THE VERY UNUSUAL PROPERTIES OF THE RESOLVENT, HEAT KERNEL, AND ZETA FUNCTION FOR THE OPERATOR $-d^2/dr^2 - 1/(4r^2)$

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Abstract. In this article we analyze the resolvent, the heat kernel and the spectral zeta function of the operator $-d^2/dr^2-1/(4r^2)$ over the finite interval. The structural properties of these spectral functions depend strongly on the chosen self-adjoint realization of the operator, a choice being made necessary because of the singular potential present. Only for the Friedrichs realization standard properties are reproduced, for all other realizations highly nonstandard properties are observed. In particular, for $k \in \mathbb{N}$ we find terms like $(\log t)^{-k}$ in the small-t asymptotic expansion of the heat kernel. Furthermore, the zeta function has $s = 0$ as a logarithmic branch point.

1. Introduction

1.1. **Zeta functions and an unusual example.** It is well-known that the zeta function of a Laplacian over a smooth compact manifold, with or without boundary, defines a meromorphic function on $\mathbb{C}$ with simple poles at prescribed half-integer values depending on the dimension of the manifold [22]. (For a manifold with boundary, we put local boundary conditions, e.g. Dirichlet conditions.) These properties have far reaching applications in physics as well as mathematics, e.g. in the context of Casimir energies, effective actions and analytic torsion; see, for example, [12, 13, 14, 26, 29, 43].

Surprisingly, there is a completely natural example of a zeta function for which the described properties break down and which has no meromorphic extension to $\mathbb{C}$. Let $\Omega \subset \mathbb{R}^2$ be any compact region and take polar coordinates $(x, y) \longleftrightarrow (r, \theta)$ centered at any fixed point in Ω. Then in these coordinates, the standard Laplacian on $\mathbb{R}^2$ takes the form

$$\Delta_{\mathbb{R}^2} = -\partial_x^2 - \partial_y^2 = -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\theta^2,$$

and the measure transforms to $dxdy = rdrd\theta$. A short computation shows that

$$\Delta_{\mathbb{R}^2}\phi = \left(-\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\theta^2\right)\phi = \mathcal{R}^{-1}\Big(-\partial_r^2 + \frac{1}{r^2}\big(-\partial_\theta^2 - \frac{1}{4}\big)\Big)\mathcal{R}\phi,$$

where $\mathcal{R}$ is the multiplication map by $r^{1/2}$, which is an isometry from $L^2(\Omega, rdrd\theta)$ to $L^2(\Omega, drd\theta)$. Hence, the following two operators are equivalent under $\mathcal{R}$:

$$\Delta_{\mathbb{R}^2} \quad \longleftrightarrow \quad -\partial_r^2 + \frac{1}{r^2}A, \qquad \text{where} \quad A = -\partial_\theta^2 - \frac{1}{4}.$$

In the zero eigenspace of $-\partial_\theta^2$, we obtain the operator of the form $-\partial_r^2 - \frac{1}{4r^2}$. Then this Laplace type operator has many different self-adjoint realizations parameterized by angles $\theta \in [0, \pi)$; the angle $\theta = \pi/2$ corresponds to the so-called Friedrichs realization. Each realization has a discrete spectrum [37]. Consider any one of the realizations, say Δ_θ, with $\theta \neq \pi/2$ and form the corresponding zeta function

$$\zeta(s, \Delta_\theta) := \sum_{\lambda_j \neq 0} \frac{1}{\lambda_j^s},$$

where the λ_j's are the eigenvalues of Δ_θ. The shocking fact is that *every* such zeta function corresponding to an angle $\theta \in [0, \pi)$, except $\theta = \pi/2$, does not have a meromorphic extension to $\mathbb{C}$; in fact each such zeta function has a logarithmic branch cut with $s = 0$ as the branch point.

1.2. **Self-adjoint realizations.** The properties of the Laplace operator considered above boil down to the main object of consideration in this paper,

$$\Delta := -\frac{d^2}{dr^2} - \frac{1}{4r^2} \qquad \text{over} \quad [0, R]. \tag{1.1}$$

In Section 2 we work out an explicit description of the maximal domain of Δ. In order to choose a self-adjoint realization of Δ, we first fix a boundary condition for Δ at $r = R$; it turns out that any such boundary condition for $\phi \in \mathscr{D}_{\max}(\Delta)$ must be of the form (see Section 4)

$$\cos\theta_2\, \phi'(R) + \sin\theta_2\, \phi(R) = 0. \tag{1.2}$$

In other words, the boundary conditions we can choose at $r = R$ are parameterized by angles $\theta_2 \in [0, \pi)$. Note that the Dirichlet condition is when $\theta_2 = \pi/2$ and the Neumann condition is when $\theta_2 = 0$. Let us henceforth *fix* an angle $\theta_2 \in [0, \pi)$ and consider Δ with the condition in (1.2). At $r = 0$, the operator Δ is singular and a limiting procedure $r \to 0$ must be used to define boundary conditions. As shown in Section 3 (see also Section 4), the self-adjoint realizations of Δ with the condition (1.2) are again parameterized by angles $\theta_1 \in [0, \pi)$; the article by Kochubeĭ [32] is perhaps one of the earliest references to contain such a parameterization. It turns out that $\theta_1 = \pi/2$ corresponds to the Friedrichs realization.

As we will show in Theorem 2.1, $\phi \in \mathscr{D}_{\max}(\Delta)$ if and only if it can be written in the form

$$\phi = c_1(\phi)\, r^{1/2} + c_2(\phi)\, r^{1/2} \log r + \mathcal{O}(r^{3/2}),$$

where $c_1(\phi)$ and $c_2(\phi)$ are constants depending on ϕ. In terms of these constants, given angles $\theta_1, \theta_2 \in [0, \pi)$, we consider the operator

$$\Delta_L := \Delta : \mathscr{D}_L \to L^2([0, R])$$

where

$$\mathscr{D}_L = \{\phi \in \mathscr{D}_{\max}(\Delta) \mid \cos\theta_1\, c_1(\phi) + \sin\theta_1\, c_2(\phi) = 0\ ,\ \cos\theta_2\, \phi'(R) + \sin\theta_2\, \phi(R) = 0\}.$$

Here, the subscript "L" represents the two-dimensional subspace $L \subset \mathbb{C}^4$ defined by

$$L := \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \ \mid\ \cos\theta_1\, z_1 + \sin\theta_1\, z_2 = 0\ ,\ \cos\theta_2\, z_3 + \sin\theta_2\, z_4 = 0\}.$$

This vector space is a Lagrangian subspace of $\mathbb{C}^4$ with respect to a natural Hermitian symplectic form intimately related to self-adjoint realizations of Δ; see Section 3. For general references on this relation see [24, 25, 28, 32, 33, 34, 35]. For a study of adjoints of "cone operators" (in the sense of Schulze [45]) see [20]. For properties of heat kernels and resolvents of cone operators see, for example, [19], [21], [36], [44].

1.3. **The resolvent, heat kernel, and zeta function.** When $\theta_1 = \pi/2$ (the Friedrichs realization), the following properties concerning the resolvent, heat kernel, and zeta function are well known; see for example, Brüning and Seeley [8], Falomir *et al.* [15], or Mooers [40]. With $\theta_1 = \pi/2$, the following properties hold:

Theorem 1.1 (Cf. [8, 15, 40])**.** *Fixing a boundary condition* (1.2) *at $r = R$, let Δ_L denote the corresponding Friedrichs realization (that is, take $\theta_1 = \pi/2$). Then*

(1) Let $\Lambda \subset \mathbb{C}$ be any sector (solid angle) not intersecting the positive real axis. Then as $|\lambda| \to \infty$ with $\lambda \in \Lambda$, we have

$$\mathrm{Tr}(\Delta_L - \lambda)^{-1} \sim \sum_{k=1}^{\infty} a_k \, (-\lambda)^{-k/2}.$$

(2) As $t \to 0$, we have

$$\mathrm{Tr}(e^{-t\Delta_L}) \sim \sum_{k=0}^{\infty} \beta_k \, t^{(k-1)/2}.$$

(3) The zeta function

$$\zeta(s, \Delta_L) = \mathrm{Tr}(\Delta_L^{-s})$$

extends from $\Re s > 1/2$ to a meromorphic function on $\mathbb{C}$ with poles at $s = 1/2 - k$ for $k = 0, 1, 2, \ldots$.

These properties are "usual" in that they remain valid, with appropriate changes, to Laplace-type operators on compact manifolds (with or without boundary); see for example Gilkey's book [22] for a thorough treatment. The first result of this paper shows that for *any* other realization, these properties are completely destroyed.

Theorem 1.2. *With any boundary condition* (1.2) *fixed at $r = R$, choose a self-adjoint realization Δ_L of the resulting operator that is* ***not*** *the Friedrichs realization. (That is, take $\theta_1 \neq \pi/2$). Let $\kappa = \log 2 - \gamma - \tan\theta_1$ where γ is the Euler constant. Then the following properties hold:*

(1) Let $\Lambda \subset \mathbb{C}$ be any sector (solid angle) not intersecting the positive real axis. Then as $|\lambda| \to \infty$ with $\lambda \in \Lambda$, we have

$$\mathrm{Tr}(\Delta_L - \lambda)^{-1} \sim \frac{1}{(-\lambda)(\log(-\lambda) - 2\kappa)} + \sum_{k=1}^{\infty} b_k \, (-\lambda)^{-k/2}.$$

(2) As $t \to 0$, we have (here $\Im$ denotes "imaginary part of")

$$\mathrm{Tr}(e^{-t\Delta_L}) \sim \frac{1}{\pi} \Im \left(\int_1^{\infty} e^{-tx} \frac{1}{x(\log x + i\pi - 2\kappa)} \, dx \right) + \sum_{k=0}^{\infty} \beta_k \, t^{(k-1)/2}.$$

(3) The zeta function $\zeta(s, \Delta_L)$ can be written in the form

$$\zeta(s, \Delta_L) = -\frac{e^{-2s\kappa} \sin \pi s}{\pi} \log s + \zeta_L(s),$$

where $\zeta_L(s)$ extends from $\Re s > 1/2$ to a holomorphic function on $\mathbb{C}$ with poles at $s = 1/2 - k$ for $k = 0, 1, 2, \ldots$. In particular, $\zeta(s, \Delta_L)$ has $s = 0$ as a logarithmic branch point!

Remark 1.3. The authors have never seen a natural *geometric* differential operator with discrete spectrum on a compact manifold having a spectral zeta function with properties of the sort described in this theorem.

Remark 1.4. The first term in assertion (1) can be expanded further if needed. In the formulation of this theorem we leave it in this more useful compact form.

Remark 1.5. The same kind of remark holds for the first term in assertion (2). Expanding further we obtain the following expansion: As $t \to 0$, we have

$$\operatorname{Tr}(e^{-t\Delta_L}) \sim \sum_{k=1}^{\infty} \alpha_k (\log t)^{-k} + \sum_{k=0}^{\infty} \beta_k \, t^{(k-1)/2} \tag{1.3}$$

with the α_k's depending on κ via (here $\Im$ denotes "imaginary part of")

$$\alpha_k = -\frac{1}{k\pi} \Im \left(\int_1^{\infty} e^{-x} \Big(\log x + i\pi - 2\kappa \Big)^k dx \right), \quad k = 1, 2, 3, \dots .$$

The expansion (1.3) is misleading as written because for $k > 1$, the terms $\beta_k t^{(k-1)/2}$ are sub-leading to any of the inverse log terms. However, we interpret the first sum in the expansion (1.3) to mean that for all N, we have

$$\frac{1}{\pi} \Im \left(\int_1^{\infty} e^{-tx} \frac{1}{x(\log x + i\pi - 2\kappa)} \, dx \right) = \sum_{k=1}^{N} \alpha_k (\log t)^{-k} + \mathcal{O}\Big((\log t)^{-N-1} \Big).$$

1.4. **Explicit formula for the zeta determinant.** Our second result is an explicit formula for a regularized determinant. For concreteness, we shall impose the Dirichlet boundary condition at $r = R$. That is, given an angle $\theta \in [0, \pi)$ with $\theta \neq \pi/2$, we consider the operator

$$\Delta_\theta := \Delta : \mathscr{D}_\theta \to L^2([0, R])$$

where

$$\mathscr{D}_\theta = \{ \phi \in \mathscr{D}_{\max}(\Delta) \,|\, \cos\theta \, c_1(\phi) + \sin\theta \, c_2(\phi) = 0 \ , \phi(R) = 0 \}.$$

Then from Theorem 1.2, the zeta function $\zeta(s, \Delta_\theta)$ has the following form

$$\zeta(s, \Delta_\theta) = -\frac{e^{-2s\kappa} \sin \pi s}{\pi} \log s + \zeta_\theta(s),$$

where $\zeta_\theta(s)$ extends from $\Re s > 1/2$ to a holomorphic function on $\mathbb{C}$ with poles at $s = 1/2 - k$ for $k = 0, 1, 2, \dots$. In particular, $\zeta(s, \Delta_\theta)$ has the form

$$\zeta(s, \Delta_\theta) \sim -s \log s + \mathcal{O}(s^2 \log s) + \text{holomorphic} \qquad \text{as } s \to 0. \tag{1.4}$$

In particular,

$$\zeta'(s, \Delta_\theta) \sim -\log s + \mathcal{O}(s \log s) + \text{holomorphic} \qquad \text{as } s \to 0,$$

so the ζ-regularized determinant $\det(\Delta_\theta) := \exp(-\zeta'(0, \Delta_\theta))$ is *not* defined! However, from (1.4), we see that

$$\zeta_{\text{reg}}(s, \Delta_\theta) := \zeta(s, \Delta_\theta) + s \log s$$

does have a well-defined derivative at $s = 0$. For this reason, we define

$$\det\nolimits_{\text{reg}}(\Delta_\theta) := \exp\Big(-\zeta_{\text{reg}}'(0, \Delta_\theta) \Big).$$

In the following theorem, we give a beautiful explicit formula for this regularized determinant.

