The ubiquitous ζ-function and some of its "usual" and "unusual" meromorphic properties

Klaus Kirsten

Department of Mathematics, Baylor University, Waco, TX 76798 U.S.A.

Paul Loya

Department of Mathematics, Binghamton University, Vestal Parkway East, Binghamton, NY 13902 U.S.A.

Jinsung Park

School of Mathematics, Korea Institute for Advanced Study 207-43, Cheongnyangni 2-dong, Dongdaemun-gu, Seoul 130-722 Korea

Abstract. In this contribution we announce a complete classification and new exotic phenomena of the meromorphic structure of ζ-functions associated to conic manifolds proved in [37]. In particular, we show that the meromorphic extensions of these ζ-functions have, in general, countably many logarithmic branch cuts on the nonpositive real axis and unusual locations of poles with arbitrarily large multiplicity. Moreover, we give a precise algebraic-combinatorial formula to compute the coefficients of the leading order terms of the singularities.

PACS numbers: 58J28, 58J52

1. Introduction

It is well known, that a precise understanding of the meromorphic structure of zeta functions for Laplace-type operators is very important and its applications in many areas of mathematics and physics are ubiquitous. For example, via its relation to the small t-asymptotic expansion of the heat kernel, the zeta function $\zeta(s,\Delta)$ associated with a Laplacian Δ on a smooth manifold with or without boundary encodes geometrical and topological information about the manifold, see e.g. [33]. In some detail, we have for the scalar Laplacian over a compact n-dimensional Riemannian manifold M that

$$(4\pi)^{\frac{n}{2}}\Gamma(s)\,\zeta(s,\Delta) \equiv \frac{\mathrm{Vol}(M)}{s-\frac{n}{2}} \;\pm\; \frac{\sqrt{\pi}\mathrm{Vol}(\partial M)}{2}\frac{1}{s-\frac{n-1}{2}}$$

modulo a function that is analytic at $s=\frac{n}{2}$, $s=\frac{n-1}{2}$, where the "+" sign is used for Neumann conditions, the "−" sign is used for Dirichlet conditions, and $\mathrm{Vol}(M)$, respectively $\mathrm{Vol}(\partial M)$ denote as usual the volume of M, respectively ∂M.

Furthermore, it is known in the same context that the zeta function $\zeta(s, \Delta)$ has a meromorphic extension to the whole complex plane with at most simple poles at the points $s = \frac{n-k}{2} \notin -\mathbb{N}_0$ for $k \in \mathbb{N}_0$ with $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Moreover, $\zeta(s, \Delta)$ is analytic at the points $s \in -\mathbb{N}_0$. In particular, $\zeta(s, \Delta)$ is analytic about $s = 0$ which allows to define a zeta regularized determinant. This has far reaching applications in quantum field theory, see e.g. [22, 23, 24, 35, 36] and in the context of the Reidemeister-Franz torsion [48]. There are many other examples where the meromorphic structure of zeta functions is crucial, found in, for instance, index theory, the study of the Casimir effect, the evaluation of trace anamolies, and so forth. We refer the reader to [9, 24, 33, 36, 52] for reviews. The basic properties mentioned are valid for *smooth* manifolds with local boundary conditions and are well known facts that have been exploited for decades.

The aim of this contribution is to show that the properties for zeta functions of Laplace-type operators on smooth manifolds are very special indeed and so are the applications based upon this structure. Here, we announce a new result for manifolds with conical singularities whose zeta functions possess unusual meromorphic structures unparalleled in the zeta function literature for Laplacians. Thus, the usual structure totally breaks down when the manifold has a conical singularity. We begin in Section 2 by reviewing the important subject of conic manifolds introduced by Cheeger [12, 13] and which appear in many areas of physics including when one studies the Aharonov-Bohm potential [1] (see also [3, 5, 19, 34, 43]), classical solutions of Einstein's equations [50], cosmic strings [53], global monopoles [4], and the Rindler metric [44], to name a few areas. Afterwards, in Section 3, we study the zeta function associated to general self-adjoint extensions of Laplace-type operators on conic manifolds and discuss their extraordinary properties including countably many unusual poles and logarithmic singularities. We also give an explicit algebraic-combinatorial formula to compute these singularities and show that such singularities occur even in simple examples.

Finally, we remark that one can always conjure up "*artificial*" zeta functions having unusual properties compared to the ones described at the beginning. For example, the zeta function associated with the prime numbers P,

$$\zeta(s) = \sum_{p \in P} p^{-s},$$

has a logarithmic branch cut at $s = 1$, see e.g. [49]. But for *natural* zeta functions, that is zeta functions of Laplacians on compact manifolds associated to geometric or physical problems, the unusual properties described here seem to be unique.

2. Conic manifolds

In this section we study Laplacians on conic manifolds. One way to understand operators over conic manifolds is to start with simplest conic manifolds.

