscr{D}$ is the set of complex numbers minus the negative real axis and the zeros of $F(\mu)$. Therefore (4.14) holds for all $t \in \mathscr{D}$. Note that this equality holds in general even if $\mathcal{L}_L$ has a nontrivial kernel. But to control the factor $\exp(\frac{1}{\pi i}\int_{\gamma_t}\cdot d\mu)$, we need the condition that $\ker \mathcal{L}_L = \{0\}$. Under this condition, recalling that γ_t is any curve in $\mathscr{D}$ from t to $-t$, the trick now is to let $t \to 0$ in (4.14), that is, taking $t \to 0$ in $\mathscr{D}$ from the upper half plane as shown in Figure 2, it follows that

$$\exp\Big(\frac{1}{\pi i}\int_{\gamma_t}\log\mu\frac{F'(\mu)}{F(\mu)}\,d\mu\Big) \to \exp\Big(0\Big) = 1.$$

We also have

$$F(0) = \det\begin{pmatrix} \overset{\mathcal{A}}{} & \overset{\mathcal{B}}{} \\ \begin{matrix} \mathrm{Id}_{q_0} & 0 \\ 0 & \mathbf{R}^\nu \end{matrix} & \begin{matrix} (\log R)\mathrm{Id}_{q_0} & 0 \\ 0 & \mathbf{R}^{-\nu} \end{matrix} \end{pmatrix}$$

from Proposition 4.1.

In conclusion, taking $t \to 0$ on the right side of (4.14), we see that

$$\det_\zeta(\mathcal{L}_L) = \frac{(-2e^\gamma)^{q_0-j_0}}{C} \det \begin{pmatrix} \overset{\mathcal{A}}{} & & \overset{\mathcal{B}}{} & \\ \mathrm{Id}_{q_0} & 0 & (\log R)\mathrm{Id}_{q_0} & 0 \\ 0 & \mathbf{R}^\nu & 0 & \mathbf{R}^{-\nu} \end{pmatrix}.$$

Finally, using that $C = a_{j_0\alpha_0}(2\pi R)^{-\frac{q}{2}} \prod_{j=1}^{q_1} 2^{-\nu_j}\Gamma(1-\nu_j)$, we get

$$(4.15) \quad \det_\zeta(\mathcal{L}_L) = \frac{(2\pi R)^{\frac{q}{2}}}{a_{j_0\alpha_0}} \prod_{j=1}^{q_1} \frac{2^{\nu_j}}{\Gamma(1-\nu_j)}(-2e^\gamma)^{q_0-j_0} \times$$

$$\det \begin{pmatrix} \overset{\mathcal{A}}{} & & \overset{\mathcal{B}}{} & \\ \mathrm{Id}_{q_0} & 0 & (\log R)\mathrm{Id}_{q_0} & 0 \\ 0 & \mathbf{R}^\nu & 0 & \mathbf{R}^{-\nu} \end{pmatrix}.$$

This completes the proof of Proposition 4.5. $\square$

Remark 4.6. In the case that $\mathcal{L}_L$ is not invertible, $F(t) \to F(0) = 0$ as $t \to 0$ since 0 is an eigenvalue of $\mathcal{L}_L$. On the other hand, the left side $\det_\zeta(\mathcal{L}_L)$ does not depend on t. This means that the factor $\exp(\frac{1}{\pi i}\int_{\gamma_t} \cdot d\mu)$ blows up as $t \to 0$. (Here γ_t should not contain the zero as in Figure 4.3.) Therefore, to get the value of $\det_\zeta(\mathcal{L}_L)$, we need to know the exact form of the asymptotics of $F(t)$ and $\exp(\frac{1}{\pi i}\int_{\gamma_t} \cdot d\mu)$ as $t \to 0$.

