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Exotic Expansions and Pathological Properties of ζ-Functions on Conic Manifolds

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Abstract We give a complete classification and present new exotic phenomena of the meromorphic structure of ζ-functions associated to general self-adjoint extensions of Laplace-type operators over conic manifolds. We show that the meromorphic extensions of these ζ-functions have, in general, countably many logarithmic branch cuts on the nonpositive real axis and unusual locations of poles with arbitrarily large multiplicity. The corresponding heat kernel and resolvent trace expansions also exhibit exotic behaviors with logarithmic terms of arbitrary positive and negative multiplicity. We also give a precise algebraic-combinatorial formula to compute the coefficients of the leading order terms of the singularities.

Keywords Zeta function · Conic manifold · Self-adjoint extension

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1 Introduction

In this article, we give a complete classification of the meromorphic structure of ζ-functions associated to conic manifolds; that is, general self-adjoint extensions of

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Laplace-type operators on conic manifolds introduced by Cheeger [9, 11]. In particular, we prove that such ζ-functions exhibit pathological meromorphic properties. Before giving a synopsis of these pathological properties, recall that the ζ-function $\zeta(s,\Delta)$ of a Laplacian Δ over a smooth closed manifold has a meromorphic extension to all of $\mathbb{C}$ with only simple poles at $s=\frac{n-k}{2}\notin -\mathbb{N}_0$ with n the dimension of the manifold and $k\in\mathbb{N}_0 := \{0,1,2,\dots\}$ [28, 52, 53, 61, 68]. The situation is completely different for conic manifolds. We show that the ζ-function associated to a general self-adjoint extension of a Laplace-type operator on a conic manifold has, as a general rule (except for very special cases, e.g. the Friedrichs extension), in addition to the singularities at $s=\frac{n-k}{2}\notin -\mathbb{N}_0$ for $k\in\mathbb{N}_0$, the following properties:

(1) It can have countably many poles of *arbitrarily* high multiplicity at "unusual" locations on the negative real axis; that is, at points not of the form $s=\frac{n-k}{2}$.
(2) It can have *countably* many logarithmic singularities at "unusual" locations.
(3) The singularities in (1) and (2) can occur for the *same* ζ-function and at the *same* "unusual" locations. Moreover, we also give an elementary and explicit algebraic-combinatorial recipe to compute the exact locations and leading coefficients of the "unusual" poles and logarithmic singularities.

In fact, the explicit computation of these exotic singularities is so straightforward (see Sect. 2.2) that for low dimensions we can find the structure of zeta functions quickly. We also remark that one can always conjure up *artificial* zeta functions having (1) and (2), but for *natural* (geometric) zeta functions, properties (1) and (2) seem to have no parallels in the differential geometry literature.

1.1 A Simple Example

Here is a surprising, and completely natural, example of a ζ-function which has no meromorphic extension to all of $\mathbb{C}$. We first review conic manifolds. Let M be an n-dimensional compact manifold with boundary Γ and let g be a smooth Riemannian metric on $M\setminus\Gamma$. We assume that near Γ there is a collared neighborhood

$$\mathcal{U}\cong[0,\varepsilon)_r\times\Gamma,$$

where $\varepsilon>0$ and the metric g is of product type dr^2+r^2h with h a metric over Γ. Such a metric is called a *conic metric* and M is called a *conic manifold*, ideas introduced by Cheeger [9, 11] (cf. [50]). Using a Liouville transformation over the collar $\mathcal{U}$ as in [4], we can identify $L^2(M,dg)$ with $L^2(M,dr\,dh)$ and the scalar Laplacian Δ_g can be identified with

$$\Delta_g|_{\mathcal{U}}=-\partial_r^2+\frac{1}{r^2}A_\Gamma,\quad \text{with } A_\Gamma=\Delta_\Gamma+\left(\frac{1-n}{2}\right)\left(1+\frac{1-n}{2}\right),\tag{1.1}$$

where Δ_Γ is the Laplacian over Γ. Notice that $A_\Gamma\geq-\frac14$ because the function $x(1+x)$ has the minimum value $-\frac14$ (when $x=-\frac12$). Let us now assume that $n=2$ so that:

$$\Delta_g|_{\mathcal{U}}=-\partial_r^2+\frac{1}{r^2}A_\Gamma,\quad \text{with } A_\Gamma=\Delta_\Gamma-\frac14.$$

We remark that the term $-\frac{1}{4r^2}$ can be considered a "singular potential," and such Laplacians and their self-adjoint extensions have been studied by physicists since the 70's [6, 14, 20, 57]. Note that Δ_Γ always has the eigenvalue 0. Let us assume that 0 is the only eigenvalue (counting multiplicity) of Δ_Γ in the interval $[0, 1)$; this is the case for the Euclidean Laplacian on a punctured region in $\mathbb{R}^2$, for in this case, Δ_Γ is just the Laplacian on the unit circle which has eigenvalues $\{k^2 \mid k \in \mathbb{Z}\}$. Then A_Γ has exactly one eigenvalue in the interval $[-\frac{1}{4}, \frac{3}{4})$, the eigenvalue $-\frac{1}{4}$, and (see Sect. 3) Δ_g has many different self-adjoint extensions, each of which is parameterized by an angle $\theta \in [0, \pi)$ (cf. [24, 26, 34, 38]). It turns out that $\theta = \frac{\pi}{2}$ corresponds to the so-called Friedrichs extension [4, 24], and any extension has a discrete spectrum [44]. Consider any one of the extensions, say Δ_θ with $\theta \in [0, \pi)$, and form the corresponding ζ-function:

$$\zeta(s, \Delta_\theta) := \sum_{\mu_j \neq 0} \frac{1}{\mu_j^s},$$

where the μ_j's are the eigenvalues of Δ_θ. The surprising fact is that the meromorphic extension of *every* such ζ-function corresponding to an angle $\theta \in [0, \pi)$ *except* $\theta = \frac{\pi}{2}$ (the Friedrich's extension) has a logarithmic singularity at $s = 0$. More precisely, as a consequence of Theorem 2.4, for $\theta \neq \frac{\pi}{2}$, we can write:

$$\zeta(s, \Delta_\theta) = \zeta_{\mathrm{reg}}(s, \Delta_\theta) - \frac{\sin(\pi s)}{\pi} e^{-2s\kappa} \log s, \tag{1.2}$$

where $\kappa = \log 2 - \gamma - \tan\theta$ with γ the Euler-Mascheroni constant and $\zeta_{\mathrm{reg}}(s, \Delta_\theta)$ has a meromorphic extension over $\mathbb{C}$ with the "regular" simple poles at the well-known values $s = \frac{1}{2} - k$ for $k \in \mathbb{N}_0$.

1.2 Operators on Conic Manifolds

Brüning and Seeley's regular singular operators [4] generalize the example (1.1) of the Laplacian on a conic manifold as follows. Let M be an n-dimensional compact manifold with boundary Γ and we assume that near Γ there is a collared neighborhood $\mathcal{U}$ such that:

$$\mathcal{U} \cong [0, \varepsilon)_r \times \Gamma,$$

where $\varepsilon > 0$ and the metric of M is of product type $dr^2 + h$ with h a metric over Γ. Let E be a Hermitian vector bundle over M and let Δ be a second order *regular singular operator* acting on $C_c^\infty(M \setminus \Gamma, E)$; this means that Δ is an elliptic symmetric nonnegative second order differential operator such that the restriction of Δ to $\mathcal{U}$ has the form:

$$-\partial_r^2 + \frac{1}{r^2} A_\Gamma, \tag{1.3}$$

where A_Γ is a Laplace-type operator over Γ with $A_\Gamma \geq -\frac{1}{4}$. (The condition $A_\Gamma \geq -\frac{1}{4}$ is necessary otherwise Δ is not bounded below [4, 7].) Laplacians on forms and squares of Dirac operators on conic manifolds [4, 9, 11, 12, 35, 43, 49] are examples of second order regular singular operators. We can also deal with the case when M has boundary components up to which Δ is smooth; at such components, put local boundary conditions.

From Von Neumann's theory of self-adjoint extensions (cf. Cheeger [9–11], Gil and Mendoza [24], Lesch [44], Mooers [54]), the self-adjoint extensions of Δ are parameterized by Lagrangian subspaces in the eigenspaces of A_Γ with eigenvalues in the interval $[-\frac{1}{4}, \frac{3}{4})$. To describe these extensions, denote by

$$-\frac{1}{4} = \underbrace{\lambda_1 = \lambda_2 = \cdots = \lambda_{q_0}}_{=-\frac{1}{4}} < \underbrace{\lambda_{q_0+1} \leq \lambda_{q_0+2} \leq \cdots \leq \lambda_{q_0+q_1}}_{-\frac{1}{4}<\lambda_\ell<\frac{3}{4}}$$

the spectrum of A_Γ in $[-\frac{1}{4}, \frac{3}{4})$ and by $\{\phi_\ell\}$ the associated eigenvectors, and define

$$V \cong \bigoplus_{-\frac{1}{4}\leq\lambda_\ell<\frac{3}{4}} E_\ell \oplus E_\ell \cong \mathbb{C}^{2q} \quad \text{with } E_\ell := \operatorname{span}\{\phi_\ell\} \text{ and } q = q_0 + q_1;$$

see Sect. 3 for a more precise description of V. We can endow V with a symplectic structure as described in Sect 3. Then the self-adjoint extensions of Δ are in a one-to-one correspondence to the Lagrangian subspaces in V. Given a Lagrangian subspace L in V, we denote by Δ_L the self-adjoint extension corresponding to L.

One of the natural questions for a given self-adjoint extension Δ_L is whether the ζ-function of Δ_L, $\zeta(s, \Delta_L)$, would have a meromorphic extension over $\mathbb{C}$ and if so, what the pole structure is. Here the ζ-function of Δ_L is defined by:

$$\zeta(s, \Delta_L) = \sum_{\mu_j\neq 0} \frac{1}{\mu_j^s} \tag{1.4}$$

for $\Re s \gg 0$ where μ_j's are the eigenvalues of Δ_L. The ζ-function has been studied in many articles for the Friedrich's extension or with the condition $A_\Gamma \geq \frac{3}{4}$, which implies that Δ is essentially self-adjoint [2–4, 7, 8, 10, 11, 13, 21, 22, 45, 46, 48, 49, 64]. For scalable extensions (extensions for which the domain is invariant under $r \mapsto cr$), the ζ-function has been studied in [44]. There are no "unusual" phenomena with these cases. Finally, for arbitrary self-adjoint extensions with $A_\Gamma \geq -\frac{1}{4}$, the ζ-function has been studied by Falomir, Muschietti and Pisani [18], see also joint work with Seeley [17] (cf. [19]), for one-dimensional Laplace-type operators over $[0, 1]$ and by Mooers [54] who was the first to study the general case over manifolds and who noticed the presence of "unusual" poles. However, the works [18, 54] only imply the existence of *simple* "unusual" poles and do not imply the existence of poles of arbitrary order nor of logarithmic singularities of the ζ-function.

We now outline this article. We begin in Sect. 2 by giving the statement of our main result, Theorem 2.1, and we also illustrate the ease of applying the main result by giving examples; in particular, we re-derive the main result of [18] and we show that poles of arbitrary order and countably many logarithmic singularities show up even for simple situations. We also show how our theorem simplifies when we make assumptions on the self-adjoint extensions and we present corresponding resolvent and heat kernel expansions. The description of self-adjoint extensions as Lagrangian subspaces plays a key role in the proof of our main result and because of this reason, in Section 3 we briefly review this important topic. The main technical task of the proof of our main theorem is the explicit form of the parametrix of the resolvent of Δ_L near the boundary. This is handled by solving model problems explicitly over a finite interval employing Theorem 6.1, which is a new representation of the resolvent in terms of an implicit eigenvalue equation, and the contour integration method [35–37]. The presentation and details of the solutions of the model problems are given in Sects. 4, 5 and 6. The results for the model problems and a parametrix construction then enable us to prove all the theorems listed in Sect. 2 below. This is described in Sect. 7. The appendix by Boris Vertman gives a careful analysis of the implicit eigenvalue equation and proves that the zeta function can be written as a contour integral in terms of the implicit eigenvalue equation.

Finally, it is a pleasure to thank Mattias Lesch for remarks that greatly improved the exposition of this article.

2 Statement and Examples of Results

2.1 Statement of Main Result

Fix a Lagrangian $L \subset V$ and hence a self-adjoint extension Δ_L of Δ (we use the notation from Sect. 1.2). In Sect. 3 we show that L can be described by $q \times q$ matrices $\mathcal{A}$ and $\mathcal{B}$ having the property that the rank of the $q \times 2q$ matrix $(\mathcal{A}\,\mathcal{B})$ is q and $\mathcal{A}'\mathcal{B}^*$ is self adjoint where $\mathcal{A}'$ is the matrix $\mathcal{A}$ with the first q_0 columns multiplied by -1 (conversely, any such $\mathcal{A}$ and $\mathcal{B}$ define a Lagrangian). Before stating the main result which describes the exact structure of $\zeta(s, \Delta_L)$, we apply a straightforward three-step algorithm to $\mathcal{A}$ and $\mathcal{B}$ that we need for the statement.

Step 1: First, we define the function:

$$p(x, y) := \det \begin{array}{c} \begin{array}{ccc} \qquad\qquad\qquad \mathcal{A} & \qquad\qquad\qquad & \mathcal{B} \end{array} \\ \begin{pmatrix} x\,\mathrm{Id}_{q_0} & 0 & 0 & 0 & \\ 0 & \tau_1 y^{2\nu_1} & 0 & 0 & \\ & & & & \mathrm{Id}_q \\ 0 & 0 & \ddots & 0 & \\ 0 & 0 & 0 & \tau_{q_1} y^{2\nu_{q_1}} & \end{pmatrix} \end{array}, \tag{2.1}$$

where Id_k denotes the $k \times k$ identity matrix and where

$$\nu_j := \sqrt{\lambda_{q_0+j} + \frac{1}{4}}, \qquad \tau_j = 2^{2\nu_j} \frac{\Gamma(1+\nu_j)}{\Gamma(1-\nu_j)}, \qquad j = 1, \dots, q_1.$$

For specific $\mathcal{A}$ and $\mathcal{B}$, $p(x, y)$ is explicitly computable "by hand"; we shall give some examples in Sect. 2.2. Expanding the determinant using one's favorite method, we can write $p(x, y)$ as a "polynomial":

$$p(x, y) = \sum a_{j\alpha} x^j y^{2\alpha},$$

where the α's are linear combinations of $\nu_1, \dots, \nu_{q_1}$ and the $a_{j\alpha}$'s are constants. Let α_0 be the smallest of all α's with $a_{j\alpha} \neq 0$ and let j_0 be the smallest of all j's amongst the $a_{j\alpha_0} \neq 0$. Then factoring out the term $a_{j_0\alpha_0} x^{j_0} y^{2\alpha_0}$ in $p(x, y)$ we can write $p(x, y)$ in the form:

$$p(x, y) = a_{j_0\alpha_0} x^{j_0} y^{2\alpha_0} \Big(1 + \sum b_{k\beta} x^k y^{2\beta}\Big) \tag{2.2}$$

for some constants $b_{k\beta}$ (equal to $a_{k\alpha}/a_{j_0\alpha_0}$).

Step 2: Second, using the power series $\log(1+z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} z^k$ with $z = \sum b_{k\beta} x^k y^{2\beta}$ for a sufficiently small $|z|$, we can write:

$$\log\Big(1 + \sum b_{k\beta} x^k y^{2\beta}\Big) = \sum c_{\ell\xi} x^\ell y^{2\xi} \tag{2.3}$$

for some constants $c_{\ell\xi}$. We emphasize that for specific $\mathcal{A}$ and $\mathcal{B}$, all the coefficients $c_{\ell\xi}$ are explicitly computable "by hand" when q is small (see the examples in Sect. 2.2) and easily with a computer for q large. With a little thought, one can see that the ξ's appearing in (2.3) are nonnegative, countable and approach $+\infty$ unless $\beta = 0$ is the only β occurring in (2.3), in which case $\xi = 0$ occurs in (2.3). Also, the ℓ's with $c_{\ell\xi} \neq 0$ for a fixed ξ are bounded below.

Step 3: Third, for each ξ appearing in (2.3), define:

$$p_\xi := \min\{\ell \leq 0 \mid c_{\ell\xi} \neq 0\} \quad \text{and} \quad \ell_\xi := \min\{\ell > 0 \mid c_{\ell\xi} \neq 0\}, \tag{2.4}$$

when these numbers are actually defined, that is, whenever the sets $\{\ell \leq 0 \mid c_{\ell\xi} \neq 0\}$ and $\{\ell > 0 \mid c_{\ell\xi} \neq 0\}$, respectively, are nonempty. We now define:

$$\mathcal{P} := \{\xi \mid p_\xi \text{ is defined}\} \quad \text{and} \quad \mathcal{L} := \{\xi \mid \ell_\xi \text{ is defined}\}. \tag{2.5}$$

The following theorem is our main result.

Theorem 2.1 *Let $L \subset V$ be an arbitrary Lagrangian subspace of V and define $\mathcal{P}$ and $\mathcal{L}$ as in (2.5) from the matrices $\mathcal{A}$ and $\mathcal{B}$ defining L. Then the ζ-function $\zeta(s, \Delta_L)$ extends from $\Re s > \frac{n}{2}$ to a holomorphic function on $\mathbb{C} \setminus (-\infty, 0]$. Moreover, $\zeta(s, \Delta_L)$ can be written in the form*

$$\zeta(s, \Delta_L) = \zeta_{\mathrm{reg}}(s, \Delta_L) + \zeta_{\mathrm{sing}}(s, \Delta_L),$$

where $\zeta_{\mathrm{reg}}(s, \Delta_L)$ has possible "regular" poles at the "usual" locations $s = \frac{n-k}{2} \notin -\mathbb{N}_0$ for $k \in \mathbb{N}_0$ and at $s = 0$ if $\dim \Gamma > 0$, and where $\zeta_{\mathrm{sing}}(s, \Delta_L)$ has the following expansion:

$$\begin{aligned}\zeta_{\mathrm{sing}}(s, \Delta_L) = \frac{\sin(\pi s)}{\pi} \Bigg\{ & (j_0 - q_0) e^{-2s(\log 2 - \gamma)} \log s \\ & + \sum_{\xi \in \mathcal{P}} \frac{f_\xi(s)}{(s+\xi)^{|p_\xi|+1}} + \sum_{\xi \in \mathcal{L}} g_\xi(s) \log(s+\xi) \Bigg\},\end{aligned} \tag{2.6}$$

where j_0 appears in (2.2) and $f_\xi(s)$ and $g_\xi(s)$ are entire functions of s such that:

$$f_\xi(-\xi) = (-1)^{|p_\xi|+1} c_{p_\xi \xi}\, \xi \frac{|p_\xi|!}{2^{|p_\xi|}}$$

and near $s = -\xi$,

$$g_\xi(s) = \begin{cases} c_{\ell_0,0}\frac{2^{\ell_0}}{(\ell_0-1)!}s^{\ell_0} + \mathcal{O}(s^{\ell_0+1}) & \textit{if } \xi = 0, \\ -c_{\ell_\xi\xi}\frac{\xi 2^{\ell_\xi}}{(\ell_\xi-1)!}(s+\xi)^{\ell_\xi-1} + \mathcal{O}((s+\xi)^{\ell_\xi}) & \textit{if } \xi > 0. \end{cases}$$

Remark 2.2 The zeta function $\zeta_{\mathrm{reg}}(s, \Delta_L)$ will only have possible poles at $s = \frac{n-k}{2} \notin -\mathbb{N}_0$ in the case that Γ is the only boundary component of M and the residue of $\zeta_{\mathrm{reg}}(s, \Delta_L)$ at $s = 0$ is given by $\mathrm{Res}_{s=0}\zeta_{\mathrm{reg}}(s, \Delta_L) = -\frac{1}{2}\mathrm{Res}_{s=-\frac{1}{2}}\zeta(s, A_\Gamma)$; in particular, this vanishes if $\zeta(s, A_\Gamma)$ is in fact analytic at $s = -\frac{1}{2}$. Later in Theorems 2.5 and 2.7, we shall present corresponding resolvent and heat kernel expansions. In general, $\mathcal{L}$ may contain the origin, so that the logarithm part in (2.6), which has the branch cut at $s = 0$, is given by:

$$\frac{\sin(\pi s)}{\pi}((j_0 - q_0)e^{-2s(\log 2-\gamma)} + g_0(s))\log s$$

where $g_0(s)$ depends on $\mathcal{A}, \mathcal{B}$. Also, it is easy to check that when there are no $-\frac{1}{4}$ eigenvalues, then there are no logarithmic singularities and the "unusual" poles occur with multiplicity at most one. Finally, the expansion (2.6) means that for any $N \in \mathbb{N}$:

$$\zeta_{\mathrm{sing}}(s, \Delta_L) = \frac{\sin(\pi s)}{\pi}\Bigg\{(j_0 - q_0)e^{-2s(\log 2-\gamma)}\log s + \sum_{\xi\in\mathcal{P},\,\xi\le N}\frac{f_\xi(s)}{(s+\xi)^{|p_\xi|+1}} \\ + \sum_{\xi\in\mathcal{L},\,\xi\le N} g_\xi(s)\log(s+\xi)\Bigg\} + F_N(s),$$

where $F_N(s)$ is holomorphic for $\Re s \ge -N$.

2.2 Examples

Via examples we show the ease and efficiency at which Theorem 2.1 computes the exact meromorphic structure of $\zeta_{\mathrm{sing}}(s, \Delta_L)$ (note that $\zeta_{\mathrm{reg}}(s, \Delta_L)$ is "uninteresting," which is why we focus on $\zeta_{\mathrm{sing}}(s, \Delta_L)$).