2.1. Regions in $\mathbb{R}^2$ minus points

Let $\Omega \subset \mathbb{R}^2$ be any compact region and take polar coordinates $(x, y) \longleftrightarrow (r, \theta)$ centered at any fixed point in Ω. In these coordinates, the metric takes the form $dx^2 + dy^2 = dr^2 + r^2\, d\theta^2$, which is called a *conic metric*. The standard Laplacian on $\mathbb{R}^2$ takes the form

$$\Delta_{\mathbb{R}^2} = -\partial_x^2 - \partial_y^2 = -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\theta^2,$$

and, finally, the measure transforms to $dxdy = rdrd\theta$. Writing $\phi \in L^2(\Omega, rdrd\theta)$ as

$$\phi = r^{-1/2}\widetilde{\phi}, \tag{1}$$

where $\widetilde{\phi} := r^{1/2}\phi$, we have

$$\int_\Omega \phi(r,\theta)\, \psi(r,\theta)\, rdrd\theta = \int_\Omega \widetilde{\phi}(r,\theta)\, \widetilde{\psi}(r,\theta)\, drd\theta.$$

A short computation shows that

$$\Delta_{\mathbb{R}^2}\phi = \left(-\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\theta^2\right)\phi = r^{-1/2}\Delta\, \widetilde{\phi},$$

where $\Delta := -\partial_r^2 + \frac{1}{r^2}A_{\mathbb{S}^1}$ with $A_{\mathbb{S}^1} := -\partial_\theta^2 - \frac{1}{4}$. In conclusion: Under the isomorphism (1) (called a *Liouville transformation*), $L^2(\Omega, rdrd\theta)$ is identified with $L^2(\Omega, drd\theta)$, and

$$\Delta_{\mathbb{R}^2} \quad \longleftrightarrow \quad -\partial_r^2 + \frac{1}{r^2}A_{\mathbb{S}^1}, \qquad \text{where} \quad A_{\mathbb{S}^1} = -\partial_\theta^2 - \frac{1}{4}. \tag{2}$$

Notice that the eigenvalues of $A_{\mathbb{S}^1}$ are given by $\{k^2 - \frac{1}{4} \mid k \in \mathbb{Z}\}$, in particular, $A_{\mathbb{S}^1} \geq -\frac{1}{4}$.

2.2. Conic manifolds

Let M be a n-dimensional compact manifold with boundary Γ and let g be a smooth Riemannian metric on $M \setminus \partial M$. We assume that near Γ there is a collared neighborhood $\mathcal{U} \cong [0, \varepsilon)_r \times \Gamma$, where $\varepsilon > 0$ and the metric g is of product type $dr^2 + r^2 h$ with h a metric over Γ. Such a metric is called a *conic metric* and M is called a *conic manifold*, concepts introduced by Cheeger [12, 13]. As in the $\mathbb{R}^2$ case, using a Liouville transformation over the collar $\mathcal{U}$, $L^2(M, dg)$ is identified with $L^2(M, drdh)$ and the scalar Laplacian Δ_g is identified with

$$\Delta_g|_{\mathcal{U}} = -\partial_r^2 + \frac{1}{r^2}A_\Gamma, \qquad \text{where} \quad A_\Gamma = \Delta_\Gamma + \Big(\frac{1-n}{2}\Big)\Big(1 + \frac{1-n}{2}\Big) \tag{3}$$

and Δ_Γ is the Laplacian over Γ. Notice that $A_\Gamma \geq -\frac{1}{4}$ because the function $x(1+x)$ has the minimum value $-\frac{1}{4}$ (when $x = -\frac{1}{2}$).

Regular singular operators [8] generalize the example (3) as follows. Let E be a Hermitian vector bundle over M, let g be a metric on M of product-type $g = dr^2 + h$ over $\mathcal{U}$, and let Δ be a second order elliptic differential operator over $M \setminus \partial M$ that is symmetric on $C_c^\infty(M \setminus \partial M, E)$ such that the restriction of Δ to $\mathcal{U}$ has the "singular" form

$$\Delta|_{\mathcal{U}} = -\partial_r^2 + \frac{1}{r^2}A_\Gamma, \tag{4}$$

where A_Γ is a Laplace-type operator over Γ with $A_\Gamma \geq -\frac{1}{4}$.‡ The operator Δ is called

‡ This condition is needed for technical reasons; if $A_\Gamma \not\geq -\frac{1}{4}$, then Δ is not bounded below [8, 10].

a second order *regular singular operator*. We remark that the manifold M may have boundary components up to which Δ is smooth; at such components, we put local boundary conditions such as the Dirichlet or Neumann boundary conditions but we will not belabor this point. In view of (2) and (3), the Laplacian on a punctured region in $\mathbb{R}^2$ and the scalar Laplacian on a conic manifold are regular singular operators. Other examples include the Laplacian on forms and squares of Dirac operators on conic manifolds [8, 12, 13, 14, 36, 41, 45].

2.3. Self-adjoint extensions

In this kind of a setting there are different self-adjoint extensions

$$\Delta_{\mathfrak{D}} := \Delta : \mathfrak{D} \to L^2(M, E)$$

possible, where $\mathfrak{D} \subset \mathfrak{D}_{\max} := \{\phi \in L^2(M, E) \,|\, \Delta\phi \in L^2(M, E)\}$; for general references on self-adjoint extensions of Laplacians and their applications to physics see, e.g., [2, 6]. From Von Neumann's theory of self-adjoint extensions [12, 13, 32, 42, 47], the self-adjoint extensions of Δ are parameterized by Lagrangian subspaces in the eigenspaces of A_Γ with eigenvalues in the interval $[-\frac{1}{4}, \frac{3}{4})$. To describe these extensions, denote by

$$-\frac{1}{4} = \underbrace{\lambda_1 = \lambda_2 = \cdots = \lambda_{q_0}}_{=-\frac{1}{4}} < \underbrace{\lambda_{q_0+1} \leq \lambda_{q_0+2} \leq \cdots \leq \lambda_{q_0+q_1}}_{-\frac{1}{4}<\lambda_\ell<\frac{3}{4}} \tag{5}$$