Recall that the *Neumann extension* is given by choosing $\mathcal{A}$ and $\mathcal{B}$ to be the diagonal matrices with the $q_0 + 1, \ldots, q$ entries in $\mathcal{A}$ equal to 1 and the $1, \ldots, q_0$ entries in $\mathcal{B}$ equal to 1 with the rest of the entries 0. Then the resulting operator $\mathcal{L}_\mathcal{N}$ has the trivial kernel. This can be shown as follows: First, by the simple form of $\mathcal{A}, \mathcal{B}$, we may assume that $q_0 = 1, q_1 = 0$ or $q_0 = 0, q_1 = 1$. For the first case, the solution of $\mathcal{L}_L\phi = 0$ should have the form $\phi = c_1 r^{\frac{1}{2}}$ if it exists since the term $r^{\frac{1}{2}}\log r$ should vanish by the condition of $\mathcal{A}, \mathcal{B}$ at $r = 0$. But, the Dirichlet condition at $r = R$ implies that $\phi = c_1 r^{\frac{1}{2}}$ can not be the solution of $\mathcal{L}_L$ either. The second case can be treated in a similar way. Now we have

Corollary 4.7. *The following equality holds*

$$\det_\zeta(\mathcal{L}_\mathcal{N}) = (2\pi R)^{\frac{q}{2}} \prod_{j=1}^{q_1} \frac{2^{\nu_j} R^{-\nu_j}}{\Gamma(1-\nu_j)}.$$

Proof. This proof is just a direct application of the formula (4.15). Observe that for $\mathcal{A}$ and $\mathcal{B}$ defining the Neumann extension,

$$p(x,y) := \det \begin{pmatrix} \overset{\mathcal{A}}{} & & & & \overset{\mathcal{B}}{} \\ x\,\mathrm{Id}_{q_0} & 0 & 0 & 0 & \\ 0 & \tau_1 y^{2\nu_1} & 0 & 0 & \\ & & & & \mathrm{Id}_q \\ 0 & 0 & \ddots & 0 & \\ 0 & 0 & 0 & \tau_{q_1} y^{2\nu_{q_1}} & \end{pmatrix}$$

$$= \det \begin{pmatrix} 0 & \mathrm{Id}_{q_0} \\ x\,\mathrm{Id}_{q_0} & \mathrm{Id}_{q_0} \end{pmatrix} \cdot \det \begin{pmatrix} \mathrm{Id}_{q_1} & & & 0 \\ \tau_1 y^{2\nu_1} & 0 & 0 & \\ 0 & \ddots & 0 & \mathrm{Id}_{q_1} \\ 0 & 0 & \tau_{q_1} y^{2\nu_{q_1}} & \end{pmatrix}$$

$$= (-1)^{q_0} x^{q_0}.$$

Therefore, $j_0 = q_0$, $\alpha_0 = 0$, and $a_{j_0\alpha_0} = (-1)^{q_0}$ for the Neumann extension. In the same way we simplified $p(x, y)$, we can simplify

$$\det \begin{pmatrix} \overset{\mathcal{A}}{\begin{matrix} \mathrm{Id}_{q_0} & 0 \\ 0 & \mathbf{R}^{\nu} \end{matrix}} & \overset{\mathcal{B}}{\begin{matrix} (\log R)\mathrm{Id}_{q_0} & 0 \\ 0 & \mathbf{R}^{-\nu} \end{matrix}} \end{pmatrix}$$

$$= \det \begin{pmatrix} 0 & \mathrm{Id}_{q_0} \\ \mathrm{Id}_{q_0} & (\log R)\mathrm{Id}_{q_0} \end{pmatrix} \cdot \det \begin{pmatrix} \mathrm{Id}_{q_1} & 0 \\ \mathbf{R}^{\nu} & \mathbf{R}^{-\nu} \end{pmatrix}$$

$$= (-1)^{q_0} \prod_{j=1}^{q_1} R^{-\nu_j}.$$

Therefore, by (4.15), we have

$$\det_\zeta(\mathcal{L}_\mathcal{N}) = \frac{(2\pi R)^{\frac{q}{2}}}{(-1)^{q_0}} \prod_{j=1}^{q_1} \frac{2^{\nu_j}}{\Gamma(1-\nu_j)} (-2e^\gamma)^0 (-1)^{q_0} \prod_{j=1}^{q_1} R^{-\nu_j} = (2\pi R)^{\frac{q}{2}} \prod_{j=1}^{q_1} \frac{2^{\nu_j} R^{-\nu_j}}{\Gamma(1-\nu_j)}.$$