Theorem 1.6. *For any $\theta \in [0, \pi)$ with $\theta \neq \pi/2$, we have*

$$\det_{\mathrm{reg}}(\Delta_\theta) = \begin{cases} 2\sqrt{2\pi R}\, e^{\gamma}(\tan\theta - \log R) & \tan\theta \neq \log R \\ \sqrt{\frac{\pi R}{2}}\, e^{\gamma} R^2 & \tan\theta = \log R. \end{cases}$$

We remark that when $\theta = \pi/2$, the zeta function $\zeta(s, \Delta_\theta)$ is regular at $s = 0$ and we can also compute the (usual) ζ-regularized determinant: For $\theta = \pi/2$, we have

$$\det(\Delta_\theta) = \sqrt{2\pi R},$$

a well known result, see e.g. Theorem 2.3 of [38], Proposition 5.2 of [39].

We now outline this article. In Sections 2–4 we study the self-adjoint realizations of our main operator using the Hermitian symplectic theory due to Gelfand [41, p. 1]; cf. also [15, 32, 33, 34, 39, 40, 42]. Although some of this material can be found piecemeal throughout the literature, we present all the details here in order to keep our article elementary, self-contained, and "user-friendly". In Sections 5–8 we prove Theorem 1.2 in the special case that the Dirichlet boundary condition is chosen at $r = R$ and in Section 9 we prove Theorem 1.6, all using the contour integration method developed in [3, 4, 5]. In Section 10 we prove Theorem 1.2 in full generality. Finally, in Appendix A, we explicitly calculate the resolvent of Δ_θ, which is needed at various places in our analysis.

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2. The maximal domain

Our first order of business is to characterize the self-adjoint realizations of the operator in (1.1); for general references on self-adjoint realizations and their applications to physics see, e.g., [2, 6, 10, 16, 17, 18, 32, 33, 34, 35, 42, 47]. To do so, we first need to determine the maximal domain of Δ:

$$\mathscr{D}_{\max}(\Delta) := \{\phi \in L^2([0,R]) \,|\, \Delta\phi \in L^2([0,R])\}.$$

For a quick review, $\Delta\phi$ is understood in the distributional sense; thus, $\Delta\phi$ is the functional on test functions $C_c^\infty((0,R))$ defined by

$$(\Delta\phi)(\xi) := \int_0^R \Delta\xi(r)\,\overline{\phi(r)}\,dr \quad \text{for all} \quad \xi \in C_c^\infty((0,R)).$$

Then $\Delta\phi \in L^2([0,R])$ means that the distribution $\Delta\phi : C_c^\infty((0,R)) \to \mathbb{C}$ is represented by an L^2 function in the sense that there is a function $f \in L^2([0,R])$ such that

$$\int_0^R \Delta\xi(r)\,\overline{\phi(r)}\,dr = \langle \xi, f\rangle \quad \text{for all} \quad \xi \in C_c^\infty((0,R))$$

where $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product (conjugate linear in the second slot) on $L^2([0,R])$. The following theorem is inspired by Falomir *et al.* [15, Lem. 2.1].

Theorem 2.1. *$\phi \in \mathscr{D}_{\max}(\Delta)$ if and only if ϕ can be written in the form*

$$\phi = c_1(\phi)\, r^{1/2} + c_2(\phi)\, r^{1/2}\log r + \widetilde{\phi}, \tag{2.1}$$

where $c_1(\phi), c_2(\phi)$ are constants and $\widetilde{\phi}$ is a continuously differentiable function on $[0,R]$ such that $\widetilde{\phi}(r) = \mathcal{O}(r^{3/2})$, $\widetilde{\phi}'(r) = \mathcal{O}(r^{1/2})$, and $\Delta\widetilde{\phi} \in L^2([0,R])$.

Proof. Since

$$\Delta\big(c_1\, r^{1/2} + c_2\, r^{1/2}\log r\big) = 0,$$

it follows that any ϕ of the stated form is in $\mathscr{D}_{\max}(\Delta)$. Now let $\phi \in \mathscr{D}_{\max}(\Delta)$; then $\Delta\phi = f \in L^2([0,R])$. Let us define $\psi := r^{-1/2}\phi$ so that $\phi = r^{1/2}\psi$. Then

$$f = -\phi'' - \frac{1}{4r^2}\phi = \frac{1}{4}r^{-3/2}\psi - r^{-1/2}\psi' - r^{1/2}\psi'' - \frac{1}{4}r^{-3/2}\psi = -r^{-1/2}\psi' - r^{1/2}\psi''.$$

After multiplication by $r^{1/2}$, we get

$$\psi' + r\psi'' = -r^{1/2}f \quad \Longrightarrow \quad (r\psi')' = -r^{1/2}f.$$

Since $r^{1/2}$ and f are in $L^2([0,R])$, by the Cauchy-Schwartz inequality, we know that $r^{1/2}f$ is in $L^1([0,R])$, therefore we can conclude that

$$\psi' = \frac{c_2}{r} - \frac{1}{r}\int_0^r t^{1/2}\, f(t)\, dt. \tag{2.2}$$

Notice that by Cauchy-Schwartz,

$$\left|\int_0^r t^{1/2}\, f(t)\, dt\right| \leq \sqrt{\int_0^r t\, dt}\cdot \|f\|_2 = \frac{r}{\sqrt{2}}\|f\|_2. \tag{2.3}$$

Thus, the second term on the right in (2.2) is in $L^1([0,R])$. Therefore, from (2.2) we see that

$$\psi(r) = c_1 + c_2\, \log r - \int_0^r \frac{1}{x}\int_0^x t^{1/2}\, f(t)\, dt\, dx,$$

or, since $\phi = r^{1/2}\psi$, we get

$$\phi(r) = c_1\, r^{1/2} + c_2\, r^{1/2}\log r + \widetilde{\phi}\;\;, \qquad \widetilde{\phi} := -r^{1/2}\int_0^r \frac{1}{x}\int_0^x t^{1/2}\, f(t)\, dt\, dx.$$

By (2.3), we have

$$\left|\int_0^r \frac{1}{x}\int_0^x t^{1/2}\, f(t)\, dt\, dx\right| \leq \int_0^r \frac{1}{\sqrt{2}}\|f\|_2\, dx = \frac{r}{\sqrt{2}}\|f\|_2.$$

From this estimate, it follows that $\widetilde{\phi}(r) = \mathcal{O}(r^{3/2})$ and $\widetilde{\phi}'(r) = \mathcal{O}(r^{1/2})$. $\square$

3. Self-adjoint realizations

Choosing a linear subspace $\mathscr{D} \subset \mathscr{D}_{\max}(\Delta)$, we say that

$$\Delta_{\mathscr{D}} := \Delta : \mathscr{D} \to L^2([0,R])$$

is *self-adjoint* (in which case $\Delta_{\mathscr{D}}$ is called a *self-adjoint realization* of Δ) if

$$\{\psi \in \mathscr{D}_{\max}(\Delta) \mid \langle \Delta\phi, \psi\rangle = \langle \phi, \Delta\psi\rangle \;\text{ for all }\; \phi \in \mathscr{D}\} = \mathscr{D};$$

in other words, Δ is symmetric on $\mathscr{D}$ and adding any elements to $\mathscr{D}$ will destroy this symmetry.

In order to determine if Δ has any self-adjoint realization, we need to analyze the quadratic form

$$\langle \phi, \Delta\psi\rangle - \langle \Delta\phi, \psi\rangle \quad \text{for} \quad \phi, \psi \in \mathscr{D}_{\max}(\Delta).$$

It turns out that this difference is related to finite-dimensional symplectic linear algebra. Let us define

$$\omega : \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}$$

by

$$\omega(v,w) := v_1\,\overline{w_2} - v_2\,\overline{w_1} + v_3\,\overline{w_4} - v_4\,\overline{w_3}. \tag{3.1}$$

The function ω is Hermitian antisymmetric and non-degenerate; for this reason, ω is called a *Hermitian symplectic form.*

Theorem 3.1. *Let $\phi, \psi \in \mathscr{D}_{\max}(\Delta)$ be written in the form (2.1), i.e.*

$$\phi = c_1(\phi)\,r^{1/2} + c_2(\phi)\,r^{1/2}\log r + \widetilde{\phi},$$

where $\widetilde{\phi}$ is continuously differentiable with $\widetilde{\phi}(r) = \mathcal{O}(r^{3/2})$, $\widetilde{\phi}'(r) = \mathcal{O}(r^{1/2})$, and $\Delta\widetilde{\phi} \in L^2([0,R])$, and with a similar formula holding for ψ. Then,

$$\langle \phi, \Delta\psi\rangle - \langle \Delta\phi, \psi\rangle = \omega(\vec{\phi}, \vec{\psi}),$$

where ω is the Hermitian symplectic form defined above and $\vec{\phi}, \vec{\psi} \in \mathbb{C}^4$ are the vectors

$$\vec{\phi} := (c_1(\phi), c_2(\phi), \phi'(R), \phi(R)) \quad , \quad \vec{\psi} := (c_1(\psi), c_2(\psi), \psi'(R), \psi(R)).$$

Proof. We have

$$\begin{aligned}
\langle \phi, \Delta\psi\rangle - \langle \Delta\phi, \psi\rangle &= \lim_{\varepsilon\to 0}\int_\varepsilon^R \Big(\phi(r)\,\overline{\Delta\psi(r)} - \Delta\phi(r)\,\overline{\psi(r)}\Big)\,dr \\
&= \lim_{\varepsilon\to 0}\int_\varepsilon^R \frac{d}{dr}\Big(-\phi(r)\,\overline{\psi'(r)} + \phi'(r)\,\overline{\psi(r)}\Big)\,dr \\
&= \lim_{\varepsilon\to 0}\Big(\phi(\varepsilon)\,\overline{\psi'(\varepsilon)} - \phi'(\varepsilon)\,\overline{\psi(\varepsilon)}\Big) + \Big(\phi'(R)\,\overline{\psi(R)} - \phi(R)\,\overline{\psi'(R)}\Big). \qquad (3.2)
\end{aligned}$$

Recall that

$$\phi = c_1(\phi)\,r^{1/2} + c_2(\phi)\,r^{1/2}\log r + \widetilde{\phi} \quad , \quad \psi = c_1(\psi)\,r^{1/2} + c_2(\psi)\,r^{1/2}\log r + \widetilde{\psi},$$

where $\widetilde{\phi}$ and $\widetilde{\psi}$ are continuously differentiable functions on $[0,R]$ such that $\widetilde{\phi}(r), \widetilde{\psi}(r) = \mathcal{O}(r^{3/2})$, $\widetilde{\phi}'(r), \widetilde{\psi}'(r) = \mathcal{O}(r^{1/2})$. Taking derivatives, we get

$$\phi' = \frac{c_1(\phi)}{2}\,r^{-1/2} + \frac{c_2(\phi)}{2}r^{-1/2}(\log r + 2) + \widetilde{\phi}'$$

and similarly for ψ'. It follows that

$$\begin{aligned}
\phi(\varepsilon)\,\overline{\psi'(\varepsilon)} = \frac{c_1(\phi)\,\overline{c_1(\psi)}}{2} &+ \frac{c_1(\phi)\,\overline{c_2(\psi)}}{2}(\log\varepsilon + 2) \\
&+ \frac{c_2(\phi)\,\overline{c_1(\psi)}}{2}\,\log\varepsilon + \frac{c_2(\phi)\,\overline{c_2(\psi)}}{2}\,\log\varepsilon(\log\varepsilon + 2) + o(1)
\end{aligned}$$

and similarly for $\phi'(\varepsilon)\,\overline{\psi(\varepsilon)}$. Subtracting, we get

$$\phi(\varepsilon)\,\overline{\psi'(\varepsilon)} - \phi'(\varepsilon)\,\overline{\psi(\varepsilon)} = c_1(\phi)\,\overline{c_2(\psi)} - c_2(\phi)\,\overline{c_1(\psi)} + o(1).$$

Combining this with (3.2) proves our result. □

Recall that a subspace $L \subset \mathbb{C}^4$ is called *Lagrangian* if $L^{\perp_\omega} = L$ where $L^{\perp_\omega}$ is the orthogonal complement of L with respect to ω; explicitly, L is Lagrangian means

$$\{w \in \mathbb{C}^4 \mid \omega(v,w) = 0 \ \text{ for all } \ v \in L\} = L.$$

We now have our main result.