the spectrum of A_Γ in the finite interval $[-\frac{1}{4}, \frac{3}{4})$ where each eigenvalue is counted according to its multiplicity. Then the self-adjoint extensions of Δ are in a one-to-one correspondence to the Lagrangian subspaces in $\mathbb{C}^{2q}$ where $q = q_0 + q_1$. We note that (see, e.g. [40]), a subspace $L \subset \mathbb{C}^{2q}$ is Lagrangian if and only if there exists $q \times q$ matrices $\mathcal{A}$ and $\mathcal{B}$ such that the rank of the $q \times 2q$ matrix $(\mathcal{A} \quad \mathcal{B})$ is q, $\mathcal{A}'\mathcal{B}^*$ is self-adjoint where $\mathcal{A}'$ is the matrix $\mathcal{A}$ with the first q_0 columns multiplied by -1, and $L = \{\phi \in \mathbb{C}^{2q} \,|\, (\mathcal{A} \quad \mathcal{B})\phi = 0\}$. Given such a subspace $L \subset \mathbb{C}^{2q}$ there exists a *canonically* associated domain $\mathfrak{D}_L \subset \mathfrak{D}_{\max}$ such that $\Delta_L := \Delta : \mathfrak{D}_L \to L^2(M, E)$ is self-adjoint. Any such self-adjoint extension has a discrete spectrum [42] and hence, if $\{\mu_j\}$ denotes the spectrum of Δ_L, then we can form the corresponding zeta function

$$\zeta(s, \Delta_L) := \sum_{\mu_j \neq 0} \frac{1}{\mu_j^s}.$$

For special self-adjoint extensions, like the Friedrichs extension, the zeta function has been studied by many people going back to the 70's [7, 8, 10, 11, 13, 15, 16, 20, 21, 28, 42, 45, 51]; the properties are similar to those for the smooth case described in the Introduction except perhaps for an additional pole at $s = 0$. On the other hand, for general self-adjoint extensions, the zeta function $\zeta(s, \Delta_L)$ has, in general, very pathological properties that remained unobserved and that we shall describe in the next section.

Zeta functions have also been studied for more general "cone operators," which generalize regular singular operators, see e.g. Gil [29]. For recent and ongoing work involving resolvents of general self-adjoint extensions of cone operators, which is the first step to a full understanding of zeta functions, see Gil et al. [30, 31] and Coriasco et al. [17].

3. Pathological zeta functions on conic manifolds

In this section we state our theorem that completely classifies the meromorphic structure of zeta functions $\zeta(s, \Delta_L)$ and we give concrete examples of the theorem.

3.1. The main theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be $q \times q$ matrices defining a Lagrangian $L \subset \mathbb{C}^{2q}$. Before stating the main result, we apply a straightforward three-step algebraic-combinatorial algorithm to $\mathcal{A}$ and $\mathcal{B}$ that we need for the statement.

Step 1: We define the function

$$p(x,y) := \det \begin{pmatrix} & \mathcal{A} & & & \mathcal{B} \\ x\,\mathrm{Id}_{q_0} & 0 & 0 & 0 & \\ 0 & \tau_1\, y^{2\nu_1} & 0 & 0 & \\ 0 & 0 & \ddots & 0 & \mathrm{Id}_q \\ 0 & 0 & 0 & \tau_{q_1}\, y^{2\nu_{q_1}} & \end{pmatrix}, \tag{6}$$

where Id_k denotes the $k \times k$ identity matrix and where

$$\nu_j := \sqrt{\lambda_{q_0+j} + \frac{1}{4}}\,, \qquad \tau_j = 2^{2\nu_j} \frac{\Gamma(1+\nu_j)}{\Gamma(1-\nu_j)}, \qquad j = 1, \dots, q_1.$$

Here, q_0, q_1, λ_j are explained in (5). Evaluating the determinant, we can write $p(x,y)$ as a "polynomial"

$$p(x,y) = \sum a_{j\alpha}\, x^j\, y^{2\alpha},$$

where the α's are linear combinations of $\nu_1, \dots, \nu_{q_1}$ and the $a_{j\alpha}$'s are constants. Let α_0 be the smallest of all α's with $a_{j\alpha} \neq 0$ and let j_0 be the smallest of all j's amongst the $a_{j\alpha_0} \neq 0$. Then factoring out the term $a_{j_0\alpha_0}\, x^{j_0}\, y^{2\alpha_0}$ in $p(x,y)$ we can write $p(x,y)$ in the form

$$p(x,y) = a_{j_0\alpha_0}\, x^{j_0}\, y^{2\alpha_0} \Big(1 + \sum b_{k\beta}\, x^k\, y^{2\beta}\Big) \tag{7}$$

for some constants $b_{k\beta}$ (equal to $a_{k\beta}/a_{j_0\alpha_0}$).

Step 2: Using formal power series expansion, we can write

$$\log\Big(1 + \sum b_{k\beta} x^k y^{2\beta}\Big) = \sum c_{\ell\xi}\, x^\ell\, y^{2\xi} \tag{8}$$

for some constants $c_{\ell\xi}$. The ξ's appearing in (8) are nonnegative, countable, and approach $+\infty$ unless $\beta = 0$ is the only β occurring in (7), in which case only $\xi = 0$ occurs in (8). Also, the ℓ's with $c_{\ell\xi} \neq 0$ for a fixed ξ are bounded below.