$\square$

This corollary agrees with the result in [85]. In particular, for an extension L with $\ker \mathcal{L}_L = \{0\}$, we have

$$\frac{\det_\zeta(\mathcal{L}_L)}{\det_\zeta(\mathcal{L}_\mathcal{N})} = \frac{(-2e^\gamma)^{q_0-j_0}}{a_{j_0\alpha_0}} \det \begin{pmatrix} \overset{\mathcal{A}}{\begin{matrix} \mathrm{Id}_{q_0} & 0 \\ 0 & \mathbf{R}^{2\nu} \end{matrix}} & \overset{\mathcal{B}}{\begin{matrix} (\log R)\mathrm{Id}_{q_0} & 0 \\ 0 & \mathrm{Id}_{q_1} \end{matrix}} \end{pmatrix},$$

and this formula completes the proof of Theorem 2.3.

5. SPECIAL CASES OF THEOREM 2.3

In this section we derive various consequences of Theorem 2.3.

5.1. **Row and column conditions.** We begin by proving Theorem 2.4. Actually, the proof of Theorem 2.4 follows directly from Theorem 2.3 and the following lemma, which computes $a_{j_0\alpha_0}$ in (2.6) explicitly under the row and columns condition of Theorem 2.4.

Lemma 5.1. *Let $q - r = \mathrm{rank}(\mathcal{A})$ and assume that $\mathcal{A}$ has r rows and columns identically zero. Let $i_1, ..., i_q$ be a permutation of the numbers $1, ..., q$ such that the rows and columns $i_1, ..., i_r$ of $\mathcal{A}$ are zero. Choose $j_0 \in \{0, 1, ..., r\}$ such that*

$$1 \le i_1 < i_2 < \cdots < i_{j_0} \le q_0 < i_{j_0+1} < \cdots < i_r \le q.$$

Let I_r denote the $q \times q$ matrix which is zero everywhere except along the diagonal where the entries $i_1, ..., i_r$ equal 1, and let I_{q-r} denote the $q \times q$ matrix which is zero everywhere except along the diagonal where the entries $i_{r+1}, ..., i_q$ equal 1. Then $\det \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ I_r & I_{q-r} \end{pmatrix} \ne 0$ and

$$p(x, y) = a_{j_0,\alpha_0} x^{j_0} y^{2\alpha_0} \big(1 + \mathcal{O}(|(x, y)|)\big)$$

where

$$a_{j_0\alpha_0} = \prod_{j=j_0+1}^{r} 2^{2\nu_{i_j}} \frac{\Gamma(1+\nu_{i_j})}{\Gamma(1-\nu_{i_j})} \cdot \det \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ I_r & I_{q-r} \end{pmatrix}$$

and $\alpha_0 = \nu_{i_{j_0+1}} + \nu_{i_{j_0+2}} \cdots + \nu_{i_r}$.

Proof. Assume for the moment that $j_0 \geq 1$. Let $\mathcal{A}_1$ denote the matrix $\mathcal{A}$ with the i_1-th column removed, let $J_1(x, y)$ denote the matrix

$$(5.1) \qquad \begin{pmatrix} x\,\mathrm{Id}_{q_0} & 0 & 0 & 0 \\ 0 & \tau_1\,y^{2\nu_1} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \tau_{q_1}\,y^{2\nu_{q_1}} \end{pmatrix}$$

with the i_1 column and row removed, and finally, let $\mathcal{C}_1$ denote the $q_1 \times q_1$ identity matrix with the i_1-th row removed. Then expanding the determinant of the matrix in the definition of $p(x, y)$:

$$p(x,y) := \det \begin{pmatrix} \overset{\mathcal{A}}{} & & & & \overset{\mathcal{B}}{} \\ x\,\mathrm{Id}_{q_0} & 0 & 0 & 0 & \\ 0 & \tau_1\,y^{2\nu_1} & 0 & 0 & \mathrm{Id}_q \\ 0 & 0 & \ddots & 0 & \\ 0 & 0 & 0 & \tau_{q_1}\,y^{2\nu_{q_1}} & \end{pmatrix},$$

about the i_1-th column, recalling that the i_1-th column of $\mathcal{A}$ is zero, we get

$$(5.2) \qquad p(x,y) = \pm x \det \begin{pmatrix} \mathcal{A}_1 & \mathcal{B} \\ J_1(x,y) & \mathcal{C}_1 \end{pmatrix}$$

(for an appropriate choice of sign, which happens to equal $(-1)^{2i_1+q}$ in this case). Assume for the moment that $j_0 \geq 2$. Let $\mathcal{A}_2$ denote the matrix $\mathcal{A}$ with the i_1 and i_2 columns removed, let $J_2(x, y)$ denote the matrix (5.1) with the i_1 and i_2 columns and rows removed, and finally, let $\mathcal{C}_2$ denote the $q_1 \times q_1$ identity matrix with the i_1 and i_2 rows removed. Then expanding the determinant of the matrix in (5.2) about the column containing the zero i_2-th column of $\mathcal{A}$, we get

$$(5.3) \qquad p(x,y) = \pm x^2 \det \begin{pmatrix} \mathcal{A}_2 & \mathcal{B} \\ J_2(x,y) & \mathcal{C}_2 \end{pmatrix}.$$

At this point, we see the general pattern: We expand the determinant in (5.3) about the column containing the zero i_3-th column of $\mathcal{A}$ and then we continue the process of expanding about each column containing the zero $i_4, i_5, i_6, \ldots, i_r$ columns of $\mathcal{A}$. At the end, we arrive at

$$(5.4) \qquad p(x,y) = \pm \tilde{\tau}\, x^{j_0} y^{2\tilde{\nu}} \det \begin{pmatrix} \mathcal{A}_r & \mathcal{B} \\ J_r(x,y) & \mathcal{C}_r \end{pmatrix},$$

where $\mathcal{A}_r$ denotes the matrix $\mathcal{A}$ with the $i_1, \ldots, i_r$ columns removed, $J_r(x, y)$ denotes the matrix (5.1) with the $i_1, \ldots, i_r$ columns and rows removed, and $\mathcal{C}_r$ denotes the $q \times q$ identity matrix with the $i_1, \ldots, i_r$ rows removed.

Now observe that

$$\pm \det \begin{pmatrix} \mathcal{A}_r & \mathcal{B} \\ J_r(0,0) & \mathcal{C}_r \end{pmatrix} = \pm \det \begin{pmatrix} \mathcal{A}_r & \mathcal{B} \\ 0 & \mathcal{C}_r \end{pmatrix} = \det \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ I_r & I_{q-r} \end{pmatrix};$$

indeed, the first equality is obvious because $J_r(0, 0)$ is the zero matrix while the second equality can be easily verified by expanding the determinant $\det \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ I_r & I_{q-r} \end{pmatrix}$ about the zero $i_1, i_2, \ldots, i_r$ columns of $\mathcal{A}$ just as we did in the previous paragraph.

It remains to prove that $\det \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ I_r & I_{q-r} \end{pmatrix} \neq 0$. To see this, recall that the $i_1, \ldots, i_r$ rows of $\mathcal{A}$ are identically zero. This implies that, since the rank of $\mathcal{A}$ is $q-r$,