Example 1 (Taken from [18]) The article by Falomir *et al.* [18] (along with Seeley *et al.* [17] and Mooers' [54]) is in many ways the inspiration for our article and is the very first article to find explicit formulas for the "unusual" poles of Laplacians; cf. [19] for the infinite interval. [18] studies the operator

$$\Delta = -\frac{d^2}{dr^2} + \frac{1}{r^2}\lambda \quad \text{over } [0, 1]$$

with $-\frac{1}{4} \le \lambda < \frac{3}{4}$. In this case, $V = \mathbb{C}^2$, therefore Lagrangians $L \subset \mathbb{C}^2$ are determined by 1×1 matrices (numbers) $\mathcal{A} = \alpha$ and $\mathcal{B} = \beta$. Fix such an (α, β); we shall determine the strange singularity structure of $\zeta(s, \Delta_L)$. Let us assume that $-\frac{1}{4} < \lambda < \frac{3}{4}$

so there are no $-\frac{1}{4}$ eigenvalues (we will come back to the $\lambda = -\frac{1}{4}$ in a moment). Then with $\nu := \sqrt{\lambda + \frac{1}{4}}$ and $\tau := 2^{2\nu}\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}$,

$$p(x,y) := \det\begin{pmatrix} \alpha & \beta \\ \tau y^{2\nu} & 1 \end{pmatrix} = \alpha - \beta\tau y^{2\nu} = \alpha\left(1 - \frac{\tau\beta}{\alpha}y^{2\nu}\right),$$

where we assume that $\alpha, \beta \neq 0$ (the $\alpha = 0$ or $\beta = 0$ cases can be handled easily), and we write $p(x,y)$ as in (2.2). Forming the power series (2.3), we see that

$$\log\left(1 - \frac{\tau\beta}{\alpha}y^{2\nu}\right) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(-\frac{\tau\beta}{\alpha}y^{2\nu}\right)^k = \sum_{k=1}^{\infty} c_{0,\nu k}x^0 y^{2\nu k},$$

where $c_{0,\nu k} = -\frac{1}{k}(\frac{\tau\beta}{\alpha})^k$. Using the definitions (2.4) and (2.5) for p_ξ, ℓ_ξ, $\mathscr{P}$, and $\mathscr{L}$, we immediately see that $\ell_{\nu k}$ never exists so $\mathscr{L} = \varnothing$, while

$$p_{\nu k} = \min\{\ell \le 0 \mid c_{\ell,\nu k} \neq 0\} = 0, \qquad \mathscr{P} = \{\nu k \mid k \in \mathbb{N}\}.$$

Therefore, by Theorem 2.1:

$$\zeta_{\text{sing}}(s, \Delta_L) = \frac{\sin(\pi s)}{\pi}\sum_{k=1}^{\infty}\frac{f_k(s)}{s + \nu k}$$

with $f_k(s)$ an entire function of s such that $f_k(-\nu k) = \nu(\frac{\tau\beta}{\alpha})^k$. In particular, $\zeta_{\text{sing}}(s, \Delta_L)$ has possible poles at each $s = -\nu k$ with the residue equal to:

$$\operatorname{Res}_{s=-\nu k}\zeta_{\text{sing}}(s, \Delta_L) = \frac{\sin(\pi(-\nu k))}{\pi}\nu\left(\frac{\tau\beta}{\alpha}\right)^k = -\frac{\nu\sin(\pi\nu k)}{\pi}\left(\frac{\tau\beta}{\alpha}\right)^k,$$

which is the main result of [18] (see (7.11) of loc. cit.).

Assume now that $\lambda = -\frac{1}{4}$. In this case:

$$p(x,y) := \det\begin{pmatrix} \alpha & \beta \\ x & 1 \end{pmatrix} = \alpha - \beta x = \alpha\left(1 - \frac{\beta}{\alpha}x\right),$$

where we assume that $\alpha, \beta \neq 0$ (the $\alpha = 0$ or $\beta = 0$ cases can be handled easily), and we write $p(x,y)$ as in (2.2). Forming the power series (2.3), we see that

$$\log\left(1 - \frac{\beta}{\alpha}x\right) = \sum_{\ell=1}^{\infty} c_{\ell,0}x^\ell y^{2\cdot 0}, \quad c_{\ell,0} = -\frac{1}{\ell}\left(\frac{\beta}{\alpha}\right)^\ell.$$

Using the definitions (2.4) and (2.5) for p_ξ, ℓ_ξ (there is only one "ξ" in the present situation, $\xi = 0$), $\mathscr{P}$, and $\mathscr{L}$, we immediately see that p_0 never exists so $\mathscr{P} = \varnothing$, while

$$\ell_0 = \min\{\ell > 0 \mid c_{\ell,0} \neq 0\} = 1, \qquad \mathscr{L} = \{0\}.$$

Therefore, by Theorem 2.1,

$$\zeta_{\text{sing}}(s, \Delta_L) = \frac{\sin(\pi s)}{\pi}\{-e^{-2s(\log 2 - \gamma)}\log s + g_0(s)\log s\},$$

$g_0(s)$ is an entire function of s such that $g_0(s) = \mathcal{O}(s)$. Hence $\zeta(s, \Delta_L)$ has a *genuine* logarithmic singularity at $s = 0$. This corrects unfortunate errors from the beautiful article [18] (and [54]), which states that $\zeta(s, \Delta_L)$ has the "usual" meromorphic structure.[1] When $\beta = 0$, one can easily see that we still have a logarithmic singularity at $s = 0$, and when $\alpha = 0$, one can easily check that there is only the "regular" part $\zeta_{\text{reg}}(s, \Delta_L)$ and no "singular" part; in fact, the case $\alpha = 0$ corresponds to the Friedrichs extension (see [4]); thus we can see that $\zeta(s, \Delta_L)$ has a logarithmic singularity for all extensions except the Friedrichs.

Example 2 (The Laplacian on $\mathbb{R}^2$) If Δ is the Laplacian on a compact region in $\mathbb{R}^2$, then as we saw before in Sect. 1.1, A_Γ has a $-\frac{1}{4}$ eigenvalue of multiplicity one and no eigenvalues in $(-\frac{1}{4}, \frac{3}{4})$. Therefore, the exact same argument we used in the $\lambda = -\frac{1}{4}$ case of the previous example shows that $\zeta(s, \Delta_L)$ has a logarithmic singularity for all extensions except the Friedrichs.

Example 3 Consider now the case of a regular singular operator Δ over a compact manifold and suppose that A_Γ has two eigenvalues in $[-\frac{1}{4}, \frac{3}{4})$, the eigenvalue $-\frac{1}{4}$ and another eigenvalue $-\frac{1}{4} < \lambda < \frac{3}{4}$, both of multiplicity one. In this case, $V = \mathbb{C}^4$, therefore Lagrangians $L \subset \mathbb{C}^4$ are determined by 2×2 matrices $\mathcal{A}$ and $\mathcal{B}$. Consider the specific examples:

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{B} = \mathrm{Id}.$$

Then with $\nu := \sqrt{\lambda + \frac{1}{4}}$ and $\tau := 2^{2\nu} \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}$, we have:

$$p(x, y) := \det \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ x & 0 & 1 & 0 \\ 0 & \tau y^{2\nu} & 0 & 1 \end{pmatrix} = 1 + \tau x y^{2\nu}.$$

Forming the power series (2.3), we see that:

$$\log(1 + \tau x y^{2\nu}) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (\tau x y^{2\nu})^k = \sum_{k=1}^{\infty} c_{k,\nu k} x^k y^{2\nu k},$$

where $c_{k,\nu k} = (-1)^{k-1} \frac{\tau^k}{k}$. Using the definitions (2.4) and (2.5) for p_ξ, ℓ_ξ, $\mathcal{P}$, and $\mathcal{L}$, we immediately see that $p_{\nu k}$ never exists so $\mathcal{P} = \varnothing$, while

$$\ell_{\nu k} = \min\{\ell > 0 \mid c_{\ell, \nu k} \neq 0\} = k, \qquad \mathcal{L} = \{\nu k \mid k \in \mathbb{N}\}.$$

Therefore, by Theorem 2.1:

$$\zeta_{\text{sing}}(s, \Delta_L) = \frac{\sin(\pi s)}{\pi} \left\{ -e^{-2s(\log 2 - \gamma)} \log s + \sum_{k=1}^{\infty} g_k(s) \log(s + \nu k) \right\},$$

[1] The error in [18] occurs in (A13) where certain antiderivatives (specifically, $x N_1(x)$ and $x^2 N_1(x)^2$) were accidentally set equal to zero at $x = 0$.

with $g_k(s)$ an entire function of s such that near $s = -\nu k$,

$$g_k(s) = (-1)^k \frac{\tau^k 2^k \nu}{(k-1)!}(s+\nu k)^{k-1} + \mathcal{O}((s+\nu k)^k).$$

In particular, $\zeta_{\text{sing}}(s, \Delta_L)$ has *countably* many logarithmic singularities!

Example 4 With the same situation as considered in Example 3, consider

$$\mathcal{A} = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}.$$

Then with $\nu := \sqrt{\lambda + \frac{1}{4}}$ and $\tau := 2^{2\nu} \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}$, we have

$$p(x, y) := \det \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ x & 0 & 1 & 0 \\ 0 & \tau y^{2\nu} & 0 & 1 \end{pmatrix} = x - \tau y^{2\nu} = x(1 - \tau x^{-1} y^{2\nu}).$$

Forming the power series (2.3), we see that:

$$\log(1 - \tau x^{-1} y^{2\nu}) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (-\tau x^{-1} y^{2\nu})^k = \sum_{k=1}^{\infty} c_{-k,\nu k} x^{-k} y^{2\nu k},$$

where $c_{-k,\nu k} = -\frac{\tau^k}{k}$. Using the definitions (2.4) and (2.5) for p_ξ, ℓ_ξ, $\mathcal{P}$, and $\mathcal{L}$, we immediately see that $\ell_{\nu k}$ never exists so $\mathcal{L} = \varnothing$, while

$$p_{\nu k} = \min\{\ell \leq 0 \mid c_{-\ell,\nu k} \neq 0\} = -k, \qquad \mathcal{P} = \{\nu k \mid k \in \mathbb{N}\}.$$

Therefore, by Theorem 2.1:

$$\zeta_{\text{sing}}(s, \Delta_L) = \frac{\sin(\pi s)}{\pi} \sum_{k=1}^{\infty} \frac{f_k(s)}{(s + \nu k)^{k+1}}$$

with $f_k(s)$ an entire function of s such that $f_k(-\nu k) = (-1)^k \frac{\tau^k k! \nu}{2^k}$. In particular, $\zeta_{\text{sing}}(s, \Delta_L)$ has poles of *arbitrarily* large order!

Example 5 Consider one last example, the case of a regular singular operator Δ over a compact manifold such that A_Γ has three eigenvalues in $[-\frac{1}{4}, \frac{3}{4})$, the eigenvalue $-\frac{1}{4}$ with multiplicity two and another eigenvalue $-\frac{1}{4} < \lambda < \frac{3}{4}$ of multiplicity one. In this case, $V = \mathbb{C}^6$ and Lagrangians $L \subset \mathbb{C}^6$ are determined by 3×3 matrices $\mathcal{A}$ and $\mathcal{B}$. Consider the specific examples:

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \mathcal{B} = \text{Id}.$$

Then with $\nu := \sqrt{\lambda + \frac{1}{4}}$ and $\tau := 2^{2\nu}\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}$, we have

$$
\begin{aligned}
p(x,y) := \det \begin{pmatrix} 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ x & 0 & 0 & 1 & 0 & 0 \\ 0 & x & 0 & 0 & 1 & 0 \\ 0 & 0 & \tau y^{2\nu} & 0 & 0 & 1 \end{pmatrix} \\
= -x + \tau y^{2\nu} - \tau x^2 y^{2\nu} = -x(1 + (x - x^{-1})\tau y^{2\nu}).
\end{aligned}
$$

Forming the power series (2.3), we see that:

$$
\begin{aligned}
&\log(1 + (x - x^{-1})\tau y^{2\nu}) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \tau^k (x - x^{-1})^k y^{2\nu k} \\
&= \sum_{k=1}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{j-1}}{k} \tau^k \binom{k}{j} x^{2j-k} y^{2\nu k} = \sum_{\ell,k} c_{\ell,\nu k} x^{\ell} y^{2\nu k},
\end{aligned} \tag{2.7}
$$

where for each k, ℓ runs from $-k$ to k. Using the definitions (2.4) and (2.5) for p_ξ, ℓ_ξ, $\mathcal{P}$, and $\mathcal{L}$, we immediately see that:

$$
\begin{aligned}
p_{\nu k} &= \min\{\ell \le 0 \mid c_{\ell,\nu k} \neq 0\} = -k, \\
\ell_{\nu k} &= \min\{\ell > 0 \mid c_{\ell,\nu k} \neq 0\} = \begin{cases} 1 & \text{if } k \text{ is odd}, \\ 2 & \text{if } k \text{ is even}, \end{cases}
\end{aligned}
$$

and $\mathcal{P} = \mathcal{L} = \{\nu k \mid k \in \mathbb{N}\}$. Therefore, by Theorem 2.1:

$$
\begin{aligned}
\zeta_{\text{sing}}(s, \Delta_L) = \frac{\sin(\pi s)}{\pi} \Bigg\{ &-e^{-2s(\log 2 - \gamma)} \log s + \sum_{k=1}^{\infty} \frac{f_k(s)}{(s + \nu k)^{k+1}} \\
&+ \sum_{k=1}^{\infty} g_k(s) \log(s + \nu k) \Bigg\},
\end{aligned} \tag{2.8}
$$

where $f_k(s)$ and $g_k(s)$ are entire functions of s such that $f_k(-\nu k) = (-1)^k \frac{\tau^k k! \nu}{2^k}$ and

$$
\begin{aligned}
g_k(s) = 2\nu(-1)^{m+1}\tau^k \binom{k}{m+1} \\
\times \begin{cases} 1 + \mathcal{O}((s + \nu k)) & \text{if } k = 2m + 1 \text{ is odd}, \\ 2(s + \nu k) + \mathcal{O}((s + \nu k)^2) & \text{if } k = 2m \text{ is even}. \end{cases}
\end{aligned}
$$

In particular, $\zeta_{\text{sing}}(s, \Delta_L)$ has poles of *arbitrarily* high orders and in addition to a logarithmic singularity at the origin, *countably* many logarithmic singularities at the same locations of the poles!

2.3 Special Lagrangian Subspaces

Theorem 2.1 simplifies considerably when the conditions given by the Lagrangian L over $-\frac{1}{4}$ eigenspaces are separated from the conditions over λ eigenspaces with λ in $(-\frac{1}{4}, \frac{3}{4})$. We shall call a Lagrangian subspace $L \subset V$ *decomposable* if $L = L_0 \oplus L_1$ where L_0 is an arbitrary Lagrangian subspace of $\bigoplus_{\lambda_\ell = -\frac{1}{4}} E_\ell \oplus E_\ell$ and L_1 is an arbitrary Lagrangian subspace of $\bigoplus_{-\frac{1}{4}<\lambda_\ell<\frac{3}{4}} E_\ell \oplus E_\ell$. As described in Proposition 3.3, the Lagrangian subspace L_0 is determined by two $q_0 \times q_0$ matrices $\mathcal{A}_0$, $\mathcal{B}_0$ where $q_0 = \dim L_0$, that is, the multiplicity of the eigenvalues $\lambda_\ell = -\frac{1}{4}$. Similarly, the Lagrangian subspace L_1 is determined by two $q_1 \times q_1$ matrices $\mathcal{A}_1$, $\mathcal{B}_1$ where $q_1 = \dim L_1$, that is, the multiplicity of the eigenvalues λ_ℓ with $-\frac{1}{4} < \lambda_\ell < \frac{3}{4}$.

Two polynomials which are explicitly determined by the matrices $\mathcal{A}_0, \mathcal{B}_0$ and $\mathcal{A}_1, \mathcal{B}_1$ play peculiar roles in the statement of our result. First, consider the polynomial $p_0(z)$ in the single variable z defined by:

$$p_0(z) := \det \begin{pmatrix} \mathcal{A}_0 & \mathcal{B}_0 \\ \mathrm{Id}_{q_0} & (\log 2 - \gamma - z)\mathrm{Id}_{q_0} \end{pmatrix}. \tag{2.9}$$

Using the definition of determinant, it is easy to see that $p_0(z)$ is a polynomial of degree at most q_0 in z. Since the degree of $p_0'(z)$ is one less than the degree of $p_0(z)$, we can write:

$$\frac{p_0'(z)}{p_0(z)} = \sum_{k=1}^{\infty} \frac{\beta_k}{z^k}, \quad \beta_k \in \mathbb{C}, \tag{2.10}$$

where the series on the right is absolutely convergent for $|z|$ sufficiently large. Second, consider the polynomial in (2.1) (and (2.2)) using $\mathcal{A}_1$ and $\mathcal{B}_1$ in place of $\mathcal{A}$ and $\mathcal{B}$:

$$p_1(y) := \det \begin{pmatrix} & \mathcal{A}_1 & & \mathcal{B}_1 \\ \tau_1 y^{2\nu_1} & 0 & 0 & \\ 0 & \ddots & 0 & \mathrm{Id}_{q_1} \\ 0 & 0 & \tau_{q_1} y^{2\nu_{q_1}} & \end{pmatrix} = a_{\alpha_0} y^{2\alpha_0} \Big(1 + \sum b_\beta y^{2\beta}\Big), \tag{2.11}$$

where the β's are positive. Then as in (2.3), write

$$\log\Big(1 + \sum b_\beta y^{2\beta}\Big) = \sum c_\xi y^{2\xi}, \tag{2.12}$$

and let $\mathcal{P} := \{\xi \mid c_\xi \neq 0\}$. Then Theorem 2.1 simplifies to the following.

Theorem 2.3 *For an arbitrary decomposable Lagrangian $L \subset V$, the ζ-function $\zeta(s, \Delta_L)$ has the following form*:

$$\zeta(s, \Delta_L) = \zeta_{\mathrm{reg}}(s, \Delta_L) + \zeta_{\mathrm{sing}}(s, \Delta_L), \tag{2.13}$$

where $\zeta_{\mathrm{reg}}(s, \Delta_L)$ has the "regular" poles at the "usual" locations $s = \frac{n-k}{2} \notin -\mathbb{N}_0$ for $k \in \mathbb{N}_0$ and at $s = 0$ if $\dim \Gamma > 0$, and where $\zeta_{\mathrm{sing}}(s, \Delta_L)$ has the following ex-

pansion:

$$\zeta_{\text{sing}}(s, \Delta_L) = -\frac{\sin(\pi s)}{\pi} f(s) \log s + \frac{\sin(\pi s)}{\pi} \sum_{\xi \in \mathcal{P}} \frac{f_\xi(s)}{s + \xi}, \tag{2.14}$$

where $f(s)$ is the entire function defined explicitly by $f(s) = \sum_{k=1}^{\infty} \beta_k \frac{(-2s)^{k-1}}{(k-1)!}$, and the $f_\xi(s)$'s are entire functions of s such that

$$f_\xi(-\xi) = -c_\xi \xi.$$

For certain types of Lagrangians, the formula for $f(s)$ becomes very simple. We shall call the Lagrangian L_0 *split-type* if it can be written as $\bigoplus_{\lambda_\ell = -\frac{1}{4}} L_\ell$ with L_ℓ a Lagrangian subspace of $E_\ell \oplus E_\ell$. As explained in Proposition 3.6, each component L_ℓ of L_0 is determined by an angle $\theta_\ell \in [0, \pi)$. Moreover, in this case, the coefficients β_k in (2.10) are given by (as follows from Corollary 5.4)

$$\beta_k = \sum_{\theta_\ell \neq \frac{\pi}{2}} \kappa_\ell^{k-1}, \quad k = 1, 2, 3, 4, \ldots \text{ (for split-type } L_0) \tag{2.15}$$

where $\kappa_\ell = \log 2 - \gamma - \tan \theta_\ell$ with θ_ℓ the angle defining L_ℓ in L_0 and γ is the Euler-Mascheroni constant. Then when L_0 is of split-type, we have the following.

Theorem 2.4 *For a decomposable Lagrangian $L \subset V$ such that L_0 is of split-type, $\zeta(s, \Delta_L)$ has the form as in (2.13) with $f(s) = \sum_{\theta_\ell \neq \frac{\pi}{2}} e^{-2s\kappa_\ell}$ in (2.14).*

Example 6 Consider the case when A_Γ has exactly one eigenvalue in $[-\frac{1}{4}, \frac{3}{4})$, the eigenvalue $-\frac{1}{4}$. Then Theorem 2.4 shows the result stated in (1.2).

2.4 Unusual Resolvent and Heat Kernel Expansions

Besides establishing exotic ζ-expansions, we also derive equally exotic resolvent and heat kernel expansions.

Theorem 2.5 *Let $\Lambda \subset \mathbb{C}$ be any sector (solid angle) not intersecting the positive real axis and choose $N \in \mathbb{N}$ with $N \geq \frac{n}{2}$. Then for an arbitrary Lagrangian L, as $|\lambda| \to \infty$ with $\lambda \in \Lambda$ we have*:

$$\begin{aligned}\operatorname{Tr}(\Delta_L - \lambda)^{-N-1} \sim &\sum_{k=0}^{\infty} a_k (-\lambda)^{\frac{n-k}{2} - N - 1} + b(-\lambda)^{-N-1} \log(-\lambda) \\ &+ \frac{1}{N!} \frac{d^N}{d\lambda^N} \left\{ \frac{q_0 - j_0}{(-\lambda)(\log(-\lambda) - 2\widetilde{\gamma})} \right\} \\ &- \frac{1}{N!} \frac{d^{N+1}}{d\lambda^{N+1}} \left\{ \sum 2^\ell c_{\ell\xi} (-\lambda)^{-\xi} (2\widetilde{\gamma} - \log(-\lambda))^{-\ell} \right\}\end{aligned} \tag{2.16}$$

where the a_k and b coefficients are independent of L, the $c_{\ell\xi}$'s are the coefficients in (2.3), and $\widetilde{\gamma} = \log 2 - \gamma$.

From the explicit formula (2.16) and from the binomial theorem for $\ell > 0$:

$$(2\widetilde{\gamma} - \log(-\lambda))^{-\ell} = (-\log(-\lambda))^{-\ell}\left(1 - \frac{2\widetilde{\gamma}}{\log(-\lambda)}\right)^{-\ell}$$
$$= (-\log(-\lambda))^{-\ell}\sum_{j=0}^{\infty}\binom{-\ell}{j}\frac{(-2\widetilde{\gamma})^j}{(\log(-\lambda))^j}, \qquad (2.17)$$

it is obvious that when A_Γ has $-\frac{1}{4}$ eigenvalues, the resolvent trace expansion has, in general, $\log(-\lambda)$ terms of *arbitrarily* high multiplicity and inverse powers $(\log(-\lambda))^{-1}$ with *infinite* multiplicity! This phenomenon is new and even for pseudodifferential operators on compact manifolds, with or without boundary, and even conic, "regular" (not inverse powers of) $\log(-\lambda)$ terms occur with at most multiplicity two [21–23, 29, 30, 47, 48]. See [25, 42, 58] for studies of resolvents for closed extensions of general cone operators in the sense of B.W. Schulze [59, 60]. Here is a concrete example illustrating this discussion.