Step 3: For each ξ appearing in (8), define

$$p_\xi := \min\{\ell \le 0 \,|\, c_{\ell\xi} \neq 0\} \qquad \text{and} \qquad \ell_\xi := \min\{\ell > 0 \,|\, c_{\ell\xi} \neq 0\}, \tag{9}$$

whenever the sets $\{\ell \le 0 \,|\, c_{\ell\xi} \neq 0\}$ and $\{\ell > 0 \,|\, c_{\ell\xi} \neq 0\}$, respectively, are nonempty. Let P, respectively L, denote the set of ξ values for which the respective sets are nonempty. The following theorem is our main result [37].

Theorem 3.1: For an arbitrary Lagrangian L, the ζ-function $\zeta(s, \Delta_L)$ extends from $\Re s > \frac{n}{2}$ to a holomorphic function on $\mathbb{C} \setminus (-\infty, 0]$. Moreover, $\zeta(s, \Delta_L)$ can be written in the form

$$\zeta(s, \Delta_L) = \zeta_{\rm reg}(s, \Delta_L) + \zeta_{\rm sing}(s, \Delta_L),$$

where $\zeta_{\rm reg}(s, \Delta_L)$ has possible simple poles at the usual locations $s = \frac{n-k}{2}$ with $s \notin -\mathbb{N}_0$ for $k \in \mathbb{N}_0$ and at $s = 0$ if $\dim \Gamma > 0$, and where $\zeta_{\rm sing}(s, \Delta_L)$ has the following expansion:

$$\zeta_{\rm sing}(s, \Delta_L) = \frac{\sin(\pi s)}{\pi} \Big\{ (j_0 - q_0) e^{-2s(\log 2 - \gamma)} \log s + \sum_{\xi \in {\rm P}} \frac{f_\xi(s)}{(s+\xi)^{|p_\xi|+1}} + \sum_{\xi \in {\rm L}} g_\xi(s) \log(s+\xi) \Big\}, \tag{10}$$

where j_0 appears in (7) and $f_\xi(s)$ and $g_\xi(s)$ are entire functions of s such that

$$f_\xi(-\xi) = (-1)^{|p_\xi|+1} c_{p_\xi \xi} \frac{|p_\xi|!}{2^{|p_\xi|}} \xi$$

and

$$g_\xi(s) = \begin{cases} c_{\ell_0,0} \frac{2^{\ell_0}}{(\ell_0 - 1)!} s^{\ell_0} + \mathcal{O}(s^{\ell_0+1}) & \text{if } \xi = 0, \\ -c_{\ell_\xi \xi} \frac{\xi\, 2^{\ell_\xi}}{(\ell_\xi - 1)!} (s+\xi)^{\ell_\xi - 1} + \mathcal{O}((s+\xi)^{\ell_\xi}) & \text{if } \xi > 0. \end{cases} \tag{11}$$

Remark 3.2: This theorem is very simple to use in practice and gives precise results *immediately* as we show in the following subsection. The regular part $\zeta_{\rm reg}(s, \Delta_L)$ will only have possible poles at $s = \frac{n}{2} - k \notin -\mathbb{N}_0$ in the case that Γ is the only boundary component of M and the residue of $\zeta_{\rm reg}(s, \Delta_L)$ at $s = 0$ is given by

$${\rm Res}_{s=0} \zeta_{\rm reg}(s, \Delta_L) = -\frac{1}{2} {\rm Res}_{s=-\frac{1}{2}} \zeta(s, A_\Gamma);$$

in particular, this vanishes if $\zeta(s, A_\Gamma)$ is in fact analytic at $s = -\frac{1}{2}$. The expansion (10) means that for any $N \in \mathbb{N}$,

$$\zeta_{\rm sing}(s, \Delta_L) = \frac{\sin(\pi s)}{\pi} \Big\{ (j_0 - q_0) e^{-2s(\log 2 - \gamma)} \log s + \sum_{\xi \in {\rm P},\, \xi \le N} \frac{f_\xi(s)}{(s+\xi)^{|p_\xi|+1}} + \sum_{\xi \in {\rm L},\, \xi \le N} g_\xi(s) \log(s+\xi) \Big\} + F_N(s),$$

where $F_N(s)$ is holomorphic for $\Re s \ge -N$. Finally, for arbitrary self-adjoint extensions with $A_\Gamma \ge -\frac{1}{4}$, the ζ-function has been studied by Falomir, Muschietti and Pisani [27] (see also [25] and joint work with Seeley [26]) for one-dimensional Laplace-type operators over $[0, 1]$ and by Mooers [47] who studied the general case of operators over manifolds and who was the first to notice the presence of unusual poles.

Remark 3.3: There are equally pathological heat operator and resolvent trace expansions with exotic behaviors such as logarithmic terms of arbitrary positive and negative multiplicity; we refer the reader to [37] for the details.

3.2. Examples of Theorem 3.1

Example 1: Falomir *et al.* [27] study the operator

$$\Delta = -\frac{d^2}{dr^2} + \frac{1}{r^2}\,\lambda \qquad \text{over} \;\; [0,1]$$

with the Dirichlet or Neumann condition at $r = 1$ and $-\frac{1}{4} \leq \lambda < \frac{3}{4}$; thus, in this example, "A_Γ" is the number "λ." In this case, $V = \mathbb{C}^2$, therefore Lagrangians $L \subset \mathbb{C}^2$ are determined by 1×1 matrices (numbers) $\mathcal{A} = \alpha$ and $\mathcal{B} = \beta$, not both zero, such that $\alpha\overline{\beta}$ is real. Fix such an (α, β) and let us assume that $-\frac{1}{4} < \lambda < \frac{3}{4}$ so there is no $-\frac{1}{4}$ eigenvalue (we will come back to the $\lambda = -\frac{1}{4}$ in a moment). Then with $\nu := \sqrt{\lambda + \frac{1}{4}}$ and $\tau := 2^{2\nu}\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}$,