Theorem 3.2. *Self-adjoint realizations of* Δ *are in one-to-one correspondence with Lagrangian subspaces of* $\mathbb{C}^4$ *in the sense that given any Lagrangian subspace* $L \subset \mathbb{C}^4$, *defining*

$$\mathscr{D}_L := \{\phi \in \mathscr{D}_{\max}(\Delta) \mid \vec{\phi} \in L\}$$

the operator

$$\Delta_L := \Delta : \mathscr{D}_L \to L^2([0,R])$$

is self-adjoint and any self-adjoint realization of Δ *is of the form* $\mathscr{D}_L$ *for some Lagrangian* $L \subset \mathbb{C}^4$.

Proof. By definition,

$$\Delta_{\mathscr{D}} := \Delta : \mathscr{D} \to L^2([0,R])$$

is *self-adjoint* means

$$\{\psi \in \mathscr{D}_{\max}(\Delta) \mid \langle \Delta\phi, \psi\rangle = \langle \phi, \Delta\psi\rangle \ \text{ for all } \ \phi \in \mathscr{D}\} = \mathscr{D}.$$

By Theorem 3.1, we can write this as: $\Delta_{\mathscr{D}}$ is self-adjoint if and only if

$$\omega(\vec{\phi}, \vec{\psi}) = 0 \ \text{ for all } \ \phi \in \mathscr{D} \quad \Longleftrightarrow \quad \psi \in \mathscr{D}. \tag{3.3}$$

Suppose that $\Delta_{\mathscr{D}}$ is self-adjoint and define $L := \{\vec{\phi} \in \mathbb{C}^4 \mid \phi \in \mathscr{D}\}$; we shall prove that L is Lagrangian. Let $w \in L$ and choose $\psi \in \mathscr{D}$ such that $\vec{\psi} = w$. Then by (3.3), $\omega(\vec{\phi}, w) = 0$ for all $\phi \in \mathscr{D}$. Therefore, $\omega(v,w) = 0$ for all $v \in L$. Conversely, let $w \in \mathbb{C}^4$ and assume that $\omega(v,w) = 0$ for all $v \in L$. Choose $\psi \in \mathscr{D}_{\max}(\Delta)$ such that $\vec{\psi} = w$; e.g. if $w = (w_1, w_2, w_3, w_4)$, then

$$\psi := w_1\, r^{1/2} + w_2\, r^{1/2}\log r + (w_3 - w_4)(r - R) + w_4 r \tag{3.4}$$

will do. Then $\omega(v,w) = 0$ for all $v \in L$ implies that $\omega(\vec{\phi}, \vec{\psi}) = 0$ for all $\phi \in \mathscr{D}$, which by (3.3), implies that $\psi \in \mathscr{D}$, which further implies that $w = \vec{\phi} \in L$.

Now let $L \subset \mathbb{C}^4$ be Lagrangian; we shall prove that Δ_L is self-adjoint, that is, (3.3) holds. Let $\psi \in \mathscr{D}_L$. Then, since L is Lagrangian, we automatically have $\omega(\vec{\phi}, \vec{\psi}) = 0$ for all $\phi \in \mathscr{D}_L$. Conversely, let $\psi \in \mathscr{D}_{\max}(\Delta)$ and assume that $\omega(\vec{\phi}, \vec{\psi}) = 0$ for all $\phi \in \mathscr{D}_L$. By the construction (3.4) given any $v \in L$ we can find a $\phi \in \mathscr{D}_{\max}(\Delta)$ such that $\vec{\phi} = v$. Therefore, $\omega(\vec{\phi}, \vec{\psi}) = 0$ for all $\phi \in \mathscr{D}_L$ implies that $\omega(v, \vec{\psi}) = 0$ for all $v \in L$, which by the Lagrangian condition on L, implies that $\vec{\psi} \in L$. This shows that $\psi \in \mathscr{D}_L$ and our proof is complete. □

4. More on Lagrangian subspaces

The symplectic form $\omega : \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}$ defined in (3.1) is naturally separated into two parts:

$$\omega(v,w) = \omega_0((v_1,v_2),(w_1,w_2)) + \omega_0((v_3,v_4),(w_3,w_4)) \tag{4.1}$$

where

$$\omega_0 : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C} \ \text{ is defined by } \ \omega_0(v,w) = v_1\,\overline{w_2} - v_2\,\overline{w_1}.$$

The first ω_0 appearing in (4.1) corresponds to the singularity at $r = 0$ and the second ω_0 in (4.1) corresponds to the boundary $r = R$. For this reason, it is natural to focus on Lagrangian subspaces $L \subset \mathbb{C}^4$ of the form $L = L_1 \oplus L_2$ where $L_i \subset \mathbb{C}^2$ is Lagrangian with respect to ω_0. With this in mind, let us characterize all such Lagrangian subspaces of $\mathbb{C}^2$. First, we observe that

Lemma 4.1. *We can write*

$$\omega_0(v,w) = \langle Gv, w\rangle \quad \text{for all} \;\; v,w \in \mathbb{C}^2,$$

where $\langle\ ,\ \rangle$ *denotes the inner product on* $\mathbb{C}^2$ *and* $G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Recalling that $L \subset \mathbb{C}^2$ is Lagrangian means that

$$\{w \in \mathbb{C}^2 \mid \omega_0(v,w) = 0 \;\; \text{for all} \;\; v \in L\} = L,$$

from this lemma, it is straightforward to show that

$$L \subset \mathbb{C}^2 \;\; \text{is Lagrangian if and only if} \;\; GL^\perp = L,$$

where $L^\perp$ is the orthogonal complement of L with respect to the inner product $\langle\ ,\ \rangle$. From this, one can easily prove the following main result in this section.

Theorem 4.2. $L \subset \mathbb{C}^2$ *is Lagrangian if and only if* $L = L_\theta$ *for some* $\theta \in \mathbb{R}$ *where*

$$L_\theta = \{(x,y) \in \mathbb{C}^2 \mid \cos\theta\, x + \sin\theta\, y = 0\}.$$

Notice that we can restrict to $0 \le \theta < \pi$ in Theorem 4.2. Let θ_1, θ_2 be two such angles and put $L := L_{\theta_1} \oplus L_{\theta_2}$. As in (2.1), we write $\phi \in \mathscr{D}_{\max}(\Delta)$ as

$$\phi = c_1(\phi)\, r^{1/2} + c_2(\phi)\, r^{1/2} \log r + \widetilde{\phi},$$

where $\widetilde{\phi}$ is continuously differentiable with $\widetilde{\phi}(r) = \mathcal{O}(r^{3/2})$, $\widetilde{\phi}'(r) = \mathcal{O}(r^{1/2})$, and $\Delta\widetilde{\phi} \in L^2([0,R])$. Then as a consequence of Theorem 3.2, we know that

$$\Delta_L := \Delta : \mathscr{D}_L \to L^2([0,R]) \tag{4.2}$$

is self-adjoint, where

$$\mathscr{D}_L = \{\phi \in \mathscr{D}_{\max}(\Delta) \mid \cos\theta_1\, c_1(\phi) + \sin\theta_1\, c_2(\phi) = 0\ ,\ \cos\theta_2\, \phi'(R) + \sin\theta_2\, \phi(R) = 0\}.$$

When $\theta_1 = \pi/2$, then we are requiring $c_2(\phi)$ vanish so that near $r = 0$, we have

$$\phi = c_1(\phi)\, r^{1/2} + \widetilde{\phi};$$

that is, no log terms; in [8], Brüning and Seeley prove that $\theta_1 = \pi/2$ is the Friedrichs realization of the operator Δ acting on smooth functions supported away from $r = 0$ with the boundary condition $\cos\theta_2\, \phi'(R) + \sin\theta_2\, \phi(R) = 0$ at $r = R$. As seen in Theorem 1.1 in the Introduction, this self-adjoint realization gives rise to the "usual" resolvent, heat kernel, and zeta function properties. When $\theta_1 \neq \pi/2$, we get very pathological properties as shown in Theorem 1.2. In the following sections we enter in the proof of Theorem 1.2.

5. Eigenvalues with Dirichlet conditions at $r = R$

As shown in detail in Section 10, the strange behaviors depicted in Theorem 1.2 do not depend on the choice of the Lagrangian L_2 (that is, the choice of boundary condition at $r = R$). For this reason, we shall use $\theta_2 = 0$ for the Lagrangian L_2 in (4.2); thus, we shall consider the self-adjoint operator $\Delta_\theta := \Delta : \mathscr{D}_\theta \to L^2([0,R])$, where $0 \le \theta < \pi$ and $\theta \neq \pi/2$, and

$$\mathscr{D}_\theta = \{\phi \in \mathscr{D}_{\max}(\Delta) \mid \cos\theta\, c_1(\phi) + \sin\theta\, c_2(\phi) = 0\ ,\ \phi(R) = 0\};$$

so we are simply imposing the Dirichlet condition at $r = R$.

We now find an explicit formula for the eigenfunctions and a transcendental equation, which determines the spectrum of Δ_θ. We begin with the following eigenvalue equation:

$$(\Delta_\theta - \mu^2)\phi = 0 \quad \Longleftrightarrow \quad \phi'' + \frac{1}{4r^2}\phi + \mu^2\phi = 0.$$

We can turn this into a Bessel equation via the usual trick by setting $\phi = r^{1/2}\psi(\mu r)$. Then,

$$\phi'' = -\frac{1}{4}r^{-3/2}\psi(\mu r) + \mu r^{-1/2}\psi'(\mu r) + \mu^2 r^{1/2}\psi''(\mu r),$$

so

$$\phi'' + \frac{1}{4r^2}\phi + \mu^2\phi = 0 \quad \Longleftrightarrow \quad \mu r^{-1/2}\psi'(\mu r) + \mu^2 r^{1/2}\psi''(\mu r) + \mu^2 r^{1/2}\psi(\mu r) = 0,$$

or

$$(\mu r)^2\,\psi''(\mu r) + (\mu r)\,\psi'(\mu r) + (\mu r)^2\,\psi(\mu r) = 0.$$

For fixed μ, the solutions to this equation are linear combinations of J_0 and Y_0 (with Y_0 the Bessel function of the second kind), so

$$\phi = C_1 r^{1/2} J_0(\mu r) + C_2 r^{1/2} Y_0(\mu r).$$

Using that [1, p. 360]

$$\frac{\pi}{2}Y_0(z) := \big(\log z - \log 2 + \gamma\big)J_0(z) - \sum_{k=1}^{\infty}\frac{H_k(-\frac{1}{4}z^2)^k}{(k!)^2}, \tag{5.1}$$

where $H_k := 1+\frac{1}{2}+\cdots+\frac{1}{k}$, the form (2.1) for $\phi \in \mathscr{D}_{\max}(\Delta)$ is obtained by choosing the constants C_1 and C_2 in such a way that

$$\phi = c_1(\phi)\,r^{1/2}\,J_0(\mu r) + c_2(\phi)\,r^{1/2}\left(\frac{\pi}{2}Y_0(\mu r) - (\log\mu - \log 2 + \gamma)\,J_0(\mu r)\right).$$

By definition of the Bessel function [1, p. 360], we have as $z \to 0$,

$$\begin{aligned} J_v(z) &= \frac{z^v}{2^v}\sum_{k=0}^{\infty}\frac{(-\frac{1}{4}z^2)^k}{k!\,\Gamma(v+k+1)} \\ &= \frac{z^v}{2^v\Gamma(1+v)}\left(1 - \frac{z^2}{4(1+v)} + \frac{z^4}{32(1+v)(2+v)} - + \cdots\right) \end{aligned} \tag{5.2}$$

and by (5.1), we see that

$$\phi = c_1(\phi)\,r^{1/2} + c_2(\phi)\,r^{1/2}\,\log r + \mathcal{O}((\mu r)^2),$$

where $\mathcal{O}((\mu r)^2)$ is a power series in $(\mu r)^2$ vanishing like $(\mu r)^2$ as $r \to 0$. Therefore, by definition of $\mathscr{D}_\theta$, we have

$$\cos\theta\, c_1(\phi) + \sin\theta\, c_2(\phi) = 0. \tag{5.3}$$

To satisfy the Dirichlet condition at $r = R$, we must have

$$c_1(\phi)\,J_0(\mu R) + c_2(\phi)\left(\frac{\pi}{2}Y_0(\mu R) - (\log\mu - \log 2 + \gamma)\,J_0(\mu R)\right) = 0.$$

It follows that

$$\det\begin{pmatrix} \cos\theta & \sin\theta \\ J_0(\mu R) & \frac{\pi}{2}Y_0(\mu R) - (\log\mu - \log 2 + \gamma)\,J_0(\mu R) \end{pmatrix} = 0,$$

or $\frac{\pi}{2}Y_0(\mu R) - (\log\mu - \log 2 + \gamma)\,J_0(\mu R) = \tan\theta\, J_0(\mu R)$. We summarize our findings in the following proposition.

Proposition 5.1. *The transcendental equation*

$$F(\mu) := \frac{\pi}{2} Y_0(\mu R) - (\log \mu - \kappa)\, J_0(\mu R) = 0\ , \qquad \kappa = \log 2 - \gamma - \tan\theta \tag{5.4}$$

determines the eigenvalues of Δ_θ.

In the following theorem we state various properties of the eigenvalues of Δ_θ; note that in [15, p. 4572] it is stated that there are no negative eigenvalues; however, it turns out that for example when $\pi/2 < \theta < \pi$ and $R \geq 1$, there is *always* a negative eigenvalue.