Example 7 For the self-adjoint extension Δ_L considered in Example 5, from the explicit formula (2.7), we immediately get:

$$\begin{aligned}\operatorname{Tr}(\Delta_L - \lambda)^{-N-1} \sim{}& \sum_{k=0}^{\infty} a_k(-\lambda)^{\frac{n-k}{2}-N-1} + b(-\lambda)^{-N-1}\log(-\lambda)\\ &+ \frac{1}{N!}\frac{d^N}{d\lambda^N}\left\{\frac{1}{(-\lambda)(\log(-\lambda) - 2\widetilde{\gamma})}\right\}\\ &- \frac{1}{N!}\frac{d^{N+1}}{d\lambda^{N+1}}\left\{\sum_{k=1}^{\infty}\sum_{j=0}^{k}\frac{2^{2j-k}(-1)^{j-1}}{k}\tau^k\binom{k}{j}\right.\\ &\left.\times(-\lambda)^{-2\nu k}\big(2\widetilde{\gamma} - \log(-\lambda)\big)^{k-2j}\right\}.\end{aligned}$$

In this very simple example, we see unusual powers $(-\lambda)^{-2\nu k - N-1}$ (after taking $N+1$ derivatives) and log terms $\log(-\lambda)$ of *arbitrarily* high multiplicity (each unusual power $(-\lambda)^{-2\nu k-N-1}$ with a log term of highest power $(\log(-\lambda))^k$), and inverse powers $(\log(-\lambda))^{-1}$ with *infinite* multiplicity because of the formula (2.17).

When L is decomposable, the last two terms in (2.16) can be made very explicit.

Theorem 2.6 *Let $\Lambda \subset \mathbb{C}$ be any sector* (*solid angle*) *not intersecting the positive real axis and choose $N \in \mathbb{N}$ with $N \geq \frac{n}{2}$. Then for an arbitrary decomposable Lagrangian L, as $|\lambda| \to \infty$ with $\lambda \in \Lambda$ we have*:

$$\operatorname{Tr}(\Delta_L - \lambda)^{-N-1} \sim \sum_{k=0}^{\infty} a_k(-\lambda)^{\frac{n-k}{2}-N-1} + b(-\lambda)^{-N-1}\log(-\lambda)$$

$$
-\frac{1}{N!}\frac{d^{N+1}}{d\lambda^{N+1}}\left\{\sum_{\xi\in\mathcal{P}} c_\xi(-\lambda)^{-\xi}\right\}
$$

$$
+\frac{1}{N!}\frac{d^{N}}{d\lambda^{N}}\left\{(-\lambda)^{-1}\sum_{k=1}^{\infty}\frac{2^{k-1}\beta_k}{(\log(-\lambda))^k}\right\}, \tag{2.18}
$$

where the a_k and b coefficients are independent of L, the β_k's are the coefficients in (2.10), *and the c_ξ's are the coefficients in* (2.12).

In the case when L_0 is of split-type, the second-to-last term in (2.18) can be made even more explicit because of the formula (2.15) for β_k. We also prove a corresponding heat kernel expansion.

Theorem 2.7 *For an arbitrary Lagrangian L, the heat kernel $e^{-t\Delta_L}$ has the following trace expansion as $t \to 0$:*

$$
\mathrm{Tr}(e^{-t\Delta_L}) \sim \sum_{k=0}^{\infty}\widetilde{a}_k t^{\frac{-n+k}{2}} + b\log t + \sum_{k=0}^{\infty}\widetilde{b}_k(\log t)^{-1-k}
$$

$$
+\sum_{\xi\in\mathcal{P}}\sum_{k=0}^{|p_\xi|+1}\widetilde{c}_{\xi k}t^{\xi}(\log t)^k + \sum_{\xi\in\mathcal{L}}\sum_{k=0}^{\infty}\widetilde{d}_{\xi k}t^{\xi}(\log t)^{-\ell_\xi-k},
$$

with $\widetilde{c}_{10} = 0$ and $\widetilde{c}_{\xi(|p_\xi|+1)} = 0$ for $\xi \notin \mathbb{N}_0$.

Thus, the heat trace expansion, in general, has powers of $\log t$ with finite multiplicity and inverse powers $(\log t)^{-1}$ with *infinite* multiplicity. The $\widetilde{c}_{\xi k}$ and $\widetilde{d}_{\xi k}$ coefficients can be expressed in terms of the coefficients in the resolvent expansion (2.16) but not so explicitly. For decomposable Lagrangians we have the following.

Theorem 2.8 *For an arbitrary decomposable Lagrangian L, the heat kernel $e^{-t\Delta_L}$ has the following trace expansion as $t \to 0$:*

$$
\mathrm{Tr}(e^{-t\Delta_L}) \sim \sum_{k=0}^{\infty}\widetilde{a}_k\, t^{\frac{-n+k}{2}} + \widetilde{b}\,\log t + \sum_{\xi\in\mathcal{P}}\widetilde{c}_\xi\, t^{\xi} + \sum_{k=1}^{\infty}\widetilde{d}_k(\log t)^{-k}.
$$

3 Hermitian Symplectic Formalism of Self-Adjoint Extensions

To orient the reader to the various terminologies used throughout this article, in this section we briefly review well known results from the classical theory of self-adjoint extensions. For more on this classical theory, see for example [32, 33, 41, 55, 66, 67] and for its use to analyze second order regular singular operators, see [10, 11, 38–40, 49, 54, 56], and see Gil and Mendoza [24] and Lesch [44] for the analysis of more general Fuchs type or cone operators in the sense of Schulze [59, 60]; finally, see [16] for applications of self-adjoint extensions to quantum physics.

3.1 The maximal Domain and Self-Adjoint Extensions

To analyze self-adjoint extensions of Δ, the idea is to find the "largest" domains $\mathfrak{D} \subset \mathrm{dom}_{\max}(\Delta)$ that make the Hermitian quadratic form:

$$\langle \Delta\phi, \psi\rangle - \langle \phi, \Delta\psi\rangle, \quad \phi, \psi \in \mathfrak{D},$$

vanish. To determine the extensions of Δ, we first describe $\mathrm{dom}_{\max}(\Delta)$. Recall that $\Delta|_{\mathcal{U}} = -\partial_r^2 + \frac{1}{r^2} A_\Gamma$ (see (1.3)), where $\mathcal{U} \cong [0, \varepsilon)_r \times \Gamma$ denotes a tubular neighborhood of Γ and A_Γ is a Laplace-type operator over Γ such that $A_\Gamma \geq -\frac{1}{4}$. Recall that:

$$-\frac{1}{4} = \underbrace{\lambda_1 = \lambda_2 = \cdots = \lambda_{q_0}}_{=-\frac{1}{4}} < \underbrace{\lambda_{q_0+1} \leq \lambda_{q_0+2} \leq \cdots \leq \lambda_{q_0+q_1}}_{-\frac{1}{4} < \lambda_\ell < \frac{3}{4}}$$

denotes the eigenvalues of A_Γ in $[-\frac{1}{4}, \frac{3}{4})$ with corresponding orthonormal eigenvectors $\{\phi_\ell\}$. Then, as shown by Cheeger [10, 11], we have the following.

Proposition 3.1 *A section $\phi \in L^2(M, E)$ is in*

$$\mathrm{dom}_{\max}(\Delta) := \{\phi \in L^2(M, E) \mid \Delta\phi \in L^2(M, E)\},$$

where "$\Delta\phi \in L^2(M, E)$" is in the distributional sense, if and only if ϕ is in H^2 away from the boundary Γ, and near Γ we can write:

$$\begin{aligned}\phi &= \sum_{\ell=1}^{q_0} \{c_\ell^+(\phi) r^{\frac{1}{2}} \phi_\ell + c_\ell^-(\phi) r^{\frac{1}{2}} \log r \phi_\ell\} \\ &\quad + \sum_{\ell=1}^{q_1} \left\{ \frac{c_{q_0+\ell}^+(\phi)}{\sqrt{2\nu_\ell}} r^{\nu_\ell+\frac{1}{2}} \phi_{q_0+\ell} + \frac{c_{q_0+\ell}^-(\phi)}{\sqrt{2\nu_\ell}} r^{-\nu_\ell+\frac{1}{2}} \phi_{q_0+\ell} \right\} + \widetilde{\phi},\end{aligned} \tag{3.1}$$

where the $c_\ell^\pm(\phi)$'s are constants, $\nu_\ell := \sqrt{\lambda_{q_0+\ell} + \frac{1}{4}} > 0$, $\widetilde{\phi} \in H^2$ and $\widetilde{\phi} = \mathcal{O}(r^{\frac{3}{2}})$.

To state the well-known correspondence between self-adjoint extensions and Lagrangian subspaces we define $\psi_\ell^+ := r^{\frac{1}{2}+\nu_\ell} \phi_\ell$ for $0 \leq \nu_\ell < 1$, and

$$\psi_\ell^- := \begin{cases} r^{\frac{1}{2}-\nu_\ell} \phi_\ell & \text{for } 0 < \nu_\ell < 1, \\ r^{\frac{1}{2}} \log r\, \phi_\ell & \text{for } \nu_\ell = 0, \end{cases}$$

and furthermore we define

$$V := \bigoplus_{-\frac{1}{4} \leq \lambda_\ell < \frac{3}{4}} E_\ell^+ \oplus E_\ell^-$$

where $E_\ell^\pm := \langle \psi_\ell^\pm \rangle$ with $\langle \psi_\ell^\pm \rangle := \mathrm{span}_{\mathbb{C}}\{\psi_\ell^\pm\}$. We endow V with the symplectic structure $\omega : V \times V \to \mathbb{C}$ defined by:

$$\omega(\psi_\ell^\pm, \psi_\ell^\mp) = \begin{cases} \mp 1 & \text{when } \nu_\ell = 0 \\ \pm 1 & \text{when } \nu_\ell \neq 0; \end{cases} \qquad \omega(\psi_\ell^\pm, \psi_j^\pm) = 0 \quad \text{otherwise,} \tag{3.2}$$

and extending to $V \times V$ linearly in the first factor and conjugate linearly in the second factor. Then we have:

$$\langle \Delta\phi, \psi \rangle - \langle \phi, \Delta\psi \rangle = \omega(\vec{\phi}, \vec{\psi})$$

where $\vec{\phi} := \sum \left\{ c_\ell^+(\phi)\psi_\ell^+ + c_\ell^-(\phi)\psi_\ell^- \right\} \in V$, $\vec{\psi}$ is defined similarly, and the following theorem is an easy consequence.

Theorem 3.2 *Self-adjoint extensions of Δ are in one-to-one correspondence with Lagrangian subspaces of V in the sense that given any Lagrangian subspace $L \subset V$, defining*:

$$\mathrm{dom}_L(\Delta) := \{\phi \in \mathrm{dom}_{\max}(\Delta) \mid \vec{\phi} \in L\},$$

the operator

$$\Delta_L := \Delta : \mathrm{dom}_L(\Delta) \to L^2(M, E)$$

is self-adjoint and any self-adjoint extension of Δ is of the form Δ_L for some Lagrangian $L \subset V$.

3.2 Characterizations of Lagrangian Subspaces

Recall now the following classical characterization of all Lagrangian subspaces in complex Euclidean space.

Proposition 3.3 *A subset $L \subset \mathbb{C}^{2k}$ is Lagrangian (with respect to the standard Euclidean symplectic form) if and only if it can be described by a system of equations*:

$$L = \{\vec{c} \in \mathbb{C}^{2k} \mid (\mathcal{A}\ \mathcal{B})\vec{c} = 0\} \subset \mathbb{C}^{2k},$$

where $\mathcal{A}$ and $\mathcal{B}$ are $k \times k$ matrices such that the rank of $(\mathcal{A}\ \mathcal{B})$ is k and $\mathcal{A}\mathcal{B}^* = \mathcal{B}\mathcal{A}^*$.

As seen in (3.2), V can be identified with $\mathbb{C}^{2q} = \mathbb{C}^{2q_0} \times \mathbb{C}^{2q_1}$, where $q = q_0 + q_1$, with minus the standard symplectic form on the $\mathbb{C}^{2q_0}$ factor and the standard symplectic form on the $\mathbb{C}^{2q_1}$ factor. Using this fact, we have the following.

Corollary 3.4 *A Lagrangian subspace $L \subset V$ can be characterized by $q \times q$ matrices $\mathcal{A}$ and $\mathcal{B}$ via*

$$L \cong \{\vec{c} \in \mathbb{C}^{2q} \mid (\mathcal{A}\ \mathcal{B})\vec{c} = 0\} \subset \mathbb{C}^{2q},$$

where $(\mathcal{A}\ \mathcal{B})$ has rank q and $\mathcal{A}'\mathcal{B}^*$ is self-adjoint where $\mathcal{A}'$ is the matrix $\mathcal{A}$ with the first q_0 columns of $\mathcal{A}$ multiplied by -1.

As seen in the formula (3.1) of Proposition 3.1, the $\lambda_\ell = -\frac{1}{4}$ eigenvalues of A_Γ and the $-\frac{1}{4} < \lambda_\ell < \frac{3}{4}$ eigenvalues of A_Γ give rise to rather distinct components of $\mathrm{dom}_{\max}(\Delta)$. For this reason, it is natural to separate Lagrangian subspaces of V into $\lambda_\ell = -\frac{1}{4}$ components and $-\frac{1}{4} < \lambda_\ell < \frac{3}{4}$ components. With this discussion in mind, we call a Lagrangian subspace $L \subset V$ *decomposable* if $L = L_0 \oplus L_1$ where L_0 is an arbitrary Lagrangian subspace of $\bigoplus_{\lambda_\ell = -\frac{1}{4}} E_\ell^+ \oplus E_\ell^-$ and L_1 is an arbitrary Lagrangian subspace of $\bigoplus_{-\frac{1}{4} < \lambda_\ell < \frac{3}{4}} E_\ell^+ \oplus E_\ell^-$.

The characterization of all such L_0, L_1 follows from Proposition 3.3.

Corollary 3.5 *The components L_0 and L_1 of a decomposable Lagrangian $L = L_0 \oplus L_1 \subset V$ can be characterized by matrices $(\mathcal{A}_0\ \mathcal{B}_0)$ (with $\mathcal{A}_0$ and $\mathcal{B}_0$ $q_0 \times q_0$ matrices) and $(\mathcal{A}_1\ \mathcal{B}_1)$ (with $\mathcal{A}_1$ and $\mathcal{B}_1$ $q_1 \times q_1$ matrices) via*

$$L_0 \cong \{\vec{c} \in \mathbb{C}^{2q_0} \mid (\mathcal{A}_0\ \mathcal{B}_0)\vec{c} = 0\} \subset \mathbb{C}^{2q_0},$$

and

$$L_1 \cong \{\vec{c} \in \mathbb{C}^{2q_1} \mid (\mathcal{A}_1\ \mathcal{B}_1)\vec{c} = 0\} \subset \mathbb{C}^{2q_1}$$

where the matrix $(\mathcal{A}_0\ \mathcal{B}_0)$ has rank q_0 and $\mathcal{A}_0\mathcal{B}_0^* = \mathcal{B}_0\mathcal{A}_0^*$, and $(\mathcal{A}_1\ \mathcal{B}_1)$ has rank q_1 and $\mathcal{A}_1\mathcal{B}_1^* = \mathcal{B}_1\mathcal{A}_1^*$.

In the Introduction we discussed split-type Lagrangians. Here, we say that the Lagrangian $L_0 \subset \bigoplus_{\lambda_\ell = -\frac{1}{4}} E_\ell^+ \oplus E_\ell^-$ is of *split-type* if it can be written as $\bigoplus_{\lambda_\ell = -\frac{1}{4}} L_\ell$ with L_ℓ a Lagrangian subspace of $E_\ell^+ \oplus E_\ell^-$. Such Lagrangians L_ℓ, restricting Proposition 3.3 to $k = 1$, are characterized as follows.

Proposition 3.6 *$L \subset \mathbb{C}^2$ is Lagrangian if and only if $L = L_\theta$ for some $\theta \in [0, \pi)$ where*

$$L_\theta = \{(x, y) \in \mathbb{C}^2 \mid \cos\theta x + \sin\theta y = 0\}.$$

4 The Model Problems

For the rest of this article, unless stated otherwise, *we fix an arbitrary Lagrangian L in V*. In this section we analyze the eigenvalue equation for the model problem.

4.1 The Model Operator

In the last section we saw that only the eigenvalues of A_Γ in the interval $[-\frac{1}{4}, \frac{3}{4})$ are involved in the various self-adjoint extensions of Δ. For this reason, in this section as a first step to prove our main results we shall analyze the projection of:

$$\Delta|_{\mathcal{U}} = -\frac{d^2}{dr^2} + \frac{1}{r^2} A_\Gamma$$

onto the eigenspaces of A_Γ with eigenvalues in $[-\frac{1}{4}, \frac{3}{4})$. Recall from Corollary 3.4, with $q = q_0 + q_1$, that the Lagrangian L can be identified with the null space of a $q \times 2q$ matrix $(\mathcal{A}\ \mathcal{B})$ of full rank with $\mathcal{A}$ and $\mathcal{B}$ $q \times q$ matrices such that $\mathcal{A}'\mathcal{B}^*$ is self-adjoint. Then writing A_Γ as a diagonal matrix with respect to its eigenfunctions with eigenvalues in $[-\frac{1}{4}, \frac{3}{4})$, we shall consider the operator:

$$\mathcal{L} := -\frac{d^2}{dr^2} + \frac{1}{r^2}A \quad \text{over } [0, R],$$

where $R > 0$ is arbitrary, but fixed, and A is the $q \times q$ matrix

$$A = \begin{pmatrix} -\frac{1}{4}\mathrm{Id}_{q_0} & 0 \\ 0 & \begin{matrix} \lambda_{q_0+1} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_{q_0+2} & 0 & \cdots & 0 \\ 0 & 0 & \lambda_{q_0+3} & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \lambda_{q_0+q_1} \end{matrix} \end{pmatrix}.$$

We put Dirichlet conditions at the right end $r = R$ of the interval $[0, R]$. Then according to Proposition 3.1, we have the following.

Proposition 4.1 *$\phi \in \mathfrak{D}_{\max}$, the maximal domain of $\mathcal{L}$, if and only if $\phi(R) = 0$ and ϕ has the following form*:

$$\phi = \sum_{\ell=1}^{q_0} \{c_\ell(\phi) r^{\frac{1}{2}} e_\ell + c_{q+\ell}(\phi) r^{\frac{1}{2}} \log r e_\ell\}$$
$$+ \sum_{\ell=1}^{q_1} \{c_{q_0+\ell}(\phi) r^{\nu_\ell+\frac{1}{2}} e_{q_0+\ell} + c_{q+q_0+\ell}(\phi) r^{-\nu_\ell+\frac{1}{2}} e_{q_0+\ell}\} + \widetilde{\phi}, \tag{4.1}$$

where $\nu_j := \sqrt{\lambda_{q_0+j} + \frac{1}{4}} > 0$, e_ℓ is the column vector with 1 in the ℓ-th slot and 0's elsewhere, the $c_j(\phi)$'s are constants, and the $\widetilde{\phi}$ is continuously differentiable on $[0, R]$ such that $\widetilde{\phi}(r) = \mathcal{O}(r^{\frac{3}{2}})$ and $\widetilde{\phi}'(r) = \mathcal{O}(r^{\frac{1}{2}})$ near $r = 0$, and $\mathcal{L}\widetilde{\phi} \in L^2([0, R], \mathbb{C}^q)$.

We dropped the factors $\frac{1}{\sqrt{2\nu_\ell}}$, which appear in the statement of Proposition 3.1, from the terms in (4.1) for ϕ. Then as a consequence of Theorem 3.2, we know that:

$$\mathcal{L}_L : \mathfrak{D}_L \to L^2([0, R], \mathbb{C}^q) \quad \text{is self-adjoint,} \ \mathfrak{D}_L = \{\phi \in \mathfrak{D}_{\max} \mid \vec{\phi} \in L\},$$

where ϕ has the form in (4.1) with $\vec{\phi} = (c_1(\phi), c_2(\phi), \ldots, c_{2q}(\phi))^t$. In terms of the matrices $\mathcal{A}$ and $\mathcal{B}$, we can also write

$$\mathfrak{D}_L = \{\phi \in \mathfrak{D}_{\max} \mid (\mathcal{A}\ \mathcal{B})\vec{\phi} = 0\}. \tag{4.2}$$

4.2 Eigenvalue Equation

To analyze the ζ-function of $\mathcal{L}_L$, we derive an equation for the eigenvalues of $\mathcal{L}_L$. For this, we first find solutions to the equation:

$$(\mathcal{L}_L - \mu^2)\phi = 0.$$

As the reader can easily check, this is just a system of Bessel equations as described in [1, p. 362], whose solution (after judiciously choosing the constants for later convenience) can be taken to be of the form:

$$\begin{aligned}\phi = &\sum_{\ell=1}^{q_0} \{c_\ell(\phi) r^{\frac{1}{2}} J_0(\mu r) e_\ell + c_{q+\ell}(\phi) r^{\frac{1}{2}} \widetilde{J}_0(\mu r) e_\ell\} \\ &+ \sum_{\ell=1}^{q_1} \{2^{\nu_\ell} \Gamma(1+\nu_\ell) c_{q_0+\ell}(\phi) \mu^{-\nu_\ell} r^{\frac{1}{2}} J_{\nu_\ell}(\mu r) e_{q_0+\ell} \\ &+ 2^{-\nu_\ell} \Gamma(1-\nu_\ell) c_{q+q_0+\ell}(\phi)\, \mu^{\nu_\ell} r^{\frac{1}{2}} J_{-\nu_\ell}(\mu r) e_{q_0+\ell}\},\end{aligned} \tag{4.3}$$

where $J_v(z)$ denotes the Bessel function of the first kind and

$$\widetilde{J}_0(\mu r) := \frac{\pi}{2} Y_0(\mu r) - (\log \mu - \log 2 + \gamma) J_0(\mu r), \tag{4.4}$$

with $Y_0(z)$ the Bessel function of the second kind. For notational convenience let us introduce $J_{+0}(\mu R) = J_0(\mu R)$ and $J_{-0}(\mu R) = \widetilde{J}_0(\mu R)$.

Define $q \times q$ matrices $J_+(\mu)$, $J_-(\mu)$ by

$$J_\pm(\mu) := \begin{pmatrix} J_{\pm 0}(\mu R)\mathrm{Id}_{q_0} & 0 & \cdots & 0 \\ 0 & 2^{\pm\nu_1} \Gamma(1\pm\nu_1)\mu^{\mp\nu_1} J_{\pm\nu_1}(\mu R) & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & & 0 \\ 0 & 0 & \cdots & 2^{\pm\nu_{q_1}} \Gamma(1\pm\nu_{q_1})\mu^{\mp\nu_{q_1}} J_{\pm\nu_{q_1}}(\mu R) \end{pmatrix}.$$

In the following proposition, we determine an eigenvalue equation for the μ's.