$$p(x,y) := \det \begin{pmatrix} \alpha & \beta \\ \tau\, y^{2\nu} & 1 \end{pmatrix} = \alpha - \beta\,\tau\, y^{2\nu} = \alpha\Big(1 - \frac{\tau\beta}{\alpha} y^{2\nu}\Big),$$

where we assume that $\alpha, \beta \neq 0$ (the $\alpha = 0$ or $\beta = 0$ cases can be handled easily), and we write $p(x,y)$ as (7). Forming the power series (8), we see that

$$\log\Big(1 - \frac{\tau\beta}{\alpha} y^{2\nu}\Big) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\Big(-\frac{\tau\beta}{\alpha} y^{2\nu}\Big)^k = \sum_{k=1}^{\infty} c_{0,\nu k}\, x^0 y^{2\nu k},$$

where $c_{0,\nu k} = -\frac{1}{k}\left(\frac{\tau\beta}{\alpha}\right)^k$ and where the ξ's in (8) are given by the νk's and the ℓ's in (8) are all 0. Using the definition (9) for p_ξ and ℓ_ξ, we immediately see that $\ell_{\nu k}$ is never defined, while

$$p_{\nu k} = \min\{\ell \leq 0 \,|\, c_{\ell,\nu k} \neq 0\} = 0$$

exists for all $k \in \mathbb{N}$. Therefore, by Theorem 3.1,

$$\zeta_{\rm sing}(s, \Delta_L) = \frac{\sin(\pi s)}{\pi} \sum_{k=1}^{\infty} \frac{f_k(s)}{s + \nu k}$$

with $f_k(s)$ an entire function of s such that

$$f_k(-\nu k) = -c_{0,\nu k}\frac{0!}{2^0}\nu k = \nu\Big(\frac{\tau\beta}{\alpha}\Big)^k.$$

In particular, $\zeta_{\rm sing}(s, \Delta_L)$ has possible poles at each $s = -\nu k$ with residue equal to

$$\mathrm{Res}_{s=-\nu k}\zeta_{\rm sing}(s, \Delta_L) = \frac{\sin(\pi(-\nu k))}{\pi}\nu\Big(\frac{\tau\beta}{\alpha}\Big)^k = -\frac{\nu \sin \pi\nu k}{\pi}\Big(\frac{\tau\beta}{\alpha}\Big)^k,$$

which is the main result of [27] (see Equation (7.11) of loc. cit.).

Assume now that $\lambda = -\frac{1}{4}$. In this case,

$$p(x,y) := \det \begin{pmatrix} \alpha & \beta \\ x & 1 \end{pmatrix} = \alpha - \beta\, x = \alpha\Big(1 - \frac{\beta}{\alpha} x\Big),$$

where we assume that $\alpha, \beta \neq 0$ (the $\alpha = 0$ or $\beta = 0$ cases can be handled easily). Proceeding as before, by Theorem 3.1,

$$\zeta_{\rm sing}(s, \Delta_L) = \frac{\sin(\pi s)}{\pi}\Big\{ - e^{-2s(\log 2 - \gamma)} \log s + g_0(s) \log s\Big\},$$

$g_0(s)$ being an entire function of s such that $g_0(s) = \mathcal{O}(s)$. In particular, $\zeta(s, \Delta_L)$ has a *genuine* logarithmic singularity at $s = 0$. When $\beta = 0$, one can easily check that we still have a logarithmic singularity at $s = 0$ and when $\alpha = 0$, we only have the part $\zeta_{\rm reg}(s, \Delta_L)$ and no $\zeta_{\rm sing}(s, \Delta_L)$; one can easily show that (see [8]) $\alpha = 0$ corresponds to the Friedrichs extension; thus we can see that $\zeta(s, \Delta_L)$ has a logarithmic singularity for all extensions except the Friedrichs.

Example 2: (The Laplacian on $\mathbb{R}^2$) If Δ is the Laplacian on a compact region in $\mathbb{R}^2$, then as we saw before in Section 2.1, A_Γ has a $-\frac{1}{4}$ eigenvalue of multiplicity one and no eigenvalues in $(-\frac{1}{4}, \frac{3}{4})$. Therefore, the exact same argument we used in the $\lambda = -\frac{1}{4}$ case of the previous example shows that $\zeta(s, \Delta_L)$ has a logarithmic singularity for all extensions except the Friedrichs.

Example 3: Consider now the case of a regular singular operator Δ over a compact manifold and suppose that A_Γ has two eigenvalues in $[-\frac{1}{4}, \frac{3}{4})$, the eigenvalue $-\frac{1}{4}$ and another eigenvalue $-\frac{1}{4} < \lambda < \frac{3}{4}$, both of multiplicity one. This situation occurs, for example, in the two-dimensional flat cone in $\mathbb{R}^3$ with $\Gamma = \mathbb{S}^1_\nu$ where $\mathbb{S}^1_\nu$ is the circle with metric $d\theta/\nu$ where $\frac{1}{2} < \nu < 1$; indeed, after a Liouville transformation, we have

$$A_\Gamma = -\nu^2 \partial_\theta^2 - \frac{1}{4},$$

which only has the eigenvalues $-\frac{1}{4}$ and $\lambda = \nu^2 - \frac{1}{4}$ in the interval $[-\frac{1}{4}, \frac{3}{4})$. In this case, $q = 2$ and Lagrangians $L \subset \mathbb{C}^4$ are determined by 2×2 matrices $\mathcal{A}$ and $\mathcal{B}$ such that $(\mathcal{A} \quad \mathcal{B})$ has full rank and $\mathcal{A}'\mathcal{B}^*$ is self-adjoint. Consider the specific examples