Theorem 5.2. *For* $0 \leq \theta < \pi$ *with* $\theta \neq \frac{\pi}{2}$,
(1) Δ_θ *has a zero eigenvalue if and only if* $\log R = \tan\theta$.
(2) Δ_θ *has a unique negative eigenvalue if and only if* $\tan\theta < \log R$.

Proof. Using (5.1) and the expansion

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}z^2)^k}{(k!)^2}, \tag{5.5}$$

the eigenvalue equation $\frac{\pi}{2} Y_0(\mu R) - (\log \mu - \log 2 + \gamma)\, J_0(\mu R) = \tan\theta\, J_0(\mu R)$ can be written as

$$-\sum_{k=1}^{\infty} \frac{H_k(-\frac{1}{4}\mu^2 R^2)^k}{(k!)^2} = (-\log R + \tan\theta) \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}\mu^2 R^2)^k}{(k!)^2}. \tag{5.6}$$

Thus, $\mu = 0$ solves this equation if and only if $\log R = \tan\theta$.

Now Δ_θ has a negative eigenvalue means that $\mu = ix$ for x real solves (5.6):

$$-\sum_{k=1}^{\infty} \frac{H_k(\frac{1}{4}x^2 R^2)^k}{(k!)^2} = (-\log R + \tan\theta) \sum_{k=0}^{\infty} \frac{(\frac{1}{4}x^2 R^2)^k}{(k!)^2}. \tag{5.7}$$

If $(-\log R + \tan\theta) > 0$, then (5.7) has no solutions because the right-hand side of (5.7) will be strictly positive for all real x while the left-hand side of (5.7) is nonpositive. Thus, we may assume that $\alpha := \log R - \tan\theta > 0$. Then we can write (5.7) as

$$f(xR) = 0 \quad \text{where} \quad f(x) = \sum_{k=1}^{\infty} \frac{H_k\, x^{2k}}{4^k (k!)^2} - \sum_{k=0}^{\infty} \frac{\alpha x^{2k}}{4^k (k!)^2};$$

thus, we just have to prove that $f(x) = 0$ has a unique solution. To prove this, observe that since the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges, we can choose $N \in \mathbb{N}$ such that $H_N > \alpha > H_{N-1}$. We now write

$$f(x) = \sum_{k=N}^{\infty} \frac{(H_k - \alpha)\, x^{2k}}{4^k (k!)^2} - \left(\sum_{k=0}^{N-1} \frac{(\alpha - H_k)\, x^{2k}}{4^k (k!)^2} \right),$$

where $H_0 := 0$, and note that $f(x) = 0$ if and only if $g(x) = 0$ where

$$g(x) := x^{-2N} f(x) = \sum_{k=N}^{\infty} \frac{(H_k - \alpha)\, x^{2(k-N)}}{4^k (k!)^2} - \left(\sum_{k=0}^{N-1} \frac{(\alpha - H_k)}{4^k (k!)^2\, x^{2(N-k)}} \right).$$

Because of the powers of x in the denominator the second sum on the right, we see that $g(0+) = -\infty$ while because of the first sum on the right, we see that

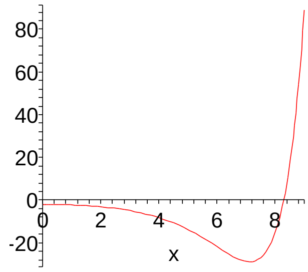

FIGURE 1. Graph of $f(x)$ when $R = 1$ and $\tan\theta = -2$.

$\lim_{x\to\infty} g(x) = \infty$. In particular, by the intermediate value theorem, $g(x) = 0$ for some $0 < x < \infty$. Since

$$g'(x) = \sum_{k=N+1}^{\infty} \frac{(H_k - \alpha)\, 2(k-N)x^{2(k-N)-1}}{4^k (k!)^2} + \left(\sum_{k=0}^{N-1} \frac{2(N-k)(\alpha - H_k)}{4^k (k!)^2\, x^{2(N-k)+1}} \right) > 0$$

the function g is strictly increasing, so there is only one $x > 0$ such that $g(x) = 0$. It follows that $f(x) = 0$ for a unique $x > 0$ and our proof is now complete. A graph of $f(x)$ for $R = 1$ and $\tan\theta = -2$ is shown in Figure 1. □

6. The ζ-function with Dirichlet conditions at $r = R$

Let $0 \le \theta < \pi$ with $\theta \neq \pi/2$. We now analyze the zeta function using the contour integral techniques developed in [4, 5, 29].

In Appendix A, Theorem A.1, we have shown that

$$\operatorname{Tr}(\Delta_\theta - \mu^2)^{-1} = -\frac{1}{2\mu}\frac{d}{d\mu}\log F(\mu).$$

Therefore, for $\Re s > 1/2$, by *definition* the zeta function is given by

$$\zeta(s, \Delta_\theta) = \frac{1}{2\pi i}\int_\gamma \mu^{-2s}\frac{d}{d\mu}\log F(\mu) d\mu = \frac{1}{2\pi i}\int_\gamma \mu^{-2s}\frac{F'(\mu)}{F(\mu)} d\mu, \tag{6.1}$$

where γ is a contour in the plane shown in Figure 2.

To analyze properties of the zeta function, we need the following technical lemma.

Lemma 6.1. *Let $0 \le \theta < \pi$ with $\theta \neq \frac{\pi}{2}$ and $\Upsilon \subset \mathbb{C}$ be a sector (closed angle) in the right-half plane. Then as $|x| \to \infty$ with $x \in \Upsilon$, we have*

$$F(ix) \sim -\frac{1}{\sqrt{2\pi}}(\log x - \kappa)(xR)^{-\frac{1}{2}} e^{xR}\Big(1 + \frac{1}{8xR} + \frac{9}{2(8xR)^2} + \mathcal{O}(x^{-3})\Big), \tag{6.2}$$

where $\mathcal{O}(x^{-3})$ is a power series in x^{-1} starting from x^{-3}, and

$$\frac{d}{dx}\log F(ix) \sim \frac{1}{x(\log x - \kappa)} + R - \frac{1}{2x} - \frac{1}{8x^2 R} + \mathcal{O}(x^{-3}), \tag{6.3}$$

with the same meaning for $\mathcal{O}(x^{-3})$. Finally, $F(\mu)$ is an even function of μ, and as $\mu \to 0$, we have

$$F(\mu) \sim (\log R - \tan\theta) + \frac{1}{4}\mu^2 R^2 (1 + \tan\theta - \log R) + \mathcal{O}(\mu^4). \tag{6.4}$$

Proof. From (5.1), we have

$$\begin{aligned}\frac{\pi}{2}Y_0(ix) &= \big(\log(ix) - \log 2 + \gamma\big)J_0(ix) - \sum_{k=1}^{\infty}\frac{H_k(-\frac{1}{4}(ix)^2)^k}{(k!)^2}\\ &= \big(\log x + i\frac{\pi}{2} - \log 2 + \gamma\big)I_0(x) - \sum_{k=1}^{\infty}\frac{H_k(\frac{1}{4}x^2)^k}{(k!)^2}\\ &= i\frac{\pi}{2}I_0(x) - K_0(x),\end{aligned}$$

where $I_0(x)$ is the modified Bessel function of the first kind, and

$$K_0(x) := -\big(\log x - \log 2 + \gamma\big)I_0(x) + \sum_{k=1}^{\infty}\frac{H_k(\frac{1}{4}x^2)^k}{(k!)^2}$$

is the modified Bessel function of the second kind. Therefore,

$$\begin{aligned}F(ix) &= \frac{\pi}{2}Y_0(ixR) - (\log(ix) - \kappa)J_0(ixR)\\ &= i\frac{\pi}{2}I_0(xR) - K_0(xR) - \Big(\log x + i\frac{\pi}{2} - \kappa\Big)I_0(xR)\\ &= -(\log x - \kappa)I_0(xR) - K_0(xR).\end{aligned}$$

By [1, p. 377], as $|x| \to \infty$ for $x \in \Upsilon$, we have

$$I_0(x) \sim \frac{e^x}{\sqrt{2\pi x}}\Big(1 + \frac{1}{8x} + \frac{9}{2(8x)^2} + \mathcal{O}(x^{-3})\Big)$$

where $\mathcal{O}(x^{-3})$ is a power series in x^{-1} starting with x^{-3}; furthermore [1, p. 378], as $|x| \to \infty$ for $x \in \Upsilon$,

$$K_0(x) \sim \sqrt{\frac{2}{\pi x}}e^{-x}\Big(1 - \frac{1}{8x} + \frac{9}{2(8x)^2} + \mathcal{O}(x^{-3})\Big).$$

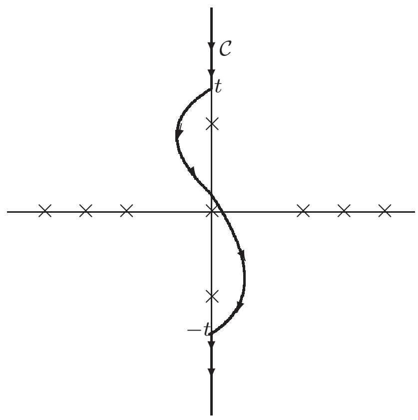

FIGURE 2. The contour γ for the zeta function. The $\times$'s represent the zeros of $F(\mu)$. The squares of the $\times$'s on the imaginary axis represent the negative eigenvalues of Δ_θ. Here, t is on the imaginary axis and is larger in absolute value than the absolute value of the negative eigenvalue of Δ_θ (if one exists). The contour γ_t goes from t to $-t$.

Therefore, as $|x| \to \infty$ for $x \in \Upsilon$, we have

$$\begin{aligned} F(ix) &\sim -(\log x - \kappa) I_0(xR) \\ &\sim -\frac{1}{\sqrt{2\pi}}(\log x - \kappa)(xR)^{-\frac{1}{2}} e^{xR}\Big(1 + \frac{1}{8xR} + \frac{9}{2(8xR)^2} + \mathcal{O}(x^{-3})\Big), \end{aligned}$$

which proves (6.2). Taking logarithms, we see that as $|x| \to \infty$ for $x \in \Upsilon$, we have

$$\log F(ix) \sim c + \log(\log x - \kappa) - \frac{1}{2}\log x + xR + \log\Big(1 + \frac{1}{8xR} + \frac{9}{2(8xR)^2} + \mathcal{O}(x^{-3})\Big),$$

where c is a constant. Since $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - + \cdots$, we have

$$\log F(ix) \sim c + \log(\log x - \kappa) - \frac{1}{2}\log x + xR + \frac{1}{8xR} + \mathcal{O}(x^{-2}).$$

Taking the derivative of this we get (6.3).

To determine the asymptotics as $\mu \to 0$, recalling that $\kappa = \log 2 - \gamma - \tan\theta$, we see that

$$\begin{aligned} F(\mu) &= \frac{\pi}{2} Y_0(\mu R) - (\log \mu - \kappa) J_0(\mu R) \\ &= \frac{\pi}{2} Y_0(\mu R) - (\log \mu - \log 2 + \gamma) J_0(\mu R) - \tan\theta J_0(\mu R) \\ &= \frac{1}{4}\mu^2 R^2 + (\log R - \tan\theta) + (\tan\theta - \log R)\cdot \frac{1}{4}\mu^2 R^2 + \mathcal{O}(\mu^4) \\ &= (\log R - \tan\theta) + \frac{1}{4}\mu^2 R^2 (1 + \tan\theta - \log R) + \mathcal{O}(\mu^4), \end{aligned}$$

where we used (5.1) and (5.5) in passing from the second to the third line. In particular, the second line with (5.1) and (5.5) shows that $F(\mu)$ is an even function of μ. This completes our proof. □

We need one more lemma.

Lemma 6.2. *We have*

$$\int_{|t|}^{\infty} x^{-2s} \frac{1}{x(\log x - \kappa)}\, dx = -e^{-2s\kappa}\log s - e^{-2s\kappa}\Big(\gamma + \log(2(\log|t| - \kappa)) + \mathcal{O}(s)\Big),$$

where $\mathcal{O}(s)$ is an entire function of s that is $\mathcal{O}(s)$ at $s = 0$.