the rows of $\mathcal{A}$ complementary to $i_1, \ldots, i_r$, namely the $i_{r+1}, \ldots, i_q$ rows where we use the notation as in the statement of this lemma, are linearly independent. Therefore, since the matrix $\begin{pmatrix} \mathcal{A} & \mathcal{B} \end{pmatrix}$ has rank q, the $i_1, \ldots, i_r$ rows of $\mathcal{A}$ are identically zero, and the $i_{r+1}, \ldots, i_q$ rows of $\mathcal{A}$ are linearly independent, it follows that the $i_1, \ldots, i_r$ rows of $\mathcal{B}$ are linearly independent and these rows, together with the $i_{r+1}, \ldots, i_q$ rows of $\mathcal{A}$ span all of $\mathbb{C}^q$. Now recall that the $i_1, \ldots, i_r$ columns of $\mathcal{A}$ are identically zero; in particular, the span of the $i_{r+1}, \ldots, i_q$ rows of $\mathcal{A}$ does not contain any $e_{i_1}, \ldots, e_{i_r}$, where e_j denote the unit vector in $\mathbb{C}^q$ with j-th slot equal to 1 and 0's elsewhere. It follows that the span of the $i_{r+1}, \ldots, i_q$ rows of $\mathcal{A}$ (which are linearly independent) is contained in the span of $e_{i_{r+1}}, \ldots, e_{i_q}$. Therefore, by the property of dimension,

$$(5.5) \qquad \text{the span of the } i_{r+1}, \ldots, i_q \text{ rows of } \mathcal{A} = \text{the span of } e_{i_{r+1}}, \ldots, e_{i_q}.$$

Hence, as the $i_1, \ldots, i_r$ rows of $\mathcal{B}$ plus the $i_{r+1}, \ldots, i_q$ rows of $\mathcal{A}$ span all of $\mathbb{C}^q$, it follows that

$$(5.6) \qquad \text{the span of the } i_1, \ldots, i_r \text{ rows of } \mathcal{B} = \text{the span of } e_{i_1}, \ldots, e_{i_r}.$$

We are now ready to prove our lemma. The nonzero rows of

$$\begin{pmatrix} \mathcal{A} \\ I_r \end{pmatrix}$$

are linearly independent by (5.5). The rows in the matrix

$$\begin{pmatrix} \mathcal{B} \\ I_{q-r} \end{pmatrix}$$

that are complementary to the nonzero rows of $\begin{pmatrix} \mathcal{A} \\ I_r \end{pmatrix}$ are therefore linearly independent by (5.6). It follows that the matrix $\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ I_r & I_{q-r} \end{pmatrix}$ has full rank, which is equivalent to $\det \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ I_r & I_{q-r} \end{pmatrix} \neq 0$. Now the formula of $a_{j_0 \alpha_0}$ follows from (2.6) and (5.4). This completes the proof. $\square$

5.2. **Decomposable Lagrangians.** Because the $-\frac{1}{4}$ eigenvalues and the eigenvalues in $(-\frac{1}{4}, \frac{3}{4})$ of A_Γ result in rather different analytic properties, it is natural to separate these eigenvalues. With this discussion in mind, we shall call a Lagrangian subspace $L \subset V$ *decomposable* if $L = L_0 \oplus L_1$ where L_0 is a Lagrangian subspace of $\bigoplus_{\lambda_\ell = -\frac{1}{4}} E_\ell \oplus E_\ell$ and L_1 is a Lagrangian subspace of $\bigoplus_{-\frac{1}{4} < \lambda_\ell < \frac{3}{4}} E_\ell \oplus E_\ell$. As described in Proposition 3.2, the Lagrangian subspace L_0 is determined by two $q_0 \times q_0$ matrices $\mathcal{A}_0, \mathcal{B}_0$ where $q_0 = \dim L_0$, that is, the multiplicity of the eigenvalues $\lambda_\ell = -\frac{1}{4}$. Similarly, the Lagrangian subspace L_1 is determined by two $q_1 \times q_1$ matrices $\mathcal{A}_1, \mathcal{B}_1$ where $q_1 = \dim L_1$, that is, the multiplicity of the eigenvalues λ_ℓ

with $-\frac{1}{4} < \lambda_\ell < \frac{3}{4}$. Thus, the function $p(x, y)$ in (2.5) takes the form