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{B} = \mathrm{Id}.$$

Then with $\tau := 2^{2\nu} \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}$ where $\nu = \sqrt{\lambda + \frac{1}{4}}$, we have

$$p(x, y) := \det \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ x & 0 & 1 & 0 \\ 0 & \tau\, y^{2\nu} & 0 & 1 \end{pmatrix} = 1 + \tau\, x\, y^{2\nu}.$$

Forming the power series (8), we see that

$$\log\Big(1 + \tau\, x\, y^{2\nu}\Big) = \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} \Big(\tau\, x\, y^{2\nu}\Big)^k = \sum_{k=1}^\infty c_{k,\nu k}\, x^k y^{2\nu k},$$

where $c_{k,\nu k} = (-1)^{k-1} \frac{\tau^k}{k}$. Using the definition (9) for p_ξ and ℓ_ξ, we immediately see that $p_{\nu k}$ is never defined, while each $\ell_{\nu k}$ is defined:

$$\ell_{\nu k} = \min\{\ell > 0 \,|\, c_{\ell, \nu k} \neq 0\} = k.$$

Therefore, by Theorem 3.1,

$$\zeta_{\rm sing}(s, \Delta_L) = \frac{\sin(\pi s)}{\pi} \Big\{ - e^{-2s(\log 2 - \gamma)} \log s + \sum_{k=1}^\infty g_k(s) \log(s + \nu k) \Big\},$$

with $g_k(s)$ an entire function of s such that

$$g_k(s) = (-1)^k \frac{\tau^k 2^k \nu}{(k-1)!}(s+\nu k)^{k-1} + \mathcal{O}((s+\nu k)^k).$$

In particular, $\zeta_{\rm sing}(s, \Delta_L)$ has *countably* many logarithmic singularities!

Example 4: With the same situation as in the previous example, consider

$$\mathcal{A} = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix},$$

so that

$$p(x,y) := \det \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ x & 0 & 1 & 0 \\ 0 & \tau\, y^{2\nu} & 0 & 1 \end{pmatrix} = x - \tau\, y^{2\nu} = x\Big(1 - \tau\, x^{-1} y^{2\nu}\Big).$$

Proceedings as before, by Theorem 3.1,

$$\zeta_{\rm sing}(s, \Delta_L) = \frac{\sin(\pi s)}{\pi} \sum_{k=1}^{\infty} \frac{f_k(s)}{(s+\nu k)^{k+1}},$$

with $f_k(s)$ an entire function of s such that

$$f_k(-\nu k) = (-1)^{k+1} c_{-k,\nu k} \frac{|-k|!}{2^{|-k|}}\, \nu k = (-1)^k \frac{\tau^k k! \nu}{2^k}.$$

In particular, $\zeta_{\rm sing}(s, \Delta_L)$ has poles of *arbitrarily* large order!

Example 5: Consider now the case of a regular singular operator Δ over a compact manifold such that A_Γ has three eigenvalues in $[-\frac{1}{4}, \frac{3}{4})$, the eigenvalue $-\frac{1}{4}$ with multiplicity two and another eigenvalue $-\frac{1}{4} < \lambda < \frac{3}{4}$ of multiplicity one. This situation occurs, for example, in the two-dimensional flat cone in $\mathbb{R}^3$ with $\Gamma = \mathbb{S}^1 \sqcup \mathbb{S}^1_\nu$, the disjoint union of the standard circle with metric $d\theta$ and the circle with metric $d\theta/\nu$ where $\frac{1}{2} < \nu < 1$; indeed, after a Liouville transformation, we have

$$A_\Gamma = \Big(-\partial_\theta^2 - \frac{1}{4}\Big) \oplus \Big(-\nu^2 \partial_\theta^2 - \frac{1}{4}\Big),$$

where in the interval $[-\frac{1}{4}, \frac{3}{4})$, the first operator has only the $-\frac{1}{4}$ eigenvalue and the second operator has only the eigenvalues $-\frac{1}{4}$ and $\lambda = \nu^2 - \frac{1}{4}$. In this case, $V = \mathbb{C}^6$ and Lagrangians $L \subset \mathbb{C}^6$ are determined by 3×3 matrices $\mathcal{A}$ and $\mathcal{B}$. Consider the specific examples

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{B} = \mathrm{Id}.$$

Then with $\nu = \sqrt{\lambda + \frac{1}{4}}$ and $\tau := 2^{2\nu} \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}$, using the procedure outlined several times, we find

$$\zeta_{\rm sing}(s, \Delta_L) = \frac{\sin(\pi s)}{\pi} \Big\{ - e^{-2s(\log 2 - \gamma)} \log s + \sum_{k=1}^{\infty} \frac{f_k(s)}{(s+\nu k)^{k+1}} + \sum_{k=1}^{\infty} g_k(s) \log(s+\nu k) \Big\},$$

where $f_k(s)$ and $g_k(s)$ are entire functions of s such that

$$f_k(-\nu k) = (-1)^{k+1} c_{-k,\nu k} \frac{k!}{2^k} \nu k = \frac{(-1)^k}{k} \tau^k \frac{k!}{2^k} \nu k = (-1)^k \frac{\tau^k k! \nu}{2^k}$$

and

$$g_k(s) = 2\nu(-1)^{m+1}\tau^k \binom{k}{m+1} \times \begin{cases} 1 + \mathcal{O}((s+\nu k)) & \text{if } k = 2m+1 \text{ is odd,} \\ 2(s+\nu k) + \mathcal{O}((s+\nu k)^2) & \text{if } k = 2m \text{ is even.} \end{cases}$$

In particular, $\zeta_{\rm sing}(s, \Delta_L)$ has poles of *arbitrarily* high orders and in addition to a logarithmic singularity at the origin, *countably* many logarithmic singularities at the same locations of the poles!