Proposition 4.2 *μ^2 is an eigenvalue of $\mathcal{L}_L$ if and only if*

$$F(\mu) := \det \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ J_+(\mu) & J_-(\mu) \end{pmatrix} = 0.$$

Proof Imposing the Dirichlet condition at $r = R$ on ϕ of the form (4.3), we obtain

$$c_\ell(\phi)\, J_0(\mu R) + c_{q+\ell}(\phi)\, \widetilde{J}_0(\mu R) = 0, \quad \ell = 1, \ldots, q_0,$$

and

$$\begin{aligned}&2^{\nu_\ell} \Gamma(1+\nu_\ell) c_{q_0+\ell}(\phi) \mu^{-\nu_\ell} J_{\nu_\ell}(\mu R) \\ &\quad + 2^{-\nu_\ell} \Gamma(1-\nu_\ell) c_{q+q_0+\ell}(\phi) \mu^{\nu_\ell} J_{-\nu_\ell}(\mu R) = 0, \quad \ell = 1, \ldots, q_1.\end{aligned}$$

We can summarize these two equations as:

$$(J_+(\mu)\ J_-(\mu))\vec{\phi} = 0, \tag{4.5}$$

where $\vec{\phi} = (c_1(\phi), c_2(\phi), \ldots, c_{2q}(\phi))^t$. Now recall that [1, p. 360]

$$z^{-v} J_v(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{v+2k} k! \Gamma(v+k+1)} \sim \frac{1}{2^v \Gamma(1+v)} \left(1 - \frac{z^2}{4(1+v)} + \cdots\right), \tag{4.6}$$

and

$$\frac{\pi}{2} Y_0(z) = (\log z - \log 2 + \gamma) J_0(z) - \sum_{k=1}^{\infty} \frac{H_k(-\frac{1}{4}z^2)^k}{(k!)^2}, \tag{4.7}$$

where $H_k := 1 + \frac{1}{2} + \cdots + \frac{1}{k}$. Combining (4.4), (4.6) with $v = 0$, and (4.7), we get:

$$\widetilde{J}_0(\mu r) = (\log r) J_0(\mu r) - \sum_{k=1}^{\infty} \frac{H_k(-\frac{1}{4}(\mu r)^2)^k}{(k!)^2} = \log r + \mathcal{O}(r). \tag{4.8}$$

From (4.3), (4.6) and (4.8), it follows that:

$$\begin{aligned}\phi &\sim \sum_{\ell=1}^{q_0} \{c_\ell(\phi) r^{\frac{1}{2}} e_\ell + c_{q+\ell}(\phi) r^{\frac{1}{2}} \log r e_\ell\} \\ &\quad + \sum_{\ell=1}^{q_1} \{c_{q_0+\ell}(\phi) r^{\nu_\ell + \frac{1}{2}} e_{q_0+\ell} + c_{q+q_0+\ell}(\phi) r^{-\nu_\ell+\frac{1}{2}} e_{q_0+\ell}\} \quad \text{near } r = 0.\end{aligned}$$

In particular, by (4.2), ϕ in $\mathfrak{D}_L$ satisfies $(\mathcal{A}\ \mathcal{B})\vec{\phi} = 0$, and therefore, in view of (4.5), we conclude that:

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ J_+(\mu) & J_-(\mu) \end{pmatrix} \vec{\phi} = 0.$$

For nontrivial $\vec{\phi}$, this equation can hold if and only if the matrix in front of $\vec{\phi}$ is singular. This completes our proof. □

4.3 Asymptotics of $F(\mu)$

In order to find relevant properties of the resolvent, the heat-trace and the ζ-function, we shall need the asymptotics of $F(\mu)$ as $|\mu| \to \infty$.

Proposition 4.3 *Let $\Upsilon \subset \mathbb{C}$ be a sector (closed angle) in the right-half plane. Then as $|x| \to \infty$ with $x \in \Upsilon$, we have:*

$$\begin{aligned} F(ix) &\sim (2\pi R)^{-\frac{q}{2}} \prod_{j=1}^{q_1} 2^{-\nu_j} \Gamma(1-\nu_j) x^{|\nu| - \frac{q}{2}} e^{qxR} (\widetilde{\gamma} - \log x)^{q_0} \\ &\quad \times p((\widetilde{\gamma} - \log x)^{-1}, x^{-1})(1 + \mathcal{O}(x^{-1})), \end{aligned} \tag{4.9}$$

where $\widetilde{\gamma} = \log 2 - \gamma$, $p(x, y)$ *is given in* (2.1), $\mathcal{O}(x^{-1})$ *is a power series in* x^{-1}, *and*

$$\begin{aligned}\frac{d}{dx}\log F(ix) \sim qR &+ \frac{q_0 - j_0}{x(\log x - \widetilde{\gamma})} \\ &+ \sum c_{\ell\xi} x^{-2\xi-1}\{\ell(\widetilde{\gamma} - \log x)^{-\ell-1} - 2\xi(\widetilde{\gamma} - \log x)^{-\ell}\} \\ &+ \mathcal{O}(x^{-1}),\end{aligned} \tag{4.10}$$

with the same meaning for $\mathcal{O}(x^{-1})$ *and where the* $c_{\ell\xi}$*'s are the constants in* (2.3).

Proof Using the identity $(iz)^{-v} J_v(iz) = z^{-v} I_v(z)$, where $I_v(z)$ is the modified Bessel function of the first kind, we can write

$$F(ix) = \det\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ J_+(ix) & J_-(ix) \end{pmatrix}$$

where (we use the notation $I_{\pm 0}(xR) = J_{\pm 0}(ixR)$)

$$J_\pm(ix) = \begin{pmatrix} I_{\pm 0}(xR)\mathrm{Id}_{q_0} & 0 & \cdots & 0 \\ 0 & 2^{\pm\nu_1}\Gamma(1\pm\nu_1)x^{\mp\nu_1}I_{\pm\nu_1}(xR) & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & & 0 \\ 0 & 0 & \cdots & 2^{\pm\nu_{q_1}}\Gamma(1\pm\nu_{q_1})x^{\mp\nu_{q_1}}I_{\pm\nu_{q_1}}(xR) \end{pmatrix}.$$

Factoring out $2^{-\nu_j}\Gamma(1-\nu_j)x^{\nu_j}I_{-\nu_j}(xR)$ from the $(q+q_0+j)$-th row of the matrix

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ J_+(ix) & J_-(ix) \end{pmatrix}$$

we obtain

$$\begin{aligned} F(ix) &= \rho \prod_{j=1}^{q_1} x^{\nu_j} I_{-\nu_j}(xR) \\ &\quad \times \det\begin{pmatrix} & \mathcal{A} & & \mathcal{B} & \\ I_0(xR)\mathrm{Id}_{q_0} & 0 & \widetilde{J}_0(ixR)\mathrm{Id}_{q_0} & 0 \\ 0 & A(x) & 0 & \mathrm{Id}_{q_1} \end{pmatrix}, \end{aligned} \tag{4.11}$$

where $\rho = \prod_{j=1}^{q_1} 2^{-\nu_j}\Gamma(1-\nu_j)$ and

$$A(x) = \begin{pmatrix} \tau_1 x^{-2\nu_1}\frac{I_{\nu_1}(xR)}{I_{-\nu_1}(xR)} & 0 & \cdots & 0 \\ 0 & \tau_2 x^{-2\nu_2}\frac{I_{\nu_2}(xR)}{I_{-\nu_2}(xR)} & \cdots & 0 \\ & & & \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \tau_{q_1} x^{-2\nu_{q_1}}\frac{I_{\nu_{q_1}}(xR)}{I_{-\nu_{q_1}}(xR)} \end{pmatrix},$$

with $\tau_j = 2^{2\nu_j}\frac{\Gamma(1+\nu_j)}{\Gamma(1-\nu_j)}$. In order to find the asymptotics of $F(ix)$ in (4.11), we shall determine the asymptotics of $A(x)$ and then of $\widetilde{J}_0(x)$. To determine the asymptotics of $A(x)$, we recall (see [1, p. 377]) that as $z \to \infty$ with $z \in \Upsilon$, we have:

$$I_v(z) \sim \frac{e^z}{\sqrt{2\pi z}}\left(1 - \frac{4v^2-1}{8z} + \mathcal{O}(z^{-2})\right) \sim \frac{e^z}{\sqrt{2\pi z}}\left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right), \tag{4.12}$$

where $\mathcal{O}(\frac{1}{z})$ is a power series in $\frac{1}{z}$ and where only v^2's occur in $\mathcal{O}(\frac{1}{z})$. In particular, as $x \to \infty$ with $x \in \Upsilon$, we have $\frac{I_{\nu_j}(xR)}{I_{-\nu_j}(xR)} \sim 1$. Therefore,

$$A(x) \sim \begin{pmatrix} \tau_1 x^{-2\nu_1} & 0 & \cdots & 0 \\ 0 & \tau_2 x^{-2\nu_2} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \tau_{q_1} x^{-2\nu_{q_1}} \end{pmatrix}. \tag{4.13}$$

To determine the asymptotics of $\widetilde{J}_0(ix)$, note that $J_0(iz) = I_0(z)$ and

$$\begin{aligned}
\frac{\pi}{2} Y_0(iz) &= (\log(iz) - \log 2 + \gamma) J_0(iz) - \sum_{k=1}^{\infty} \frac{H_k(-\frac{1}{4}(iz)^2)^k}{(k!)^2} \\
&= \left(\log z + i\frac{\pi}{2} - \log 2 + \gamma\right) I_0(z) - \sum_{k=1}^{\infty} \frac{H_k(\frac{1}{4}z^2)^k}{(k!)^2} \\
&= i\frac{\pi}{2} I_0(z) - K_0(z),
\end{aligned}$$

where

$$K_0(z) := -(\log z - \log 2 + \gamma) I_0(z) + \sum_{k=1}^{\infty} \frac{H_k(\frac{1}{4}z^2)^k}{(k!)^2}$$

is the modified Bessel function of the second kind. Thus, we can write:

$$\begin{aligned}
\widetilde{J}_0(ixR) &= \frac{\pi}{2} Y_0(ixR) - (\log(ix) - \log 2 + \gamma) J_0(ixR) \\
&= i\frac{\pi}{2} I_0(xR) - K_0(xR) - \left(i\frac{\pi}{2} + \log x - \log 2 + \gamma\right) I_0(xR) \\
&= -(\log x - \widetilde{\gamma}) I_0(xR) - K_0(xR).
\end{aligned}$$

By [1, p. 378], $K_0(x)$ is exponentially decaying as $x \to \infty$ in Υ, so

$$\widetilde{J}_0(ixR) = -(\log x - \widetilde{\gamma}) I_0(xR) - K_0(xR) \sim (\widetilde{\gamma} - \log x) I_0(xR).$$

Summarizing our work so far, we see from (4.11) that:

$$\begin{aligned}
F(ix) &\sim \rho x^{|\nu|} \prod_{j=1}^{q_1} I_{-\nu_j}(xR) \\
&\quad \times \det \begin{pmatrix} \overset{\mathcal{A}}{I_0(xR)\mathrm{Id}_{q_0}} & 0 & \overset{\mathcal{B}}{(\widetilde{\gamma} - \log x) I_0(xR)\mathrm{Id}_{q_0}} & 0 \\ 0 & A(x) & 0 & \mathrm{Id}_{q_1} \end{pmatrix},
\end{aligned}$$

where $\rho = \prod_{j=1}^{q_1} 2^{-\nu_j}\Gamma(1-\nu_j)$, $|\nu| = \nu_1 + \cdots + \nu_{q_1}$, and $A(x)$ satisfies (4.13). Now factoring out $(\widetilde{\gamma} - \log x)I_0(xR)$ from $F(ix)$ and using the definition of $p(x, y)$ in (2.1) (with "x" replaced with $(\widetilde{\gamma} - \log x)^{-1}$ and "y" replaced with x^{-1}), we obtain:

$$F(ix) \sim \rho x^{|\nu|} \prod_{j=1}^{q_1} I_{-\nu_j}(xR) I_0(xR)^{q_0} (\widetilde{\gamma} - \log x)^{q_0} p((\widetilde{\gamma} - \log x)^{-1}, x^{-1}).$$

In view of the asymptotics (4.12) for $I_v(z)$, we get:

$$\begin{aligned} F(ix) \sim \rho x^{|\nu|} ((2\pi)^{-\frac{1}{2}} (xR)^{-\frac{1}{2}} e^{xR})^{q_0+q_1} (\widetilde{\gamma} - \log x)^{q_0} \\ \times p((\widetilde{\gamma} - \log x)^{-1}, x^{-1})(1 + \mathcal{O}(x^{-1})), \end{aligned}$$

which is equivalent to:

$$\begin{aligned} F(ix) \sim (2\pi R)^{-\frac{q}{2}} \prod_{j=1}^{q_1} 2^{-\nu_j}\Gamma(1-\nu_j) x^{|\nu| - \frac{q}{2}} e^{qxR} (\widetilde{\gamma} - \log x)^{q_0} \\ \times p((\widetilde{\gamma} - \log x)^{-1}, x^{-1})(1 + \mathcal{O}(x^{-1})), \end{aligned}$$

and the proof of our first asymptotic formula is complete. To prove our second formula, recall from (2.2) that:

$$p(x, y) = a_{j_0\alpha_0} x^{j_0} y^{2\alpha_0} \Big(1 + \sum b_{k\beta} x^k y^{2\beta}\Big),$$

so that (with "x" replaced with $(\widetilde{\gamma} - \log x)^{-1}$ and "y" replaced with x^{-1})

$$\begin{aligned} F(ix) \sim \text{const} \times x^{|\nu| - \frac{q}{2} - 2\alpha_0} e^{qxR} (\widetilde{\gamma} - \log x)^{q_0 - j_0} \\ \times \Big(1 + \sum b_{k\beta} (\widetilde{\gamma} - \log x)^{-k} x^{-2\beta}\Big)(1 + \mathcal{O}(x^{-1})). \end{aligned}$$

As in (2.3), $\log(1 + \sum b_{k\beta} x^k y^{2\beta}) = \sum c_{\ell\xi} x^\ell y^{2\xi}$ so taking the logarithm of $F(ix)$ we see that

$$\begin{aligned} \log F(ix) \sim \text{const} + qxR + \left(|\nu| - \frac{q}{2} - 2\alpha_0\right) \log x + \big(q_0 - j_0\big) \log(\widetilde{\gamma} - \log x) \\ + \sum c_{\ell\xi} (\widetilde{\gamma} - \log x)^{-\ell} x^{-2\xi} + \mathcal{O}(x^{-1}), \end{aligned}$$

and taking the derivative of both sides completes our proof. □

4.4 The Log Terms Only Case

Suppose that $q_1 = 0$ so that the only eigenvalues of A_Γ in the critical interval $[-\frac{1}{4}, \frac{3}{4})$ are the $-\frac{1}{4}$ eigenvalues. In this case, we shall denote $\mathcal{A}$ by $\mathcal{A}_0$ and $\mathcal{B}$ by $\mathcal{B}_0$ so that:

$$F(\mu) = \det \begin{pmatrix} \mathcal{A}_0 & \mathcal{B}_0 \\ J_0(\mu R)\mathrm{Id}_{q_0} & \widetilde{J}_0(\mu R)\mathrm{Id}_{q_0} \end{pmatrix}.$$

Recall from Sect. 2.3 (see (2.9)) the polynomial:

$$p_0(z) := \det \begin{pmatrix} \mathcal{A}_0 & \mathcal{B}_0 \\ \mathrm{Id}_{q_0} & (\widetilde{\gamma} - z)\mathrm{Id}_{q_0} \end{pmatrix}, \quad \text{where } \widetilde{\gamma} = \log 2 - \gamma,$$

which is a polynomial in the complex variable $z \in \mathbb{C}$ of degree at most q_0. Then we can write:

$$\frac{p_0'(z)}{p_0(z)} = \sum_{k=1}^{\infty} \frac{\beta_k}{z^k},$$

where the series is absolutely convergent for $|z|$ sufficiently large and where $\beta_1 = \deg p_0$. In the case that $q_1 = 0$, Proposition 4.3 can be written as follows.

Proposition 4.4 *Suppose that $q_1 = 0$ and let $\Upsilon \subset \mathbb{C}$ be a sector (closed angle) in the right-half plane. Then as $|x| \to \infty$ with $x \in \Upsilon$, we have:*

$$F(ix) \sim (2\pi x R)^{-\frac{q_0}{2}} e^{q_0 x R} p_0(\log x)(1 + \mathcal{O}(x^{-1})), \tag{4.14}$$

where $\mathcal{O}(x^{-1})$ is a power series in x^{-1}, and

$$\frac{d}{dx} \log F(ix) \sim \sum_{k=1}^{\infty} \frac{\beta_k}{x(\log x)^k} + q_0 R - \frac{q_0}{2x} + \mathcal{O}(x^{-2}), \tag{4.15}$$

where $\mathcal{O}(x^{-2})$ is a power series in x^{-1} starting from x^{-2}.

Proof A direct application of (4.9) in Proposition 4.3 with $q_1 = 0$ gives

$$F(ix) \sim (2\pi x R)^{-\frac{q_0}{2}} e^{q_0 x R} (\widetilde{\gamma} - \log x)^{q_0} p((\widetilde{\gamma} - \log x)^{-1})(1 + \mathcal{O}(x^{-1})),$$

where $\mathcal{O}(x^{-1})$ is a power series in x^{-1} and

$$p(x) := \det \begin{pmatrix} \mathcal{A}_0 & \mathcal{B}_0 \\ x\,\mathrm{Id}_{q_0} & \mathrm{Id}_{q_0} \end{pmatrix}.$$

By definition of p_0, we have $(\widetilde{\gamma} - \log x)^{q_0} p((\widetilde{\gamma} - \log x)^{-1}) = p_0(\log x)$. This proves (4.14) and then taking the logarithmic derivative of (4.14) gives (4.15). □

5 The Zeta Function for the Model Problems

Working with the fixed Lagrangian L, we now analyze the zeta-function of $\mathcal{L}_L$. The Appendix by Boris Vertman proves by a careful asymptotic analysis the integral representation of the zeta-function

$$\zeta(s, \mathcal{L}_L) = \frac{1}{2\pi i} \int_{\mathcal{C}} \mu^{-2s} \frac{d}{d\mu} \log F(\mu) d\mu = \frac{1}{2\pi i} \int_{\mathcal{C}} \mu^{-2s} \frac{F'(\mu)}{F(\mu)} d\mu,$$

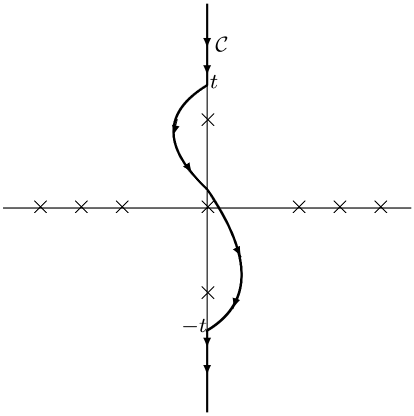


Fig. 1 The contour $\mathcal{C}$ for the zeta function. The $\times$'s represent the zeros of $F(\mu)$. The squares of the $\times$'s on the imaginary axis represent the negative eigenvalues of $\mathcal{L}_L$. Here, t is on the imaginary axis and $|t|^2$ is larger than the absolute value of the negative eigenvalue of $\mathcal{L}_L$ (if one exists). The contour $\mathcal{C}_t$ goes from t to $-t$

where $\mathcal{C}$ is a contour in the plane shown in Fig. 1. It is only formally an application of the Argument Principle, since the contour $\mathcal{C}$ is not closed. Here we used that μ^2 is an eigenvalue of $\mathcal{L}_L$ if and only if μ is a zero of $F(\mu)$. By Proposition 4.3, the zeta function $\zeta(s, \mathcal{L}_L)$ is well-defined for $\Re s > \frac{1}{2}$.

5.1 A Basic Lemma

In order to determine the exact structure of the analytic continuation of $\zeta(s, \mathcal{L}_L)$, we need the following fundamental result.

Lemma 5.1 *Let c be a constant and let $|t|$ be sufficiently large so that $\log x > c$ for $x \geq |t|$. Then for any $k \geq 0$ we can write*:

$$\int_{|t|}^{\infty} x^{-2s-2\xi-1}(c - \log x)^k dx \equiv \sum_{j=0}^{k} \frac{\sigma_{kj}}{(s+\xi)^{j+1}}, \tag{5.1}$$

modulo an entire function, *where $\sigma_{kj} = (-1)^k \binom{k}{j} \frac{j!}{2^{j+1}}(-c)^{k-j}$*, *and for any $k > 0$ we can write*:

$$\int_{|t|}^{\infty} x^{-2s-2\xi-1}(c - \log x)^{-k} dx \equiv \frac{2^{k-1}}{(k-1)!}(s+\xi)^{k-1} e^{-2(s+\xi)c} \log(s+\xi) \tag{5.2}$$

modulo an entire function.

Proof Replacing s by $s - \xi$, we can assume that $\xi = 0$ from the start. To analyze the first integral we first expand $(c - \log x)^k$ using the binomial theorem:

$$\int_{|t|}^{\infty} x^{-2s-1}(c - \log x)^k dx = \sum_{j=0}^{k} (-1)^k \binom{k}{j} (-c)^{k-j} \int_{|t|}^{\infty} x^{-2s-1}(\log x)^j dx.$$

Thus, we are left to prove that:

$$\int_{|t|}^{\infty} x^{-2s-1}(\log x)^j\,dx \equiv \frac{j!}{2^{j+1}}\cdot\frac{1}{s^{j+1}} \tag{5.3}$$

modulo an entire function. However, since the integral $\int_1^{|t|} x^{-2s-1}(\log x)^j\,dx$ is entire, we can assume that the lower limit of the integral in (5.3) is 1. Now taking j derivatives of both sides of the equality $\int_1^{\infty} x^{-2s-1}dx = \frac{1}{2s}$ with respect to s, we obtain:

$$(-2)^j\int_1^{\infty} x^{-2s-1}(\log x)^j\,dx = \frac{(-1)^j j!}{2s^{j+1}},$$

which proves (5.3).