$$p(x, y) := \det \begin{pmatrix} & & \mathcal{A}_0 & 0 & & & \mathcal{B}_0 & 0 \\ & & 0 & \mathcal{A}_1 & & & 0 & \mathcal{B}_1 \\ x\,\mathrm{Id}_{q_0} & 0 & 0 & 0 & & & & \\ 0 & \tau_1\,y^{2\nu_1} & 0 & 0 & & & \mathrm{Id}_q & \\ 0 & 0 & \ddots & 0 & & & & \\ 0 & 0 & 0 & \tau_{q_1}\,y^{2\nu_{q_1}} & & & & \end{pmatrix}$$

$$= \det \begin{pmatrix} \mathcal{A}_0 & \mathcal{B}_0 \\ x\,\mathrm{Id}_{q_0} & \mathrm{Id}_{q_0} \end{pmatrix} \cdot \det \begin{pmatrix} & \mathcal{A}_1 & & & \mathcal{B}_1 \\ \tau_1\,y^{2\nu_1} & 0 & 0 & & \\ 0 & \ddots & 0 & & \mathrm{Id}_{q_1} \\ 0 & 0 & \tau_{q_1}\,y^{2\nu_{q_1}} & & \end{pmatrix}$$

$$=: p_0(x) \cdot p_1(y),$$

where p_0 and p_1 are the corresponding determinants in the second line. Expanding the determinants, we can write

$$(5.7) \qquad p_0(x) = \sum a_j\,x^j \qquad \text{and} \qquad p_1(y) = \sum b_\alpha\,y^{2\alpha}.$$

The next theorem follows immediately from Proposition 4.5 and Theorem 2.3.

Theorem 5.2. *For a decomposable Lagrangian $L \subset \mathbb{C}^{2q}$ such that $\ker \mathcal{L}_L = \{0\}$, we have*

$$(5.8) \quad \det_\zeta(\mathcal{L}_L) = \frac{(2\pi R)^{\frac{q}{2}}}{a_{j_0} b_{\alpha_0}} \prod_{j=1}^{q_1} \frac{2^{\nu_j}}{\Gamma(1 - \nu_j)} (-2e^\gamma)^{q_0 - j_0} \times$$

$$\det \begin{pmatrix} \mathcal{A}_0 & \mathcal{B}_0 \\ \mathrm{Id}_{q_0} & (\log R)\mathrm{Id}_{q_0} \end{pmatrix} \det \begin{pmatrix} \mathcal{A}_1 & \mathcal{B}_1 \\ \mathbf{R}^\nu & \mathbf{R}^{-\nu} \end{pmatrix},$$

where a_{j_0} and b_{α_0} are the coefficients in (5.7) corresponding to the smallest j and α with a nonzero coefficient in $p_0(x)$ and $p_1(y)$, respectively. In particular, for the generalized cone we have

$$\frac{\det_\zeta(\Delta_L)}{\det_\zeta(\Delta_\mathcal{N})} = \frac{(-2e^\gamma)^{q_0 - j_0}}{a_{j_0} b_{\alpha_0}} \det \begin{pmatrix} \mathcal{A}_0 & \mathcal{B}_0 \\ \mathrm{Id}_{q_0} & (\log R)\mathrm{Id}_{q_0} \end{pmatrix} \det \begin{pmatrix} \mathcal{A}_1 & \mathcal{B}_1 \\ \mathbf{R}^{2\nu} & \mathrm{Id}_{q_1} \end{pmatrix}.$$

5.3. **The one-dimensional case.** Consider now the one-dimensional operator

$$\mathcal{L} := -\frac{d^2}{dr^2} + \frac{1}{r^2}\lambda \quad \text{over } [0, R], \text{ where } -\frac{1}{4} \le \lambda < \frac{3}{4}.$$