Example 6: From the previous examples, we can see that by looking at flat cones in $\mathbb{R}^3$ whose boundaries are disjoint unions of circles of various circumferences, one can easily come up with completely natural (that is, geometric) zeta functions having as wild singularities involving unusual poles and logarithmic singularities as the mind can image.

3.3. Conclusion and final remarks

In this paper we have considered zeta functions of self-adjoint extensions of Laplace-type operators over conic manifolds. We have presented a theorem that gives the exact structure of zeta functions for arbitrary self-adjoint extensions of Laplace-type operators over manifolds with conical singularities. As we have seen, the structure found can be dramatically different from the standard one. Using this exact structure, with a suitable redefinition, functional determinants of Laplacians on generalized cones can still be obtained [38].

The ideas presented here can equally well be applied to the Dirac operator [39]. In the presence of a Dirac delta magnetic field [46], different self-adjoint extensions are considered as manifestations of different physics within the vortex [18]. The physics represented by the self-adjoint extensions described by $\mathcal{A}$ and $\mathcal{B}$ and the implications of the meromorphic structure of the zeta functions found are very interesting questions to pursue.

Acknowledgements

KK acknowledges support by the Baylor University Summer Sabbatical Program and by the Baylor University Research Committee.

References

[1] Y. Aharonov and D. Bohm, *Significance of electromagnetic potentials in the quantum theory*, Phys. Rev. (2) **115** (1959), 485–491.

[2] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable models in quantum mechanics*, Texts and Monographs in Physics, Springer-Verlag, New York, 1988.

[3] M.G. Alford and F. Wilczek, *Aharonov-Bohm interaction of cosmic strings with matter*, Phys. Rev. Lett. **62** (1989), 1071–1074.

[4] M. Barriola and A. Vilenkin, *Gravitational field of a global monopole*, Phys. Rev. Lett. **63** (1989), no. 4, 341–343.

[5] C.G. Beneventano, M. De Francia, K. Kirsten, and E.M. Santangelo, *Casimir energy of massive MIT fermions in a Aharonov-Bohm background*, Phys. Rev. **D61** (2000), 085019.

[6] G. Bonneau, J. Faraut, and G. Valent, *Self-adjoint extensions of operators and the teaching of quantum mechanics*, Am. J. Phys. **69** (2001), 322–331.

[7] M. Bordag, S. Dowker, and K. Kirsten, *Heat-kernels and functional determinants on the generalized cone*, Comm. Math. Phys. **182**, no. 2 (1996), 371–393.

[8] J. Brüning and R. Seeley, *The resolvent expansion for second order regular singular operators*, J. Funct. Anal. **73**, no. 2 (1987), 369–429.

[9] A.A. Bytsenko, G. Cognola, L. Vanzo, and S. Zerbini, *Quantum fields and extended objects in space-times with constant curvature spatial section*, Phys. Rept. **266** (1996), 1–126.

[10] C. Callias, *The heat equation with singular coefficients. I. Operators of the form $-d^2/dx^2 + \kappa/x^2$ in dimension* 1, Comm. Math. Phys. **88**, no. 3 (1983), 357–385.

[11] C. Callias, *The resolvent and the heat kernel for some singular boundary problems*, Comm. Partial Differential Equations **13** (1988), no. 9, 1113–1155.

[12] J. Cheeger, *On the spectral geometry of spaces with cone-like singularities*, Proc. Nat. Acad. Sci. U.S.A. **76** (1979), no. 5, 2103–2106.

[13] J. Cheeger, *Spectral geometry of singular Riemannian spaces*, J. Differential Geom. **18** no. 4 (1983), 575–657.

[14] A. Chou, *The Dirac operator on spaces with conical singularities and positive scalar curvatures*, Trans. Amer. Math. Soc. **289** (1985), no. 1, 1–40.

[15] G. Cognola, K. Kirsten, and L. Vanzo, *Free and selfinteracting scalar fields in the presence of conical singularities*, Phys. Rev. **D49** (1994), 1029–1038.

[16] G. Cognola and S. Zerbini, *Zeta-function on a generalised cone*, Lett. Math. Phys. **42** (1997), no. 1, 95–101.

[17] S. Coriasco, E. Schrohe and J. Seiler, *H_∞-calculus for differential operators on conic manifolds with boundary*, Comm. Partial Differential Equations, to appear; arXiv: math.AP/0507081.

[18] P. de Sousa Gerbert, *Fermions in an Aharonov-Bohm field and cosmic strings*, Phys. Rev. **D40** (1989), 1346–1349.

[19] P. de Sousa Gerbert and R. Jackiw, *Classical and quantum scattering on a spinning cone*, Commun. Math. Phys. **124** (1989), 229–260.

[20] J. S. Dowker, *Quantum field theory on a cone*, J. Phys. **10** (1977), no. 1, 115–124.