Proof. In the integral we assume that $\log|t| > \kappa$ so that the integral is well-defined. Now to analyze this integral we make the change of variables $u = \log x - \kappa$ or $x = e^{\kappa} e^{u}$, and obtain

$$\int_{|t|}^{\infty} x^{-2s} \frac{1}{x(\log x - \kappa)}\, dx = e^{-2s\kappa} \int_{\log|t| - \kappa}^{\infty} e^{-2su} \frac{du}{u}.$$

Making the change of variables $y = 2su$, we get

$$\int_{|t|}^{\infty} x^{-2s} \frac{1}{x(\log x - \kappa)}\, dx = e^{-2s\kappa} \int_{2s(\log|t| - \kappa)}^{\infty} e^{-y} \frac{dy}{y}.$$

Recall that the *exponential integral* is defined by (see [23, Sec. 8.2])

$$\mathrm{Ei}(z) := -\int_{-z}^{\infty} e^{-y} \frac{dy}{y}.$$

Therefore,

$$\int_{|t|}^{\infty} x^{-2s} \frac{1}{x(\log x - \kappa)}\, dx = -e^{-2s\kappa} \mathrm{Ei}\big(- 2s(\log|t| - \kappa)\big).$$

Also from [23, p. 877], we have

$$\mathrm{Ei}(z) = \gamma + \log(-z) + \sum_{k=1}^{\infty} \frac{z^k}{k \cdot k!},$$

therefore

$$\begin{aligned} \int_{|t|}^{\infty} x^{-2s} \frac{1}{x(\log x - \kappa)}\, dx &= -e^{-2s\kappa}\Big(\gamma + \log(2s(\log|t| - \kappa)) + \mathcal{O}(s)\Big) \\ &= -e^{-2s\kappa} \log s - e^{-2s\kappa}\Big(\gamma + \log(2(\log|t| - \kappa)) + \mathcal{O}(s)\Big), \end{aligned} \tag{6.5}$$

where $\mathcal{O}(s)$ is an entire function of s that is $\mathcal{O}(s)$ at $s = 0$. This completes our proof. □

We now determine the structure of the zeta function.

Proposition 6.3. *The zeta function $\zeta(s, \Delta_\theta)$ can be written in the form*

$$\zeta(s, \Delta_\theta) = -\frac{e^{-2s\kappa} \sin \pi s}{\pi} \log s + \zeta_\theta(s),$$

where $\kappa = \log 2 - \gamma - \tan\theta$ and $\zeta_\theta(s)$ extends from $\Re s > 1/2$ to a holomorphic function on $\mathbb{C}$ with poles at $s = 1/2 - k$ for $k = 0, 1, 2, \ldots$. In particular, $\zeta(s, \Delta_\theta)$ has $s = 0$ as a logarithmic branch point!

Proof. Recalling (6.1), we write

$$\int_\gamma = -\int_t^{0+i\infty} + \int_{-t}^{0-i\infty} + \int_{\gamma_t},$$

where γ_t is the part of γ from t to $-t$, and using that

$$i^{-2s} = (e^{i\pi/2})^{-2s} = e^{-i\pi s} \quad \text{and} \quad (-i)^{-2s} = (e^{-i\pi/2})^{-2s} = e^{i\pi s},$$

we obtain the integral

$$\begin{aligned} \zeta(s, \Delta_\theta) &= \frac{1}{2\pi i} \int_\gamma \mu^{-2s} \frac{d}{d\mu} \log F(\mu)\, d\mu \\ &= \frac{1}{2\pi i}\Big\{ - \int_{|t|}^{\infty} (ix)^{-2s} \frac{d}{dx} \log F(ix)\, dx + \int_{|t|}^{\infty} (-ix)^{-2s} \frac{d}{dx} \log F(-ix)\, dx \Big\} \\ &\qquad + \frac{1}{2\pi i} \int_{\gamma_t} \mu^{-2s} \frac{F'(\mu)}{F(\mu)}\, d\mu \\ &= \frac{1}{2\pi i}\Big(- e^{-i\pi s} + e^{i\pi s}\Big) \int_{|t|}^{\infty} x^{-2s} \frac{d}{dx} \log F(ix)\, dx + \frac{1}{2\pi i} \int_{\gamma_t} \mu^{-2s} \frac{F'(\mu)}{F(\mu)}\, d\mu, \end{aligned}$$

or,

$$\zeta(s, \Delta_\theta) = \frac{\sin \pi s}{\pi} \int_{|t|}^{\infty} x^{-2s} \frac{d}{dx} \log F(ix)\, dx + \frac{1}{2\pi i} \int_{\gamma_t} \mu^{-2s} \frac{F'(\mu)}{F(\mu)}\, d\mu, \tag{6.6}$$

a formula that will be analyzed in a moment. The second integral here is over a finite contour so an entire function of $s \in \mathbb{C}$, so we are left to analyze the analytic properties of the first integral. To do so, recall from (6.3) that for $x \to \infty$, we have

$$\frac{d}{dx} \log F(ix) \sim \frac{1}{x(\log x - \kappa)} + \sum_{k=0}^{\infty} \beta_k x^{-k},$$

for some constants β_k. Since

$$\frac{\sin \pi s}{\pi} \int_{|t|}^{\infty} x^{-2s-k}\, dx = \frac{\sin \pi s}{\pi} \frac{x^{-2s-k+1}}{-2s-k+1}\bigg|_{x=|t|}^{\infty} = \frac{\sin \pi s}{\pi} \frac{|t|^{-2s-k+1}}{2s+k-1}$$

which has poles at $s = (1-k)/2$ for $s \notin \mathbb{Z}$, it follows that the expansion $\sum_{k=0}^{\infty} \beta_k x^{-k}$ will contribute to the function $\zeta_\theta(s)$ in the statement of this proposition where $\zeta_\theta(s)$ extends from $\Re s > 1/2$ to a holomorphic function on $\mathbb{C}$ with poles at $s = 1/2 - k$ for $k = 0, 1, 2, \ldots$. Lemma 6.2 applied to the integral

$$\frac{\sin \pi s}{\pi} \int_{|t|}^{\infty} x^{-2s} \frac{1}{x(\log x - \kappa)}\, dx$$

now completes our proof. □

Remark 6.4. *The existence of the logarithmic branch point at $s = 0$ has been missed in* [15]. *The error occurs in equation (A13) where certain antiderivatives (specifically $xY_1(x)$ and $x^2 Y_1^2$) were accidentally set equal to zero at $s = 0$.*

7. Trace of the resolvent with Dirichlet conditions at $r = R$

Using the Theorem A.1, we can now prove Theorem 1.2 (1) for Δ_θ. We have chosen to present the results in a form where the first term has been expanded further; Theorem 1.2 (1) is contained in equation (7.1) of the proof and the explanation of the meaning of the expansion is similar to that found in Remark 1.5.

Proposition 7.1. *Let $\theta \neq \pi/2$ and $\kappa = \log 2 - \gamma - \tan\theta$, furthermore let $\Lambda \subset \mathbb{C}$ be any sector (solid angle) not intersecting the positive real axis. Then as $|\lambda| \to \infty$ with $\lambda \in \Lambda$, we have*

$$\mathrm{Tr}(\Delta_\theta - \lambda)^{-1} \sim (-\lambda)^{-1} \sum_{k=0}^{\infty} a_k \,(\log(-\lambda))^{-k-1} + \sum_{k=1}^{\infty} b_k \,(-\lambda)^{-k/2},$$

where $a_k = (2\kappa)^k$ for $k = 0, 1, 2, \ldots$ (in particular, $a_0 = 1 \neq 0$).

Proof. Setting $\lambda = -x^2$ with $x \in \Upsilon \subset \mathbb{C}$ a sector in the right-half plane, it suffices to prove that as $|x| \to \infty$ with $x \in \Upsilon$, we have

$$\mathrm{Tr}(\Delta_\theta + x^2)^{-1} \sim x^{-2} \sum_{k=0}^{\infty} \frac{\kappa^k}{2} \,(\log x)^{-k-1} + x^{-1} \sum_{k=0}^{\infty} b_k \, x^{-k},$$

or after multiplication by $2x$, we just have to prove that

$$2x \,\mathrm{Tr}(\Delta_\theta + x^2)^{-1} \sim x^{-1} \sum_{k=1}^{\infty} \kappa^k \,(\log x)^{-k-1} + \sum_{k=0}^{\infty} \beta_k \, x^{-k},$$

To prove this, we recall from Theorem A.1 that

$$2x \,\mathrm{Tr}(\Delta_\theta + x^2)^{-1} = \frac{d}{dx} \log F(ix).$$

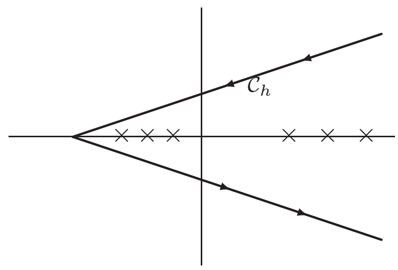

FIGURE 3. The contour γ.

By Lemma 6.1 (see (6.3)) we know that as $|x| \to \infty$ for $x \in \Upsilon$,

$$\frac{d}{dx} \log F(ix) \sim \frac{1}{x(\log x - \kappa)} + \sum_{k=0}^{\infty} \beta_k x^{-k}. \tag{7.1}$$

Finally, the expansion

$$\begin{aligned} \frac{1}{x(\log x - \kappa)} &= \frac{1}{x \log x} \cdot \frac{1}{(1 - \kappa(\log x)^{-1})} \\ &= \frac{1}{x \log x} \sum_{k=0}^{\infty} \kappa^k (\log x)^{-k} = x^{-1} \sum_{k=0}^{\infty} \kappa^k (\log x)^{-k-1}, \end{aligned}$$

concludes our result. □

As shown in the proof, for $|x| \to \infty$ with $x \in \Upsilon$, where Υ is a sector in the right-half plane, we have

$$2x \operatorname{Tr}(\Delta_\theta + x^2)^{-1} = \frac{d}{dx} \log F(ix) \sim \frac{1}{x(\log x - \kappa)} + \sum_{k=0}^{\infty} \beta_k x^{-k}, \tag{7.2}$$

or with $\lambda = -x^2$, as $|\lambda| \to \infty$ with $\lambda \in \Lambda \subset \mathbb{C}$, a sector not intersecting the positive real axis, we have

$$\operatorname{Tr}(\Delta_\theta - \lambda)^{-1} \sim \frac{1}{(-\lambda)(\log(-\lambda) - 2\kappa)} + \sum_{k=1}^{\infty} b_k (-\lambda)^{-k/2}. \tag{7.3}$$

This fact will be used in the next section.

8. THE HEAT TRACE WITH DIRICHLET CONDITIONS AT $r = R$

To determine the small-time heat asymptotics, we write

$$\operatorname{Tr}(e^{-t\Delta_\theta}) = \frac{i}{2\pi} \int_\gamma e^{-t\lambda} \operatorname{Tr}(\Delta_\theta - \lambda)^{-1} d\lambda$$

where γ is a counter-clockwise contour in the plane surrounding the eigenvalues of Δ_θ; see Figure 3. This is the starting point to show Theorem 1.2 (2) for Δ_θ. Again we have chosen to present the results in a form where the first term has been expanded further. This makes the actual structure of the small-t expansion more explicit; Theorem 1.2 (2) is contained in equation (8.2) of the proof.

Proposition 8.1. *As $t \to 0$, we have*

$$\operatorname{Tr}(e^{-t\Delta_\theta}) \sim \sum_{k=1}^{\infty} \alpha_k (\log t)^{-k} + \sum_{k=0}^{\infty} \beta_k \, t^{(k-1)/2}.$$

with the α_k's depending on κ via

$$\alpha_k = -\frac{1}{k\pi} \Im \left(\int_0^\infty e^{-x} \Big(\log x + i\pi - 2\kappa \Big)^k dx \right).$$

Proof. The small-time asymptotics are determined by the large-spectral parameter asymptotics of the resolvent. Now recall from (7.3) that as $|\lambda| \to \infty$ with λ in a sector not intersecting the positive real axis, we have

$$\operatorname{Tr}(\Delta_\theta - \lambda)^{-1} \sim \frac{1}{(-\lambda)(\log(-\lambda) - 2\kappa)} + \sum_{k=1}^{\infty} b_k (-\lambda)^{-k/2}.$$

Since (making the change of variables $\lambda \mapsto t^{-1}\lambda$)

$$\int e^{-t\lambda} (-\lambda)^{-k/2} \, d\lambda = t^{-1+k/2} \int e^{-\lambda} (-\lambda)^{-k/2} \, d\lambda$$

the series $\sum_{k=1}^{\infty} b_k \, (-\lambda)^{-k/2}$ gives rise to a small time expansion

$$\sum_{k=1}^{\infty} \beta_k \, t^{-1+k/2}.$$

Therefore, we just have to analyze the behavior of

$$\frac{i}{2\pi} \int_\gamma e^{-t\lambda} \frac{1}{(-\lambda)(\log(-\lambda) - 2\kappa)} d\lambda.$$

Deforming γ to the real line, the integral here is, modulo a term that is a smooth function of t at $t = 0$, equal to

$$-\int_1^\infty e^{-tx} \frac{1}{(-x)(\log(-(x+i0)) - 2\kappa)} dx + \int_1^\infty e^{-tx} \frac{1}{(-x)(\log(-(x-i0)) - 2\kappa)} dx,$$

or after simplification, this sum becomes

$$\int_1^\infty e^{-tx} \frac{1}{x(\log x - i\pi - 2\kappa)} dx - \int_1^\infty e^{-tx} \frac{1}{x(\log x + i\pi - 2\kappa)} dx.$$

(The reason we start at $x = 1$ is that $1/(x[\log x \pm i\pi - 2\kappa])$ is not integrable near $x = 0$.) Since for any complex number z, we have $i(\overline{z} - z) = 2 \Im z$, we see that modulo a term that is a smooth function of t at $t = 0$,

$$\begin{aligned} \frac{i}{2\pi} \int_\gamma e^{-t\lambda} \frac{1}{(-\lambda)(\log(-\lambda) - 2\kappa)} d\lambda &\equiv \frac{1}{\pi} \Im \left(\int_1^\infty e^{-tx} \frac{1}{x(\log x + i\pi - 2\kappa)} dx \right) \\ &= \frac{1}{\pi} \Im \, \ell(t), \end{aligned} \tag{8.1}$$

where

$$\ell(t) := \int_1^\infty e^{-tx} \frac{1}{x(\log x + i\pi - 2\kappa)} dx.$$

In summary, we have proved that

$$\operatorname{Tr}(e^{-t\Delta_\theta}) \sim \frac{1}{\pi} \Im \, \ell(t) + \sum_{k=0}^{\infty} \beta_k \, t^{(k-1)/2}, \tag{8.2}$$

which is exactly the statement of Theorem 1.2 (2).