In this one-dimensional case, Lagrangians are given by two 1×1 matrices (numbers) $\mathcal{A} = \alpha$ and $\mathcal{B} = \beta$ where $\alpha\overline{\beta} \in \mathbb{R}$. One can check that (see e.g. [76, prop. 3.7] that we can take $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 = 1$. We shall compute $\det_\zeta(\mathcal{L}_L)$ using Theorem 5.2 under the assumption $\ker \mathcal{L}_L = \{0\}$. Assume that $\lambda = -\frac{1}{4}$. Then

$$p_0(x) = \det \begin{pmatrix} \alpha & \beta \\ x & 1 \end{pmatrix} = \alpha - \beta\,x,$$

which implies that $j_0 = 0$ and $a_{j_0} = \alpha$ if $\alpha \neq 0$ and $j_0 = 1$ and $a_{j_0} = -\beta$ if $\alpha = 0$, and by (5.8), we have

$$
\begin{aligned}
\det_\zeta(\mathcal{L}_L) &= \frac{(2\pi R)^{\frac{1}{2}}}{a_{j_0}}(-2e^\gamma)^{1-j_0}\det\begin{pmatrix} \alpha & \beta \\ 1 & (\log R) \end{pmatrix} \\
&= \frac{\sqrt{2\pi R}}{a_{j_0}}(-2e^\gamma)^{1-j_0}\Big(\alpha\log R - \beta\Big).
\end{aligned}
$$

In conclusion, we see that in the case $\lambda = -\frac{1}{4}$, we have

$$
\det_\zeta(\mathcal{L}_L) = \begin{cases} 2\sqrt{2\pi R}\, e^\gamma\Big(\frac{\beta}{\alpha} - \log R\Big) & \text{if } \alpha \neq 0 \\ \sqrt{2\pi R} & \text{if } \alpha = 0. \end{cases}
$$

Assume now that $-\frac{1}{4} < \lambda < \frac{3}{4}$. Then with $\nu := \sqrt{\lambda + \frac{1}{4}}$ and $\tau = 2^{2\nu}\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}$, we have

$$
p_1(y) = \det\begin{pmatrix} \alpha & \beta \\ \tau\, y^{2\nu} & 1 \end{pmatrix} = \alpha - \beta\,\tau\,y^{2\nu},
$$

which implies that $\alpha_0 = 0$ and $b_{\alpha_0} = \alpha$ if $\alpha \neq 0$ and $\alpha_0 = 2\nu$ and $b_{\alpha_0} = -\beta\tau$ if $\alpha = 0$, and by (5.8), we have

$$
\det_\zeta(\mathcal{L}_L) = \frac{(2\pi R)^{\frac{1}{2}}}{b_{\alpha_0}}\frac{2^\nu}{\Gamma(1-\nu)}\det\begin{pmatrix} \alpha & \beta \\ R^\nu & R^{-\nu} \end{pmatrix} = \frac{\sqrt{2\pi R}}{b_{\alpha_0}}\frac{2^\nu}{\Gamma(1-\nu)}\Big(\alpha R^{-\nu} - \beta R^\nu\Big).
$$

In conclusion, we see that in the case $-\frac{1}{4} < \lambda < \frac{3}{4}$, we have

$$
\det_\zeta(\mathcal{L}_L) = \begin{cases} 2^{\nu+1/2}\sqrt{\pi R}\,\Gamma(1-\nu)^{-1}\Big(R^{-\nu} - \frac{\beta}{\alpha}R^\nu\Big) & \text{if } \alpha \neq 0 \\ 2^{-\nu+1/2}\sqrt{\pi R}\,\Gamma(1+\nu)^{-1}R^\nu & \text{if } \alpha = 0. \end{cases}
$$

6. Conclusions and final remarks

In this article we have considered zeta functions and zeta regularized determinants for arbitrary self-adjoint extensions of Laplace-type operators over conic manifolds. In general, the zeta function will have a logarithmic branch point as well as a simple pole at $s = 0$. In order to get a well-defined notion of a determinant we propose to use the natural prescription (2.3). Within this prescription, Theorem 2.3 is the central theorem proven in this article. It gives a closed form for the determinant of the Laplacian over the cone associated with an arbitrary self-adjoint extension. As we have seen, it is easily applied to particular cases and known results have been easily reproduced.

For convenience we have chosen to work with Dirichlet boundary conditions at $r = R$, emphasizing the role of the self-adjoint extension for the analytic structure of the zeta function and for the determinant. Equally well other boundary conditions at $r = R$ can be considered along the same lines.

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