[21] J. S. Dowker, *Heat kernels on curved cones*, Classical Quantum Gravity **11** (1994), no. 11, L137–L140.

[22] J. S. Dowker and R. Critchley, *Effective Lagrangian and energy momentum tensor in de Sitter space*, Phys. Rev. D **13** (1976), 3224-3232.

[23] G. V. Dunne, J. Hur, C. Lee and H. Min, *Precise quark mass dependence of instanton determinant*, Phys. Rev. Lett. **94** (2005), 072001.

[24] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko, and S. Zerbini, *Zeta regularization techniques with applications*, World Scientific Publishing Co. Inc., River Edge, NJ, 1994.

[25] H. Falomir, P. A. G. Pisani and A. Wipf, *Pole structure of the Hamiltonian ζ-function for a singular potential*, J. Phys. A **35** (2002), no. 26, 5427–5444.

[26] H. Falomir, M. A. Muschietti, P. A. G. Pisani and R. T. Seeley, *Unusual poles of the ζ-functions for some regular singular differential operators*, J. Phys. A **36**, no. 39 (2003), 9991–10010.

[27] H. Falomir, M. A. Muschietti and P. A. G. Pisani *On the resolvent and spectral functions of a second order differential operator with a regular singularity*, J. Math. Phys. **45**, no. 12 (2004), 4560–4577.

[28] D. V. Fursaev, *Spectral geometry and one-loop divergences on manifolds with conical singularities*,

Phys. Lett. B **334** (1994), no. 1-2, 53–60.

[29] J. B. Gil, *Full asymptotic expansion of the heat trace for non-self-adjoint elliptic cone operators*, Math. Nachr. **250** (2003), 25–57.

[30] J. B. Gil, T. Krainer and G. A. Mendoza, *Resolvents of elliptic cone operators*, J. Funct. Anal. **241** (2006), 1–55.

[31] J. B. Gil, T. Krainer and G. A. Mendoza, *On rays of minimal growth for elliptic cone operators*, Oper. Theory Adv. Appl. **172** (2007), 33–50.

[32] J. B. Gil and G. Mendoza, *Adjoints of elliptic cone operators*, Am. J. Math. **125** (2003), 357–408.

[33] P. B. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, second ed., CRC Press, Boca Raton, FL, 1995.

[34] C.R. Hagen, *Aharonov-Bohm scattering of particles with spin*, Phys. Rev. Lett. **64** (1990), 503–506.

[35] S. W. Hawking, *Zeta function regularization of path integrals in curved spacetime*, Comm. Math. Phys. **55** (1977), no. 2, 133–148.

[36] K. Kirsten, *Spectral functions in mathematics and physics*, Chapman & Hall/CRC Press, Boca Raton, 2001.

[37] K. Kirsten, P. Loya, and J. Park, *Exotic expansions and pathological properties of ζ-functions on conic manifolds*; arXiv: math/0511185.

[38] K. Kirsten, P. Loya, and J. Park, *Functional determinants for general self-adjoint extensions of Laplace-type operators resulting from the generalized cone*, Manuscripta Mathematica, to appear; arXiv: 0709.1232.

[39] K. Kirsten, P. Loya, and J. Park, *On the spectral functions and their invariants for self-adjoint extensions of Dirac operators on a cone*, in preparation.

[40] V. Kostrykin and R. Schrader, *Kirchhoff's rule for quantum wires*, J. Phys. A **32** (1999), no. 4, 595–630.

[41] A. Legrand and S. Moroianu, *On the L^p index of spin Dirac operators on conical manifolds*, Studia Math. **177** (2006), 97–112.

[42] M. Lesch, *Operators of Fuchs type, conical singularities, and asymptotic methods*, B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1997.

[43] S. Leseduarte and A. Romeo, *Influence of a magnetic fluxon on the vacuum energy of quantum fields confined by a bag*, Commun. Math. Phys. **193** (1998), 317–336.

[44] T. Levi-Civita, *ds^2 Einsteiniani in campi Newtoniani. ii: Condizioni di integrabilit'a e comportamento geometrico spaziale (italian)*, Rom. Acc. L. Rend. **5** (1918), no. 27, 3–12.

[45] P. Loya, P. McDonald and J. Park, *Zeta Regularized Determinants for Conic Manifolds*, J. Funct. Anal. **242** (2007), 195–229.

[46] C. Manuel and R. Tarrach, *Contact interactions and Dirac anyons*, Phys. Lett. **B301** (1993), 72–76.

[47] E. Mooers, *Heat kernel asymptotics on manifolds with conic singularities*, J. Anal. Math. **78** (1999), 1–36.

[48] D. B. Ray and I. M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Advances in Math. **7** (1971), 145–210.

[49] J.-P. Serre, *A course in Arithmetic*, Springer-Verlag, Heidelberg, 1973.

[50] D.D. Sokolov and A. A. Starobinskii, *The structure of the curvature tensor at conical singularities (Russian, English)*, J Sov. Phys., Dokl. **22** (1977), 312–313, translation from Dokl. Akad. Nauk SSSR 234 (1977) 1043–1046.

[51] M. Spreafico, *Zeta function and regularized determinant on a disc and on a cone*, J. Geom. Phys. **54** (2005), no. 3, 355–371.

[52] D.V. Vassilevich, *Heat kernel expansion: User's manual*, Phys. Rept. **388** (2003), 279–360.

[53] A. Vilenkin, *Cosmic strings and domain walls*, Phys. Rep. **121** (1985), no. 5, 263–315.