To prove the second claim in this proposition, we make the change of variables $y = 2s(\log x - c)$ or $x = e^c e^{y/2s}$, and obtain:

$$\int_{|t|}^{\infty} x^{-2s-1}(c-\log x)^{-k}dx = (-1)^k e^{-2sc}(2s)^{k-1}\int_{2sC}^{\infty} e^{-y}\frac{dy}{y^k},$$

where $C := \log|t| - c$. Recall that the *exponential integral* is defined by (see [1, p. 228] or [27, Sec. 8.2])

$$\mathrm{Ei}_k(z) := \int_1^{\infty} e^{-zu}\frac{du}{u^k} = z^{k-1}\int_z^{\infty} e^{-y}\frac{dy}{y^k}.$$

Therefore,

$$\int_{|t|}^{\infty} x^{-2s-1}(c-\log x)^{-k}dx = \frac{(-1)^k e^{-2sc}}{C^{k-1}}\mathrm{Ei}_k(2sC). \tag{5.4}$$

Also from [1, p. 229] or [27, p. 877], we have:

$$\mathrm{Ei}_k(z) = \frac{(-z)^{k-1}}{(k-1)!}\{-\log z + \psi(k)\} - \sum_{j=0,\ j\neq k-1}^{\infty}\frac{(-z)^{j-1}}{(j-k+1)j!},$$

where $\psi(1) := -\gamma$ and $\psi(k) := -\gamma + \sum_{j=1}^{k-1}\frac{1}{j}$ for $k > 1$. Hence,

$$\mathrm{Ei}_k(2sC) = C^{k-1}\frac{(-2s)^{k-1}}{(k-1)!}\{-\log(2sC) + \psi(k)\} - \sum_{j=0,\ j\neq k-1}^{\infty}\frac{(-2sC)^{j-1}}{(j-k+1)j!}.$$

Replacing this into (5.4) and simplifying, we obtain:

$$\int_{|t|}^{\infty} x^{-2s-1}(c-\log x)^{-k}dx \equiv \frac{(2s)^{k-1}}{(k-1)!}e^{-2sc}\log s$$

modulo an entire function. This completes our proof. $\square$

5.2 The ζ-Function

We now prove the "model problem version" of Theorem 2.1 via the contour integration method [35–37].

Proposition 5.2 *Let $L \subset V$ be an arbitrary Lagrangian subspace of $\mathbb{C}^{2q}$ and define $\mathcal{P}$ and $\mathcal{L}$ as in (2.5) from the matrices $\mathcal{A}$ and $\mathcal{B}$ defining L. Then the ζ-function $\zeta(s, \mathcal{L}_L)$ extends from $\Re s > \frac{1}{2}$ to a holomorphic function on $\mathbb{C} \setminus (-\infty, 0]$. Moreover, $\zeta(s, \mathcal{L}_L)$ can be written in the form:*

$$\zeta(s, \mathcal{L}_L) = \zeta_{\mathrm{reg}}(s, \mathcal{L}_L) + \zeta_{\mathrm{sing}}(s, \mathcal{L}_L),$$

where $\zeta_{\mathrm{reg}}(s, \mathcal{L}_L)$ has the "regular" poles at the "usual" locations $s = \frac{1}{2} - k$ for $k \in \mathbb{N}_0$, and where $\zeta_{\mathrm{sing}}(s, \mathcal{L}_L)$ has the following expansion:

$$\zeta_{\mathrm{sing}}(s, \mathcal{L}_L) = \frac{\sin(\pi s)}{\pi} \Bigg\{ (j_0 - q_0) e^{-2s(\log 2 - \gamma)} \log s + \sum_{\xi \in \mathcal{P}} \frac{f_\xi(s)}{(s+\xi)^{|p_\xi|+1}}$$
$$+ \sum_{\xi \in \mathcal{L}} g_\xi(s) \log(s+\xi) \Bigg\},$$

where j_0 appears in (2.2) and $f_\xi(s)$ and $g_\xi(s)$ are entire functions of s such that

$$f_\xi(-\xi) = (-1)^{|p_\xi|+1} c_{p_\xi \xi} \xi \frac{|p_\xi|!}{2^{|p_\xi|}}$$

and

$$g_\xi(s) = \begin{cases} c_{\ell_0, 0} \frac{2^{\ell_0}}{(\ell_0 - 1)!} s^{\ell_0} + \mathcal{O}(s^{\ell_0+1}) & \text{if } \xi = 0, \\ -c_{\ell_\xi \xi} \frac{\xi 2^{\ell_\xi}}{(\ell_\xi - 1)!} (s+\xi)^{\ell_\xi - 1} + \mathcal{O}((s+\xi)^{\ell_\xi}) & \text{if } \xi > 0, \end{cases}$$

where the $c_{\ell \xi}$'s are the coefficients in (2.3).

Proof With Fig. 1 in mind, we write:

$$\int_{\mathcal{C}} = -\int_t^{0+i\infty} + \int_{-t}^{0-i\infty} + \int_{\mathcal{C}_t},$$

where $\mathcal{C}_t$ is the curvy part of $\mathcal{C}$ from t to $-t$, and second, using that:

$$i^{-2s} = (e^{i\pi/2})^{-2s} = e^{-i\pi s} \quad \text{and} \quad (-i)^{-2s} = (e^{-i\pi/2})^{-2s} = e^{i\pi s},$$

we obtain the integral:

$$\zeta(s, \mathcal{L}_L) = \frac{1}{2\pi i} \int_{\mathcal{C}} \mu^{-2s} \frac{d}{d\mu} \log F(\mu) d\mu$$
$$= \frac{1}{2\pi i} \Bigg\{ -\int_{|t|}^{\infty} (ix)^{-2s} \frac{d}{dx} \log F(ix) dx$$

$$
\begin{aligned}
&+ \int_{|t|}^{\infty} (-ix)^{-2s} \frac{d}{dx} \log F(-ix) dx \Big\} + \frac{1}{2\pi i} \int_{\mathcal{C}_t} \mu^{-2s} \frac{F'(\mu)}{F(\mu)} d\mu \\
&= \frac{1}{2\pi i} (-e^{-i\pi s} + e^{i\pi s}) \int_{|t|}^{\infty} x^{-2s} \frac{d}{dx} \log F(ix) dx \\
&+ \frac{1}{2\pi i} \int_{\mathcal{C}_t} \mu^{-2s} \frac{F'(\mu)}{F(\mu)} d\mu,
\end{aligned}
$$

or,

$$
\zeta(s, \mathcal{L}_L) = \frac{\sin(\pi s)}{\pi} \int_{|t|}^{\infty} x^{-2s} \frac{d}{dx} \log F(ix) dx + \frac{1}{2\pi i} \int_{\mathcal{C}_t} \mu^{-2s} \frac{F'(\mu)}{F(\mu)} d\mu, \tag{5.5}
$$

a formula that will be analyzed in a moment. The second integral here is over a bounded contour so is an entire function of $s \in \mathbb{C}$, so we are left to analyze the analytic properties of the first integral in (5.5). To do so, recall the asymptotics (4.10) in Proposition 4.3, which states that for $x \to \infty$ we have:

$$
\frac{d}{dx} \log F(ix) \sim \frac{q_0 - j_0}{x(\log x - \widetilde{\gamma})} + G_1(x) + G_2(x) + G_3(x), \tag{5.6}
$$

where $\widetilde{\gamma} = \log 2 - \gamma$, $G_3(x)$ is a power series in x^{-1} starting with the constant term qR,

$$
G_1(x) := \sum_{\xi} \sum_{\ell \le 0} c_{\ell\xi} x^{-2\xi-1} \{\ell(\widetilde{\gamma} - \log x)^{-\ell-1} - 2\xi(\widetilde{\gamma} - \log x)^{-\ell}\},
$$

and

$$
G_2(x) := \sum_{\xi} \sum_{\ell > 0} c_{\ell\xi} x^{-2\xi-1} \{\ell(\widetilde{\gamma} - \log x)^{-\ell-1} - 2\xi(\widetilde{\gamma} - \log x)^{-\ell}\}.
$$

Since

$$
\frac{\sin(\pi s)}{\pi} \int_{|t|}^{\infty} x^{-2s-k} dx = \frac{\sin(\pi s)}{\pi} \frac{x^{-2s-k+1}}{-2s-k+1}\Big|_{x=|t|}^{\infty} = \frac{\sin(\pi s)}{\pi} \frac{|t|^{-2s-k+1}}{2s+k-1}
$$

which has poles at $s = \frac{1-k}{2}$ for $s \notin \mathbb{Z}$, it follows that

$$
\frac{\sin(\pi s)}{\pi} \int_{|t|}^{\infty} x^{-2s} \frac{d}{dx} \log G_3(ix) dx \tag{5.7}
$$

will contribute to the function $\zeta_{\mathrm{reg}}(s, \mathcal{L}_L)$ in the statement of this proposition. Setting $\xi = 0$ and $k = 1$ in (5.2) in Lemma 5.1 we see that:

$$
\int_{|t|}^{\infty} x^{-2s} \frac{q_0 - j_0}{x(\log x - \widetilde{\gamma})} dx \equiv -(q_0 - j_0) e^{-2s\widetilde{\gamma}} \log s
$$

modulo an entire function, which gives us the first term in $\zeta_{\mathrm{sing}}(s, \mathcal{L}_L)$.

We now analyze $\int_{|t|}^{\infty} x^{-2s} G_2(x)\,dx$. To do so, we apply (5.2) term-by-term to

$$\int_{|t|}^{\infty} x^{-2s} G_2(x) dx$$
$$= \int_{|t|}^{\infty} x^{-2s} \sum_{\xi} \sum_{\ell>0} c_{\ell\xi}\, x^{-2\xi-1} \{\ell(\widetilde{\gamma} - \log x)^{-\ell-1} - 2\xi(\widetilde{\gamma} - \log x)^{-\ell}\} dx,$$

and we see that, modulo an entire function,

$$\int_{|t|}^{\infty} x^{-2s} G_2(x) dx$$
$$\equiv \sum_{\xi} \sum_{\ell>0} c_{\ell\xi} \left\{ \frac{\ell 2^{\ell}}{\ell!} e^{-2(s+\xi)\widetilde{\gamma}} (s+\xi)^{\ell} \log(s+\xi) \right.$$
$$\left. - \frac{\xi 2^{\ell}}{(\ell-1)!} e^{-2(s+\xi)\widetilde{\gamma}} (s+\xi)^{\ell-1} \log(s+\xi) \right\}$$
$$= \sum_{\xi} \left(e^{-2(s+\xi)\widetilde{\gamma}} \sum_{\ell>0} c_{\ell\xi} \left\{ \frac{\ell 2^{\ell}}{\ell!} (s+\xi)^{\ell} - \frac{\xi 2^{\ell}}{(\ell-1)!} (s+\xi)^{\ell-1} \right\} \right) \log(s+\xi),$$

which can be written in the form $\sum_{\xi} g_{\xi}(s) \log(s+\xi)$ where

$$g_{\xi}(s) = e^{-2(s+\xi)\widetilde{\gamma}} \sum_{\ell>0} c_{\ell\xi} \left\{ \frac{2^{\ell}}{(\ell-1)!} (s+\xi)^{\ell} - \frac{\xi 2^{\ell}}{(\ell-1)!} (s+\xi)^{\ell-1} \right\}.$$

From this explicit formula for $g_{\xi}(s)$, we see that:

$$g_{\xi}(s) = \begin{cases} c_{\ell_0,0} \frac{2^{\ell_0}}{(\ell_0-1)!} s^{\ell_0} + \mathcal{O}(s^{\ell_0+1}) & \text{if } \xi = 0, \\ -c_{\ell_{\xi}\xi} \frac{\xi 2^{\ell_{\xi}}}{(\ell_{\xi}-1)!} (s+\xi)^{\ell_{\xi}-1} + \mathcal{O}((s+\xi)^{\ell_{\xi}}) & \text{if } \xi > 0, \end{cases}$$

where we recall that $\ell_{\xi} := \min\{\ell > 0 \mid c_{\ell\xi} \neq 0\}$.

We now analyze $\int_{|t|}^{\infty} x^{-2s} G_1(x)\,dx$. With $\sigma_{kj} = (-1)^k \binom{k}{j} \frac{j!}{2^{j+1}} (-\widetilde{\gamma})^{k-j}$, from (5.1) in Lemma 5.1 we can write, modulo an entire function,

$$\int_{|t|}^{\infty} x^{-2s} \left(\sum_{\ell \le 0} c_{\ell\xi}\, x^{-2\xi-1} \{\ell(\widetilde{\gamma} - \log x)^{-\ell-1} - 2\xi(\widetilde{\gamma} - \log x)^{-\ell}\} \right) dx$$
$$= \sum_{\ell \le 0} c_{\ell\xi} \left\{ \sum_{j=0}^{|\ell|-1} \frac{\ell \sigma_{|\ell|-1,j}}{(s+\xi)^{j+1}} - \sum_{j=0}^{|\ell|} \frac{2\xi \sigma_{|\ell| j}}{(s+\xi)^{j+1}} \right\} = \frac{f_{\xi}(s)}{(s+\xi)^{|p_{\xi}|+1}}, \tag{5.8}$$

where

$$p_{\xi} := \min\{\ell \le 0 \mid c_{\ell\xi} \neq 0\} \quad \Longrightarrow \quad |p_{\xi}| = \max\{|\ell| \mid \ell \le 0 \text{ and } c_{\ell\xi} \neq 0\}$$

and

$$f_\xi(s) := \sum_{\ell\le 0} c_{\ell\xi}\left\{\sum_{j=0}^{|\ell|-1} \ell\sigma_{|\ell|-1,j}(s+\xi)^{|p_\xi|-j} - \sum_{j=0}^{|\ell|} 2\xi\sigma_{|\ell|j}(s+\xi)^{|p_\xi|-j}\right\}$$

is entire. It follows that:

$$\begin{aligned}&\int_{|t|}^{\infty} x^{-2s} G_1(x)dx\\ &\quad = \int_{|t|}^{\infty} x^{-2s}\sum_{\xi}\sum_{\ell\le 0} c_{\ell\xi}\, x^{-2\xi-1}\{\ell(\widetilde{\gamma}-\log x)^{-\ell-1} - 2\xi(\widetilde{\gamma}-\log x)^{-\ell}\}dx\\ &\quad = \sum_{\xi}\frac{f_\xi(s)}{(s+\xi)^{|p_\xi|+1}}.\end{aligned}$$

Moreover, from the above explicit formula for $f_\xi(s)$ we see that:

$$f_\xi(-\xi) = -2c_{p_\xi\xi}\xi\sigma_{|p_\xi|,|p_\xi|} = -2c_{p_\xi\xi}\xi(-1)^{|p_\xi|}\frac{|p_\xi|!}{2^{|p_\xi|+1}} = (-1)^{|p_\xi|+1}c_{p_\xi\xi}\xi\frac{|p_\xi|!}{2^{|p_\xi|}}.$$

This completes our proof. □

5.3 The Decomposable Case

Suppose now that $L = L_0 \oplus L_1$ is decomposable where as in Corollary 3.5 L_0 is given by $q_0\times q_0$ matrices $\mathcal{A}_0$ and $\mathcal{B}_0$ and L_1 is given by $q_1\times q_1$ matrices $\mathcal{A}_1$ and $\mathcal{B}_1$. Let us recall the polynomial $p_0(z)$ introduced in (2.9) in Sect. 2.3 and consider the following result.

Lemma 5.3 *For $|t|$ sufficiently large so that $p_0(\log x)$ has no zeros for $x \ge |t|$, we can write*:

$$\int_{|t|}^{\infty} x^{-2s}\frac{p_0'(\log x)}{xp_0(\log x)}dx \equiv -f(s)\log s$$

modulo an entire function, where $f(s)$ is the entire function given explicitly by:

$$f(s) = \sum_{k=1}^{\infty}\beta_k\frac{(-2s)^{k-1}}{(k-1)!}$$

with the β_k's the coefficients of the expansion of $\frac{p_0'(z)}{p_0(z)} = \sum_{k=1}^{\infty}\frac{\beta_k}{z^k}$ in (2.10).

Proof Using the expansion $\frac{p_0'(z)}{p_0(z)} = \sum_{k=1}^{\infty}\frac{\beta_k}{z^k}$, we can write:

$$\int_{|t|}^{\infty} x^{-2s}\frac{p_0'(\log x)}{xp_0(\log x)}dx = \sum_{k=1}^{\infty}\beta_k\int_{|t|}^{\infty} x^{-2s}\frac{1}{x(\log x)^k}dx. \tag{5.9}$$

To analyze this integral we put $\xi = c = 0$ in formula (5.2) from Lemma 5.1 to see

$$\int_{|t|}^{\infty} x^{-2s} \frac{1}{x(\log x)^k} dx \equiv -\frac{(-2)^{k-1}}{(k-1)!} s^{k-1} \log s = -\frac{(-2s)^{k-1}}{(k-1)!} \log s,$$

modulo an entire function. Replacing this formula into (5.9) and simplifying, we obtain our result. □

Let us apply this theorem to the case when $q_0 = 1$. In this case, by Proposition 3.6 we have $\mathcal{A}_0 = \cos\theta$ and $\mathcal{B}_0 = \sin\theta$ for an angle $\theta \in [0, \pi)$, therefore:

$$p_0(z) := \det \begin{pmatrix} \cos\theta & \sin\theta \\ 1 & (\widetilde{\gamma} - z) \end{pmatrix} = -\cos\theta \cdot z + \widetilde{\gamma} \cdot \cos\theta - \sin\theta.$$

Hence, with $\kappa := \widetilde{\gamma} - \tan\theta = \log 2 - \gamma - \tan\theta$, we have:

$$\frac{p_0'(z)}{p_0(z)} = \frac{-\cos\theta}{-\cos\theta \cdot z + \widetilde{\gamma} \cdot \cos\theta - \sin\theta} = \frac{1}{z - \widetilde{\gamma} + \tan\theta} = \sum_{k=1}^{\infty} \frac{\kappa^{k-1}}{z^k}. \tag{5.10}$$

Thus, $\beta_k = \kappa^{k-1}$, so:

$$f(s) = \sum_{k=1}^{\infty} \beta_k \frac{(-2s)^{k-1}}{(k-1)!} = \sum_{k=1}^{\infty} \frac{(-2s\kappa)^{k-1}}{(k-1)!} = e^{-2s\kappa}.$$

Therefore, Lemma 5.3 reduces to the following.

Corollary 5.4 *Suppose that $q_0 = 1$, $q_1 = 0$ and $\theta \neq \frac{\pi}{2}$. Then the coefficients β_k in the expansion $\frac{p_0'(z)}{p_0(z)}$ are given by $\beta_k = \kappa^{k-1}$. In particular, for $|t|$ sufficiently large so that $p_0(\log x)$ has no zeros for $x \geq |t|$, we can write*

$$\int_{|t|}^{\infty} x^{-2s} \frac{p_0'(\log x)}{x p_0(\log x)} dx \equiv -e^{-2s\kappa} \log s$$

modulo an entire function, where $\kappa = \log 2 - \gamma - \tan\theta$.

From (2.11) and (2.12) in Sect. 2.3, let us recall that the polynomial $p_1(y)$ has the expression:

$$p_1(y) = a_{\alpha_0}\, y^{2\alpha_0} \Big(1 + \sum b_\beta y^{2\beta}\Big),$$

where the β's are positive and

$$\log\Big(1 + \sum b_\beta y^{2\beta}\Big) = \sum c_\xi\, y^{2\xi}, \tag{5.11}$$

and let $\mathcal{P} := \{\xi \mid c_\xi \neq 0\}$. We now prove the model problem versions of Theorems 2.3 and 2.4.

Proposition 5.5 *For an arbitrary decomposable Lagrangian $L \subset V$, the ζ-function $\zeta(s, \mathcal{L}_L)$ has the following form*:

$$\zeta(s, \mathcal{L}_L) = \zeta_{\text{reg}}(s, \mathcal{L}_L) + \zeta_{\text{sing}}(s, \mathcal{L}_L)$$

where $\zeta_{\text{reg}}(s, \mathcal{L})$ has the "regular" poles at the "usual" locations $s = \frac{1}{2} - k$ for $k \in \mathbb{N}_0$, and where $\zeta_{\text{sing}}(s, \mathcal{L}_L)$ has the following expansion:

$$\zeta_{\text{sing}}(s, \mathcal{L}_L) = -\frac{\sin(\pi s)}{\pi} f(s) \log s + \frac{\sin(\pi s)}{\pi} \sum_{\xi \in \mathcal{P}} \frac{f_\xi(s)}{s + \xi},$$

where $f(s)$ is the entire function defined explicitly by $f(s) = \sum_{k=1}^{\infty} \beta_k \frac{(-2s)^{k-1}}{(k-1)!}$, and the $f_\xi(s)$'s are entire functions such that $f_\xi(-\xi) = -c_\xi \xi$ with the c_ξ's the coefficients in (5.11).

Proof Since $L = L_0 \oplus L_1$ is decomposable, it follows that

$$\zeta(s, \mathcal{L}_L) = \zeta(s, \mathcal{L}_{L_0}) + \zeta(s, \mathcal{L}_{L_1}),$$

where $\mathcal{L}_{L_0}$ is the operator $\mathcal{L}_L$ restricted to the $-\frac{1}{4}$ eigenspaces of A and $\mathcal{L}_{L_1}$ is the operator $\mathcal{L}_L$ restricted to the eigenspaces of A in $(-\frac{1}{4}, \frac{3}{4})$. From (5.11), we can observe that $p_\xi = 0$ for any ξ and $\mathcal{L} = \emptyset$ for the operator $\mathcal{L}_{L_1}$. Hence, there are only terms $G_1(x)$ with $\ell = 0$ and $G_3(x)$ in (5.6), that is,

$$\frac{d}{dx} \log F(ix) \sim \sum_{k=0}^{\infty} b_k x^{-k} + \sum -2\xi c_\xi x^{-2\xi - 1} \tag{5.12}$$

where c_ξ's are the coefficients in (5.11). It follows from the proof of Proposition 5.2, in particular, (5.8) with $\ell = 0$ that:

$$\zeta(s, \mathcal{L}_{L_1}) = \zeta_{\text{reg}}(s, \mathcal{L}_{L_1}) + \zeta_{\text{sing}}(s, \mathcal{L}_{L_1})$$

where $\zeta_{\text{reg}}(s, \mathcal{L}_{L_1})$ has poles at $s = \frac{1}{2} - k$ for $k \in \mathbb{N}_0$ and

$$\zeta_{\text{sing}}(s, \mathcal{L}_{L_1}) = \frac{\sin(\pi s)}{\pi} \sum_{\xi \in \mathcal{P}} \frac{f_\xi(s)}{s + \xi},$$

where the $f_\xi(s)$'s are entire functions of s such that $f_\xi(-\xi) = -c_\xi \xi$.