We shall compute the asymptotics of $\ell(t)$ as $t \to 0$. To do so, observe that

$$\ell'(t) := -\int_1^\infty e^{-tx} \frac{1}{(\log x + i\pi - 2\kappa)} dx$$

Now $1/\log x$ is integrable near $x = 0$, so we can write

$$\ell'(t) = f(t) + g(t),$$

where

$$f(t) := -\int_0^\infty e^{-tx} \frac{1}{(\log x + i\pi - 2\kappa)} dx \quad , \quad g(t) := \int_0^1 e^{-tx} \frac{1}{(\log x + i\pi - 2\kappa)} dx.$$

Note that $g(t)$ is smooth at $t = 0$. We will now determine the asymptotics of $f(t)$ near $t = 0$. To this end, we make the change of variables $x \mapsto t^{-1}x$:

$$\begin{aligned} f(t) &= -t^{-1} \int_0^\infty e^{-x} \frac{1}{(\log(x/t) + i\pi - 2\kappa)} dx \\ &= t^{-1} (\log t)^{-1} \int_0^\infty e^{-x} \frac{1}{1 - \frac{\log x + i\pi - 2\kappa}{\log t}} dx. \end{aligned}$$

Since $(1-r)^{-1} = \sum_{k=0}^N r^k + r^{N+1}(1-r)^{-1}$ for any $N \in \mathbb{N}$, we see that for any $N \in \mathbb{N}$,

$$\begin{aligned} f(t) = t^{-1}(\log t)^{-1} \sum_{k=0}^N (\log t)^{-k} \int_0^\infty e^{-x} \Big(\log x + i\pi - 2\kappa \Big)^k dx \quad + \\ t^{-1}(\log t)^{-1} \cdot (\log t)^{-N-1} \int_0^\infty e^{-x} \frac{\Big(\log x + i\pi - 2\kappa \Big)^{N+1}}{1 - \frac{\log x + i\pi - 2\kappa}{\log t}} dx. \end{aligned}$$

The last integral here is bounded in t as $t \to 0$. Since N is arbitrary, it follows that

$$f(t) \sim t^{-1} \sum_{k=0}^\infty (\log t)^{-k-1} \int_0^\infty e^{-x} \Big(\log x + i\pi - 2\kappa \Big)^k dx.$$

Therefore, since $\ell'(t) = f(t) + g(t)$, as $t \to 0$ we have

$$\ell'(t) \sim t^{-1} \sum_{k=0}^\infty (\log t)^{-k-1} \int_0^\infty e^{-x} \Big(\log x + i\pi - 2\kappa \Big)^k dx + \sum_{k=0}^\infty \gamma_k t^k.$$

Integrating both sides, using that

$$\int t^{-1} (\log t)^{-1}\, dt = \log|\log t| \quad , \quad \int t^{-1} (\log t)^{-k-1} dt = -\frac{1}{k} (\log t)^{-k} \quad \text{for} \quad k > 0,$$

we get

$$\ell(t) \sim \log|\log t| - \sum_{k=1}^\infty \frac{1}{k} (\log t)^{-k} \int_0^\infty e^{-x} \Big(\log x + i\pi - 2\kappa \Big)^k dx + \sum_{k=0}^\infty \delta_k t^k.$$

Finally, in view of (8.1), we see that as $t \to 0$,

$$\frac{i}{2\pi} \int_\gamma e^{-t\lambda} \frac{1}{(-\lambda)(\log(-\lambda) - 2\kappa)} d\lambda \sim \frac{1}{\pi} \Im \, \ell(t) \sim$$
$$- \sum_{k=1}^\infty \frac{1}{k\pi} (\log t)^{-k} \, \Im \left(\int_0^\infty e^{-x} \Big(\log x + i\pi - 2\kappa \Big)^k dx \right) + \sum_{k=0}^\infty \epsilon_k t^k.$$

□

9. The zeta determinant

By Proposition 6.3, we have

$$\zeta(s, \Delta_\theta) = - \frac{e^{-2s\kappa} \sin \pi s}{\pi} \log s + \zeta_\theta(s)$$

where $\zeta_\theta(s)$ extends from $\Re s > 1/2$ to a holomorphic function on $\mathbb{C}$ with poles at $s = 1/2 - k$ for $k = 0, 1, 2, \ldots$. Since $\frac{e^{-2s\kappa} \sin \pi s}{\pi} = s + \mathcal{O}(s^2)$, it follows that

$$\zeta_{\mathrm{reg}}(s, \Delta_\theta) := \zeta(s, \Delta_\theta) + s \log s$$

has a derivative at $s = 0$. Therefore, we can define

$$\det_{\mathrm{reg}}(\Delta_\theta) := \exp(-\zeta'_{\mathrm{reg}}(0, \Delta_\theta)),$$

which is computed in this section.

Recall that $0 \le \theta < \pi$ with $\theta \ne \frac{\pi}{2}$. The idea here is to make the first term in

$$\zeta(s, \Delta_\theta) = \frac{\sin \pi s}{\pi} \int_{|t|}^\infty x^{-2s} \frac{d}{dx} \log F(ix) \, dx + \frac{1}{2\pi i} \int_{\gamma_t} \mu^{-2s} \frac{F'(\mu)}{F(\mu)} \, d\mu$$

regular at $s = 0$, as the second term (being entire) is already regular at $s = 0$. In order to analytically continue the first term, we add and subtract off the leading asymptotics of $F(ix)$. Thus, recalling Lemma 6.1 (see (6.2))

$$F(ix) \sim C_0 (\log x - \kappa) x^{-\frac{1}{2}} e^{xR} \left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right) \quad \text{as } x \to \infty,$$

where $C_0 = -\frac{1}{\sqrt{2\pi R}}$, we consider

$$\int_{|t|}^\infty x^{-2s} \frac{d}{dx} \log F(ix) \, dx = \int_{|t|}^\infty x^{-2s} \frac{d}{dx} \log \Big(\frac{F(ix)}{C_0(\log x - \kappa) x^{-\frac{1}{2}} e^{xR}} \Big) \, dx$$
$$+ \int_{|t|}^\infty x^{-2s} \frac{d}{dx} \log \Big(C_0 (\log x - \kappa) x^{-\frac{1}{2}} e^{xR} \Big) \, dx.$$

The second integral can be computed explicitly:

$$\int_{|t|}^\infty x^{-2s} \frac{d}{dx} \log \Big(C_0 (\log x - \kappa) x^{-\frac{1}{2}} e^{xR} \Big) \, dx$$
$$= \int_{|t|}^\infty x^{-2s} \frac{1}{x(\log x - \kappa)} \, dx - \frac{|t|^{-2s}}{4s} + \frac{|t|^{-2s+1}}{2s-1} R.$$

Therefore,

$$\begin{aligned}\zeta(s,\Delta_\theta) = \frac{\sin\pi s}{\pi}\int_{|t|}^{\infty} x^{-2s}\frac{d}{dx}\log\Big(\frac{F(ix)}{C_0(\log x-\kappa)x^{-\frac{1}{2}}e^{xR}}\Big)\,dx \\ + \frac{\sin\pi s}{\pi}\int_{|t|}^{\infty} x^{-2s}\frac{1}{x(\log x-\kappa)}\,dx - \frac{\sin\pi s}{\pi}\frac{|t|^{-2s}}{4s} \\ + \frac{\sin\pi s}{\pi}\frac{|t|^{-2s+1}}{2s-1}R + \frac{1}{2\pi i}\int_{\gamma_t}\mu^{-2s}\frac{F'(\mu)}{F(\mu)}\,d\mu.\end{aligned}$$

Hence, as $\zeta_{\mathrm{reg}}(s,\Delta_\theta) = \zeta(s,\Delta_\theta) + s\log s$, we see that

$$\begin{aligned}\zeta_{\mathrm{reg}}(s,\Delta_\theta) = \frac{\sin\pi s}{\pi}\int_{|t|}^{\infty} x^{-2s}\frac{d}{dx}\log\Big(\frac{F(ix)}{C_0(\log x-\kappa)x^{-\frac{1}{2}}e^{xR}}\Big)\,dx \\ + \frac{\sin\pi s}{\pi}\int_{|t|}^{\infty} x^{-2s}\frac{1}{x(\log x-\kappa)}\,dx + s\log s \\ - \frac{\sin\pi s}{\pi}\frac{|t|^{-2s}}{4s} + \frac{\sin\pi s}{\pi}\frac{|t|^{-2s+1}}{2s-1}R + \frac{1}{2\pi i}\int_{\gamma_t}\mu^{-2s}\frac{F'(\mu)}{F(\mu)}\,d\mu.\end{aligned}$$

By Lemma 6.2, we have

$$\begin{aligned}\frac{\sin\pi s}{\pi}\int_{|t|}^{\infty} x^{-2s}\frac{1}{x(\log x-\kappa)}\,dx \\ = -\frac{e^{-2s\kappa}\sin\pi s}{\pi}\log s - \frac{e^{-2s\kappa}\sin\pi s}{\pi}\Big(\gamma+\log(2(\log|t|-\kappa)) + \mathcal{O}(s)\Big) \\ = -s\log s - s\Big(\gamma + \log(2(\log|t|-\kappa)) + \mathcal{O}(s\log s)\Big).\end{aligned}$$

Thus,

$$\begin{aligned}\zeta_{\mathrm{reg}}(s,\Delta_\theta) = \frac{\sin\pi s}{\pi}\int_{|t|}^{\infty} x^{-2s}\frac{d}{dx}\log\Big(\frac{F(ix)}{C_0(\log x-\kappa)x^{-\frac{1}{2}}e^{xR}}\Big)\,dx \\ - s\Big(\gamma+\log(2(\log|t|-\kappa)) + \mathcal{O}(s^2\log s)\Big) \\ - \frac{\sin\pi s}{\pi}\frac{|t|^{-2s}}{4s} + \frac{\sin\pi s}{\pi}\frac{|t|^{-2s+1}}{2s-1}R + \frac{1}{2\pi i}\int_{\gamma_t}\mu^{-2s}\frac{F'(\mu)}{F(\mu)}\,d\mu.\end{aligned}$$

The first integral on the right is regular at $s=0$ due to the asymptotics found in Lemma 6.1. Therefore, using that

$$\frac{\sin(\pi s)}{\pi}\Big|_{s=0} = 0\ ,\ \frac{d}{ds}\frac{\sin(\pi s)}{\pi}\Big|_{s=0} = 1\ ,\ \frac{\sin(\pi s)}{\pi s}\Big|_{s=0} = 1\ ,\ \frac{d}{ds}\frac{\sin(\pi s)}{\pi s}\Big|_{s=0} = 0,$$

we see that

$$\begin{aligned}\zeta'_{\mathrm{reg}}(0,\Delta_\theta) = \int_{|t|}^{\infty}\frac{d}{dx}\log\Big(\frac{F(ix)}{C_0(\log x-\kappa)x^{-\frac{1}{2}}e^{xR}}\Big)\,dx \\ -\Big(\gamma+\log(2(\log|t|-\kappa)\Big) + \frac{1}{2}\log|t| - |t|R - \frac{1}{\pi i}\int_{\gamma_t}\log\mu\frac{F'(\mu)}{F(\mu)}\,d\mu \\ = -\log\Big(\frac{F(i|t|)}{C_0}\Big) - \gamma - \log 2 - \frac{1}{\pi i}\int_{\gamma_t}\log\mu\frac{F'(\mu)}{F(\mu)}\,d\mu.\end{aligned}$$

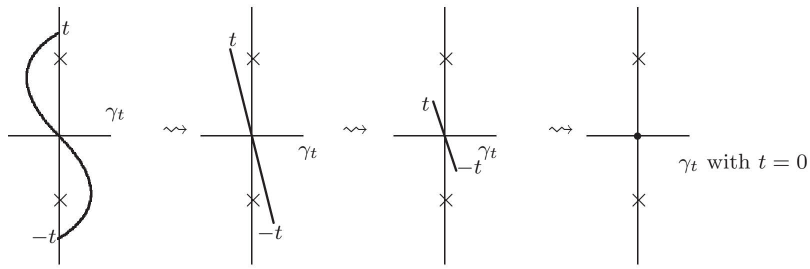

FIGURE 4. The contour γ_t as we let $t \to 0$ in $\mathscr{D}$ from the upper half plane.