Thus, it remains to analyze $\zeta(s, \mathcal{L}_{L_0})$. To do so, we follow the proof of Proposition 5.2 up to (5.5), for $|t| \gg 0$ we can write:

$$\zeta(s, \mathcal{L}_{L_0}) \equiv \frac{\sin(\pi s)}{\pi} \int_{|t|}^{\infty} x^{-2s} \frac{d}{dx} \log F_0(ix) dx$$

modulo an entire function, where $F_0(\mu)$ is the function $F(\mu)$ defined in Proposition 4.2 in the case that $q_1 = 0$, $\mathcal{A} = \mathcal{A}_0$, and $\mathcal{B} = \mathcal{B}_0$. By (4.15) in Proposition 4.4, we have:

$$\frac{d}{dx} \log F_0(ix) \sim \frac{p_0'(\log x)}{x\, p_0(\log x)} + G(x),$$

where $G(x)$ is a power series in x^{-1} starting with a constant term. Just as we noticed in (5.7) for $G_3(x)$ (which has the same asymptotics as $G(x)$) in the proof of Proposition 4.4, the integral $\frac{\sin(\pi s)}{\pi}\int_{|t|}^{\infty} x^{-2s} G(ix)\,dx$ will contribute to the function $\zeta_{\mathrm{reg}}(s, \mathcal{L}_{L_0})$ in the statement of this proposition. Finally, invoking Lemma 5.3:

$$\int_{|t|}^{\infty} x^{-2s} \frac{p_0'(\log x)}{x p_0(\log x)} dx \equiv -f(s)\log s$$

modulo an entire function, where $f(s) = \sum_{k=1}^{\infty} \beta_k \frac{(-2s)^{k-1}}{(k-1)!}$, completes the proof. □

Corollary 5.4 implies

Corollary 5.6 *Suppose that* $q_0 = 1$, $q_1 = 0$ *and* $\theta \neq \frac{\pi}{2}$. *Then the zeta function* $\zeta(s, \Delta_\theta)$ *can be written in the form*

$$\zeta(s, \Delta_\theta) = -\frac{\sin(\pi s)}{\pi} e^{-2s\kappa} \log s + \zeta_\theta(s),$$

where $\kappa = \log 2 - \gamma - \tan\theta$ *and* $\zeta_\theta(s)$ *extends from* $\Re s > \frac{1}{2}$ *to a holomorphic function on* $\mathbb{C}$ *with poles at* $s = \frac{1}{2} - k$ *for* $k \in \mathbb{N}_0$. *In particular,* $\zeta(s, \Delta_\theta)$ *has* $s = 0$ *as a logarithmic branch point. In the case that* $\theta = \frac{\pi}{2}$, *the* ζ*-function* $\zeta(s, \Delta_\theta)$ *has the properties of* $\zeta_\theta(s)$.

The last statement for $\theta = \frac{\pi}{2}$ in Corollary 5.6 follows from the results in [18].

6 The Resolvent and Heat Kernel for the Model Problems

In this section we analyze the resolvent and heat kernel expansions for the model problems, which will be of great use for the general case.

6.1 The Resolvent

Using the new contour $\mathcal{C}$ shown in Fig. 2, we see that if $\{\mu_j^2\}$ denote the eigenvalues of $\mathcal{L}_L$, then by an application of the Argument Principle, we have

$$2\,\mathrm{Tr}(\mathcal{L}_L + x^2)^{-1} = 2\sum_{j=1}^{\infty} \frac{1}{\mu_j^2 + x^2} = \frac{1}{2\pi i}\int_\gamma (\mu^2 + x^2)^{-1} \frac{d}{d\mu} \log F(\mu) d\mu,$$

where $|x|^2$ is larger than the absolute value of the negative eigenvalues of $\mathcal{L}_L$ (if one exists). The factor of 2 on the left hand side is a result of all eigenvalues being enclosed twice. Using this formula we can express the trace of the resolvent in terms of $F(ix)$ in the following theorem.

Theorem 6.1 *We have*:

$$2x\,\mathrm{Tr}(\mathcal{L}_L + x^2)^{-1} = \frac{d}{dx} \log F(ix)$$

for all complex $x \in \mathbb{C}$ *for which either* (*and hence both*) *sides make sense.*

Fig. 2 The new contour $\mathcal{C}$

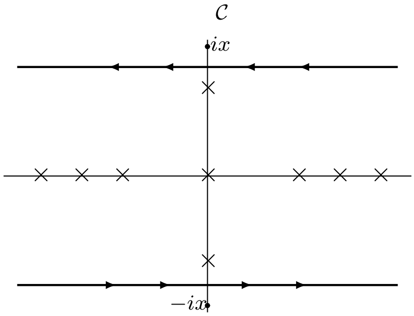
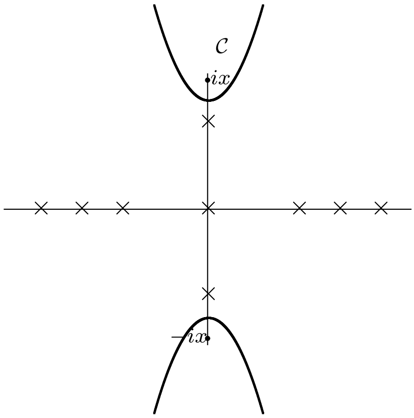


Fig. 3 Deforming the contour $\mathcal{C}$



Proof Deforming the contour as in Fig. 3 and using Cauchy's formula, we obtain:

$$
\begin{aligned}
2\,\mathrm{Tr}(\mathcal{L}_L + x^2)^{-1} &= \frac{1}{2\pi i}\int_\gamma \frac{1}{(\mu - ix)(\mu + ix)}\frac{F'(\mu)}{F(\mu)}d\mu \\
&= -\frac{1}{2ix}\frac{F'(ix)}{F(ix)} - \frac{1}{-2ix}\frac{F'(-ix)}{F(-ix)} = \frac{i}{x}\frac{F'(ix)}{F(ix)} = \frac{1}{x}\frac{d}{dx}\log F(ix),
\end{aligned}
$$

where we used the fact that $F(\mu)$ is an even function of μ. Indeed, to see this observe that, by definition, $F(\mu)$ is expressed in terms of $\mu^v J_{-v}(\mu R)$ with appropriate v's and the function $\widetilde{J}_0(\mu R)$, which are even functions by (4.4), (4.6) and (4.7). This proves that $2x\,\mathrm{Tr}(\mathcal{L}_L + x^2)^{-1} = \frac{d}{dx}\log F(ix)$ at least when x is real and $x \gg 0$. However, by analytic continuation, both sides must still be equal for all complex x for which both sides are defined. $\square$

Using this theorem, we can now prove the following.

Proposition 6.2 *Let $L \subset V$ be an arbitrary Lagrangian subspace of $\mathbb{C}^{2q}$ and let $\Lambda \subset \mathbb{C}$ be any sector (solid angle) not intersecting the positive real axis. Then as $|\lambda| \to \infty$ with $\lambda \in \Lambda$, we have:*

$$\operatorname{Tr}(\mathcal{L}_L - \lambda)^{-1} \sim \sum_{k=1}^{\infty} a_k(-\lambda)^{-\frac{k}{2}} + \frac{q_0 - j_0}{(-\lambda)(\log(-\lambda) - 2\widetilde{\gamma})} - \frac{d}{d\lambda}\Big\{\sum 2^{\ell} c_{\ell\xi}\,(-\lambda)^{-\xi}(2\widetilde{\gamma} - \log(-\lambda))^{-\ell}\Big\},$$

where the a_k coefficients are independent of L and the $c_{\ell\xi}$'s are given in (2.3).

Proof By Proposition 4.3 (see (4.10)), we have:

$$\frac{d}{dx}\log F(ix) \sim \sum_{k=0}^{\infty} b_k x^{-k} + \frac{q_0 - j_0}{x(\log x - \widetilde{\gamma})} + \sum c_{\ell\xi}\, x^{-2\xi-1}\{\ell(\widetilde{\gamma} - \log x)^{-\ell-1} - 2\xi(\widetilde{\gamma} - \log x)^{-\ell}\}$$

for some coefficients b_k that, by the proof of Proposition 4.3, are independent of L. Therefore, by Theorem 6.1, we have:

$$2x\operatorname{Tr}(\mathcal{L}_L + x^2)^{-1} \sim \sum_{k=0}^{\infty} b_k x^{-k} + \frac{q_0 - j_0}{x(\log x - \widetilde{\gamma})} + \sum c_{\ell\xi}\, x^{-2\xi-1}\{\ell(\widetilde{\gamma} - \log x)^{-\ell-1} - 2\xi(\widetilde{\gamma} - \log x)^{-\ell}\}.$$

Dividing by $2x$ and then setting $x = (-\lambda)^{\frac{1}{2}}$, with $a_k = (1/2)b_{k-1}$ we obtain:

$$\operatorname{Tr}(\mathcal{L}_L - \lambda)^{-1} \sim \sum_{k=1}^{\infty} a_k(-\lambda)^{-\frac{k}{2}} + \frac{q_0 - j_0}{(-\lambda)(\log(-\lambda) - 2\widetilde{\gamma})} - \frac{d}{d\lambda}\left\{\sum c_{\ell\xi}(-\lambda)^{-\xi}\left(\widetilde{\gamma} - \frac{1}{2}\log(-\lambda)\right)^{-\ell}\right\},$$

which is equivalent to our desired result. □

For decomposable L, we have the following.

Proposition 6.3 *Let $\Lambda \subset \mathbb{C}$ be any sector (solid angle) not intersecting the positive real axis. Then for an arbitrary decomposable Lagrangian L, as $|\lambda| \to \infty$ with $\lambda \in \Lambda$ we have:*

$$\operatorname{Tr}(\mathcal{L}_L - \lambda)^{-1} \sim \sum_{k=1}^{\infty} a_k(-\lambda)^{-\frac{k}{2}} - \frac{d}{d\lambda}\left\{\sum_{\xi\in\mathcal{P}} c_{\xi}\,(-\lambda)^{-\xi}\right\} + \left\{(-\lambda)^{-1}\sum_{k=1}^{\infty}\frac{2^{k-1}\beta_k}{(\log(-\lambda))^k}\right\},$$

where the a_k coefficients are independent of L, the c_ξ's are the coefficients in (2.12) *and the β_k's are the coefficients in* (2.10).

Proof Since $L = L_0 \oplus L_1$ is decomposable, it follows that

$$(\mathcal{L}_L - \lambda)^{-1} = (\mathcal{L}_{L_0} - \lambda)^{-1} + (\mathcal{L}_{L_1} - \lambda)^{-1}$$

where $\mathcal{L}_{L_0}$ is the operator $\mathcal{L}_L$ restricted to the $-\frac{1}{4}$ eigenspaces of A and $\mathcal{L}_{L_1}$ is the operator $\mathcal{L}_L$ restricted to the eigenspaces of A in $(-\frac{1}{4}, \frac{3}{4})$. For the operator $\mathcal{L}_{L_1}$, recall (5.12),

$$\frac{d}{dx} \log F(ix) \sim \sum_{k=0}^{\infty} b_k x^{-k} + \sum -2\xi c_\xi\, x^{-2\xi-1}.$$

Combining this with Theorem 6.1, we have:

$$\mathrm{Tr}(\mathcal{L}_{L_1} - \lambda)^{-1} \sim \sum_{k=1}^{\infty} d_k (-\lambda)^{-\frac{k}{2}} - \frac{d}{d\lambda} \Big\{ \sum c_\xi\, (-\lambda)^{-\xi} \Big\}$$

where the d_k coefficients are independent of L_1 and the c_ξ's are the coefficients in (2.12) or (5.11). Let $F_0(\mu)$ denote the function $F(\mu)$ in Proposition 4.2 in the case that $q_1 = 0$, $\mathcal{A} = \mathcal{A}_0$, and $\mathcal{B} = \mathcal{B}_0$ where $(\mathcal{A}_0\ \mathcal{B}_0)$ defines L_0. Then just as in the proof of Proposition 6.2, in conjunction with Proposition 4.4 (see (4.15)):

$$\frac{d}{dx} \log F_0(ix) \sim \sum_{k=1}^{\infty} \frac{\beta_k}{x(\log x)^k} + \sum_{k=0}^{\infty} e_k x^{-k},$$

where the β_k's are the coefficients in (2.10), using again Theorem 6.1, we obtain

$$\mathrm{Tr}(\mathcal{L}_{L_0} - \lambda)^{-1} \sim \sum_{k=1}^{\infty} f_k\, (-\lambda)^{-\frac{k}{2}} + \left\{ (-\lambda)^{-1} \sum_{k=1}^{\infty} \frac{2^{k-1}\beta_k}{(\log(-\lambda))^k} \right\}.$$

Combining this with $\mathrm{Tr}(\mathcal{L}_{L_1} - \lambda)^{-1}$ analyzed just before completes our proof. □

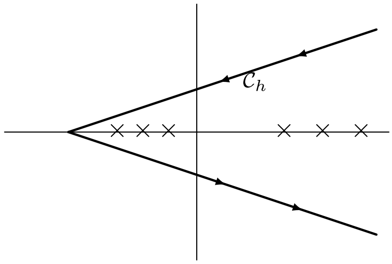
6.2 The Heat Kernel

Now we consider the asymptotics of the trace of $e^{-t\mathcal{L}_L}$ as $t \to 0$. For this, we use

$$\mathrm{Tr}(e^{-t\mathcal{L}_L}) = \frac{i}{2\pi} \int_{\mathcal{C}_h} e^{-t\lambda}\, \mathrm{Tr}(\mathcal{L}_L - \lambda)^{-1} d\lambda \tag{6.1}$$

where $\mathcal{C}_h$ is a counter-clockwise contour in the plane surrounding eigenvalues of $\mathcal{L}_L$; see Fig. 4. Then the small-time asymptotics of the heat trace is determined by the large-spectral parameter asymptotics of $\mathrm{Tr}(\mathcal{L}_L - \lambda)^{-1}$ as we will see in the following proposition.

Fig. 4 The contour $\mathcal{C}_h$



Proposition 6.4 *For an arbitrary Lagrangian subspace $L \subset \mathbb{C}^{2q}$, as $t \to 0$ we have*:

$$\mathrm{Tr}(e^{-t\mathcal{L}_L}) \sim \sum_{k=0}^{\infty} \widetilde{a}_k t^{\frac{-1+k}{2}} + \sum_{k=0}^{\infty} \widetilde{b}_k (\log t)^{-1-k}$$

$$+ \sum_{\xi \in \mathcal{P}} \sum_{k=0}^{|p_\xi|+1} \widetilde{c}_{\xi k} t^{\xi} (\log t)^k + \sum_{\xi \in \mathcal{L}} \sum_{k=0}^{\infty} \widetilde{d}_{\xi k} t^{\xi} (\log t)^{-\ell_\xi - k},$$

with $\widetilde{c}_{10} = 0$ *and* $\widetilde{c}_{\xi(|p_\xi|+1)} = 0$ *for* $\xi \notin \mathbb{N}_0$.

Proof By Proposition 6.2, we have:

$$\mathrm{Tr}(\mathcal{L}_L - \lambda)^{-1} \sim \sum_{k=1}^{\infty} a_k (-\lambda)^{-\frac{k}{2}} + \frac{q_0 - j_0}{(-\lambda)(\log(-\lambda) - 2\widetilde{\gamma})}$$

$$- \frac{d}{d\lambda}\left\{\sum 2^{\ell} c_{\ell\xi} (-\lambda)^{-\xi} (2\widetilde{\gamma} - \log(-\lambda))^{-\ell}\right\} \tag{6.2}$$

as $|\lambda| \to \infty$ with λ in a sector not intersecting the positive real axis. We use (6.1) for each term on the right hand side. For the first term, making the change of variables $\lambda \mapsto t^{-1}\lambda$,

$$\int_{\mathcal{C}_h} e^{-t\lambda} (-\lambda)^{-\frac{k}{2}} d\lambda = t^{\frac{k}{2}-1} \int_{t\mathcal{C}_h} e^{-\lambda} (-\lambda)^{-\frac{k}{2}} d\lambda.$$

The integral part depends on t via $t\mathcal{C}_h$, but is smooth at $t = 0$. Hence, the first part $\sum_{k=1}^{\infty} a_k (-\lambda)^{-\frac{k}{2}}$ contributes

$$\sum_{k=0}^{\infty} a'_k t^{\frac{-1+k}{2}}. \tag{6.3}$$

For the third term on the right-hand side of (6.2), using integration by parts, we have:

$$\int_{\mathcal{C}_h} e^{-t\lambda} \frac{d}{d\lambda} \{(-\lambda)^{-\xi} (2\widetilde{\gamma} - \log(-\lambda))^{-\ell}\} d\lambda$$

$$= t \int_{\mathcal{C}_h} e^{-t\lambda} \{(-\lambda)^{-\xi} (2\widetilde{\gamma} - \log(-\lambda))^{-\ell}\} d\lambda.$$

Deforming $\mathcal{C}_h$ to the real line, we find

$$
\begin{aligned}
& t \int_{\mathcal{C}_h} e^{-t\lambda}\{(-\lambda)^{-\xi}(2\widetilde{\gamma} - \log(-\lambda))^{-\ell}\}d\lambda \\
&= t\bigg(\int_{\infty}^{1} e^{-tx}(-(x+i0))^{-\xi}(2\widetilde{\gamma} - \log(-(x+i0)))^{-\ell}dx \\
&\quad + \int_1^{\infty} e^{-tx}(-(x-i0))^{-\xi}(2\widetilde{\gamma} - \log(-(x-i0)))^{-\ell}dx + h(t)\bigg) \\
&= t\bigg(e^{-i\pi\xi}\int_1^{\infty} e^{-tx}x^{-\xi}(2\widetilde{\gamma} - \log x - i\pi)^{-\ell}dx \\
&\quad - e^{i\pi\xi}\int_1^{\infty} e^{-tx}x^{-\xi}(2\widetilde{\gamma} - \log x + i\pi)^{-\ell}dx + h(t)\bigg),
\end{aligned}
$$

where $h(t)$ is a smooth function at $t=0$. Since for any complex number z we have $i(z-\bar{z}) = -2\Im z$, we see that modulo a term that is a smooth function of t at $t=0$,

$$
\frac{i}{2\pi}\int_{\mathcal{C}_h} e^{-t\lambda}\frac{d}{d\lambda}\{(-\lambda)^{-\xi}(2\widetilde{\gamma} - \log(-\lambda))^{-\ell}\}d\lambda = -\frac{t}{\pi}\Im\ell(t)
$$

where

$$
\ell(t) = e^{-i\pi\xi}\int_1^{\infty} e^{-tx}x^{-\xi}(2\widetilde{\gamma} - \log x - i\pi)^{-\ell}dx; \tag{6.4}
$$

we shall compute the asymptotics of $\ell(t)$ as $t \to 0$. To do so, let $j \geq \xi > j-1$, $j \in \mathbb{N}_0$; observe that the j-th derivative $\ell^{(j)}(t)$ of $\ell(t)$ is given by

$$
\ell^{(j)}(t) = e^{-i\pi\xi}(-1)^j\int_1^{\infty} e^{-tx}x^{j-\xi}(2\widetilde{\gamma} - \log x - i\pi)^{-\ell}dx.
$$

Note that $x^{j-\xi}\cdot(2\widetilde{\gamma} - \log x - i\pi)^{-\ell}$ is integrable near $x=0$, so we can write

$$
\ell^{(j)}(t) = e^{-i\pi\xi}(-1)^j(f(t)+g(t))
$$

with

$$
f(t) := \int_0^{\infty} e^{-tx}\, x^{j-\xi}(2\widetilde{\gamma} - \log x - i\pi)^{-\ell}dx
$$

and

$$
g(t) = -\int_0^1 e^{-tx}x^{j-\xi}(2\widetilde{\gamma} - \log x - i\pi)^{-\ell}dx.
$$

Note that $g(t)$ is smooth at $t=0$. We will now determine the asymptotics of $f(t)$ near $t=0$. To this end, we make the change of variables $x \mapsto t^{-1}x$:

$$
f(t) = t^{\xi-j-1}\int_0^{\infty} e^{-x}\, x^{j-\xi}(2\widetilde{\gamma} - \log x + \log t - i\pi)^{-\ell}dx.
$$

We need to consider two cases; $\ell \leq 0$ and $\ell > 0$. For $\ell \leq 0$ we use the binomial expansion to find

$$f(t) = t^{\xi-j-1} \sum_{k=0}^{|\ell|} \binom{|\ell|}{k} (\log t)^{|\ell|-k} \int_0^\infty e^{-x} x^{j-\xi} (2\widetilde{\gamma} - \log x - i\pi)^k dx$$
$$= t^{\xi-j-1} \sum_{k=0}^{|\ell|} c_{\xi,k,\ell} (\log t)^k,$$

with suitable coefficients $c_{\xi,k,\ell}$. For $\ell > 0$ we first write

$$f(t) = t^{\xi-j-1} (\log t)^{-\ell} \int_0^\infty e^{-x} x^{j-\xi} \left(1 - \frac{\log x + i\pi - 2\widetilde{\gamma}}{\log t}\right)^{-\ell} dx.$$

Since $(1-r)^{-1} = \sum_{k=0}^N r^k + r^{N+1}(1-r)^{-1}$ for any $N \in \mathbb{N}$, we see that for any $N \in \mathbb{N}$,

$$(1-r)^{-\ell} = \sum_{k=0}^{N-\ell+1} a_{k,\ell} r^k + r^{N-\ell+2} \sum_{k=0}^{\ell-1} b_{k,\ell} \frac{r^k}{(1-r)^{k+1}}.$$

For $f(t)$ this implies

$$f(t) = t^{\xi-j-1} (\log t)^{-\ell} \sum_{k=0}^{N-\ell+1} a_{k,\ell} (\log t)^{-k} \int_0^\infty e^{-x} x^{j-\xi} (\log x + i\pi - 2\widetilde{\gamma})^k dx$$
$$+ t^{\xi-j-1} (\log t)^{-\ell} (\log t)^{-(N-\ell+2)} \sum_{k=0}^{\ell-1} b_{k,\ell} (\log t)^{-k}$$
$$\times \int_0^\infty e^{-x} x^{j-\xi} \frac{(\log x + i\pi - 2\widetilde{\gamma})^{N-\ell+2+k}}{(1 - \frac{\log x + i\pi - 2\widetilde{\gamma}}{\log t})^{k+1}} dx.$$

The last integral is bounded as $t \to 0$; as $N \in \mathbb{N}$ was arbitrary, we conclude for $\ell > 0$

$$f(t) \sim t^{\xi-j-1} \sum_{k=0}^\infty \widetilde{a}_{\xi,k,\ell} (\log t)^{-k-\ell}.$$

In summary: for $\ell \leq 0$ we have shown

$$\ell^{(j)}(t) \sim e^{-i\pi\xi} \left(t^{\xi-j-1} \sum_{k=0}^{|\ell|} c_{\xi,k,\ell} (\log t)^k + \sum_{k=0}^\infty \gamma_{\xi,k,\ell} t^k \right), \tag{6.5}$$

while for $\ell > 0$, we found

$$\ell^{(j)}(t) \sim e^{-i\pi\xi} \left(t^{\xi-j-1} \sum_{k=0}^\infty \widetilde{a}_{\xi,k,\ell} (\log t)^{-k-\ell} + \sum_{k=0}^\infty \widetilde{\gamma}_{\xi,k,\ell} t^k \right). \tag{6.6}$$

In order to find the small-t asymptotics of $\ell(t)$ we need to integrate j times. Using [27], (2.722),

$$\int t^n (\log t)^m dt = \frac{t^{n+1}}{m+1} \sum_{k=0}^{m} (-1)^k (m+1) \cdot m \cdots (m-k+1) \frac{(\log t)^{m-k}}{(n+1)^{k+1}},$$

for $n \neq -1$, $m \neq -1$, and [27], (2.724), in the form:

$$\int \frac{t^n}{(\log t)^m} dt = \frac{t^{n+1}}{(n+1)(\log t)^m} + \frac{m}{n+1} \int \frac{t^n}{(\log t)^{m+1}} dt,$$

for $n \neq -1$, $m \neq 0$, in addition:

$$\int t^{-1} (\log t)^{-1} dt = \log |\log t|,$$

$$\int t^{-1} (\log t)^{-k-1} dt = -\frac{1}{k} (\log t)^{-k} \quad \text{for } k \neq 0,$$

we obtain for $\ell \leq 0$, $\xi \notin \mathbb{N}_0$, that:

$$\Im \ell(t) \sim t^{\xi-1} \sum_{k=0}^{|\ell|} c'_{\xi,k,\ell} (\log t)^k + \sum_{k=0}^{\infty} \gamma'_{\xi,k,\ell} t^k, \tag{6.7}$$

whereas for $\xi \in \mathbb{N}_0$ the first summation extends up to $|l| + 1$ and $c'_{1,0,\ell} = 0$.

For $l > 0$ the answer reads:

$$\Im \ell(t) \sim t^{\xi-1} \sum_{k=0}^{\infty} \widetilde{c}'_{\xi,k,\ell} (\log t)^{-k-\ell} + \sum_{k=0}^{\infty} \widetilde{\gamma}'_{\xi,k,\ell} t^k. \tag{6.8}$$

Contributions from the second term in (6.2) are found from (6.8) with $\xi = 1$ and $\ell = 1$,

$$\sum_{k=0}^{\infty} \widetilde{c}_k (\log t)^{-k-1} + \sum_{k=0}^{\infty} \widetilde{\gamma}_k t^k. \tag{6.9}$$

Combining (6.3), (6.7), (6.8) and (6.9) completes the proof. □

Using Proposition 6.3 and repeating the proof of Proposition 6.4, we have the following.

Proposition 6.5 *For an arbitrary decomposable Lagrangian L, the heat kernel $e^{-t\mathcal{L}_L}$ has the following trace expansion as $t \to 0$:*

$$\mathrm{Tr}(e^{-t\mathcal{L}_L}) \sim \sum_{k=0}^{\infty} \widetilde{a}_k t^{\frac{-1+k}{2}} + \sum_{\xi \in \mathcal{P}} \widetilde{c}_\xi t^\xi + \sum_{k=1}^{\infty} \widetilde{d}_k (\log t)^{-k}.$$

7 Proofs of the Main Theorems

We now prove our main results starting with the resolvent expansion.

7.1 The Resolvent Expansion—Theorems 2.5 and 2.6

We work under the assumptions of Theorem 2.5, so $\Lambda \subset \mathbb{C}$ denotes a sector not intersecting the positive real axis and $L \subset V$ denotes a given, but arbitrary, Lagrangian subspace of V. We cut the manifold M at the hypersurface $r = R$ in the collar $[0, \varepsilon)_r \times \Gamma$ with $0 < R < \varepsilon$, giving a decomposition:

$$M = X \cup Y,$$

where $X = [0, R]_r \times \Gamma$ and Y is a manifold with a collared neighborhood $[R, \varepsilon)_r \times \Gamma$ near its boundary, which we identify with Γ. Let Δ_Y denote the restriction of Δ to Y with the Dirichlet condition at $r = R$ and let $\Delta_{X,L}$ denote the restriction of Δ to X:

$$\Delta_{X,L} := -\partial_r^2 + \frac{1}{r^2} A_\Gamma$$

with domain the restriction of $\operatorname{dom}(\Delta_L)$ to X and with the Dirichlet condition at $r = R$. It is well-known that the Schwartz kernel of the resolvent $(\Delta_Y - \lambda)^{-1}(y, y')$, where $(y, y') \in Y \times Y$, is a *smooth* function of (y, y') and vanishes to infinite order as $|\lambda| \to \infty$ with $\lambda \in \Lambda$ as long as $y \neq y'$ (see for instance [63]). In the following lemma we prove a similar statement for the operator $\Delta_{X,L}$ on the generalized cone.

Lemma 7.1 *If $\varphi, \psi \in C^\infty(X)$ have disjoint supports, then for any differential operator P that vanishes near ∂M, the operator*

$$\varphi P (\Delta_{X,L} - \lambda)^{-1} \psi$$

is a trace-class operator that vanishes, with all derivatives, to infinite order (in the trace-class norm) as $|\lambda| \to \infty$ with $\lambda \in \Lambda$.

Proof If we prove this theorem for $\overline{\psi}(\Delta_{X,L} - \overline{\lambda})^{-1} P^* \overline{\varphi}$, then taking adjoints we get our theorem. Hence, we just have to prove the corresponding statement for $\varphi(\Delta_{X,L} - \lambda)^{-1} P \psi$, where $P\psi$ is the operator $f \mapsto P(\psi f)$. We prove this lemma using the heat kernel $e^{-t\Delta_{X,L}}$, whose structure is found in [54]. To this end, observe that:

$$(\Delta_{X,L} - \lambda)^{-1} = \int_0^1 e^{t\lambda} e^{-t\Delta_{X,L}}\, dt + e^\lambda (\Delta_{X,L} - \lambda)^{-1} e^{-\Delta_{X,L}}.$$

Then

$$\varphi(\Delta_{X,L} - \lambda)^{-1} P \psi = \int_0^1 e^{t\lambda} \varphi e^{-t\Delta_{X,L}} P \psi\, dt + e^\lambda \varphi (\Delta_{X,L} - \lambda)^{-1} e^{-\Delta_{X,L}} P \psi. \quad (7.1)$$

Assume for the moment that $\Lambda \subset \mathbb{C}$ is contained entirely in the left-half plane (so that $\Re\lambda \to -\infty$ as $|\lambda| \to \infty$ with $\lambda \in \Lambda$). Now, the operator $e^{-\Delta_{X,L}} P$ is of trace-class and

$(\Delta_{X,L} - \lambda)^{-1}$ is a bounded operator which decays like $|\lambda|^{-1}$ as $|\lambda| \to \infty$ with $\lambda \in \Lambda$. Therefore, since the trace-class operators form an ideal within the bounded operators, the operator

$$(\Delta_{X,L} - \lambda)^{-1} e^{-\Delta_{X,L}} P$$

is of trace-class and it decays like $|\lambda|^{-1}$ as $|\lambda| \to \infty$ with $\lambda \in \Lambda$. Hence,

$$e^{\lambda}(\Delta_{X,L} - \lambda)^{-1} e^{-\Delta_{X,L}} P$$

decays exponentially, with all derivatives, in the trace-class operators as $|\lambda| \to \infty$ with $\lambda \in \Lambda$ (recall that $\Re\lambda \to -\infty$ as $|\lambda| \to \infty$ with $\lambda \in \Lambda$). Therefore, the second operator in (7.1) decays exponentially in the trace-class operators as $|\lambda| \to \infty$ with $\lambda \in \Lambda$. By the main theorem of [54] (see also Theorem 4.1 of loc. cit.), since the supports of φ and $P\psi$ are disjoint, it follows that the operator

$$\varphi e^{-t\Delta_{X,L}} P\psi$$

is a trace-class operator that vanishes to infinite order at $t = 0$ (within the trace-class operators). Therefore, the operator $\int_0^1 e^{t\lambda} \varphi e^{-t\Delta_{X,L}} P\psi\, dt$ in (7.1) decays exponentially, with all derivatives, in the trace-class operators as $|\lambda| \to \infty$ with $\lambda \in \Lambda$.

Summarizing: We have proved our theorem when $\Lambda \subset \mathbb{C}$ is contained entirely in the left-half plane (so that $\Re\lambda \to -\infty$ as $|\lambda| \to \infty$ with $\lambda \in \Lambda$). Our proof is finished once we finish the cases when $\Lambda \subset \mathbb{C}$ is contained entirely in the upper-half plane and lower-half plane; for concreteness, let us focus on the upper-half plane. Then we can fix a complex number $a \in \mathbb{C}$ with *positive* real part (and positive imaginary part) such that $a \cdot \Lambda \subset \mathbb{C}$ is entirely contained in the left-half plane. Then one can construct the heat kernel $e^{-ta\Delta_{X,L}}$ (cf. [51, p. 282–284]) which has the same trace-class properties as $e^{-t\Delta_{X,L}}$ as described in [54, Th. 4.1]. Now we proceed as above: Just as we wrote (7.1), one can check that:

$$\varphi(\Delta_{X,L} - \lambda)^{-1} P\psi = a\int_0^1 e^{t\lambda} \varphi e^{-ta\Delta_{X,L}} P\psi dt + e^{a\lambda} \varphi(\Delta_{X,L} - \lambda)^{-1} e^{-a\Delta_{X,L}} P\psi. \tag{7.2}$$

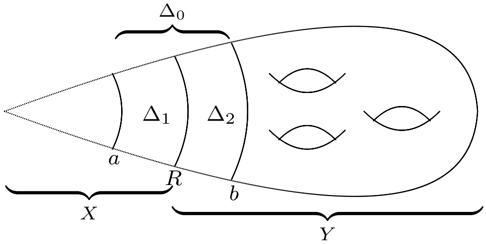
By the choice of a, note that as $|\lambda| \to \infty$ with $\lambda \in \Lambda$, we have $\Re(a\lambda) \to -\infty$ as $|\lambda| \to \infty$ with $\lambda \in \Lambda$. Therefore, analyzing (7.2) by repeating the argument we used in the previous paragraph to analyze (7.1) proves our lemma in the case when $\Lambda \subset \mathbb{C}$ is contained entirely in the upper-half plane. □

Let us *fix* $0 < a < R$ and $R < b < \varepsilon$, and define:

$$M_0 := [a, b] \times \Gamma, \qquad M_1 := [a, R] \times \Gamma, \qquad M_2 := [R, b] \times \Gamma.$$

For $j = 0, 1, 2$, let Δ_j denote the Laplacian on M_j with the Dirichlet boundary condition at the boundaries of M_j; see Fig. 5. The importance of the operators Δ_0, Δ_1, Δ_2 is that they are *smooth* (not singular) Laplace-type operators on compact manifolds with boundary with local boundary conditions, the properties of which are completely understood [62, 63]. The idea to prove Theorem 2.5 is to compare the resolvents on M, X, and Y to those on M_0, M_1, and M_2.

Fig. 5 The maps $\Delta_0, \Delta_1, \Delta_2$



Lemma 7.2 *The differences of resolvents*

$$\begin{aligned}\mathcal{S}(\lambda) := (\Delta_L - \lambda)^{-1} - (\Delta_{X,L} - \lambda)^{-1} - (\Delta_Y - \lambda)^{-1} \\ - ((\Delta_0 - \lambda)^{-1} - (\Delta_1 - \lambda)^{-1} - (\Delta_2 - \lambda)^{-1})\end{aligned}$$

is trace-class and vanishes, with all derivatives, to infinite order (in the trace-class norm) as $|\lambda| \to \infty$ *with* $\lambda \in \Lambda$.

Proof Let $\varrho(r) \in C^\infty(\mathbb{R})$ be a non-decreasing function such that $\varrho(r) = 0$ for $r \leq 1/4$ and $\varrho(r) = 1$ for $r \geq 3/4$. For real numbers $\alpha < \beta$, we define $\varrho_{\alpha,\beta}(r) := \varrho(\frac{r-\alpha}{\beta-\alpha})$. The main properties of $\varrho_{\alpha,\beta}$ we will use below are that $\varrho_{\alpha,\beta}(r) = 0$ on a neighborhood of $\{r \leq \alpha\}$ and $\varrho_{\alpha,\beta}(r) = 1$ on a neighborhood of $\{r \geq \beta\}$. Let us choose real numbers a_1, a_2, b_1, b_2 such that:

$$a < a_1 < a_2 < R < b_1 < b_2 < b.$$

We define

$$\begin{aligned}&\psi_1(r) := 1 - \varrho_{a_1,a_2}(r), \qquad \psi_2(r) := \varrho_{b_1,b_2}(r), \\ &\psi_0(r) := 1 - \psi_1(r) - \psi_2(r),\end{aligned}$$

and

$$\begin{aligned}&\varphi_1(r) := 1 - \varrho_{a_2,R}(r), \qquad \varphi_2(r) := \varrho_{R,b_1}(r), \\ &\varphi_0(r) := 1 - \varrho_{a,a_1}(r) - \varrho_{b_2,b}(r).\end{aligned}$$

The functions $\{\psi_i\}$, $\{\varphi_i\}$ extend either by 0 or 1 to define smooth functions on all of M and $\{\psi_i\}$ forms a partition of unity of M such that $\varphi_i = 1$ on $\mathrm{supp}(\psi_i)$. Now to prove this lemma, we first claim that each of the following equalities holds modulo a trace-class operator vanishing to infinite order as $|\lambda| \to \infty$ with $\lambda \in \Lambda$:

$$\begin{aligned}(\Delta_L - \lambda)^{-1} &= \varphi_1(\Delta_{X,L} - \lambda)^{-1}\psi_1 + \varphi_0(\Delta_0 - \lambda)^{-1}\psi_0 + \varphi_2(\Delta_Y - \lambda)^{-1}\psi_2, \\ (\Delta_{X,L} - \lambda)^{-1} &= \varphi_1(\Delta_{X,L} - \lambda)^{-1}\psi_1 + \varphi_0(\Delta_1 - \lambda)^{-1}\psi_0, \qquad (7.3) \\ (\Delta_Y - \lambda)^{-1} &= \varphi_0(\Delta_2 - \lambda)^{-1}\psi_0 + \varphi_2(\Delta_Y - \lambda)^{-1}\psi_2.\end{aligned}$$

For instance, let us verify the first claim in (7.3); the other claims are verified using a similar argument. Define:

$$Q(\lambda) := \varphi_1(\Delta_{X,L} - \lambda)^{-1}\psi_1 + \varphi_0(\Delta_0 - \lambda)^{-1}\psi_0 + \varphi_2(\Delta_Y - \lambda)^{-1}\psi_2.$$

Then observe that $(\Delta_L - \lambda)Q(\lambda) = \mathrm{Id} + K(\lambda)$ where

$$\begin{aligned}K(\lambda) &= [(\Delta_L - \lambda), \varphi_1](\Delta_{X,L} - \lambda)^{-1}\psi_1\\ &\quad + [(\Delta_L - \lambda), \varphi_0](\Delta_0 - \lambda)^{-1}\psi_0 + [(\Delta_L - \lambda), \varphi_2](\Delta_Y - \lambda)^{-1}\psi_2,\end{aligned}$$

where [,] denotes the "commutator". Now, because the supports of

$$[(\Delta_L - \lambda), \varphi_i] = -[\partial_r^2, \varphi_i] = -(\varphi_i'' + 2\varphi' \, \partial_r)$$

and ψ_i are disjoint, it follows that each of the three operators making up $K(\lambda)$ is trace-class and vanishes to infinite order (in the trace-class norm) as $|\lambda| \to \infty$ with $\lambda \in \Lambda$; indeed, this statement for $[(\Delta_L - \lambda), \varphi_1](\Delta_{X,L} - \lambda)^{-1}\psi_1$ follows from Lemma 7.1 and the statements for $[(\Delta_L - \lambda), \varphi_0](\Delta_0 - \lambda)^{-1}\psi_0$ and $[(\Delta_L - \lambda), \varphi_2](\Delta_Y - \lambda)^{-1}\psi_2$ are well known (see, e.g., [63]). Therefore, $K(\lambda)$ is trace-class and vanishes to infinite order (in the trace-class norm) as $|\lambda| \to \infty$ with $\lambda \in \Lambda$. Now applying $(\Delta_L - \lambda)^{-1}$ to both sides of $(\Delta_L - \lambda)Q(\lambda) = \mathrm{Id} + K(\lambda)$, we obtain:

$$(\Delta_L - \lambda)^{-1} = Q(\lambda) - (\Delta_L - \lambda)^{-1}K(\lambda),$$

which establishes our claim for the first equality in (7.3). A similar argument works to prove that the other equalities in (7.3) hold modulo trace-class with infinite decay (with all derivatives as $|\lambda| \to \infty$ with $\lambda \in \Lambda$). From (7.3), it follows that modulo trace-class with infinite decay,

$$\begin{aligned}&(\Delta_L - \lambda)^{-1} - (\Delta_{X,L} - \lambda)^{-1} - (\Delta_Y - \lambda)^{-1}\\ &\quad = \varphi_0(\Delta_0 - \lambda)^{-1}\psi_0 - \varphi_0(\Delta_1 - \lambda)^{-1}\psi_0 - \varphi_0(\Delta_2 - \lambda)^{-1}\psi_0. \qquad (7.4)\end{aligned}$$

On the other hand, very similar arguments used to establish (7.3) show that modulo trace-class with infinite decay:

$$\begin{aligned}(\Delta_0 - \lambda)^{-1} &= \varphi_1(\Delta_1 - \lambda)^{-1}\psi_1 + \varphi_0(\Delta_0 - \lambda)^{-1}\psi_0 + \varphi_2(\Delta_2 - \lambda)^{-1}\psi_2,\\ (\Delta_1 - \lambda)^{-1} &= \varphi_1(\Delta_1 - \lambda)^{-1}\psi_1 + \varphi_0(\Delta_1 - \lambda)^{-1}\psi_0,\\ (\Delta_2 - \lambda)^{-1} &= \varphi_0(\Delta_2 - \lambda)^{-1}\psi_0 + \varphi_2(\Delta_2 - \lambda)^{-1}\psi_2.\end{aligned}$$

Combining these identities we can write, modulo trace-class with infinite decay,

$$\begin{aligned}&(\Delta_0 - \lambda)^{-1} - (\Delta_1 - \lambda)^{-1} - (\Delta_2 - \lambda)^{-1}\\ &\quad = \varphi_0(\Delta_0 - \lambda)^{-1}\psi_0 - \varphi_0(\Delta_1 - \lambda)^{-1}\psi_0 - \varphi_0(\Delta_2 - \lambda)^{-1}\psi_0.\end{aligned}$$

Comparing this with (7.4) completes the proof of our lemma. □

Using our standard notation, let $\{\lambda_\ell\}$ denote the set of all eigenvalues of A_Γ and let E_ℓ denote the span of the λ_ℓ-th eigenvector. Let Π and $\Pi^\perp$ denote, respectively, the orthogonal projections of $L^2(\Gamma, E_\Gamma)$ onto $W := \bigoplus_{-\frac{1}{4} \le \lambda_\ell < \frac{3}{4}} E_\ell$ and $W^\perp$. Using the isometry between

$$L^2([0, R] \times \Gamma, E) \cong L^2([0, R], L^2(\Gamma, E_\Gamma))$$

where $E_\Gamma := E|_\Gamma$, we obtain the corresponding projections on $L^2([0, R] \times \Gamma, E)$, which we denote by the same notations Π and $\Pi^\perp$. Let $\mathcal{L}_L$ denote the model operator introduced in Sect. 4 (specifically, Sect. 4.1), and define

$$\Delta'_X := -\partial_r^2 + \frac{1}{r^2} A'_\Gamma, \quad \text{where } A'_\Gamma := \begin{cases} \frac{3}{4} & \text{over } W, \\ A_\Gamma & \text{over } W^\perp. \end{cases}$$

Proposition 7.3 *We have*:

$$\begin{aligned}(\Delta_L - \lambda)^{-1} &= \Pi(\mathcal{L}_L - \lambda)^{-1}\Pi + \Pi^\perp(\Delta'_X - \lambda)^{-1}\Pi^\perp + (\Delta_Y - \lambda)^{-1} \\ &\quad + (\Delta_0 - \lambda)^{-1} - (\Delta_1 - \lambda)^{-1} - (\Delta_2 - \lambda)^{-1} + \mathcal{S}(\lambda)\end{aligned}$$

where $\mathcal{S}(\lambda)$ is trace-class and vanishes, with all derivatives, to infinite order as $|\lambda| \to \infty$ with $\lambda \in \Lambda$.