Therefore,

$$\det_{\mathrm{reg}}(\Delta_\theta) = 2e^{\gamma} \frac{F(t)}{C_0} \cdot \exp\Big(\frac{1}{\pi i} \int_{\gamma_t} \log\mu \frac{F'(\mu)}{F(\mu)}\, d\mu\Big). \tag{9.1}$$

This formula is derived, a priori, when t is on the upper half of the imaginary axis. However, the right side is a *holomorphic* function of $t \in \mathscr{D}$, where $\mathscr{D}$ is the set of complex numbers minus the negative real axis and the zeros of $F(\mu)$. Therefore (9.1) holds for all $t \in \mathscr{D}$. Here, we recall that γ_t is any curve in $\mathscr{D}$ from t to $-t$. As before, the trick now is to let $t \to 0$ in (9.1).

First, assume that $\log R - \tan\theta \neq 0$ so that Δ_θ has no zero eigenvalue by Theorem 5.2. We determine the limit as $t \to 0$ of the exponential $\exp\big(\frac{1}{\pi i}\int_{\gamma_t} \log\mu \frac{F'(\mu)}{F(\mu)}\, d\mu\big)$. Let's take $t \to 0$ in $\mathscr{D}$ from the upper half plane as shown in Figure 4. In view of Figure 4, it follows that

$$\exp\Big(\frac{1}{\pi i} \int_{\gamma_t} \log\mu \frac{F'(\mu)}{F(\mu)}\, d\mu\Big) \to \exp\Big(0\Big) = 1.$$

Recalling (6.4), as $\mu \to 0$, we have

$$F(\mu) \;\sim\; (\log R - \tan\theta) + \frac{1}{4}\mu^2 R^2 (1 + \tan\theta - \log R) + \mathcal{O}(\mu^4).$$

In this case $F(0) = \log R - \tan\theta$. In conclusion, taking $t \to 0$ in (9.1), we see that

$$\det_{\mathrm{reg}}(\Delta_\theta) = 2\sqrt{2\pi R}\, e^{\gamma} (\tan\theta - \log R). \tag{9.2}$$

Second, assume now that $\log R - \tan\theta = 0$ so that as $\mu \to 0$, we have

$$F(\mu) \;\sim\; \frac{1}{4}\mu^2 R^2 (1 + \mathcal{O}(\mu^2)).$$

Let us put

$$\widetilde{F}(\mu) := \frac{F(\mu)}{\mu^2};$$

then $\widetilde{F}(\mu)$ is nonzero at $\mu = 0$ with value $\frac{R^2}{4}$, and

$$\zeta(s, \Delta_\theta) = \frac{\sin\pi s}{\pi} \int_{|t|}^{\infty} x^{-2s} \frac{d}{dx} \log\widetilde{F}(ix)\, dx + \frac{1}{2\pi i} \int_{\gamma_t} \mu^{-2s} \frac{\widetilde{F}'(\mu)}{\widetilde{F}(\mu)}\, d\mu.$$

By Lemma 6.1 (see (6.2)), we have

$$\widetilde{F}(ix) \sim \frac{C_0(\log x - \kappa) x^{-\frac{1}{2}} e^{xR}}{-x^2} = \widetilde{C}_0 (\log x - \kappa) x^{-\frac{5}{2}} e^{xR}, \quad \text{where } \widetilde{C}_0 = \frac{1}{\sqrt{2\pi R}}.$$

Now following the argument above used to prove (9.1), we can show

$$\det_{\mathrm{reg}}(\Delta_\theta) = 2e^{\gamma}\frac{\widetilde{F}(t)}{\widetilde{C}_0}\cdot\exp\Big(\frac{1}{\pi i}\int_{\gamma_t}\log\mu\,\frac{\widetilde{F}'(\mu)}{\widetilde{F}(\mu)}\,d\mu\Big).$$

Finally, taking $t\to 0$ as we did before, yields in the $\tan\theta=\log R$ case, the result

$$\det_{\mathrm{reg}}(\Delta_\theta) = 2e^{\gamma}\frac{R^2}{4\widetilde{C}_0} = \frac{R^2}{2}e^{\gamma}\sqrt{2\pi R} = \sqrt{\frac{\pi R}{2}}e^{\gamma}R^2.$$

10. General boundary conditions at $r = R$

We now prove Theorem 1.2. Let's briefly recall the set-up. Let $0\le\theta_1,\theta_2<\pi$ with $\theta_1\neq\pi/2$ and put $L := L_{\theta_1}\oplus L_{\theta_2}$. Then as a consequence of Theorem 3.2, we know that

$$\Delta_L := \Delta : \mathscr{D}_L \to L^2([0,R])$$

is self-adjoint, where

$$\mathscr{D}_L = \{\phi\in\mathscr{D}_{\max}(\Delta)\mid \cos\theta_1\,c_1(\phi)+\sin\theta_1\,c_2(\phi)=0\ ,\ \cos\theta_2\,\phi'(R)+\sin\theta_2\,\phi(R)=0\}.$$

The trick to proving Theorem 1.2 is to write the resolvent $(\Delta_L-\lambda)^{-1}$ in terms of $(\Delta_{\theta_1}-\lambda)^{-1}$ (same self-adjoint condition at $r=0$ but with the Dirichlet condition at $r=R$). To do so, let $\varrho(r)\in C^\infty((-\infty,\infty))$ be a non-decreasing function such that $\varrho(r)=0$ for $r\le 1/4$ and $\varrho(r)=1$ for $r\ge 3/4$. Given any real numbers $\alpha<\beta$, we define

$$\varrho_{\alpha,\beta}(r) := \varrho((r-\alpha)/(\beta-\alpha)). \tag{10.1}$$

Then $\varrho_{\alpha,\beta}(r)=0$ on a neighborhood of $\{r\le\alpha\}$ and $\varrho_{\alpha,\beta}(r)=1$ on a neighborhood of $\{r\ge\beta\}$. We define

$$\begin{aligned}\psi_1(r) &= \varrho_{R/2,3R/4}(r), \quad \psi_2(r) = 1-\psi_1(r),\\ \varphi_1(r) &= \varrho_{R/4,R/2}(r), \quad \varphi_2(r) = 1-\varrho_{3R/4,R}(r).\end{aligned} \tag{10.2}$$

Let $\Delta' := -\frac{d^2}{dr^2}-\frac{1}{4r^2}$ over $[\frac{R}{4},R]$ with the Dirichlet condition at $r=R/4$ and the condition $\cos\theta_2\,\phi'(R)+\sin\theta_2\,\phi(R)=0$ at $r=R$; note that since $r\ge R/4$ over $[\frac{R}{4},R]$, the operator Δ' is a true smooth elliptic operator over this interval with no singularities. We define

$$Q(\lambda) := \varphi_1(\Delta'-\lambda)^{-1}\psi_1 + \varphi_2(\Delta_{\theta_1}-\lambda)^{-1}\psi_2. \tag{10.3}$$

It follows that $Q(\lambda)$ maps into the domain $\mathscr{D}_L$ of Δ_L, and

$$\begin{aligned}(\Delta_L-\lambda)Q(\lambda) &= (\Delta_L-\lambda)\varphi_1(\Delta'-\lambda)^{-1}\psi_1 + (\Delta_L-\lambda)\varphi_2(\Delta_{\theta_1}-\lambda)^{-1}\psi_2\\ &= \varphi_1(\Delta'-\lambda)(\Delta'-\lambda)^{-1}\psi_1 + \varphi_2(\Delta_{\theta_1}-\lambda)(\Delta_{\theta_1}-\lambda)^{-1}\psi_2 + K_0(\lambda)\\ &= \psi_1+\psi_2+K_0(\lambda) = \mathrm{Id}+K_0(\lambda),\end{aligned}$$

where

$$K_0(\lambda) = [\Delta,\varphi_1](\Delta'-\lambda)^{-1}\psi_1 + [\Delta,\varphi_2](\Delta_{\theta_1}-\lambda)^{-1}\psi_2,$$

Because the supports of $[\Delta,\varphi_i]$ and ψ_i, where $i=1,2$, are disjoint, using the explicit formula (A.3) for the resolvent $(\Delta_{\theta_1}-\lambda)^{-1}$ and the properties of the resolvent of $(\Delta'-\lambda)^{-1}$ found in the work of Seeley [46] it is straightforward to check that $K_0(\lambda)$ is trace-class operator that vanishes to infinite order as $|\lambda|\to\infty$ for λ in any sector Λ of $\mathbb{C}$ not intersecting the positive real axis; we shall fix such a sector Λ from now on. In particular, forming the Neumann series, it follows that $\mathrm{Id} + K_0(\lambda)$ is invertible for $|\lambda|$ large with $\lambda \in \Lambda$ with

$$(\mathrm{Id} + K_0(\lambda))^{-1} = \mathrm{Id} + K(\lambda),$$

where $K(\lambda)$ has the same properties as $K_0(\lambda)$. Thus, multiplying both sides of $(\Delta_L - \lambda)Q(\lambda) = \mathrm{Id} + K_0(\lambda)$ by $\mathrm{Id} + K(\lambda)$, we see that

$$(\Delta_L - \lambda)^{-1} = Q(\lambda) + Q(\lambda)K(\lambda).$$

Therefore, as $|\lambda| \to \infty$ for $\lambda \in \Lambda$, we see that Proposition 7.1 holds also for $\mathrm{Tr}(\Delta_L - \lambda)^{-1}$. Now using the resolvent asymptotics, we can proceed to copy the proofs of Proposition 8.1 and 6.3. The proof of Theorem 1.2 is now complete.

Appendix A. The resolvent with Dirichlet conditions at $r = R$

In this Appendix, we compute the trace of the resolvent by explicitly finding the Schwartz kernel of the Bessel function. To do so, recall that the resolvent kernel of the differential operator $\Delta_\theta - \mu^2$ with given boundary conditions at $r = 0$ and $r = R$ can be expressed as follows (see Lemma 4.1 in [9] or [27, Sec. 3.3] for an elementary account):

$$\frac{-1}{W(p,q)} \begin{cases} p(r,\mu)\, q(s,\mu) & \text{for} \quad r \le s \\ p(s,\mu)\, q(r,\mu) & \text{for} \quad r \ge s, \end{cases}$$

where $p(r,\mu)$ and $q(r,\mu)$ are solutions of $\big(\Delta_L - \mu^2\big)\phi = 0$ satisfying the given boundary conditions at $r = 0$ and $r = R$, respectively, and where $W(p,q)$ is the Wronskian of (p,q). Recall that the general solution to $\big(\Delta_L - \mu^2\big)\phi = 0$ is

$$\phi = c_1(\phi)\, r^{1/2}\, J_0(\mu r) + c_2(\phi)\, r^{1/2} \Big(\frac{\pi}{2} Y_0(\mu r) - (\log\mu - \log 2 + \gamma)\, J_0(\mu r)\Big),$$

To satisfy the boundary condition at $r = 0$, we must have (see (5.3)):

$$\cos\theta\, c_1(\phi) + \sin\theta\, c_2(\phi) = 0.$$

Thus, we can take $c_2(\phi) = 1$ and $c_1(\phi) = -\tan\theta$; this gives

$$p(r,\mu) = r^{1/2} \Big(\frac{\pi}{2} Y_0(\mu r) - (\log\mu - \kappa)\, J_0(\mu r)\Big), \tag{A.1}$$

where $\kappa = \log 2 - \gamma - \tan\theta$. To determine $q(r,\mu)$ we use the less fancy formulation of the general solution:

$$\phi = C_1\, r^{1/2} J_0(\mu r) + C_2\, r^{1/2} Y_0(\mu r).$$

To satisfy the Dirichlet condition at $r = R$, we therefore take

$$q(r,\mu) = r^{1/2} \Big(Y_0(\mu R)\, J_0(\mu r) - J_0(\mu R)\, Y_0(\mu r)\Big). \tag{A.2}$$

The Wronskian is easily computed using that $W(r^{1/2}J_0(\mu r), r^{1/2}Y_0(\mu r)) = \frac{2}{\pi}$ (see [48]):

$$\begin{aligned} W(p,q) = \frac{\pi}{2}\, Y_0(\mu R)\, W(r^{1/2}Y_0(\mu r), & r^{1/2}J_0(\mu r)) \\ & + (\log\mu - \kappa) J_0(\mu R)\, W(r^{1/2}J_0(\mu r), r^{1/2}Y_0(\mu r)) \\ & = -Y_0(\mu R) + \frac{2}{\pi}(\log\mu - \kappa) J_0(\mu R). \end{aligned}$$

Therefore,

$$(\Delta_\theta - \mu^2)^{-1}(r,s) = \frac{1}{F(\mu)} \begin{cases} p(r,\mu)\, q(s,\mu) & \text{for} \quad r \le s \\ p(s,\mu)\, q(r,\mu) & \text{for} \quad r \ge s, \end{cases} \tag{A.3}$$

where p and q are given in (A.1) and (A.2), respectively, and where

$$F(\mu) := Y_0(\mu R) - \frac{2}{\pi}(\log \mu - \kappa) J_0(\mu R).$$

We now need to compute $\int_0^R p(r,\mu)\, q(r,\mu)\, dr$; that is,

$$\begin{aligned} &\int_0^R r \left(\frac{\pi}{2} Y_0(\mu r) - (\log \mu - \kappa)\, J_0(\mu r)\right) \Big(Y_0(\mu R)\, J_0(\mu r) - J_0(\mu R)\, Y_0(\mu r)\Big)\, dr \\ &\qquad = \frac{\pi}{2} Y_0(\mu R) \int_0^R r\, Y_0(\mu r)\, J_0(\mu r)\, dr - \frac{\pi}{2} J_0(\mu R) \int_0^R r\, Y_0(\mu r)^2\, dr \\ &\qquad + (\log \mu - \kappa) J_0(\mu R) \int_0^R r J_0(\mu r)\, Y_0(\mu r)\, dr \\ &\qquad - (\log \mu - \kappa) Y_0(\mu R) \int_0^R r\, J_0(\mu r)^2\, dr. \end{aligned} \tag{A.4}$$