Proof Observe that:

$$(\Delta_{X,L} - \lambda)^{-1} = \Pi(\Delta_{X,L} - \lambda)^{-1}\Pi + \Pi^\perp(\Delta_{X,L} - \lambda)^{-1}\Pi^\perp,$$

since $\Delta_{X,L}$ preserves W and $W^\perp$, and

$$\Pi(\Delta_{X,L} - \lambda)^{-1}\Pi = \Pi(\mathcal{L}_L - \lambda)^{-1}\Pi.$$

Also, observe that:

$$\Pi^\perp(\Delta_{X,L} - \lambda)^{-1}\Pi^\perp = \Pi^\perp(\Delta'_X - \lambda)^{-1}\Pi^\perp.$$

Hence,

$$(\Delta_{X,L} - \lambda)^{-1} = \Pi(\mathcal{L}_L - \lambda)^{-1}\Pi + \Pi^\perp(\Delta'_X - \lambda)^{-1}\Pi^\perp.$$

Now solving for $(\Delta_L - \lambda)^{-1}$ in Lemma 7.2, we obtain:

$$\begin{aligned}(\Delta_L - \lambda)^{-1} &= (\Delta_{X,L} - \lambda)^{-1} + (\Delta_Y - \lambda)^{-1} \\ &\quad + (\Delta_0 - \lambda)^{-1} - (\Delta_1 - \lambda)^{-1} - (\Delta_2 - \lambda)^{-1} + \mathcal{S}(\lambda) \\ &= \Pi(\mathcal{L}_L - \lambda)^{-1}\Pi + \Pi^\perp(\Delta'_X - \lambda)^{-1}\Pi^\perp + (\Delta_Y - \lambda)^{-1} \\ &\quad + (\Delta_0 - \lambda)^{-1} - (\Delta_1 - \lambda)^{-1} - (\Delta_2 - \lambda)^{-1} + \mathcal{S}(\lambda)\end{aligned}$$

where $\mathcal{S}(\lambda)$ is trace-class and vanishes, with all derivatives, to infinite order as $|\lambda| \to \infty$ with $\lambda \in \Lambda$. This completes our proof. □

We can now prove Theorem 2.5. Let $N \geq \frac{n}{2}$ with $n = \dim M$. Then taking N derivatives of both sides of the preceding equality we see that

$$\begin{aligned}(\Delta_L - \lambda)^{-N-1} &= \Pi(\mathcal{L}_L - \lambda)^{-N-1}\Pi + \Pi^\perp(\Delta'_X - \lambda)^{-N-1}\Pi^\perp \\ &\quad + (\Delta_Y - \lambda)^{-N-1} + (\Delta_0 - \lambda)^{-N-1} - (\Delta_1 - \lambda)^{-N-1} \\ &\quad - (\Delta_2 - \lambda)^{-N-1} + \mathcal{S}^{(N)}(\lambda), \end{aligned} \tag{7.5}$$

where $\mathcal{S}^{(N)}(\lambda) = \frac{d^N}{d\lambda^N}\mathcal{S}(\lambda)$. We now analyze each term on the right. First, taking N derivatives in the asymptotic expression of Proposition 6.2, we know that as $|\lambda| \to \infty$ with $\lambda \in \Lambda$ we have:

$$\operatorname{Tr}(\mathcal{L}_L - \lambda)^{-N-1} \sim \sum_{k=0}^{\infty} b_k(-\lambda)^{\frac{1-k}{2}-N-1} + \frac{1}{N!}\frac{d^N}{d\lambda^N}\left\{\frac{q_0 - j_0}{(-\lambda)(\log(-\lambda) - 2\widetilde{\gamma})}\right\}$$
$$- \frac{1}{N!}\frac{d^{N+1}}{d\lambda^{N+1}}\left\{\sum 2^\ell c_{\ell\xi}\,(-\lambda)^{-\xi}(2\widetilde{\gamma} - \log(-\lambda))^{-\ell}\right\}. \quad (7.6)$$

It follows from [48] that the operator $\Pi^\perp(\Delta'_X - \lambda)^{-N-1}\Pi^\perp$ is trace-class and

$$\operatorname{Tr}(\Pi^\perp(\Delta'_X - \lambda)^{-N-1}\Pi^\perp) \sim \sum_{k=0}^{\infty} c_k(-\lambda)^{\frac{n-k}{2}-N-1} + b(-\lambda)^{-N-1}\log(-\lambda); \quad (7.7)$$

in principle (see [51, Chap. 7]), one can derive this resolvent expansion with a lot of work from the corresponding heat kernel expansion [4, 5, 11, 12]. From the work of Seeley [63], we also know that each of $(\Delta_Z - \lambda)^{-N-1}$, where $Z = Y, 0, 1, 2$, is trace class, and

$$\operatorname{Tr}((\Delta_Z - \lambda)^{-N-1}) \sim \sum_{k=0}^{\infty} c_{Z,k}(-\lambda)^{\frac{n-k}{2}-N-1}. \quad (7.8)$$

Finally, we know that $\mathcal{S}^{(N)}(\lambda)$ is trace-class and vanishes, with all derivatives, to infinite order as $|\lambda| \to \infty$ with $\lambda \in \Lambda$. In conclusion, in view of the expression (7.5) and our discussions around (7.6), (7.7) and (7.8), we see that $(\Delta_L - \lambda)^{-N-1}$ is trace-class, and

$$\operatorname{Tr}(\Delta_L - \lambda)^{-N-1} \sim \sum_{k=0}^{\infty} a_k(-\lambda)^{\frac{n-k}{2}-N-1} + b(-\lambda)^{-N-1}\log(-\lambda)$$
$$+ \frac{1}{N!}\frac{d^N}{d\lambda^N}\left\{\frac{q_0 - j_0}{(-\lambda)(\log(-\lambda) - 2\widetilde{\gamma})}\right\}$$
$$- \frac{1}{N!}\frac{d^{N+1}}{d\lambda^{N+1}}\left\{\sum 2^\ell c_{\ell\xi}\,(-\lambda)^{-\xi}(2\widetilde{\gamma} - \log(-\lambda))^{-\ell}\right\}.$$

This completes the proof of Theorem 2.5. Theorem 2.6 is established by replacing the trace expansion (7.6) with the trace expansion found in Proposition 6.3.

7.2 Proofs of Theorems 2.1, 2.3, 2.4

We now prove the ζ-function theorem. We start off with Proposition 7.3, which states that:

$$(\Delta_L - \lambda)^{-1} = \Pi(\mathcal{L}_L - \lambda)^{-1}\Pi + \Pi^\perp(\Delta'_X - \lambda)^{-1}\Pi^\perp + (\Delta_Y - \lambda)^{-1}$$
$$+ (\Delta_0 - \lambda)^{-1} - (\Delta_1 - \lambda)^{-1} - (\Delta_2 - \lambda)^{-1} + \mathcal{S}(\lambda)$$

where $\mathcal{S}(\lambda)$ is trace-class and vanishes, with all derivatives, to infinite order as $|\lambda| \to \infty$ with $\lambda \in \Lambda$. Therefore, by the definition of the ζ-function:

$$\zeta(s, \Delta_L) := \mathrm{Tr}(\Delta_L^{-s} P_0^{\perp}),$$
$$\mathrm{Tr}(\Delta_L^{-s} P_0^{\perp}) = \sum_{\lambda_\ell < 0} \lambda_\ell^{-s} + \int_{\Re\lambda=\delta} \lambda^{-s}(\Delta_L - \lambda)^{-1} d\lambda,$$

where P_0 is the orthogonal projection onto $\ker \Delta_L$ and $\delta > 0$ is any positive number sufficiently small so that the spectrum of Δ_L intersected with $(0, \delta]$ is empty, it follows that

$$\zeta(s, \Delta_L) \equiv \zeta(s, \mathcal{L}_L) + \zeta(s, \Delta'_X) + \zeta(s, \Delta_Y) + \zeta(s, \Delta_0) - \zeta(s, \Delta_1) - \zeta(s, \Delta_2) \quad (7.9)$$

modulo an entire function. The ζ-function $\zeta(s, \mathcal{L}_L)$ is studied thoroughly in Proposition 5.2. Also, by the standard relation between the asymptotics of the resolvent and the poles of the ζ-function (see e.g. [31]) it follows from the resolvent expansions (7.7) and (7.8) that $\zeta(s, \Delta'_X)$, $\zeta(s, \Delta_Y)$, $\zeta(s, \Delta_0)$, $\zeta(s, \Delta_1)$, and $\zeta(s, \Delta_2)$ have the "regular" poles at the "usual" locations $s = \frac{n-k}{2} \notin -\mathbb{N}_0$ for $k \in \mathbb{N}_0$ and, only for $\zeta(s, \Delta'_X)$, at $s = 0$ if $\dim \Gamma > 0$. These facts together with Proposition 5.2 prove Theorem 2.1. Note that Theorems 2.3 and 2.4 follow from applying Proposition 5.5 and Corollary 5.6 to $\zeta(s, \mathcal{L}_L)$ in (7.9).

7.3 Proof of Theorems 2.7 and 2.8

Finally, it remains to prove the heat expansion. As with the proof for the ζ-function, we start off with Proposition 7.3:

$$\begin{aligned}(\Delta_L - \lambda)^{-1} &= \Pi(\mathcal{L}_L - \lambda)^{-1}\Pi + \Pi^{\perp}(\Delta'_X - \lambda)^{-1}\Pi^{\perp} + (\Delta_Y - \lambda)^{-1} \\ &\quad + (\Delta_0 - \lambda)^{-1} - (\Delta_1 - \lambda)^{-1} - (\Delta_2 - \lambda)^{-1} + \mathcal{S}(\lambda)\end{aligned}$$

where $\mathcal{S}(\lambda)$ is trace-class and vanishes, with all derivatives, to infinite order as $|\lambda| \to \infty$ with $\lambda \in \Lambda$. Then by the definition of the heat operator:

$$e^{-t\Delta_L} := \frac{i}{2\pi} \int_{\mathcal{C}_h} e^{-t\lambda}(\Delta_L - \lambda)^{-1} d\lambda,$$

where $\mathcal{C}_h$ is a contour as in Fig. 4, it follows that

$$\begin{aligned}e^{-t\Delta_L} &= \Pi e^{-t\mathcal{L}_L}\Pi + \Pi^{\perp} e^{-t\Delta'_X}\Pi^{\perp} + e^{-t\Delta_Y} \\ &\quad + e^{-t\Delta_0} - e^{-t\Delta_1} - e^{-t\Delta_2} + \mathcal{T}(t)\end{aligned}$$

where $\mathcal{T}(t)$ is trace-class and smooth at $t = 0$. Hence,

$$\begin{aligned}\mathrm{Tr}(e^{-t\Delta_L}) &= \mathrm{Tr}(e^{-t\mathcal{L}_L}) + \mathrm{Tr}(e^{-t\Delta'_X}) + \mathrm{Tr}(e^{-t\Delta_Y}) \\ &\quad + \mathrm{Tr}(e^{-t\Delta_0}) - \mathrm{Tr}(e^{-t\Delta_1}) - \mathrm{Tr}(e^{-t\Delta_2})\end{aligned}$$

modulo a function that is smooth at $t=0$. The heat trace $\mathrm{Tr}(e^{-t\mathcal{L}_L})$ is studied thoroughly in Proposition 6.4 and for decomposable Lagrangians in Proposition 6.5. Also, by the standard relation between the asymptotics of the resolvent and the heat trace expansion (see e.g. [31] or Sect. 6.2) it follows from the resolvent expansions (7.7) and (7.8) that $\mathrm{Tr}(e^{-t\Delta'_X})$, $\mathrm{Tr}(e^{-t\Delta_Y})$, $\mathrm{Tr}(e^{-t\Delta_0})$, $\mathrm{Tr}(e^{-t\Delta_1})$ and $\mathrm{Tr}(e^{-t\Delta_2})$ have the "regular" expansion except that $\mathrm{Tr}(e^{-t\Delta'_X})$ may have a $\log t$ term if $\dim \Gamma > 0$. These facts together with Propositions 6.4 and 6.5 prove Theorems 2.7 and 2.8.

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Appendix A: The Contour-Integral Method on a Bounded Generalized Cone

The goal of the Appendix is to verify the validity of the contour integration method in the setup of a bounded generalized cone:

Proposition A.1 *Let L be any fixed Lagrangian and $\mathcal{L}_L$ the associated self-adjoint realization of the Laplacian $\mathcal{L}$. For $\mathcal{C}$ being the contour in Fig. 1 we have*

$$\zeta(s,\mathcal{L}_L)=\frac{1}{2\pi i}\int_{\mathcal{C}}\mu^{-2s}\frac{d}{d\mu}\log F(\mu)d\mu,\quad \Re s>\frac{1}{2}.$$

Remark A.2 The integral representation of the zeta-function above is formally an application of the Argument Principle. However the contour $\mathcal{C}$ is not closed, so the asymptotic behavior of the implicit eigenvalue function $F(z)$ needs to be analyzed.

Proof Fix an angle $\theta\in(0,\pi/2)$ and put $\Omega:=\{z\in\mathbb{C}||\arg(z)|\le\theta\}$. Then [1, 9.2.1,9.2.2] but also [27, 8.451] provide the standard asymptotic behavior of Bessel functions as $|z|\to\infty$, $z\in\Omega$. Inserting these asymptotics into the definition of $F(z)$ in Proposition 4.2 we obtain the following uniform expansion:

$$\begin{aligned}F(z)=&\prod_{l=1}^{q_1}\left\{2^{-\nu_l}\Gamma(1-\nu_l)z^{\nu_l-1/2}\sqrt{\frac{2}{\pi R}}\cos\left(zR+\frac{\nu_l\pi}{2}-\frac{\pi}{4}\right)\right\}\\&\times\left\{\sqrt{\frac{2}{\pi zR}}(\widetilde{\gamma}-\log z)\cos\left(zR-\frac{\pi}{4}\right)\right\}^{q_0}\cdot\det M(z).\end{aligned}$$

Here the matrix $M(z)$ is given as follows:

$$M(z)=\begin{pmatrix} & \mathcal{A} & & \mathcal{B}\\ b(z)\cdot \mathrm{Id}_{q_0} & 0 & c(z)\cdot\mathrm{Id}_{q_0} & 0\\ 0 & \mathrm{diag}[a_l^+(z)] & 0 & \mathrm{diag}[a_l^-(z)]\end{pmatrix},$$

where for $l=1,\dots,q_1$ we have

$$a_l^+(z)=2^{2\nu_l}\frac{\Gamma(1+\nu_l)}{\Gamma(1-\nu_l)}z^{-2\nu_l}\frac{\cos(zR-\frac{\nu_l\pi}{2}-\frac{\pi}{4})}{\cos(zR+\frac{\nu_l\pi}{2}-\frac{\pi}{4})}\cdot\left(1+\frac{f_l^+(z)}{\cos(zR-\frac{\nu_l\pi}{2}-\frac{\pi}{4})}\right),$$

$$a_l^-(z) = 1 + \frac{f_l^-(z)}{\cos(zR + \frac{\nu_l\pi}{2} - \frac{\pi}{4})}, \qquad b(z) = \frac{1}{\widetilde{\gamma} - \log z} \cdot \left(1 + \frac{f_b(z)}{\cos(zR - \frac{\pi}{4})}\right),$$

$$c(z) = 1 + \frac{f_c(z)}{\cos(zR - \frac{\pi}{4})},$$

and the functions $f_l^{\pm}(z)$ and $f_b(z)$ with their derivatives are of the asymptotics $e^{|\Im(zR)|}O(1/|z|)$ as $|z| \to \infty, z \in \Omega$. The function $f_c(z)$ and its derivative are of the asymptotics $e^{|\Im(zR)|}O(1/|\log z|)$ as $|z| \to \infty, z \in \Omega$. Similar calculations are provided in [65, Sect. 4.2].

Next put for the fixed angle $\theta \in (0, \pi/2)$ and any $a \in \mathbb{R}^+$:

$$\begin{aligned}
\delta(a) &:= \{z \in \mathbb{C} | \Re z = a, |\arg(z)| \leq \theta\}, \\
\rho(a) &:= \{z \in \mathbb{C} | |z| = a/\cos(\theta), |\arg(z)| \in [\theta, \pi/2]\}, \\
\gamma(a) &:= \delta(a) \cup \rho(a),
\end{aligned}$$

where the contour $\gamma(a)$ is oriented counter-clockwise. The logarithmic form of the asymptotics in Proposition 4.3 together with the symmetry of the implicit eigenvalue function $F(z)$, imply for $\Re s > 1/2$:

$$\int_{\rho(a_n)} z^{-2s} \frac{d}{dz} \log F(z) dz \overset{n\to\infty}{\longrightarrow} 0, \tag{A.1}$$

for any sequence $(a_n)_{n\in\mathbb{N}}$ of positive real numbers with $a_n \to \infty$ as $n \to \infty$. Thus it remains to find a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ which goes to infinity and further ensures that

$$\int_{\delta(a_n)} z^{-2s} \frac{d}{dz} \log F(z) dz \overset{n\to\infty}{\longrightarrow} 0, \tag{A.2}$$

where for each $n \in \mathbb{N}$ the integral is well-defined. In order to construct such a sequence, fix $a > 0$ subject to the following conditions

$$\cos\left(aR - \frac{\pi}{4}\right) \neq 0, \qquad \cos\left(aR \pm \frac{\nu_l\pi}{2} - \frac{\pi}{4}\right) \neq 0, \quad l = 1, \ldots, q_1. \tag{A.3}$$

Such a choice is always possible, due to discreteness of zeros of the meromorphic functions. Given such an $a > 0$, we define

$$\Delta(a) := \bigcup_{k\in\mathbb{N}} \delta\left(a + \frac{2\pi}{R}k\right).$$

Observe for any $\xi \in \mathbb{R}$ that $\cos(zR + \xi) = e^{|\Im(zR)|}O(1)$ as $|z| \to \infty, z \in \Delta(a)$, where the asymptotic term $O(1)$ is bounded with the bounds depending only on the sign of $\Im z$, $a > 0$ and $\xi \in \mathbb{R}$. Putting $\alpha = (\alpha_1, \ldots, \alpha_{q_1}) \in \{0, 1\}^{q_1}$, $q_1 = q - q_0$, we obtain for the asymptotic behavior of $\det M(z)$ as $|z| \to \infty, z \in \Delta(a)$:

$$\det M(z) = \sum_{j=0}^{q_0} \sum_{\alpha\in\{0,1\}^{q_1}} \sum_{\beta=0}^{q} \mathrm{const}(j, \alpha, \beta, L) \left[\frac{1}{\widetilde{\gamma} - \log z}\right]^j$$

$$\times \prod_{l=1}^{q_1}\left[z^{-2\nu_l}\frac{\cos(zR-\frac{\nu_l\pi}{2}-\frac{\pi}{4})}{\cos(zR+\frac{\nu_l\pi}{2}-\frac{\pi}{4})}\right]^{\alpha_l}\cdot\left[1+f_{j,\alpha,\beta}(z)\right],\ f_{j,\alpha,\beta}(z)$$
$$= O\left(\frac{1}{|\log z|}\right),$$

where the derivative $f'_{j,\alpha,\beta}(z)$ is of the asymptotics $O(1/|\log z|)$ as $|z|\to\infty, z\in\Delta(a)$.

Next we make the following auxiliary observation. Under the condition (A.3) on the choice of $a>0$, there exist constants $\mathfrak{C}_1>0$ and $\mathfrak{C}_2>0$, depending only on a, such that for $z\in\Delta(a)$ and for all $l=1,\dots,q_1$ we have:

$$\mathfrak{C}_1\le\left|\frac{\cos(zR-\frac{\nu_l\pi}{2}-\frac{\pi}{4})}{\cos(zR+\frac{\nu_l\pi}{2}-\frac{\pi}{4})}\right|\le\mathfrak{C}_2. \tag{A.4}$$

Further details are provided in [65, Sect. 4.2]. In particular the cosine terms in $\det M(z)$ are not relevant for its asymptotic behavior as $|z|\to\infty, z\in\Delta(a)$. Now let us consider the summands in $\det M(z)$ of slowest decrease as $|z|\to\infty, z\in\Delta(a)$:

$$\left[\frac{1}{\widetilde{\gamma}-\log z}\right]^{j_0}z^{-\alpha_0}\cdot\left\{\sum_{\beta=0}^{q}\sum_{\alpha\in I}\mathrm{const}(j_0,\alpha_0,\beta,L)\prod_{l=1}^{q_1}\left[\frac{\cos(zR-\frac{\nu_l\pi}{2}-\frac{\pi}{4})}{\cos(zR+\frac{\nu_l\pi}{2}-\frac{\pi}{4})}\right]^{\alpha_l}\right\}$$
$$=:\left[\frac{1}{\widetilde{\gamma}-\log z}\right]^{j_0}z^{-\alpha_0}g(z),$$
$$I:=\left\{\alpha\,\middle|\,\sum_{l=1}^{q_1}2\nu_l\alpha_l=\alpha_0\right\},$$

where the coefficients j_0,α_0 correspond to those in (2.2). By similar arguments as behind (A.4), see also [65, Sect. 4.2], we can choose $a>0$ sufficiently large, still subject to the condition (A.3), such that there exist constants $\mathfrak{C}'_1>0$ and $\mathfrak{C}'_2>0$, depending only on $a>0$ and for $z\in\Delta(a)$

$$\mathfrak{C}'_1\le|g(z)|\le\mathfrak{C}'_2. \tag{A.5}$$

Moreover (A.4) implies that $|g'(z)|$ is bounded above for $z\in\Delta(a)$. For $q=q_0$ we simply put $g(z)\equiv 1$ and the bounding statements on $g(z), g'(z)$ are then trivial. Using (A.5) we finally obtain for $\det M(z)$ as $|z|\to\infty, z\in\Delta(a)$

$$\det M(z)=\left[\frac{1}{\widetilde{\gamma}-\log z}\right]^{j_0}\frac{g(z)}{z^{\alpha_0}}\left(1+\frac{[\frac{1}{\widetilde{\gamma}-\log z}]^{j_0}\frac{g(z)}{z^{\alpha_0}}\cdot O(1/|\log(z)|)+\cdots}{[\frac{1}{\widetilde{\gamma}-\log z}]^{j_0}\frac{g(z)}{z^{\alpha_0}}+\cdots}\right)$$
$$=\left[\frac{1}{\widetilde{\gamma}-\log z}\right]^{j_0}z^{-\alpha_0}g(z)(1+f(z)),\qquad f(z)=O(1/|\log(z)|),$$

where the dots denote terms that decrease faster than the corresponding leading terms, as $|z|\to\infty, z\in\Delta(a)$. Using the asymptotics of $f_{j,\alpha,\beta}(z)$, the observation (A.4) and boundedness of $g(z), g'(z)$, we find that $f'(z)=O(1/|\log z|)$.

In total we have derived the asymptotic behavior of $F(z)$ as $|z| \to \infty, z \in \Delta(a)$:

$$F(z) = \prod_{l=1}^{q_1} \left\{ 2^{-\nu_l} \Gamma(1-\nu_l) z^{\nu_l - 1/2} \sqrt{\frac{2}{\pi R}} \cos\left(zR + \frac{\nu_l \pi}{2} - \frac{\pi}{4} \right) \right\}$$
$$\times \left\{ \sqrt{\frac{2}{\pi z R}} (\widetilde{\gamma} - \log z) \cos\left(zR - \frac{\pi}{4} \right) \right\}^{q_0} \left[\frac{1}{\widetilde{\gamma} - \log z} \right]^{j_0}$$
$$\times z^{-\alpha_0} g(z)(1 + f(z)),$$

where there exist positive constants $\mathfrak{C}_1', \mathfrak{C}_2', \mathfrak{C}''$, depending only on $a > 0$, such that for $z \in \Delta(a)$ we have $\mathfrak{C}_1' \le |g(z)| \le \mathfrak{C}_2'$, $|g'(z)| \le \mathfrak{C}''$ and $f(z) = O(1/|\log z|)$, $f'(z) = O(1/|\log z|)$ as $|z| \to \infty, z \in \Delta(a)$.

Note that for $N \in \mathbb{N}$ sufficiently large, the asymptotics above, together with the conditions (A.3) and (A.5), imply that $F(a + 2\pi k/R) \neq 0$ for all $k \in \mathbb{N}, k \ge N$. Put $a_n := a + 2\pi(N+n)/R, n \in \mathbb{N}$. Then we infer from the asymptotics of $F(z)$ above, that for $\Re s > 1/2$

$$\int_{\delta(a_n)} z^{-2s} \frac{d}{dz} \log F(z) dz \overset{n\to\infty}{\longrightarrow} 0, \tag{A.6}$$

where by construction for each $n \in \mathbb{N}$ we have $F(a_n) \neq 0$, and hence the integrals are well-defined. Together with (A.1) this finally proves the statement of the proposition by an application of the Argument Principle. □

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