We next use the indefinite integrals

$$\begin{aligned} \int r J_0(\mu r)^2\, dr &= \frac{r^2}{2}\big(J_0(\mu r)^2 + J_1(\mu r)^2\big) \\ \int r Y_0(\mu r)^2\, dr &= \frac{r^2}{2}\big(Y_0(\mu r)^2 + Y_1(\mu r)^2\big) \\ \int r Y_0(\mu r)\, J_0(\mu r)\, dr &= \frac{r^2}{2}\big(Y_0(\mu r) J_0(\mu r) + Y_1(\mu r) J_1(\mu r)\big), \end{aligned}$$

which we need to evaluate between $r = 0$ and $r = R$. Recalling (5.2), $zJ_0(z)$ and $zJ_1(z)$ vanish at $z = 0$. Also, by (5.1) $zY_0(z)$ also vanishes at $z = 0$. However, it is a remarkable fact, which may be easily overlooked, that since

$$\frac{\pi}{2} Y_1(z) = -\frac{1}{z} J_0(z) + \big(\log z - \log 2 + \gamma\big) J_1(z) - \frac{1}{2} z \sum_{k=1}^{\infty} \frac{k H_k (-\frac{1}{4} z^2)^{k-1}}{(k!)^2},$$

where we used that $Y_1(z) = -Y_0'(z)$ and $J_1(z) = -J_0'(z)$ from [1, p. 361], we have

$$\Big(zY_1(z)\Big)\Big|_{z=0} = -\frac{2}{\pi} \qquad \Longrightarrow \qquad \Big(z^2 Y_1(z)^2\Big)\Big|_{z=0} = \frac{4}{\pi^2}.$$

Therefore,

$$\begin{aligned} \int_0^R r J_0(\mu r)^2\, dr &= \frac{R^2}{2}\big(J_0(\mu R)^2 + J_1(\mu R)^2\big) \\ \int_0^R r Y_0(\mu r)^2\, dr &= \frac{R^2}{2}\Big(Y_0(\mu R)^2 + Y_1(\mu R)^2\Big) - \frac{1}{2}\Big(r^2 Y_1(\mu r)^2\Big)\Big|_{r=0}\Big) \\ &= \frac{R^2}{2}\big(Y_0(\mu R)^2 + Y_1(\mu R)^2\big) - \frac{2}{\pi^2 \mu^2} \\ \int_0^R r Y_0(\mu r)\, J_0(\mu r)\, dr &= \frac{R^2}{2}\big(Y_0(\mu R) J_0(\mu R) + Y_1(\mu R) J_1(\mu R)\big). \end{aligned}$$

Plugging these integrals into (A.4) and using the identity [1, p. 360]

$$J_1(z)\,Y_0(z) - J_0(z)\,Y_1(z) = \frac{2}{\pi z}$$

to simplify the expression obtained, we eventually arrive that

$$\begin{aligned}\int_0^R p(r,\mu)\,q(r,\mu)\,dr &= \frac{R}{2\mu}\Big(Y_1(\mu R) - \frac{2}{\pi}(\log\mu-\kappa)J_1(\mu R)\Big) + \frac{1}{\pi\mu^2}J_0(\mu R)\\ &= \frac{R}{2\mu}\Big(Y_1(\mu R) - \frac{2}{\pi}(\log\mu-\kappa)J_1(\mu R) + \frac{2}{\pi\mu R}J_0(\mu R)\Big).\end{aligned}$$

Using the fact that $J_0'(z) = -J_1(z)$ and $Y_0'(z) = -Y_1(z)$, we can write this as

$$\int_0^R p(r,\mu)\,q(r,\mu)\,dr = -\frac{1}{2\mu}\frac{d}{d\mu}\Big(Y_0(\mu R) - \frac{2}{\pi}(\log\mu-\kappa)J_0(\mu R)\Big) = -\frac{1}{2\mu}\frac{d}{d\mu}F(\mu).$$

where we recall that

$$F(\mu) := Y_0(\mu R) - \frac{2}{\pi}(\log\mu-\kappa)J_0(\mu R).$$

Since (see (A.3))

$$(\Delta_\theta - \mu^2)^{-1}(r,s) = \frac{1}{F(\mu)}\begin{cases} p(r,\mu)\,q(s,\mu) & \text{for} \quad r\le s\\ p(s,\mu)\,q(r,\mu) & \text{for} \quad r\ge s,\end{cases}$$

we have proved the following theorem,

Theorem A.1. *With $F(\mu) := Y_0(\mu R) - \frac{2}{\pi}(\log\mu-\kappa)J_0(\mu R)$, we have*

$$\begin{aligned}\mathrm{Tr}(\Delta_\theta-\mu^2)^{-1} &= -\frac{1}{2\mu}\frac{1}{F(\mu)}\frac{d}{d\mu}F(\mu)\\ &= -\frac{1}{2\mu}\frac{d}{d\mu}\log F(\mu).\end{aligned}$$

This theorem has been used to analyze the zeta function, resolvent, and heat kernel of Δ_θ in Sections 6–8.

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References

1. M. Abramowitz and I. A. Stegun (eds.), *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Dover Publications Inc., New York, 1992, Reprint of the 1972 edition.
2. J. Audretsch, U. Jasper, and V. D. Skarzhinsky, *A pragmatic approach to the problem of the self-adjoint extension of Hamilton operators with the Aharonov-Bohm potential*, J. Phys. A **28** (1995), no. 8, 2359–2367.
3. M. Bordag, B. Geyer, K. Kirsten, and E. Elizalde, *Zeta function determinant of the Laplace operator on the D-dimensional ball*, Comm. Math. Phys. **179** (1996), no. 1, 215–234.
4. M. Bordag, S. Dowker, and K. Kirsten, *Heat-kernels and functional determinants on the generalized cone*, Comm. Math. Phys. **182**, no. 2 (1996), 371–393.
5. M. Bordag, E. Elizalde, and K. Kirsten, *Heat kernel coefficients of the Laplace operator on the D-dimensional ball*, J. Math. Phys. **37** (1996), 895–916.
6. J. L. Borg and J. V. Pulé, *Pauli approximations to the self-adjoint extensions of the Aharonov-Bohm Hamiltonian*, J. Math. Phys. **44** (2003), no. 10, 4385–4410.

7. J. Brüning and R. Seeley, *Regular singular asymptotics*, Adv. in Math. **58** (1985), no. 2, 133–148.
8. ———, *The resolvent expansion for second order regular singular operators*, J. Funct. Anal. **73** (1987), no. 2, 369–429.
9. ———, *An index theorem for first order regular singular operators*, Amer. J. Math. **110** (1988), no. 4, 659–714.
10. W. Bulla and F. Gesztesy, *Deficiency indices and singular boundary conditions in quantum mechanics*, J. Math. Phys. **26** (1985), no. 10, 2520–2528.
11. S. A. Coon and B. R. Holstein, *Anomalies in quantum mechanics: the $1/r^2$ potential*, Amer. J. Phys. **70** (2002), no. 5, 513–519.
12. J. S. Dowker and R. Critchley, *Effective Lagrangian and energy momentum tensor in de Sitter space*, Phys. Rev. D **13** (1976), 3224-3232.
13. G. V. Dunne, J. Hur, C. Lee and H. Min, *Precise quark mass dependence of instanton determinant*, Phys. Rev. Lett. **94** (2005), 072001.
14. E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko, and S. Zerbini, *Zeta regularization techniques with applications*, World Scientific Publishing Co. Inc., River Edge, NJ, 1994.
15. H. Falomir, M. A. Muschietti, and P. A. G. Pisani, *On the resolvent and spectral functions of a second order differential operator with a regular singularity*, J. Math. Phys. **45** (2004), no. 12, 4560–4577.
16. H. Falomir, M. A. Muschietti, P. A. G. Pisani and R. Seeley, *Unusual poles of the ζ-functions for some regular singular differential operators*, J. Phys. A: Math. Gen. **36** (2003), 9991–10010.
17. H. Falomir and P. A. G. Pisani, *Hamiltonian self-adjoint extensions for $(2+1)$-dimensional Dirac particles*, J. Phys. A: Math. Gen. **34** (2001), no. 19, 4143–4154.
18. W. M. Frank, D. J. Land, and R. M. Spector, *Singular potentials*, Rev. Modern Phys. **43** (1971), no. 1, 36–98.
19. J.B. Gil, *Full asymptotic expansion of the heat trace for non-self-adjoint elliptic cone operators*, Math. Nachr. **250** (2003), 25–57.
20. J.B. Gil and G. Mendoza, *Adjoints of elliptic cone operators*, Am. J. Math. **125**, 357–408 (2003).
21. J.B. Gil, T. Krainer, and G. Mendoza, *Resolvents of elliptic cone operators*, preprint math.AP/0410176 at arXiv.org.
22. P. B. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, second ed., CRC Press, Boca Raton, FL, 1995.
23. I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, sixth ed., Academic Press Inc., San Diego, CA, 2000, Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger.
24. M. Harmer, *Hermitian symplectic geometry and extension theory*, J. Phys. A **33** (2000), no. 50, 9193–9203.
25. ———, *Hermitian symplectic geometry and the factorization of the scattering matrix on graphs*, J. Phys. A **33** (2000), no. 49, 9015–9032.
26. S. W. Hawking, *Zeta function regularization of path integrals in curved spacetime*, Comm. Math. Phys. **55** (1977), no. 2, 133–148.
27. F. B. Hildebrand, *Methods of applied mathematics*, second ed., Dover Publications Inc., New York, 1992.
28. B. S. Kay and U. M. Studer, *Boundary conditions for quantum mechanics on cones and fields around cosmic strings*, Comm. Math. Phys. **139** (1991), no. 1, 103–139.
29. K. Kirsten, *Spectral functions in mathematics and physics*, Chapman & Hall/CRC Press, Boca Raton, 2001.
30. K. Kirsten, P. Loya, and J. Park, *Exotic expansions and pathological properties of ζ-functions on conic manifolds*, Preprint, 2005.
31. K. Kirsten and A.J. McKane, *Functional determinants by contour integration methods*, Ann. Physics **308** (2003), no. 2, 502–527.
32. A. N. Kochubeĭ, *Extensions of a positive definite symmetric operator*, Dokl. Akad. Nauk Ukrain. SSR Ser. A (1979), no. 3, 168–171, 237.
33. ———, *Selfadjoint extensions of Schroedinger operators with singular potentials*, Order, disorder and chaos in quantum systems (Dubna, 1989), Oper. Theory Adv. Appl., vol. 46, Birkhäuser, Basel, 1990, pp. 221–227.

34. ———, *Selfadjoint extensions of the Schrödinger operator with a singular potential*, Sibirsk. Mat. Zh. **32** (1991), no. 3, 60–69.
35. V. Kostrykin and R. Schrader, *Kirchhoff's rule for quantum wires*, J. Phys. A **32** (1999), no. 4, 595–630.
36. T. Krainer, *Resolvents of elliptic boundary problems on conic manifolds*, preprint 2005.
37. M. Lesch, *Operators of Fuchs type, conical singularities, and asymptotic methods*, B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1997.
38. ———, *Determinants of Regular Singular Sturm-Liouville Operators*, Math. Nach. **194** (1998), 139–170.
39. P. Loya, P. McDonald and J. Park, *Zeta Regularized Determinants for Conic Manifolds*, Preprint (2005), http://www.math.binghamton.edu/paul/papers/.
40. E. A. Mooers, *Heat kernel asymptotics on manifolds with conic singularities*, J. Anal. Math. **78** (1999), 1–36.
41. S. P. Novikov, *Schrodinger operators on graphs and symplectic geometry*, The Arnoldfest (Toronto, ON, 1997), Fields Inst. Commun., vol. 24, Amer. Math. Soc., Providence, RI, 1999, pp. 397–413.
42. B. S. Pavlov, *The theory of extensions, and explicitly solvable models*, Uspekhi Mat. Nauk **42** (1987), no. 6(258), 99–131, 247.
43. D. B. Ray and I. M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Advances in Math. **7** (1971), 145–210.
44. E. Schrohe and J. Seiler, *The resolvent of closed extensions of cone differential operators*, to appear in Canadian Journal of Mathematics.
45. B.-W. Schulze, *Pseudo-differential operators on manifolds with singularities*, North-Holland Publishing Co., Amsterdam, 1991.
46. R. T. Seeley, *Analytic extension of the trace associated with elliptic boundary problems*, Amer. J. Math. **91** (1969), 963–983.
47. I. Tsutsui, T. Fülöp, and T. Cheon, *Connection conditions and the spectral family under singular potentials*, J. Phys. A **36** (2003), no. 1, 275–287.
48. Wolfram research, *BesselJ*, `http://functions.wolfram.com/BesselAiryStruveFunctions/BesselJ/`.

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